Rough Equality Algebras

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Abstract

The notion of *rough equality* was introduced by Pawlak in [6]. It was extensively examined in [1]. It was proved in [9] that the notion of rough equality of sets leads to a definition of a new class of algebras, the rough equality algebras. We call them here *rough-quasi S5 algebras*. We also introduce a notion of a *rough-quasi S4 algebra*. These rough-quasi S5 and S4 algebras are quasi-Boolean version of topological S4 and S5 Boolean algebras (see [3], [4]) which are algebraic models for modal logics S5 and S4, respectively ([7]). We examine and discuss some of their most interesting properties, their relationship with each other, and with the topological S4 and S5 Boolean algebras. The presented properties were chosen out of over seven hundred theorems which were discovered and proved automatically by a theorem prover `DATAC` (*Déduction Automatique dans des Théories Associatives et Commutatives*). The prover was developed at CRIN & INRIA Lorraine, Nancy (France), by the second author.

1 Introduction

This paper is a continuation of investigations of [1] and [9]. It was shown in [9] that the notion of rough equality of sets leads to a definition of a new class of algebras and that a representation theorem holds for any algebra in that class. The representation theorem establishes links between rough set theory, logic, abstract algebras and topology, and hence it provides a deeper mathematical characterization of the notion of rough set and rough equality.

The new algebras introduced in [9] are a non-classical (quasi-Boolean instead of Boolean) version of topological Boolean algebras which are algebraic models for the modal logic S5 and in which every closed element is open. In order to stress rough sets and the quasi-Boolean origins of our new class of algebras we call them here *rough-quasi S5 algebras* (RQ5). We also introduce, in a natural way its S4 correspondent, namely the *rough-quasi S4 algebras* (RQ4). We examine the properties and relationship between the two classes of algebras in section 2, and between the rough-quasi and topological Boolean algebras in section 3. The process of creating the rough-quasi algebras is the following: logic (S5), Lindenbaum-Tarski construction (modulo rough equality), rough algebra, class of rough-quasi algebras, representation theorem for rough-quasi S5 algebras. The Lindenbaum-Tarski algebra is a principal construction in *Algebraic Logic* and is used to establish a relationship between logics and corresponding abstract algebras. The algebra corresponding to classical logic is the Boolean algebra. Algebras corresponding to the modal S4 and S5 logics are S4 and S5 topological Boolean algebras. The book *An algebraic approach to non-classical logics* ([7]) formulates an algebraic approach to a carefully selected widest possible class of logics which can be approached from algebraic point of view. Our investigations lead to the inverse problem: *define a class of algebras which can be algebraic models for a logic (defined in general terms).* Our further research will hence involve the search for the answer to a natural open question: *Which are the logics (if any) corresponding to rough-quasi algebras (or their extensions)?*

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2 RQ5 and RQ4 Algebras

The rough-quasi algebras are topological extensions of quasi-Boolean algebras. They are not purely mathematical invention. The rough algebra developed and examined in [1] is an example a rough-quasi S5 algebra.

The quasi-Boolean algebras were introduced and examined in [2]. They are de Morgan lattices with a unit element and a zero element. De Morgan lattices have been introduced and examined in [5]. They are distributive lattices with an additional operation $\sim$ defined by two axioms: $\sim\sim a = a$, $\sim(a \cup b) = \sim a \cap \sim b$. The complementation $\sim$ is called a *strong complementation*. It differs from classical Boolean (set theoretical) complementation. The quasi-Boolean algebra is a rough-quasi S4 algebra (RQ4) if it contains additionally the axioms defining a topological space, i.e. the following axioms: $I(a \cap b) = Ia \cap Ib$, $Ia \subseteq Ia$, $IIa = a$, $IA = A$, where A is the universe of the algebra, I is an *interior operation*. The RQ4 algebra a rough-quasi S5 algebra (RQ5) if additionally every closed element is open, i.e. if for any $a \in A$, $CIa = Ia$, where C is a dual operation to I, i.e. $Ca = \sim I \sim a$. C is called a *closure operation*. The interior I and closure C operations correspond to modal operators $\Box$ and $\Diamond$, respectively. The formal definition of the class of rough-quasi S4 and S5 algebras is the following.

Definition 2.1 (RQ4 and RQ5 algebras) *An abstract algebra $(A, 1, \cup, \cap, \sim)$ is a rough-quasi S4 (RQ4) algebra if it is a distributive lattice and additionally for all $a, b, c \in A$ the following equations are satisfied.*

$$\begin{array}{rclrcl} \sim\sim a & = & a & \sim(a \cup b) & = & \sim a \cap \sim b \\ I(a \cap b) & = & Ia \cap Ib & Ia \cap a & = & Ia \\ IIa & = & a & I1 & = & 1 \end{array}$$

The RQ4 algebra is a RQ5 if contains additionally the following axiom:

$$CIa = Ia$$

2.1 DATAC - a tool for automatic deduction

The properties of the RQ4 and RQ5 algebras presented here are chosen from over seven hundred which were discovered and proved automatically by a theorem prover DATAC (*Déduction Automatique dans des Théories Associatives et Commutatives*) developed at CRIN & INRIA Lorraine, Nancy (France). This software implements a new technique [8] of automated deduction in first-order logic, in presence of associative and commutative operators. This technique combines an ordering strategy, a system of deduction rules based on resolution and paramodulation rules, and techniques for the deletion of redundant clauses.

