On Generalized Rough Sets

(EXTENDED ABSTRACT)

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1 Generalized Rough Sets

The idea of rough set was proposed by Pawlak in 1982 [10]. The starting point of the rough set theory is the assumption that we have initially some information (knowledge) about the elements of the universe and that some elements can be indiscernible (equivalent) in the context of the information about them, that they can be "seen" as the same. These assumptions justified the following definitions [11].

Approximation space Let U be a non-empty set called a universe, and let R be an equivalence relation on U. The tuple (U, R) is called an approximation space.

Lower, upper approximations Let (U, R) and be the approximation space and $A \subset U$. Denote by $[u]$ the equivalence class of R. The sets $\underline{R}A = \bigcup\{[u] \in A/R : [u] \subset A\}$, $\overline{R}A = \bigcup\{[u] \in A/R : [u] \cap A \neq \emptyset\}$ are called *lower* and *upper* approximations of A, respectively. This mean that any set A can be observed through its lower and upper approximations $\underline{R}A$ and $\overline{R}A$. A is observed precisely only if $\underline{R}A = \overline{R}A$. Otherwise A is observed only "roughly", or more precisely, A is called *a rough set.*

The lower and upper approximations treated as one argument operations $\underline{R}$ and $\overline{R}$ in the approximation space (U, R) have standard topological properties of interior and closure operation in topological spaces ([11]). This relationship has been studied by many authors, see for example [8], [14], [19]. We expend here on this relationship in order to justify our definition of the generalized rough set. In order to do so we recall below few basic definitions.

Topological space A tuple (X, I) is called *a topological space* ([4]) if for every $A \subset X$ there is associated a set $IA \subset X$, called the *interior of* A, in such a way thet the following conditions are satisfied for any $A, B \subset X$: $I(A \cap B) = IA \cap IB$, $IA \subset A$, $IIA = IA$, and $IX = X$. For every $A \subset X$, the set $-I-A$ is called the *closure of* A and denoted by CA. Thus by definition, $CA = -I - A$ and $IA = -C - A$.

Approximation topological space By an *approximation topological space* [14] we understand any topological space such that its interior operation is induced by an equivalence relation.

More exactly, let (U, R) be an approximation space. Let $\mathbf{B}$ be a class composed of $U, \emptyset$ and all $[u] \in U/R$. It is easy to see that $\mathbf{B}$ is a basis of uniquely defined interior operation I, namely $IA = \bigcup\{B \in \mathbf{B} : B \subset A\}$ and the triple (U, R, I) is called the *approximation topological space.* We define, as before, $CA = -I - A$.

These definitions of the interior and closure operations in the case under consideration is equivalent with $IA = \bigcup\{[u] \in A/R : [u] \subset A\}$, $CA = \bigcup\{[u] \in A/R : [u] \cap A \neq \emptyset\}$. Moreover, it is easy to prove that the upper and lower approximations in the approximation space (U, R) and interior and closure operations in the topological approximation space (U, R, I) have the following property.

Clopen sets property For any $A \subset U$, $CIA = IA$ and $ICA = CA$. Sets with such property are called *clopen sets* and the property is called a *clopen sets property.*

In 1988 Wiweger [19] proved the following theorem.

Wiweger theorem *Let X be a topological space with an interior operation I, having the clopen sets property. Then there exists an equivalence relation R in X such that the interion operation $I^{'}$, induced by R, coincides with I.*

The above definitions, properties and the Wiweger theorem justify the introduction of a following definition.

Generalized rough set Any topological space (X, I) with a clopen sets property is be called a *generalized rough set.*

2 Topological and Rough Boolean Algebras

Let (X, I) be a topological space (generalized rough set), then the field $\mathbf{P}(\mathbf{X})$ of all subsets of X under set-theoretic operations $\cup, \cap, -$ is called a topological (rough) field of sets.

