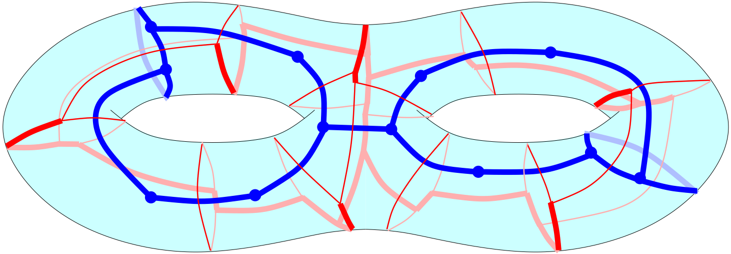
Computational topology of graphs on surfaces

Éric Colin de Verdière

CNRS, LIGM, Université Paris-Est Marne-la-Vallée, France

(joint works by/with many coauthors)



Summer School "Low-Dimensional Geometry and Topology: Discrete & Algorithmic Aspects"

Expected audience

- Graduate students and researchers,
- Computer scientists and mathematicians,
- Communities expected: computational geometry/topology; differential/Riemannian/topological geometry.

Speakers (~ 8 hours of lectures each)

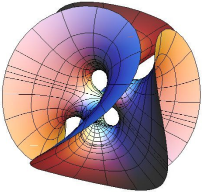
- Jeff Erickson (UIUC),
- Joel Hass (UC Davis).

Shorter talks

about 12 shorter talks will be planned.

When, where

- June 18–22, 2018,
  just after SoCG 2018 in Budapest!
- In the center of Paris (IHP).



Organizing team

Computer scientists and mathematicians at U. Paris-Est Marne-la-Vallée (Labex Bézout): É. Colin de Verdière, X. Goaoc, L. Hauswirth, A. Hubard, S. Sabourau.

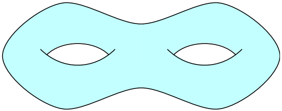
Registration

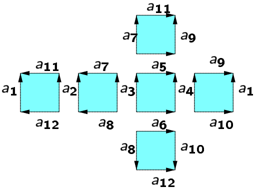
- free but mandatory!
- deadline not before March 15, 2018.

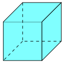
More informations

`http://geomschool2018.univ-mlv.fr/`
Feel free to contact the organizers!

Graphs on surfaces







What's a surface? Equivalent definitions:

1. A (compact, connected) 2-manifold.
2. A space obtained by gluing edges of disjoint polygons in pairs.
3. (In the orientable case): A topological space obtained from the sphere by attaching $g \geq 0$ *handles*; g is the genus.

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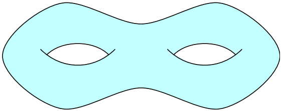

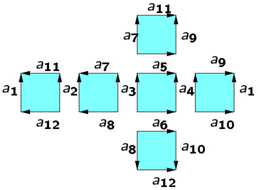

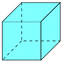

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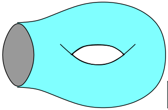
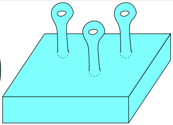
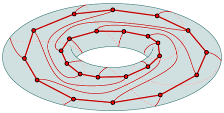

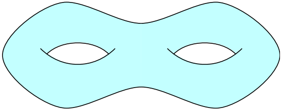

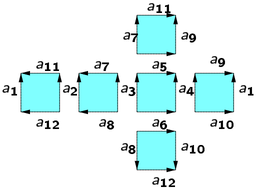

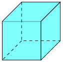

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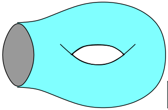
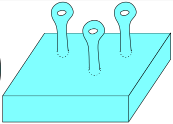
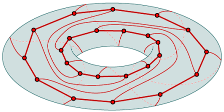






This talk: Graphs embedded (drawn without crossings) on surfaces, in the field of computational topology.