DATAC proposes either to prove properties by refutation, or by straightforward deduction from axioms. The refutation technique is a proof existence technique i.e. we add a negation of a formula we want to prove to the set of already proven formulas and we search for contradiction. In this case, if the proof exists the prover would say: "yes, it is a theorem", i.e. the prover acts as a proof existence checker. Of course, the whole system is, as a classical predicate logic, semi-decidable. In the other the prover acts as a deductive system i.e. the end product is a property with its formal proof. We have used here mainly the second technique.

2.2 Properties common to RQ4 and RQ5

The axioms of RQ4 are strictly included in the set of axioms for RQ5. Hence all properties we can prove in RQ4, we can prove in RQ5 and there are also *pure* RQ5 properties, i.e the RQ5 properties which can't be proved in RQ4. Of course in general setting the *pure* RQ5 properties are the set theoretical difference between all RQ5 properties and those which are *common* to RQ4 and RQ5. In general case there is a countably infinite number of all properties of the RQ4 and RQ5 algebras, so we never can find all pure RQ5 properties. In our case all sets are finite and we have a practical way of finding the *common* and *pure* properties. It isn't straightforward because of the nature of the prover and we discuss the results in this section for common properties and in 2.3 for the pure.

We have used the prover separately for RQ4 and RQ5 algebras. All executions have been arbitrary stopped after 5000 deductions. We stopped the experiments after deducing 407 properties for RQ4 and 294 properties for RQ5. The answer to the question which properties from 407 + 294 = 701 are common to both algebras is simple. We run a matching program for the proofs of all of 701 properties and found that 217 of them were identical in both separate runs for RQ4 and RQ5. The prover in the first steps has deduced the following properties.

$$
\begin{array}{rclrcl}
a \cup a & = & a & a \cap a & = & a \\
\sim(a \cap b) & = & \sim a \cup \sim b & & & \\
a \cap \sim 1 & = & \sim 1 & a \cup \sim 1 & = & a
\end{array}
$$

We added them to the set of the initial axioms. We have also added the duality property of operators:

$$Ca = \sim I \sim a$$

We decomposed 217 common properties into two categories. The first-one corresponds to intuitively obvious properties of topological spaces (or modal logics S4 and S5), the second category contains all other common properties. The properties of the second category seems to be really not trivial even for topological spaces with normal set theoretical operations.

Remark: As $(A, \cap, \cup)$ is a lattice, we use the natural lattices notation $a \subseteq b$, which is equivalent to $a \cup b = b$ and $a \cap b = a$.

Some intuitively obvious properties derived are:

$$
\begin{array}{rclrcl}
C1 & = & 1 & Ia & \subseteq & a \\
C \sim 1 & = & \sim 1 & a & \subseteq & Ca \\
I \sim 1 & = & \sim 1 & Ia & \subseteq & C(a) \\
C(a \cup b) & = & Ca \cup Cb & Ia \cap b & \subseteq & a \\
CCa & = & Ca & a \cap b & \subseteq & Ca \\
\sim Ca & = & I \sim a & \sim Ia & = & C \sim a
\end{array}
$$

Here are some from the second category.

$$
\begin{array}{rcl}
Ia \cap b \cap c & \subseteq & C(Ia \cap b) \cap a \\
a \cap Ib & \subseteq & C(a \cap Ib) \cap b \\
I(a \cup b) & \subseteq & (I(Ca \cup b) \cap a) \cup (I(Ca \cup b) \cap b) \\
Ia \cap Ib \cap Ic & \subseteq & a \cap b \cap C(a \cap b \cap c) \\
Ia \cap b & \subseteq & I(a \cup c) \cap C(b \cap Ia) \\
a \cap Ib & \subseteq & a \cap C(a \cap C(a \cap Ib)) \cap b \\
b \cap Ia & \subseteq & C(C(b \cap Ia) \cap a) \\
Ca \cap I(Ia \cup b) \cap c & \subseteq & Ia \cup (Ca \cap b)
\end{array}
$$

$$
\begin{array}{l}
(I(a \cup b) \cap Ic \cap C(c \cap a)) \cup (I(a \cup b) \cap Ic \cap C(c \cap b)) \\
\qquad = Ic \cap I(a \cup b) \\
(I(a \cup c) \cap I(a \cup b) \cap C(c \cap b)) \cup (I(a \cup c) \cap I(a \cup b) \cap Ca) \\
\qquad = I(a \cup b) \cap I(a \cup c)
\end{array}
$$

