What if Gauss had had a computer?

Paul Zimmermann, INRIA, Nancy, France

Celebrating 75 Years of Mathematics of Computation, ICERM,
Brown University, Providence, November 1st, 2018

Carl Friedrich Gauss, Werke, Volume 2, 1863, pages 477-502:

TAFEL

ZUR

CYKLOTECHNIE.

NACHLASS. ZERLEGBARE $aa+1$.

2	5	119	73.97	500	53.53.89	1341	73.109.113	3405	29.29.61.113
3	5	123	5.17.89	507	5.5.53.97	1385	41.149.157	3458	5.73.181.181
4	17	128	5.29.113	512	5.13.37.109	1393	5.5.197.197	3521	29 37.53.109
5	13	129	53.157	515	13.101.101	1407	5.5.17.17.137	3532	5.5.17.149.197
6	37	132	5.5.17.41	524	37.41.181	1432	5.5.5.5.17.193	3583	5.13.17.37.157
7	5.5	133	5.29.61	538	5.13.61.73	1433	5.29.73.97	3740	41.41.53.157
8	5.13	142	5.37.109	557	5.5.5.17.73	1467	5.29.41.181	3782	5.5.29.109.181
9	41	157	5.5.17.29	560	53.61.97	1477	5.13.97.173	3793	5.5.53.61.89
10	101	162	5.29.181	568	5.5.5.29.89	1560	17.37.53.73	3957	5.5.13.13.17.109
11	61	172	5.61.97	577	5.13.13.197	1567	5.41.53.113	4193	5.5.5.5.5.29.97
12	5.29	173	5.41.73	599	17.61.173	1568	5.5.5.13.17.89	4217	5.13.29.53.89
13	5.17	174	13.17.137	606	13.13.41.53	1597	5.37.61.113	4232	5.5.41.101.173
14	197	182	5.5.5.5.53	616	13.17.17.101	1607	5.5.13.29.137	4246	13.17.29.29.97
15	113	183	5.17.197	621	29.61.109	1636	17.29.61.89	4327	5.89.109.193
17	5.29	185	109.157	657	5.5.89.97	1744	137.149.149	4484	17 89.97 137
18	5.5.13	191	17.29.37	660	37.61.193	1772	5.17.17.41.53	4535	17.53.101.113
19	181	192	5.73.101	682	5.5.5.61.61	1818	5.5.5.137.193	4545	13.37.109.197
21	13.17	193	5.5.5.149	684	13.17.29.73	1823	5.17.113.173	4581	13.53.97.157
22	5.97	200	13.17.181	693	5.5.5.17.113	1832	5.5.17.53.149	4594	13.17.29.37.89
23	5.53	211	113.197	697	5.13.37.101	1893	5.5.13.37.149	4662	5.13.13.17.17.89

Page 481

[illegible]	[illegible]	[illegible]	[illegible]
16317267	5.13.17.17.61.61.101.109.173	2971354082	5.5.13.17.29.41.53.53.113.149 157.181
18378313	5.13.13.17.37.61.137.193.197	3955080927	5.13.17.17.17.17.53.53.61.61.101.149.173.197
18975991	13.17.17.17.53.61.89.97.101	8193535810	13.13.29.29.61.109.109.137.157.157.193
20198495	13.17.41.89.101.101.137.181	14033378718	5.5.13.13.17.17.61.61.61.61.73.73.157.181
22866693	5.5.5.5.41.61.73.101.113.197		

