Coding of arithmetic discrete hyperplanes and numeration systems

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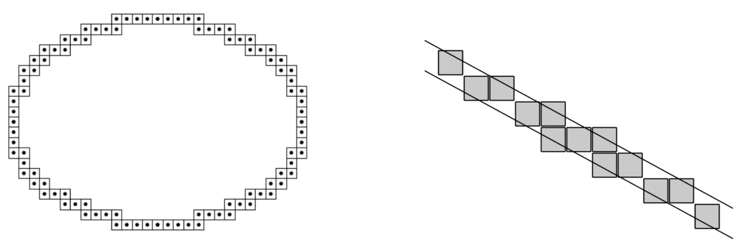
1 Context

The aim of discrete geometry is the study of discrete geometric objets (see 1). While these objects may be seen as approximations of continuous ones, their intrinsic properties, such as connexity, may also be investigated from a strictly discrete point of view [2, 8]. In dimension 2, the discrete line is a fundamental object which has been widely studied [10, 4] (cf figure 1b). It appears in many fields such as digital imagery, symbolic dynamics, or the study of arithmetic modular sequences.



(a) (b)

Figure 1: (a) Discrete curve of an ellipse. (b) A discrete line. The points are represented as unit squares the centers of which are located between the two affine lines which define the discrete line.

1.1 Discrete hyperplanes

While discrete lines are central in the study of discrete geometry in $\mathbb{Z}^2$, we are interested in a generalisation to $\mathbb{Z}^d$ where the fundamental objects are artichmetic discrete hyperplanes [1, 3] (cf figure 2). An arithmetic discrete hyperplane H is defined by two parallel affine hyperplanes H_- ans H_+ of $\mathbb{R}^d$. H is the set of integer points between these two hyperplanes. These points are ususally represented as unit hypercubes like the points in the discrete line in $\mathbb{Z}^2$ were represented as unit squares.

An arithmetic discrete hyperplane is characterized par 3 parameters:

- a vector $v \in \mathbb{R}^d$ which is the normal vector to the hyperplanes H_- and H_+.
- a *thickness* $\theta \in \mathbb{R}_+$ which defines the *distance* between H_- and H_+.
- a *shift* $\mu \in \mathbb{R}$ which is the distance from H_- to the origine.

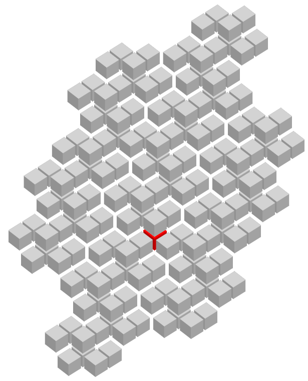


Figure 2: A portion of a discrete plane in dimension 3

Formally, the arithmetic discrete hyperplane $H(v,\mu,\theta)$ is the set $\{x \in \mathbb{Z}^d \mid 0 \leqslant \langle v, x\rangle + \mu < \theta\}$.

1.2 Connnexité

We are interested in the connectedness of these discrete hyperplanes. A part E of $\mathbb{Z}^d$ is connected if and only if between two points in E, there always exists in E a path consisting of adjacent points, where two points are adjacent if and only if their difference belongs to some fixed neighbourhood of the origin. We study connecetedness with respect to the chosen neighbourhood of the origin and the 3 parameters v, μ and θ.

Facet connectedness (related to facet-adjacency) has been completely studied. Two points x et y are facet-adjacent if and ond only if $\|x-y\|_1 \leqslant 1$. Geometrically the unit hypercubes centered at x and y share a facet, hence the name *facet-connectedness*. For instance, the hyperplane in figure 2 is facet-connected while the line in figure 1 is not.