2.3 Purely RQ5 Properties

After 5000 deduction steps, our prover has ended with 294 properties describing RQ5. We subtracted from them the common 217 properties (using the matching program again) we got 77 as probable *candidates* for *purely* RQ5 properties. Then we studied their proofs, one by one. We have looked for those which used *only* RQ4 axioms i.e. we selected the proofs in RQ5 in which only the RQ4 axioms were used. We found 27 of such proofs. We eliminated those 27 properties from the set of 77 RQ5 candidates. This process gave us *50 purely RQ5 properties.*

Here are some of them.

$$
\begin{array}{rcl}
ICa & = & Ca \\
a & \subseteq & I(Ca \cup b) \\
I(Ca \cup Ib) & = & Ca \cup Ib \\
C(Ca \cap Ib) & = & Ca \cap Ib
\end{array}
$$

2.3.1 Observations

One of the features of the prover is that it always introduces more and more variables and tries to generate new properties in this way. For example, we have found (by direct examination) that between our 50 purely RQ5 properties the following 2 variables property.

$$I(Ia \cup Ib) = Ia \cup Ib$$

We have also found the following 3 and 4 variables properties.

$$
\begin{array}{rcl}
I(Ia \cup Ib \cup Ic) & = & Ia \cup Ib \cup Ic \\
I(Ia \cup Ib \cup Ic \cup Id) & = & Ia \cup Ib \cup Ic \cup Id
\end{array}
$$

They are in fact, 3 and 4 variables generalizations of the first 2 variables property. They follow an obvious pattern listed below (where $m > 0$).

$$I(Ia_1 \cup \ldots \cup Ia_m) = Ia_1 \cup \ldots \cup Ia_m$$

The proof by mathematical induction is straightforward.

There were many other patterns. For example the following formulas

$$
\begin{array}{rclrcl}
I(Ca \cup Cb) & = & Ca \cup Cb & C(Ca \cap Cb) & = & Ca \cap Cb \\
I(Ca \cup Ib) & = & Ca \cup Ib & C(Ca \cap Ib) & = & Ca \cap Ib \\
I(Ia \cup Ib) & = & Ia \cup Ib & C(Ia \cap Ib) & = & Ia \cap Ib
\end{array}
$$

together with their 3,4 variables generalizations can be described by the following two patterns ($n+m > 0$).

$$
\begin{aligned}
&I(Ca_1 \cup \ldots \cup Ca_n \cup Ib_1 \cup \ldots \cup Ib_m) \\
&\qquad = Ca_1 \cup \ldots \cup Ca_n \cup Ib_1 \cup \ldots \cup Ib_m \\
&C(Ca_1 \cap \ldots \cap Ca_n \cap Ib_1 \cap \ldots \cap Ib_m) \\
&\qquad = Ca_1 \cap \ldots \cap Ca_n \cap Ib_1 \cap \ldots \cap Ib_m
\end{aligned}
$$

These are only few between many patterns and generalizations.

3 RQ algebras and topological Boolean algebras

It is obvious from the representation theorem for RQ5 algebras ([9]) that the principal boolean property

$$a \cup \sim a = 1$$

does not hold in neither RS5 nor in RS4. It was proved in [7] that when we add the above property to the axioms of the quasi-Boolean algebra we obtain a Boolean algebra. The quasi complementation $\sim$ becomes in this case a classical set theoretical complementation.

This proves the correctness of the following definition.

Definition 3.1 *RQ4 (RQ5) algebra* $(A, 1, \cup, \cap, \sim)$ *with one of one of the following additional axioms (where* 0 *denotes* ~ 1*)*

$$a \cup \sim a = 1 \quad \text{or} \quad a \cap \sim a = 0$$

is called a S4 (S5) topological Boolean algebra.

We added those two axioms to the set of axioms of RQ4 (RQ5), respectively and let the prover run.

The prover has derived more than 300 properties for these topological S4, S5 Boolean algebras. Here are some we found interesting.

$$
\begin{aligned}
Ia \cup C \sim a &= 1 \qquad & Ia \cap C \sim a &= 0 \\
Ca \cup I \sim a &= 1 \qquad & Ca \cap I \sim a &= 0
\end{aligned}
$$

$$
\begin{aligned}
&(Ca \cap Cb) \cup (Ca \cap c) \cup (I \sim b \cap \sim c) \cup I \sim a = 1 \\
&(Ca \cap b) \cup Cc \cup (I \sim c \cap I \sim a) \cup (I \sim c \cap \sim b) = 1 \\
&(C(a \cap I \sim b) \cap Cb) \cup (C(a \cap I \sim b) \cap \sim a) \cup (a \cap I \sim b) \\
&\qquad = C(a \cap I \sim b) \\
&(Ia \cap C(a \cap C \sim a)) \cup (a \cap C \sim a) = a \cap C(a \cap C \sim a) \\
&(I \sim a \cap Ib) \cup (\sim a \cap b \cap Ca) \cup (\sim a \cap b \cap C \sim b) = \sim a \cap b \\
&(Ca \cap \sim b) \cup (Ca \cap c) \cup (b \cap \sim c) \cup I \sim a = 1 \\
&(Ia \cap b) \cup Cc \cup (I \sim c \cap C \sim a) \cup (I \sim c \cap \sim b) = 1
\end{aligned}
$$

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