5	2. 3. 7
13	5. 8. 18. 57. 239
17	4. 13. 21. 38. 47. 268
29	12. 17. 41. 70. 99. 157. 307
37	6. 31. 43. 68. 117. 191. 302. 327. 882. 18543*
41	9. 32. 73. 132. 278. 378. 829. 993. 2943
53	23. 30. 83. 182. 242. 401. 447. 606. 931. 1143*. 1772. 6118. 34208. 44179. 85353. 485298
61	11. 50. 72. 133. 255. 438. 682. 2673. 2917. 4747*. 4952. 5257. 9466. 12943. 17557. 114669. 330182
73	27. 46. 173. 265. 319 538. 557. 684. 1068. 1560*. 2163. 2309. 2436. 3039. 5667. 8368. 14773. 48737. 72662. 478707*
89	34. 55. 123. 233. 411. 500. 568. 746. 1568. 1636*. 3793. 4217. 4594. 4662. 6107. 11981. 19703. 24263. 32807. 37770*. 45068. 51387. 99557. 157318. 260359. 24208144
97	22. 75. 119. 172. 216. 463. 507. 560. 657. 1433*. 1918. 2059. 2738. 4193. 4246. 5357. 5507. 5648. 6962. 9193*. 9872. 17923. 21124. 29757. 30383. 39307. 41688. 112595. 320078. 390112*. 617427. 1984933. 2343692. 3449051. 6225244
101	10. 91. 111. 192. 212. 293. 313. 394. 515. 616*. 697. 798. 818. 1303. 2818. 3141. 3323. 8393. 17766. 36673*. 66347. 71700. 74043. 173932. 177144. 508929. 683982. 1635786. 2478328. 2809305*. 3014557. 6367252. 18975991. 193788912. 201229582. 2189376182
109	33. 76. 142. 251. 294. 360. 512. 621. 905. 948*. 1057. 1123. 1929. 2801. 3521. 3957. 5701. 6943. 8578. 9298*.

Page 501

Machin $(1) = 4(5) - (239)$ auch Clausen

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \qquad \text{(Machin, 1706)}$$

Gauss. 1. $= 12(18) + 8(57) - 5(239)$

Gauss. 2. $= 12(38) + 20(57) + 7(239) + 24(268)$

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} \qquad \text{(Gauss, 1863)}$$

$$\frac{\pi}{4} = 12 \arctan \frac{1}{38} + 20 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} + 24 \arctan \frac{1}{268} \qquad \text{(Gauss, 1863)}$$

Plan of the talk

- how such identities can be verified
- how they can be (re)discovered
- by hand and using modern computational mathematics tools

Page 478

NACHLASS. ZERLEGBARE $aa+1$.

2	5	119	73.97	500	53.53.89	1341	73.109.113	3405	29.29.61.113
3	5	123	5.17.89	507	5.5.53.97	1385	41.149.157	3458	5.73.181.181
4	17	128	5.29.113	512	5.13.37.109	1393	5.5.197.197	3521	29.37.53.109
5	13	129	53.157	515	13.101.101	1407	5.5.17.17.137	3532	5.5.17.149.197
6	37	132	5.5.17.41	524	37.41.181	1432	5.5.5.5.17.193	3583	5.13.17.37.157
7	5.5	133	5.29.61	538	5.13.61.73	1433	5.29.73.97	3740	41.41.53.157
8	5.13	142	5.37.109	557	5.5.5.17.73	1467	5.29.41.181	3782	5.5.29.109.181
9	41	157	5.5.17.29	560	53.61.97	1477	5.13.97.173	3793	5.5.53.61.89
10	101	162	5.29.181	568	5.5.5.29.89	1560	17.37.53.73	3957	5.5.13.13.17.109
11	61	172	5.61.97	577	5.13.13.197	1567	5.41.53.113	4193	5.5.5.5.5.29.97
12	5.29	173	5.41.73	599	17.61.173	1568	5.5.5.13.17.89	4217	5.13.29.53.89
13	5.17	174	13.17.137	606	13.13.41.53	1597	5.37.61.113	4232	5.5.41.101.173
14	197	182	5.5.5.5.53	616	13.17.17.101	1607	5.5.13.29.137	4246	13.17.29.29.97
15	113	183	5.17.197	621	29.61.109	1636	17.29.61.89	4327	5.89.109.193
17	5.29	185	109.157	657	5.5.89.97	1744	137.149.149	4484	17.89.97.137
18	5.5.13	191	17.29.37	660	37.61.193	1772	5.17.17.41.53	4535	17.53.101.113
19	181	192	5.73.101	682	5.5.5.61.61	1818	5.5.5.137.193	4545	13.37.109.197
21	13.17	193	5.5.5.149	684	13.17.29.73	1823	5.17.113.173	4581	13.53.97.157
22	5.97	200	13.17.181	693	5.5.5.17.113	1832	5.5.17.53.149	4594	13.17.29.37.89
23	5.53	211	113.197	697	5.13.17.101	1893	5.5.13.37.149	4662	5.13.13.17.17.89

```
sage: a=4594; factor(a^2 + 1)
13 * 17 * 29 * 37 * 89
```