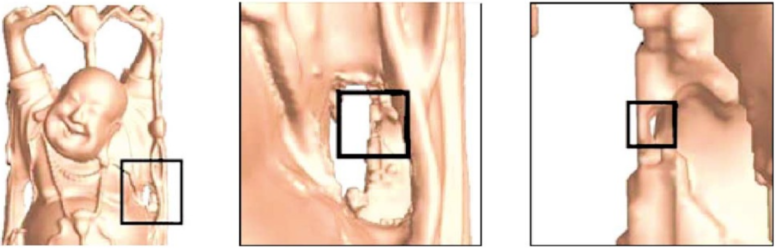
Roadmap of the talk

1. quick survey on topological graphs on surfaces in different fields of mathematics and computer science
2. decision problems: deformations of curves and graphs (homotopy/isotopy)
3. shortest non-contractible closed curves
4. topological decompositions of surfaces
5. other problems solved
6. open problems

1. Topological graphs on surfaces in general

Applications

- Topological simplification, remeshing, approximation;



[Wood et al., 2004]

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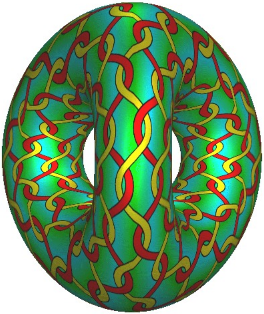

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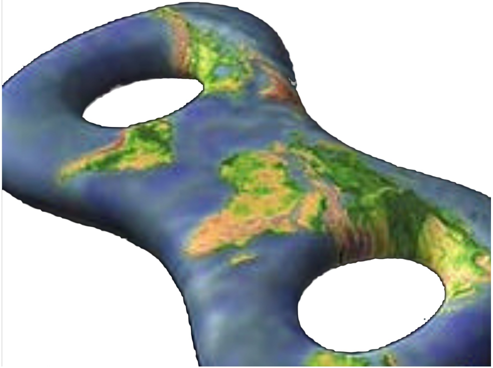

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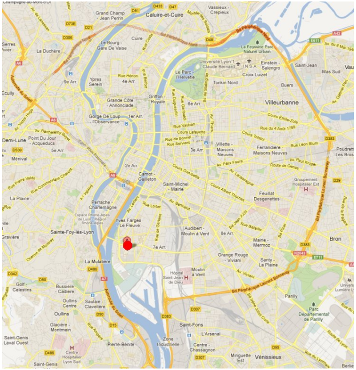

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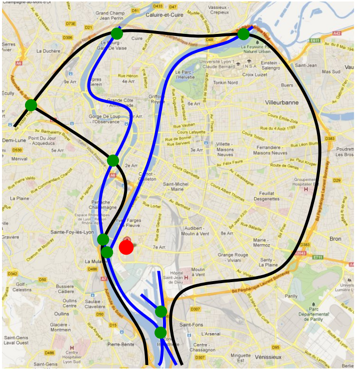

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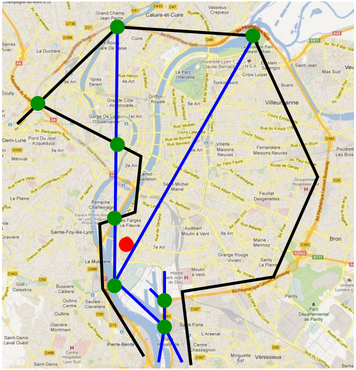

Topology

- Gigantic recent progress in 3-dimensional topology (Poincaré conjecture [Perelman, 2003], . . .);
- "computational" motivations: classify 3-manifolds, decide if a knot is trivial [Haken 1961, Kuperberg 2011, Lackenby 2016, . . .], braids, . . . ;
- algorithms on surface-based structures:
  - algebraic structures (surface groups, mapping class groups, . . .);
  - representation of curves (train tracks, curve complex, pants complex, . . .);
  - deformability of curves to make them disjoint.

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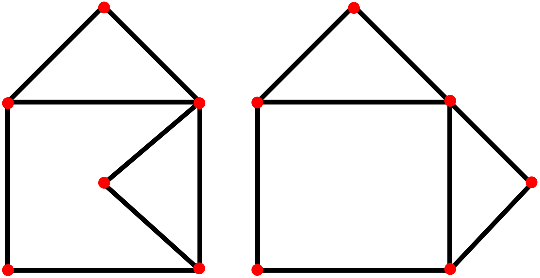
In computational topology. . .

- algorithmically more precise;
- topologically more elementary;
- more "concrete" problems (?).

Combinatorial maps (=rotation system)

- A graph is cellularly embedded if its faces are disks.
- Combinatorial maps represent cellular embeddings combinatorially.



