Rough Sets based Proofs Visualisation

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Abstract

We present here an approach we used for proving important properties of clopen topological spaces. We combine powerful theorem provers techniques (and implementations) with a graphical technique based on a graphical representation of a rough set, called Rough Diagrams. Rough Diagrams are a generalization of a classical notion of Venn Diagrams for algebra of sets to clopen topological spaces. We use them as a powerful automated technique of constructing counter-models of properties the prover has a hard time proving and the user might suspect of being false. It means we propose to add a visual tool to a prover that after some fixed number of prover deductions would start constructing a visual counter-model for a property the prover is trying to prove. A prover with the visual tool is called a visual prover. The visual prover has a completeness property: for any rough set equality we can construct its proof or its counter-model.

1. Introduction

The principle of rough sets [2] is to consider that some elements of the universe are equivalent in the context of the knowledge about them. This theory provides a complementary model to fuzzy [3] or evidence [5] models for dealing with imprecise, noisy, or incomplete information. It is also a departure point for studies in machine learning, knowledge discovery and lately in database mining [13]. It has also been applied to various domains such as medical diagnosis, information retrieval, control algorithms acquisition, and market analysis.

We have extended this notion of rough equality to generalized rough sets [10], i.e. any topological space with a clopen property.

A *topological space* is defined by a function I over a set X, called the *interior of* A, such that the following properties are satisfied, for any $A, B \in X$: $I(A \cap B) = IA \cap IB$, $IA \subset A$, $IIA = IA$, and $IX = X$. For every $A \subset X$, the set $-I-A$, where $-$ is the complement function over X, is called the *closure of* A and denoted by CA.

A topological space satisfies the *Clopen sets property* if, for any $A \subset X$, $CIA = IA$ and $ICA = CA$.

This relation between rough sets and topological spaces has been described by Wiweger [12]: rough sets represent any approximation topological space, i.e. any triple (X, I, R), where (X, I) is a topological space, R is an equivalence relation, and I is induced by R.

Topological spaces and clopen topological spaces being represented respectively by topological Boolean algebras and rough Boolean algebras [6, 1, 4, 9], we have used a theorem prover for studying these abstract algebras.

This study was motivated by the lack of information concerning these algebras. This is due to the fact that they are very difficult to study "by hand". Running an automatic tool brings this advantage to perform a fastidious and repetitive work.

The prover used is called daTac, and has been developed by one of the authors [7]. Its principle is to apply deductions in first-order logic with equality. Its specificity is to apply deductions with a built-in theory. This theory is composed with associative-commutative properties of operators (such as union and intersection for Boolean algebras), some very costly properties when explicitely used as axioms.

We have mainly used the prover as a generator of properties. The principle is to give daTac the specification of an algebra and to ask the prover to do some deductions. Then it is arbitrarily stopped after some time, and we get the set of newly generated properties.

For topological and rough Boolean algebras, we got hundreds of new properties [11]. From these properties, we have proved some patterns, and also compared algebras, isolating for instance some formulas of rough Boolean algebras that are not properties of topological Boolean algebras.

We have also studied the rough equality [10]. We have shown that there exists an algebra such that the topological equality is a congruence with respect to the operators of that algebra. This has been done by defining new union and intersection operators.

2. Limits of the Prover

We have used the prover daTac for generating hundreds of properties. This has been obtained after hours (or days, sometimes) of automatic work, impossible for a human being. But a major problem is to exploit this amount of new knowledge. Some properties may be quite large, with several variables. They may be difficult to understand. What we have done is to infer some patterns from a subset of properties, and sometimes we had got the intuition of some additional properties.

Using the prover for trying to confirm an idea is a natural step, but this is not always successful. Since there are infinitely many properties, the user cannot be sure that the prover will be able to prove something in a reasonably delay. This may take several days, or the prover may even stop by lack of memory.