Page 481

[illegible]	[illegible]
2971354082	5.5.13.17.29.41.53.53.113.149.157.181
3955080927	5.13.17.17.17.17.53.53.61.61.101.149.173.19…
8193535810	13.13.29.29.61.109.109.137.157.157.193
14033378718	5.5.13.13.17.17.61.61.61.61.73.73.157.181

```
sage: factor(14033378718^2 + 1)
5^2 * 13 * 17^2 * 61^4 * 73^2 * 157 * 181
```

Even Gauss made errors...

Page 481

5	2. 3. 7
13	5. 8. 18. 57. 239
17	4. 13. 21. 38. 47. 268
29	12. 17. 41. 70. 99. 157. 307
37	6. 31. 43. 68. 117. 191. 302. 327. 882. 18543*
41	9. 32. 73. 132. 278. 378. 829. 993. 2943
53	23. 30. 83. 182. 242. 401. 447. 606. 931. 1143*. 1772. 6118. 34208. 44179. 85353. 485298
61	11. 50. 72. 133. 255. 438. 682. 2673. 2917. 4747*. 4952. 5257. 9466. 12943. 17557. 114669. 330182
73	27. 46. 173. 265. 319 538. 557. 684. 1068. 1560*. 2163. 2309. 2436. 3039. 5667. 8368. 14773. 48737. 72662. 478707*
89	34. 55. 123. 233. 411. 500. 568. 746. 1568. 1636*. 3793. 4217. 4594. 4662. 6107. 11981. 19703. 24263. 32807. 37770*. 45068. 51387. 99557. 157318. 260359. 24208144
97	22. 75. 119. 172. 216. 463. 507. 560. 657. 1433*. 1918. 2059. 2738. 4193. 4246. 5357. 5507. 5648. 6962. 9193*. 9872. 17923. 21124. 29757. 30383. 39307. 41688. 112595. 320078. 390112*. 617427. 1984933. 2343692. 3449051. 6225244
101	10. 91. 111. 192. 212. 293. 313. 394. 515. 616*. 697. 798. 818. 1303. 2818. 3141. 3323. 8393. 17766. 36673*. 66347. 71700. 74043. 173932. 177144. 508929. 683982. 1635786. 2478328. 2809305*. 3014557. 6367252. 18975991. 193788912. 201229582. 2189376182
109	33. 76. 142. 251. 294. 360. 512. 621. 905. 948*. 1057. 1123. 1929. 2801. 3521. 3957. 5701. 6943. 8578. 9298*.

```
sage: [a for a in [1..10^4] if largest_prime(a^2+1) == 5]
[2, 3, 7]
sage: [a for a in [1..10^4] if largest_prime(a^2+1) == 13]
[5, 8, 18, 57, 239]
sage: [a for a in [1..10^4] if largest_prime(a^2+1) == 109]
[33, 76, 142, 251, 294, 360, 512, 621, 905, 948, 1057, 1123,
1929, 2801, 3521, 3957, 5701, 6943, 8578, 9298]
```