It is well known that fields of sets are typical examples of Boolean algebras, and that, by Stone ([15]) representation theorem: *every Boolean algebra is isomorphic to a field of sets.*

The topological field of sets are typical examples of *topological Boolean algebras*([5], [6], [13]) defined below.

Topological Boolean algebra By a topological Boolean algebra we mean an abstract algebra $(A, \cap, \cup, \neg, I, C)$, where $(A, \cap, \cup, \neg)$ is a Boolean algebra and the following conditions hold: $I(a \cap b) = Ia \cap Ib$, $Ia \cap a = Ia$, $IIa = Ia$, $I1 = 1$ and $Ca = -I - a$, for any $a, b \in A$.

Rough Boolean algebra A topological Boolean algebra is called a *rough Boolean algebra* iff for any its any element $a \in A$, $ICa = Ca$ and $CIa = Ia$.

Representation theorem for topological Boolean algebras ([5]) can be easly extended to the rough Boolean algebras (called also clopen topological algebras) as follows.

Representation theorem *For every topological (rough) Boolean algebra there exists a monomorphism into a topological (rough) field of all subsets of a topological space (generalized rough set).*

3 Rough Equalities

The notion of *rough equality* was introduced by Pawlak in 1982 [10] as follows.

Rough equality $A \approx_R B$ Given an approximation space (U, R) and any $A, B \subset U$. Let $\underline{R}$ and $\overline{R}$ denote the lower and upper approximation induced by R. We say that the sets A and B are *roughly equal* and denote it by $A \approx_R B$ if and only if $\underline{R}A = \underline{R}B$ and $\overline{R}A = \overline{R}B$.

The rough equality was extensively examined in 1982-85 by Novotny [7], Bronikowski [3] and most recently by Pagliani [9], Banerjee [1], Vigneron-Wasilewska [16] and others. It was examined from the logical point of view by Banerjee [1], from algebraic point of view by Wasilewska [17] and from algebraic-automated theorem proving angle by Wasilewska-Vigneron [18].

In all above cases the notion of the rough equality was restricted to the rough sets only, i.e. to the sets defined by the approximation spaces (U, R) and the lower and upper approximations induced by it.

We propose here to extend the notion of rough equality to general topological spaces (X, I) and generalized rough sets (X, I) and consequently, to topological and rough Boolean algebras, respectively.

Topological equality $\approx_t$ Given a topological space (X, I), for any two sets $A, B \subset X$ we say that they are *topologically equivalent*, and denote $A \approx_t B$ if and only if $IA = IB$ and $CA = CB$.

Given a topological Boolean algebra $(A, \cap, \cup, \neg, I, C)$, for any $a, b \in A$, $a \approx_t b$ if and only if $Ia = Ib$ and $Ca = Cb$.

Topological equality $\approx_r$ In case of a generalized rough set and rough Boolean algebras the topological equality $\approx_t$ will be denoted by $\approx_r$.

4 Rough Equalities as Congruences

Given a topological (rough) Boolean algebra $\mathcal{A}$ = $(A, \cap, \cup, \neg, I, C)$ and a topological equality $\approx_t, (\approx_r)$ the following two natural questions arise.

Q1 Is $\approx_t$ $(\approx_r)$ a congruence with respect to some (all) operations of the topological (rough) Boolean algebra?

Q2 Can we define an algebra $\mathcal{B}$ similar to $\mathcal{A}$, such that topological equality is is a congruence with respect to all its operations?

We prove the following.

Fact 4.1 *If $\mathcal{A}$ is topological Boolean algebra, then $\approx_t$ is a congruence with respect to $-$ only.*

If $\mathcal{A}$ is a rough Boolean algebra, then $\approx_r$ is a congruence with respect to $-$, I, C.

Let $\mathcal{A} = (A, \cap, \cup, \neg, I, C)$ be a topological Boolean algebra. We define in A two binary operations $\sqcup$ and $\sqcap$ in A:

$a \sqcup b = (a \cup b) \cap (a \cup Ib \cap \neg I(a \cup b))$,

$a \sqcap b = (a \cap b) \cup (a \cap Cb \cap \neg C(a \cap b))$.