According to the triple (v,μ,θ), we may charaterize whether $H(v,\mu,\theta)$ is facet-connected or not:

- from [5], there exists a critical thickness $\Omega(v,\mu)$, called *connecting thickness* such that $H(v,\mu,\theta)$ is nonempty and connected if $\theta > \Omega(v,\mu)$ and it is empty or disconnected if $\theta < \Omega(v,\mu)$. This critical thickness may be computed by means of the *fully subtractive algorithm* decribed in [11] (see algo. 1 below).
- the normal vectors v for which $H(v,0,\Omega(v,0))$ is connected may be charaterized by the fully subtractive algorithm [7]. It was shown in [9] that the set $\mathcal{K}_d$ of such vectors is Lebesgue-negligible: the hyperplane at the critical thickness is therefore almost always disconnected.
- If $v \in \mathcal{K}_d$, it is shown in [6] that the set of shifts μ which make $H(v,\mu,\Omega(v,\mu))$ connected may be characterized by means of a Büchi automaton.

We compute the connecting thickness by means of the fully subtractive algorithm [11] (*cf* algo. 1).

This algorithm terminates only if the dimension of the $\mathbb{Q}$-vector space generated by the components of v is 1. It is however always convergent. If $(v^n,\Omega^n)_{n\in\mathbb{N}}$ is the sequence generated by the algorithm then $\Omega(v) = \Omega^\infty + \|v^\infty\|_\infty$.

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Ω ← 0 ;
d ← dimension of v (number of components) ;
while d ≠ 1 do
    if v has a zero component v_k then
        v ← (v_1, ..., v_{k-1}, v_{k+1}, ..., v_d) ;
        d ← d − 1
    end
    else
        v_k ← min_{i∈⟦1,d⟧} v_i ;
        Ω ← Ω + v_k ;
        v ← (v_1 − v_k, ..., v_{k-1} − v_k, v_k, v_{k+1} − v_k, ..., v_d − v_k)
    end
end
return Ω
```

Algorithm 1: Computation of $\Omega(v, 0)$

1.3 Δ-numeration

The fully subtractive algorithm induces a sequence $\Delta = (\delta_n)_{n\in\mathbb{N}^\star}$ where $\delta_n \in [\![1, d]\!]$ is the index of a minimal component of v^n. We called θ_n this component. We may establish a recurrence relation between the θ_i's [7] and, when Δ is periodic, express them in the field $\mathbb{Q}[\beta]$ where β is the inverse of a Pisot number. We show that if $v \in \mathbb{K}_d$, then

$$\left\{\sum_{i\in\mathbb{N}} \varepsilon_i\,\theta_i \mid (\varepsilon_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\right\} = [0, \Omega].$$

We define a new numeration system, called Δ-numeration, where the sequence of integers $a_1, \dots, a_n, \dots \in \mathbb{Z}^\omega$ encodes the real number $\sum_{n\in\mathbb{N}^*} a_n\theta_n$ if it exists. By considering a bi-infinite sequence $(\theta_n)_{n\in\mathbb{Z}}$ satisfying the same recurrence relation, we obtain an canonical encoding of any real number as a sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$. The properties of the sequence $(\theta_n)_{n\in\mathbb{N}}$ allow to deduce computation algorithms for usual arithmetical operations in this new numeration system.

2 Goal of the thesis

The first goal of the thesis is to generalize the results on facet-connectedness to conectedness defined by arbitrary neighbourhood relations. Given a convexe and symetric neighbourhood $\mathcal{V}$ of 0, the aim is to study the connectedness of hyperplanes with respect to the neighbourhood relation defined by: "x and y are neighbours if and only if $x - y \in \mathcal{V}$". For instance, the line in figure 1 if connected if the adjacent squares share at leat a vertex but not if adjacent squares must share at least an edge. The goal is (1) to study in a general way the existence of a connecting thickness and, if it exists, to design an algorithm to compute it; (2) to study the connectedness and other toplogical properties of hyperplanes at this critical thickness; (3) to deduce a general incremental construction for hyperplanes.

The sequence Δ and the associated numeration system depend on the normal vector v and on the neighbourhood relation. The second goal of the thesis is (1) the study and the formalization of these numeration systems, considering an arbitrary neighbourhood relation; (2) investigate the link between these numeration systems and the β-numeration where the sequence $a_1, \dots, a_n, \dots$ encodes the number $\sum_{n\in\mathbb{Z}} a_n\beta^n$.

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