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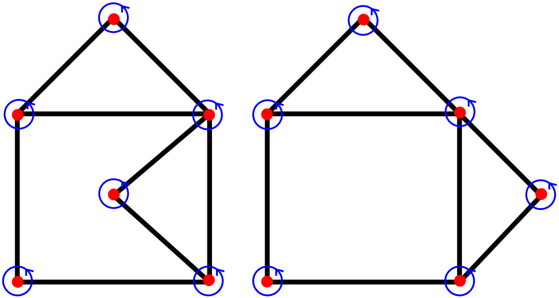

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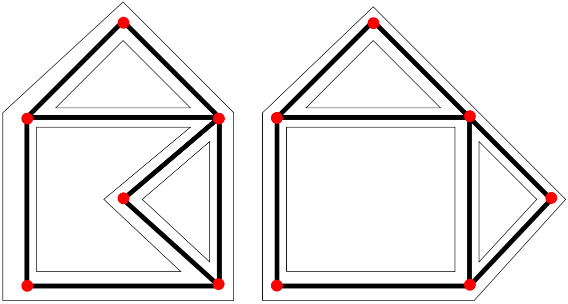

Enumerative combinatorics

Combinatorial maps (=rotation system)

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Enumeration

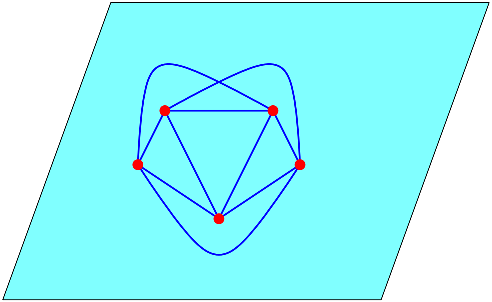
Given g, n, count (exactly or asymptotically) combinatorial maps with genus g and n vertices: rooted / triangulations or quadrangulations / cut graphs / ...

Typical and limit properties

- Properties of a random map, diameter, etc.
- Scaling limits: limits of random maps.

Topological graph theory

Natural generalization of planar graphs: Every graph can be embedded on some surface.



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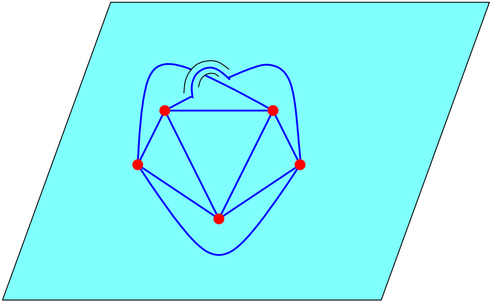

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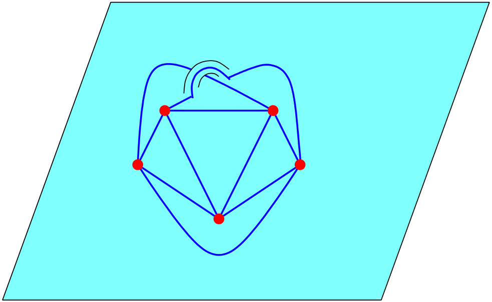


Testing whether a graph with n vertices and edges embeds on a surface of genus g:

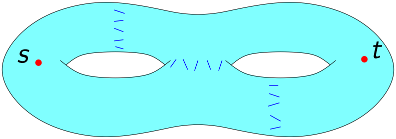
- running time $2^{\text{poly}(g)} \cdot n$ [Mohar, 1996...];
- NP-hard (no polynomial-time algorithm unless P=NP) if g is part of the input;
- space complexity, approximation of the genus, ...

Graph algorithms

General recipe

- Take any graph algorithm problem;
- study it in the specific case where the input graph is embedded in the plane;
- or more generally on a fixed surface.

Examples: Minimum cut, maximum flow, induced cycles, graph isomorphism, minimum multicut, Steiner tree, TSP, ...



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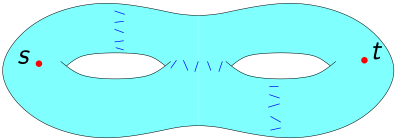


Planar graphs are rather limited (add one edge and you cannot do anything).