For instance, we have inferred the following patterns, for all n and m:

$$\begin{aligned} & I(Ca_1 \cup \cdots \cup Ca_n \cup Ib_1 \cup \cdots \cup Ib_m) \\ & \qquad = Ca_1 \cup \cdots \cup Ca_n \cup Ib_1 \cup \cdots \cup Ib_m \\ & C(Ca_1 \cap \cdots \cap Ca_n \cap Ib_1 \cap \cdots \cap Ib_m) \\ & \qquad = Ca_1 \cap \cdots \cap Ca_n \cap Ib_1 \cap \cdots \cap Ib_m \end{aligned}$$

This intuition has been due to the fact that the prover has proved some of the first elements of these patterns. For a formal proof of the whole pattern, we have had to study how the first elements have had been generated, and we have found a generalization of these proofs. For this step, the prover could not help.

Then, from these patterns and from our try to define new union and intersection operators for proving topological equality can be a congruence, we have had the intuition of the following properties:

$$\begin{array}{ll} I(a \cup Ib) = Ia \cup Ib & C(a \cup Cb) = Ca \cup Cb \\ C(a \cup Ib) = Ca \cup Ib & I(a \cup Cb) = Ia \cup Cb \end{array}$$

But the prover has been unable to prove even one of them, and we have had no idea of how to do it. All our tries have been unsuccessful, and we have doubt of the existence of these properties.

In general, this happens very often that we do have an intuition for a property, but not for the way to prove it. And sometimes we have the intuition that a property is false, but we are unable to find a counter-example.

In this case of rough sets, we have used a graphical representation for being able to study some particular cases (see Figure 1).

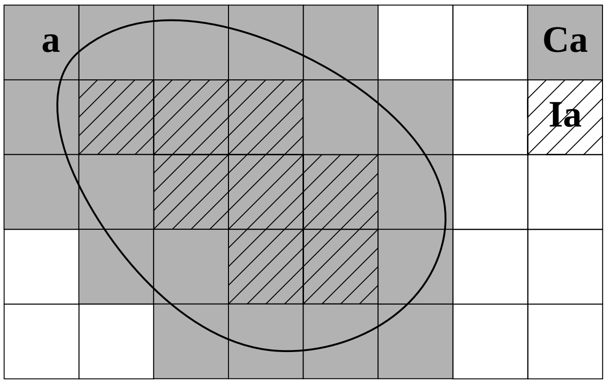

Figure 1. Rough Diagrams

3. Rough Diagrams

Rough Diagrams are a generalization of a classical notion of Venn Diagrams for algebra of sets to clopen topological spaces.

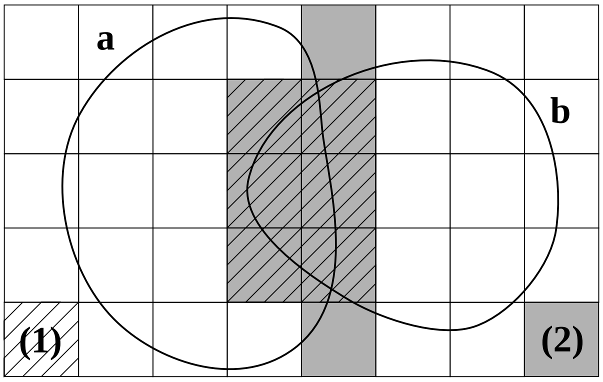

Figure 2. (1): $C(a \cap b)$**, (2):** $Ca \cap Cb$

A graphical representation has the advantage to be much more easy to understand than an algebraic formula. For instance, there are two important properties that are not satisfied by rough Boolean algebras: $C(a \cap b)$ is not equal to $Ca \cap Cb$, and $I(a \cup b)$ is not

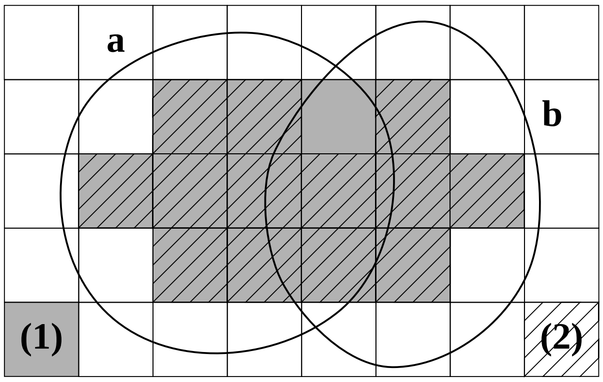

Figure 3. (1): $I(a \cup b)$**, (2):** $Ia \cup Ib$

equal to $Ia \cup Ib$. One can build counter-examples of these properties, as illustrated in Figures 2 and 3.