Page 501

Machin	$(1) = 4(5) - (239)$	auch Clausen
Euler	$= (2) + (3)$	(Euler à Goldbach 1746 Mai 28)
Vega	$= 5(7) + 2\left(\frac{79}{3}\right)$	(Vega Thesaurus logar. p. 633)
Vega	$= 2(3) + (7)$	auch Clausen (Astr. Nachr. B. 25. S. 209)
Rutherford	$= 4(5) - (70) + (99)$	(Philos. Trans. 1841. p. 283)
Dase	$= (2) + (5) + (8)$	(Crelle Journal. B. 27. S. 198)
Gauss. 1.	$= 12(18) + 8(57) - 5(239)$	
Gauss. 2.	$= 12(38) + 20(57) + 7(239) + 24(268)$	

Notation: (n) or $[n]$ denotes $\arctan \frac{1}{n}$.

Measure of an arc-tangent identity

Lehmer proposes in 1938 the following measure. For example, Machin's formula

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

has measure

$$\frac{1}{\log_{10} 5} + \frac{1}{\log_{10} 239} \approx 1.8511$$

A formula with measure say 2 needs two terms of the arc-tangent series to get one digit of π:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$$

Machin (1706, measure 1.8511):

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

Gauss (1863, measure 1.7866):

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

Gauss (1863, measure 2.0348):

$$\frac{\pi}{4} = 12 \arctan \frac{1}{38} + 20 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} + 24 \arctan \frac{1}{268}$$

Why is the arc-tangent series so popular?

$$\arctan\frac{1}{n} = \frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \cdots$$

$$10^{15}\arctan\frac{1}{239} \approx \frac{10^{15}}{239} - \frac{10^{15}}{3\cdot 239^3} + \frac{10^{15}}{5\cdot 239^5}$$

$$\left\lfloor \frac{10^{15}}{239} \right\rfloor = 4184100418410$$

$$\left\lfloor \frac{4184100418410}{239^2} \right\rfloor = 73249775, \qquad \left\lfloor \frac{73249775}{3} \right\rfloor = 24416591$$

$$\left\lfloor \frac{73249775}{239^2} \right\rfloor = 1282, \qquad \left\lfloor \frac{1282}{5} \right\rfloor = 256$$

$$10^{15}\arctan\frac{1}{239} \approx 4184100418410 - 24416591 + 256 = 4184076002075$$

2-term identities

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \quad \text{(Machin, 1706, measure 1.8511)}$$

$$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7} \quad \text{(Machin, 1706, measure 3.2792)}$$

$$\frac{\pi}{4} = 2\arctan\frac{1}{2} - \arctan\frac{1}{7} \quad \text{(Machin, 1706, measure 4.5052)}$$

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3} \quad \text{(Machin, 1706, measure 5.4178)}$$

Störmer proved in 1899 these are the only ones of the form $k\pi/4 = m\arctan(1/x) + n\arctan(1/y)$.

3-term identities

The one with best measure (with numerators 1) is due to Gauss (1863, measure 1.7866):

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

Störmer found 103 3-term identities in 1896, Wrench found two more in 1938, and Chien-lih a third one in 1993. Their exact number remains an open question.

4-term identities

The one with best measure (with numerators 1) is due to Störmer (1896, measure 1.5860):

$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943}$$

It was used by Kanada *et al.* in 2002 to compute $1,241,100,000,000$ digits of π.

The second best was found by Escott in 1896 (measure 1.6344), the third one by Arndt in 1993 (1.7108).

Computation of π

1962: Shanks and Wrench compute $100,265$ decimal digits of π using Störmer's formula (1896, measure 2.0973):

$$\frac{\pi}{4} = 6 \arctan \frac{1}{8} + 2 \arctan \frac{1}{57} + \arctan \frac{1}{239}$$

The verification was done with Gauss' formula:

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

The first check did agree only to 70,695 digits, due to an error in the computation of $6 \arctan(1/8)$!