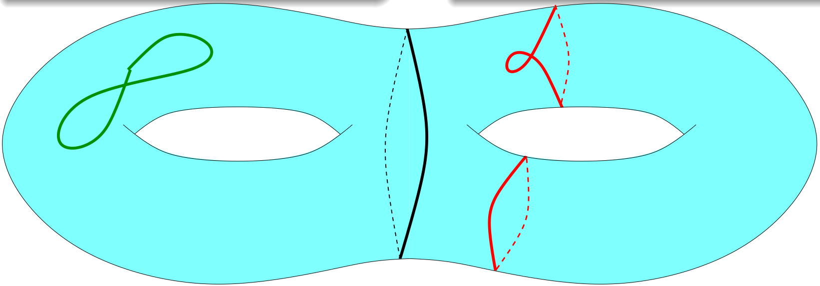
2. Decision problems: homotopy and isotopy

Testing homotopy

Let G be a graph cellularly embedded on $\mathscr{S}$.

Given a closed curve γ in G, decide whether γ is contractible in $\mathscr{S}$ (can be continuously deformed to a point).

Given two closed curves γ and δ in G, decide whether they are freely homotopic in $\mathscr{S}$ (can be deformed one into the other on $\mathscr{S}$).



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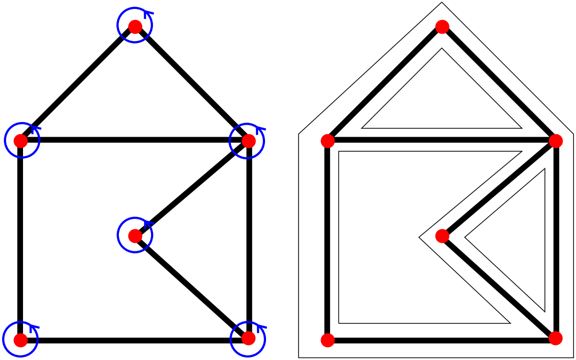
Remarks

- These problems date back to Poincaré (*word problem, conjugacy problem for surface groups*).
- In 1912, Dehn gives a solution, which translates to a polynomial-time algorithm.
- There is more to be said: These problems are solvable in linear time [Dey, Guha 1999][Lazarus, Rivaud, 2012][Erickson, Whittlesey 2013].

Data structures

Data structures

- of size linear in the number n of edges
- allowing to do reasonable operations efficiently:
  - visit the vertices/edges/faces in $O(n)$ time,
  - degree of a face/vertex in $O(\text{degree})$, ...



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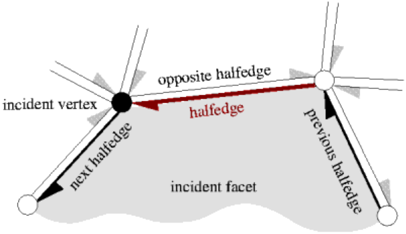
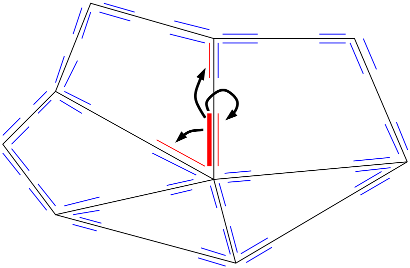




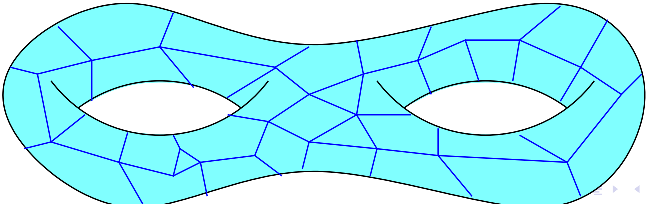
Reduction

[Lazarus, Rivaud, 2012]: WLOG, G has

- two vertices, of degree $4g$,
- $4g$ edges, and
- $2g$ faces, which are quadrilaterals.

Proof

- WLOG, there is a single vertex (edge contractions).
- WLOG, there is a single face (edge deletions).
- By Euler's formula $v - e + f = 2 - 2g$, the graph has $2g$ edges.
- The curves γ (and δ) use edges that were removed, but we can transform them by creating a new vertex.



