

## Phd thesis topic)

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**Title :** Combinatorial convexity, its generalizations  
and applications to optimization

Localization : Nancy

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## Context

The general goal of this PhD thesis is to contribute to the development of *combinatorial convexity* and its *topological generalization*, as well as their applications to *optimization* (*convex or otherwise*).

The theorems of Carathéodory, Helly and Radon initiated a combinatorial theory of convexity, often dubbed **combinatorial convexity**, that investigates the intersection patterns of families of convex sets, with a particular attention to the systems formed by the convex hulls of subsets of a finite point set. Some landmarks in this line of research are the fractional Helly theorem of Kalai and the  $(p, q)$ -theorem of Alon and Kleitman. See for instance the textbooks [Mat02, Bár21] or the introductory lectures [BGJ<sup>+</sup>20, §5] (in french).

A classical extension of combinatorial convexity relaxes the convexity assumption into more topological conditions. One way to do this is to retain only some of the topological properties of convex sets (e.g. they form *good covers*); another way is to interpret the system of convex hulls of a finite point set as the image of some simplicial complex under the *linear* map with vertices mapped to those points, and broaden our attention to the class of *continuous* maps. Both ways naturally relates combinatorial convexity to classical results in algebraic topology (e.g. the *nerve* or the *Borsuk-Ulam* theorems). A recent development in this line of research is the use of *homological minors* to deal with **set systems of bounded topological complexity** [GHP21, GPP<sup>+</sup>17, Pat20].

**Convex optimization** is a natural application area for combinatorial convexity, as the latter allows to analyze, explain, unify or generalize various algorithms or problems. For instance, the pivoting principle of the simplex algorithm (for linear programming) and the Lemke-Howson algorithm (for linear complementarity problems) both stem from Carathéodory's theorem. Another example is the bounded size of bases in *linear programming* or *chance constrained optimization*, which can be traced back to Helly's theorem and gave rise to the class of LP-type problems. See the survey [DLGMM19] for these and more examples.

## Tasks

The more advanced results from combinatorial convexity, for instance the fractional Helly theorem and the  $(p, q)$ -theorem, have found so far little application in combinatorial optimization. A first task is to investigate such applications. One concrete question is the *k-adaptability* problem [SGW20], which asks for instance, given a parameterized linear problem, to determine some  $k$  solutions such that for any choice of parameter there is one solution that is feasible (or even as good as possible). Can combinatorial convexity guarantee that this is always possible, or help deciding whether it is? The topic of *chance constrained optimization* also offers many interesting questions. Raising these problems is likely to pinpoint *algorithmic* questions related

to combinatorial convexity; for instance, how efficiently can one decide if a given collection of convex sets satisfies the assumption of the  $(p, q)$ -theorem (among any  $p$ , some  $q$  have a point in common)?

The combination of homological minors and Ramsey theory established that a large part of combinatorial convexity generalizes to set systems of bounded topological complexity, that is families of sets in  $\mathbb{R}^d$  such that the intersection of any subfamily has bounded Betti numbers. These results typically give upper bounds on parameters (*e.g.* Helly or Radon number) that are galactic due to the use of Ramsey theory. Another task would be to revisit this body of work and give quantitatively meaningful bounds; this requires to identify and carefully analyze the combinatorial problems that were initially squashed by the Ramsey hammer. Another important question is whether the topological complexity can be allowed to grow polynomially (rather than being bounded); that this is possible was asserted some 20 years ago by Kalai and Meshulam [Kal04] in a series of conjectures that outline a theory of *homological Vapnik-Chervonenkis dimension*.

These two tasks naturally lead to the third task of generalizing optimization methods to set systems that are not convex, but satisfy topological assumptions. A natural example of such sets are *geometric transversals*.

## Pre-requisites

- Msc in Computer science or in mathematics. Fluency in French or English.
- An interest in the interplay between computer science and mathematics.
- A solid background in areas such as discrete or computational geometry, algorithms, complexity theory, topology, algebra, ... is a plus.

## References

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