This was published in volume 16 of Mathematics of Computation. Pages 80-99 of the paper give the $100,000$ digits.

1973: Guilloud and Boyer compute $1,001,250$ digits using the same formulae.

Computation of π (continued)

2002: Kanada *et al.* compute $1,241,100,000,000$ digits using the self-checking pair

$$\frac{\pi}{4} = 44 \arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943},$$

and

$$\frac{\pi}{4} = 12 \arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443}.$$

How to verify such identities with a computer?

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

Let us check Machin's formula

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

```
sage: combine(x,y) = (x+y)/(1-x*y)
sage: combine(1/5,1/5)
5/12
```

Thus

$$2 \arctan \frac{1}{5} = \arctan \frac{5}{12}$$

```
sage: combine(5/12,5/12)
120/119
```

Thus

$$4 \arctan \frac{1}{5} = \arctan \frac{120}{119}$$

```
sage: combine(120/119,-1/239)
1
```

Thus

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan 1 = \frac{\pi}{4}$$

We can “multiply” an arc-tangent by a positive integer n:

```
sage: muln = lambda x,n: x if n==1 else combine(x,muln(x,n-1))
```

Then we get:

```
sage: muln(1/5,4)
120/119
```

and:

```
sage: combine(muln(1/5,4),-1/239)
1
```

Symbolic transformations

```
sage: muln(1/x,2).normalize()
2*x/(x^2 - 1)
```

$$2\arctan\frac{1}{x} = \arctan\frac{2x}{x^2-1}$$

```
sage: muln(1/x,3).normalize()
(3*x^2 - 1)/((x^2 - 3)*x)
```

$$3\arctan\frac{1}{x} = \arctan\frac{3x^2-1}{x^3-3x}$$

```
sage: muln(1/x,4).normalize()
4*(x^2 - 1)*x/(x^4 - 6*x^2 + 1)
```

$$4\arctan\frac{1}{x} = \arctan\frac{4x(x^2-1)}{x^4-6x^2+1}$$

How to discover such identities?

- experimentally with Pari/GP `lindep`
- with Gaussian integers
- a direct method using integers only

Playing with Pari/GP lindep

On page 481, Gauss writes for $p = 5, 13, \ldots$ which $a^2 + 1$ have p as largest prime factor:

5	2. 3. 7 .
13	5. 8. 18. 57. 239

We can (re)discover some identities using Pari/GP as follows:

```
? lindep([atan(1/2),atan(1/3),Pi/4])
%7 = [-1, -1, 1]~
? lindep([atan(1/5),atan(1/8),atan(1/18),Pi/4])
%9 = [-3, -2, 1, 1]~
? lindep([atan(1/8),atan(1/18),atan(1/57),Pi/4])
%11 = [-5, -2, -3, 1]~
? lindep([atan(1/18),atan(1/57),atan(1/239),Pi/4])
%13 = [-12, -8, 5, 1]~
```

Take all numbers a such that $a^2 + 1$ has all its factors ≤ 13:

```
? lindep([atan(1/2),atan(1/3),atan(1/5),atan(1/7),atan(1/8),
          atan(1/18),atan(1/57),atan(1/239),Pi/4])
%1 = [-1, 1, 0, 1, 0, 0, 0, 0, 0]~
```

Thus $\arctan(1/2) = \arctan(1/3) + \arctan(1/7)$:

```
sage: combine(1/3,1/7)
1/2
```

We can thus omit $\arctan(1/2)$.

```
? lindep([atan(1/3),atan(1/5),atan(1/7),atan(1/8),atan(1/18),
          atan(1/57),atan(1/239),Pi/4])
%2 = [-1, 1, 0, 1, 0, 0, 0, 0]~
```

Thus $\arctan(1/3) = \arctan(1/5) + \arctan(1/8)$:

```
sage: combine(1/5,1/8)
1/3
```