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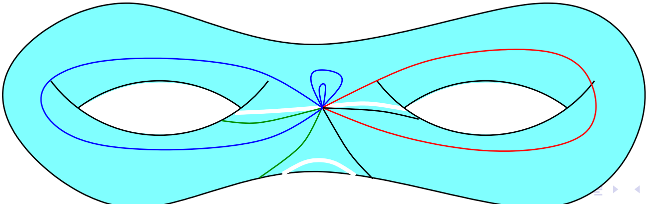

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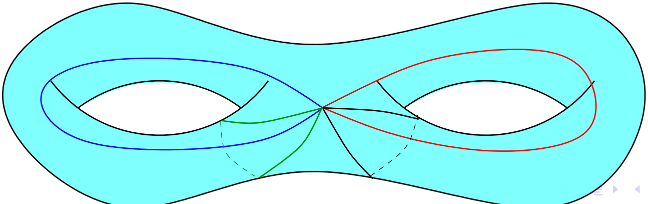

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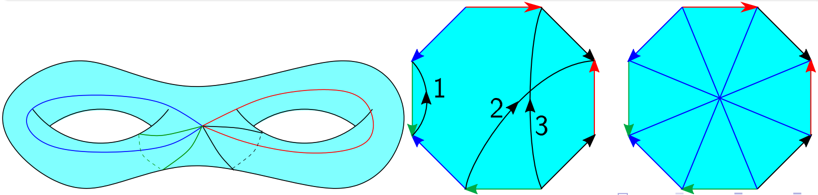

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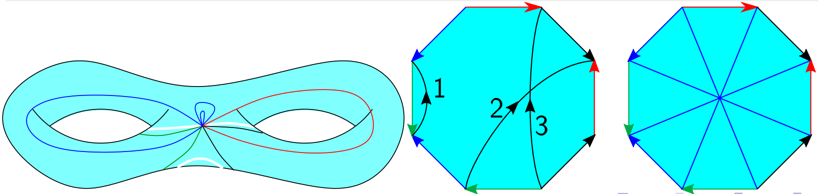

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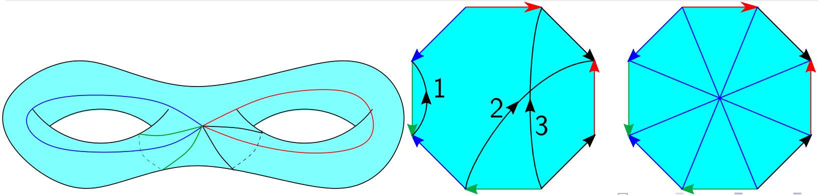

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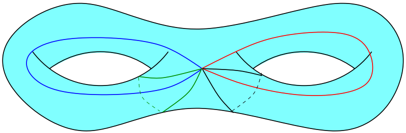

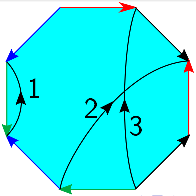

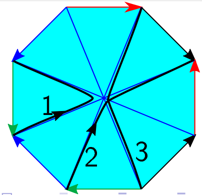

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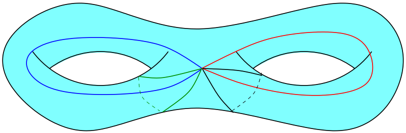
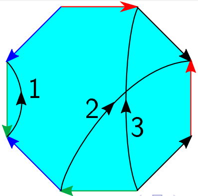
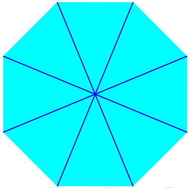


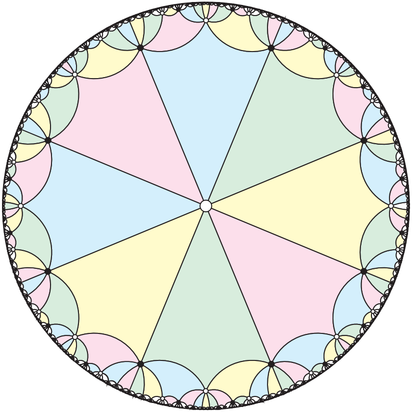


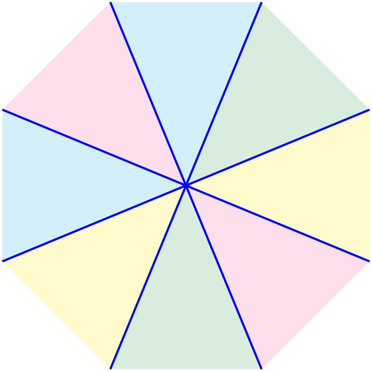


Universal cover $\tilde{\mathscr{S}}$

A regular tiling of squares meeting $4g$ at a vertex:

- Every path in $\mathscr{S}$ lifts to a path in $\tilde{\mathscr{S}}$;
- a closed curve is contractible in $\mathscr{S}$ iff it lifts to a closed curve.