Finding counter-examples by drawing sets is much more easy than doing calculi with algebraic formulas. Such graphical calculi can also be used for testing some cases of a property that looks to be satisfied, as shown in Figure 4 for $I(a \cup Ib) = Ia \cup Ib$.

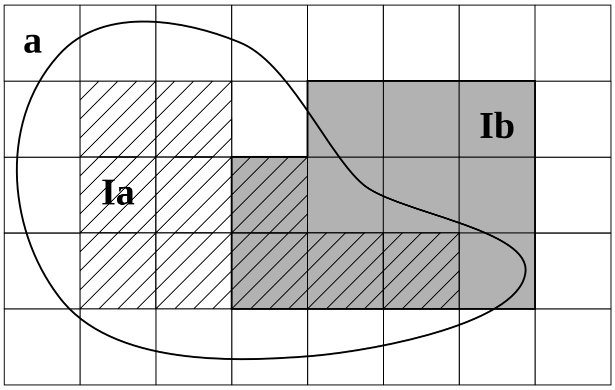

Figure 4. $I(a \cup Ib) = Ia \cup Ib$

We have used rough diagrams for guiding our search for some properties, when the prover unable to do it. We have also used them for defining new union and intersection operators [8] for defining a new algebra, and proving topological equality is a congruence. Without this graphical representation, we would never have succeeded to do it.
For example, this has been easy to show that, in the congruent algebra, $a \sqcup \neg a$ is not equivalent to 1, where $\sqcup$ is our new union operator; $I(a \sqcup \neg a)$ and $C(a \sqcup \neg a)$ would have to be equal to $I1$ and $C1$, respectively. A counter-example is show in Figure 5.

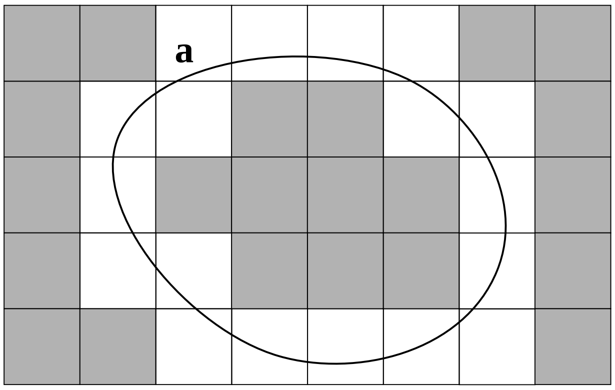

Figure 5. $I(a \sqcup \neg a)$ **(**$\neq I1 = 1$**)**

4. Towards A Visual Prover

From our experiments with Rough Diagrams, we wonder if this could be possible to define a method for building counter-examples, graphically. Such a method can be based on a database of the most current cases, for two or three sets. Looking for a counter-example would consist in trying each case, and trying new cases resulting of the combination of the initial ones.
An even more interesting result would be to define a general method, able to deal with any kind of formula, whatever the number of sets is. This should be possible by decomposing the formula for considering simple cases, and then combining these cases for considering back the whole formula.

We are currently working on this subject, and the first results are very encouraging. Our purpose is to define a visual tool, either automatic or semi-automatic. This tool would be very useful for a lot of researchers working on rough Boolean algebras, or modal logics since the S5 modal logic is a special case of these algebras.

Besides this use of the visual tool, we propose to combine it with a prover, to get a visual prover. Our study of clopen topological spaces has been done by the prover and by Rough Diagrams. We have worked on Rough Diagrams when the prover was unable to prove some properties. And we have used the prover for getting a formal proof of some properties pictured after some drawings. These two aspects are complementary. Offering the opportunity to combine them into an automatic tool is our challenge. This would lead to a complete tool, able to look for a formal proof or to find counter-examples.

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