We can thus omit $\arctan(1/3)$.

```
? lindep([atan(1/5),atan(1/7),atan(1/8),atan(1/18),atan(1/57),
         atan(1/239),Pi/4])
%3 = [-1, 1, 0, 1, 0, 0, 0]~
```

Thus $\arctan(1/5) = \arctan(1/7) + \arctan(1/18)$.

```
? lindep([atan(1/7),atan(1/8),atan(1/18),atan(1/57),atan(1/239),
         Pi/4])
%4 = [-1, 1, 0, 1, 0, 0]~
```

Thus $\arctan(1/7) = \arctan(1/8) + \arctan(1/57)$.

```
? lindep([atan(1/8),atan(1/18),atan(1/57),atan(1/239),Pi/4])
%5 = [1, -2, -1, 1, 0]~
```

$\arctan(1/8) = 2\arctan(1/18) + \arctan(1/57) - \arctan(1/239)$.

```
? lindep([atan(1/18),atan(1/57),atan(1/239),Pi/4])
%6 = [-12, -8, 5, 1]~
```

We find Gauss' 1st formula:

$$\frac{\pi}{4} = 12\arctan\frac{1}{18} + 8\arctan\frac{1}{57} - 5\arctan\frac{1}{239}$$

Reducible and irreducible arctangent

We say that $\arctan(1/n)$ is reducible if it can be expressed as a linear combination of smaller arctangents. Otherwise it is irreducible.

For $1 \le n \le 20$, we have 6 reducible arctangents:

$$[3] = [1] - [2]$$

$$[7] = -[1] + 2[2]$$

$$[8] = [1] - [2] - [5]$$

$$[13] = [1] - [2] - [4]$$

$$[17] = -[1] + 2[2] - [12]$$

$$[18] = [1] - 2[2] + [5]$$

Which primes p can divide $a^2 + 1$?

p divides $a^2 + 1$ is equivalent to $a^2 \equiv -1 \bmod p$

Thus -1 should be a quadratic residue modulo p.

In other words the Jacobi symbol $\left(\frac{-1}{p}\right)$ should be 1.

```
sage: [p for p in prime_range(3,110) if (-1).jacobi(p) == 1]
[5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109]
```

We find the primes appearing on the bottom of page 481.

By the first supplement to quadratic reciprocity, only 2 and primes of the form $4k + 1$ can appear.

How to find the $a^2 + 1$ with largest factor p?

```
sage: def largest_prime(n):
....:     l = factor(n)
....:     return l[len(l)-1][0]
sage: largest_prime(1001)
13

sage: def search(p,B):
....:     for a in range(1,B):
....:         if largest_prime(a^2+1)==p:
....:             print a
sage: search(5,10^6)
2
3
7
```

Faster way of searching: if p divides $a^2 + 1$, then $r := a \bmod p$ is one of the roots of $x^2 + 1 \bmod p$:

```
sage: def search2(p,B):
....:     r = (x^2+1).roots(ring=GF(p))
....:     for t,_ in r:
....:         for a in range(ZZ(t),B,p):
....:             if largest_prime(a^2+1)==p:
....:                 print a
sage: search2(5,10^6)
3
2
7
```

We check only 2 values out of p.

Gaussian Integers

Gaussian integers are of the form $a + ib$, with $a, b \in \mathbb{Z}$.

They form an unique factorization domain, with units $\pm 1, \pm i$.

$$17 + i = -i(1 + i)(2 + i)(5 + 2i)$$

```
sage: ZZI.<I> = GaussianIntegers()
sage: factor(17+I)
(I) * (-I - 2) * (I + 1) * (2*I + 5)
```

A Gaussian integer like $5 + 2i$ that cannot be factored is called irreducible.