Let $\mathcal{B} = (A, \sqcup, \sqcap, \neg, I, C)$.

Fact 4.2 *If $\mathcal{A}$ is a* rough Boolean algebra *then $\approx_r$ is a* congruence *with all operations of the algebra $\mathcal{B} = (A, \sqcup, \sqcap, \neg, I, C)$.*

Fact 4.3
The congruent algebra $\mathcal{B}_r = (A/\approx_r, \sqcup, \sqcap, \neg, I, C)$ has the following properties.

- **r1** $(A/\approx_r, \leq, \sqcup, \sqcap, 0, 1)$ *is a distributive lattice with* 0 *and* 1,
- **r2** *For any* $[a], [b] \in A/\approx_r$, $\neg([a] \sqcup [b]) = (\neg[a] \sqcap \neg[b])$,
- **r3** *For any* $[a] \in A/\approx_r$, $\neg\neg[a] = [a]$.
- **r4** *The congruent algebra $\mathcal{B}_r$ is not a Boolean algebra, i.e. there is an element* $[a] \in (A/\approx_r$ *such that* $\neg[a] \sqcap [a] \neq 0$ *and* $\neg[a] \sqcup [a] \neq 1$.
- **r5** *For any* $[a], [b] \in A/\approx_r$, $I([a] \sqcap [b]) = (I[a] \sqcap I[b])$, $I[a] \leq [a]$, $II[a] = I[a]$, $I1 = 1$, and $CI[a] = I[a]$, *where* $C[a] = \neg I \neg [a]$ *and* $\leq$ *is a lattice ordering.*

The rough algebra of Banerjee The rough algebra of Banerjee ([1], [2]) is a *similar algebra* to the algebra $\mathcal{B}_r = (A/\approx_r, \sqcup, \sqcap, \neg, I, C)$ for A being a set F of all formulas of the modal logic $S5$ with an equivalence relation $\approx$, called *rough equality* defined as follows: for any $A, B \in F$, $A \approx B$ if and only if $\vdash ((\Box A \Leftrightarrow \Box B) \cap (\Diamond A \Leftrightarrow \Diamond B))$. The algebra $(F/\approx, \sqcup, \sqcap, \neg, I, C, 0, 1)$, such that the operations $\sqcup$, $\sqcap$ are defined below and the operations $\neg, I, C$ are defined as $\neg[A] = [\neg A]$, $IA = [\Box A]$, and $CA = [\Diamond A]$ is called a rough algebra. For any formulas A,B of logic $S5$, $[A] \sqcup [B] = [(A \cup B) \cap (A \cup \Box A \cup \Box B \cup \neg\Box(A \cup B))]$, and $[A] \sqcap [B] = [(A \cap B) \cup (A \cap \Diamond A \cap \Diamond B \cap \neg\Diamond(A \cap B))]$.

This proves the following.

Fact 4.4 *Banerjee's rough algebra is as a particular case of the algebra $\mathcal{B}$.*

5 Automated Proofs

All proofs of facts and properties listed in the paper were done fully automatically by a theorem prover daTac (*Déduction Automatique dans des Théories Associatives et Commutatives*) developed at CRIN & INRIA Lorraine, Nancy (France), by the second author.

This fact has also an interesting ramification concerning Banerjee work and results. As one can see from her definition of rough equality and rough algebra that both of them are purely *syntactical* and deal with the notion of *formal provability* of certain formulas in modal logic $S5$. But all of her (very difficult sometimes) proofs are *semantical*, for the rough sets in the approximation spaces (U, R) defining ([8]) the semantics for the logic $S5$ under consideration. Our proofs generated by the prover are purely syntactical and can be traslated into formal proofs in the $S5$ logic. We have hence provided, starting from the most general premisses formal proofs for all of Banerjee results about her rough equality and rough algebras.

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