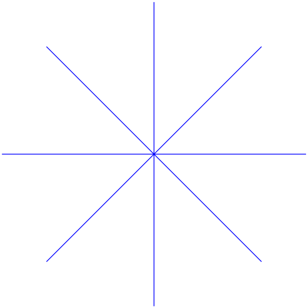
The Gaussian Integers Method

A term $\arctan \frac{b}{a}$ corresponds to the Gaussian integer $a + ib$.

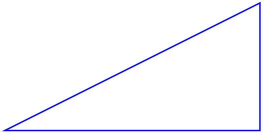
A term $k \arctan \frac{b}{a}$ corresponds to $(a + ib)^k$.

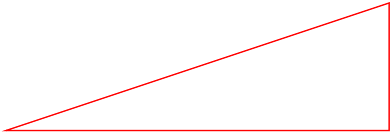
A sum $\arctan \frac{b}{a} + \arctan \frac{d}{c}$ corresponds to $(a + ib)(c + id)$.

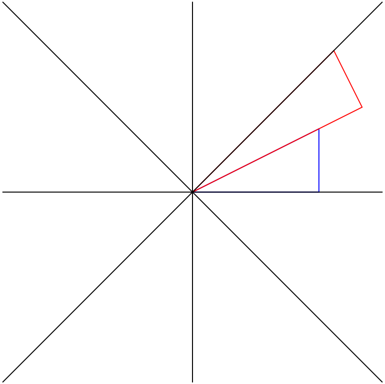
We thus want to find a product of Gaussian integers whose argument is a (non-zero) multiple of $\pi/4$.



Example: $\arctan(1/2) + \arctan(1/3) = \arctan(1)$







Machin's formula in terms of Gaussian integers

```
sage: ZZI.<I> = GaussianIntegers()
sage: factor((5+I)^4)
(-3*I - 2)^4 * (I + 1)^4
sage: factor(239+I)
(I) * (-3*I - 2)^4 * (I + 1)
```

Thus $4\arctan(1/5) - \arctan(1/239)$ corresponds to $-i(1+i)^3$, i.e., to $9\pi/4$, i.e., $\pi/4$ modulo 2π.

```
sage: (5+I)^4*(239-I)
114244*I + 114244
```

Norm of Gaussian Integers

Definition: The norm of $a + ib$ is $N(a + ib) := a^2 + b^2$.

The norm is multiplicative: if $a + ib = (b + id)(e + if)$, then $N(a + ib) = N(b + id)N(e + if)$.

$$(b + id)(e + if) = (be - df) + i(bf + de)$$

$$\begin{aligned} N((b + id)(e + if)) &= (be - df)^2 + (bf + de)^2 \\ &= (be)^2 + (df)^2 + (bf)^2 + (de)^2 \\ &= (b^2 + d^2)(e^2 + f^2) \end{aligned}$$

The Gaussian Integers Algorithm

The term $\arctan(1/a)$ corresponds to Gaussian integers $a+i$, thus to the norm a^2+1.

If a^2+1 has only few small prime divisors, then $a+i$ can have only few irreducible factors, since their norm must divide a^2+1.

Algorithm:

- Input: a set S of primes, a bound A
- factor a^2+1 for a up to some bound A;
- identify those a^2+1 with only prime divisors in S;
- factor the corresponding Gaussian integers $a+i$;
- find linear combinations to cancel the exponents of irreductible factors other that $1+i$ (up to an unit).

With $S=\{2,5,13,17\}$, there are 15 values of a up to $A=10^6$:

$$1,2,3,4,5,7,8,13,18,21,38,47,57,239,268$$

This is related to the roots of $x^2 + 1$ modulo $2, 5, 13, 17$:

```
sage: for p in [2,5,13,17]:
....:     print p, (x^2+1).roots(ring=GF(p))
2 [(1, 2)]
5 [(3, 1), (2, 1)]
13 [(8, 1), (5, 1)]
17 [(13, 1), (4, 1)]
```

$a = 268$ corresponds to the roots 3 mod 5, 8 mod 13, 13 mod 17:

```
sage: crt([3,8,13],[5,13,17])
268
sage: factor(268^2+1)
5^2 * 13^2 * 17
```

If we take the other root 4 modulo 17, we get $a = 463$, but $a^2 + 1$ has a spurious prime factor 97:

```
sage: crt([3,8,4],[5,13,17])
463
sage: factor(463^2+1)
2 * 5 * 13 * 17 * 97
```

Todd's reduction process

Idea: decompose $N + i$ into a product $(l_1 \pm i)(l_2 \pm i) \cdots (l_k \pm i)$.

Example for $N = 580$:

```
sage: factor(580^2+1)
13 * 113 * 229
```

The least integer m such that $p = 229$ divides $m^2 + 1$ is $m = l_1 = 107$.

If $N + l_1$ is divisible by p, then we take $l_1 - i$, else we take $l_1 + i$.

We compute the next residue by multiplying by the conjugate and dividing by p:

```
sage: (580+I)*(107+I)/229
3*I + 271
```

Todd's reduction process (continued)

We continue the reduction from $271 + 3i$:

```
sage: factor(271^2+3^2)
2 * 5^2 * 13 * 113
```

The least integer such that $p = 113$ divides $m^2 + 1$ is $m = 15$.
Since $271 + 3 \cdot 15$ is not divisible by 113, we take $15 + i$:

```
sage: (271+3*I)*(15-I)/113/2
-I + 18
```

At the end of Todd's reduction process we get:

$$\arctan \frac{1}{580} = -\arctan 1 + 2 \arctan \frac{1}{2} - \arctan \frac{1}{5} + \arctan \frac{1}{15} - \arctan \frac{1}{107}$$

Other identities

$$\arctan\frac{1}{n} = \arctan\frac{1}{n+1} + \arctan\frac{1}{n^2+n+1}$$

$$\arctan\frac{1}{n} = 2\arctan\frac{1}{2n} - \arctan\frac{1}{4n^3+3n}$$

If we use the latter in Machin's formula, we can replace $\arctan(1/5)$ by $2\arctan(1/10) - \arctan(1/515)$, which gives:

$$\frac{\pi}{4} = 8\arctan\frac{1}{10} - \arctan\frac{1}{239} - 4\arctan\frac{1}{515}$$

discovered by the Scottish mathematician Robert Simson in 1723.

Conclusion

Gauss' work can be reproduced using modern computational tools.

We can provide algorithms to check or discover identities.

Using computers, we can find identities with large denominators.

But some open questions still remain...

References

https://gallica.bnf.fr/ark:/12148/bpt6k99402s/f483:
Gauss' factorization tables of $a^2 + 1$, ..., $a^2 + 4$, $a^2 + 81$

On Arccotangent Relations for π, Derrick H. Lehmer, The American Mathematical Monthly, 1938.

Calculation of π to 100,000 Decimals, Daniel Shanks and John W. Wrench, Jr., Mathematics of Computation, 1962.

Gaussian Integers and Arctangent Identities for π, Jack S. Calcut, American Math. Monthly, 2009.

Search for the "best" arctan relation, Joerg Arndt, workshop Computing by the Numbers: Algorithms, Precision, and Complexity, Berlin, 2006, www.jjj.de/arctan/arctanpage.html

On the derivation of Machin-like arctangent identities for computing π, Amrik S. Nimbran, The Mathematics Student, 2010.

References (continued)

Solution complète en nombres entiers de l'équation $m\arctan(1/x) + n\arctan(1/y) = k\pi/4$, Carl Störmer, Bulletin de la SMF, 1899.

https://en.wikipedia.org/wiki/Machin-like_formula#Derivation: Machin-like formula on Wikipedia

http://www.machination.eclipse.co.uk/: a database of Machin-like identities, by Hwang Chien-lih and Michael R. Wetherfield