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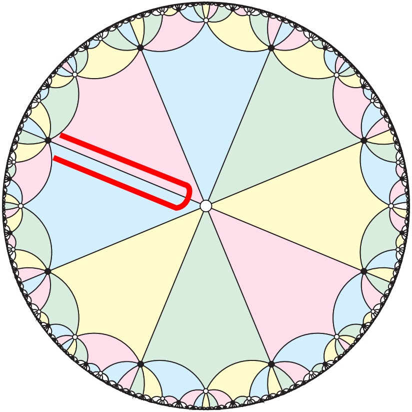

Result from geometric group theory [Gersten, Short, 1990]

In this tiling, every non-trivial closed curve has either a spur or a bracket.

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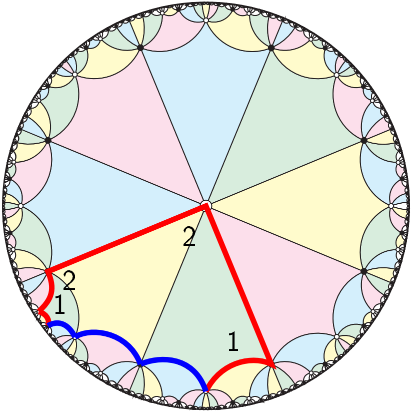

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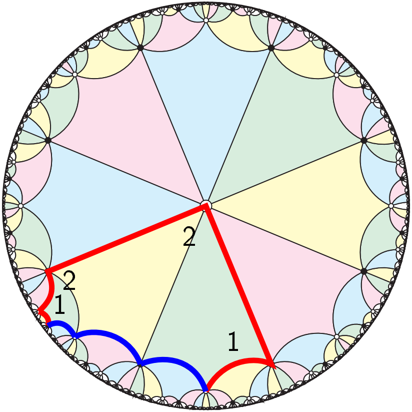

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In this tiling, every non-trivial closed curve has either a spur or a bracket.

Algorithm [Erickson, Whittlesey 2013]

Remove iteratively spurs and brackets whenever possible!

Extension: Geometric intersection numbers

The game

- Given a curve γ, move it continuously (by a homotopy) to minimize its number of crossings.
- Given two curves γ and δ, move them continuously (by a homotopy) to minimize the number of crossings between them.

Result [Despré, Lazarus, 2017]

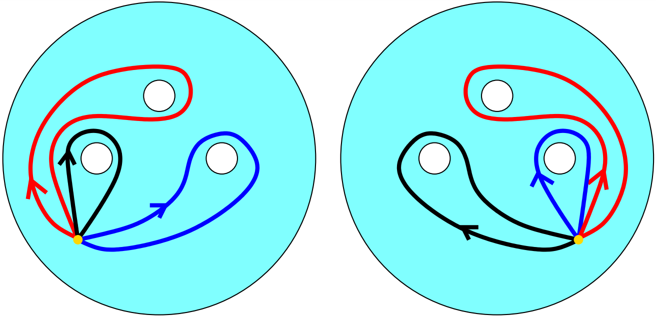
Computing these numbers is doable in near-linear time.

Proof

Similar spirit (ask Vincent at the coffee break).

Testing isotopy

An isotopy of an embedded curve or graph is a homotopy (deformation) that remains crossing-free at all times.



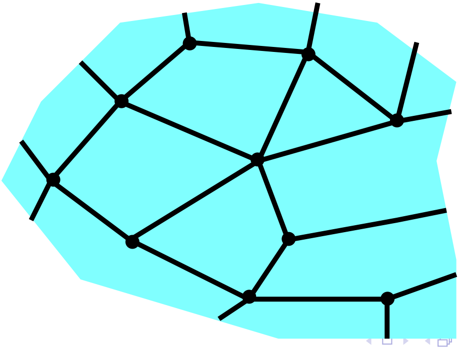
Problem

Given an abstract graph G embedded in two different ways, G_1 and G_2, on $\mathscr{S}$, does there exist a continuous family of embeddings between G_1 and G_2?

This is possible in linear in the input size [CdV, de Mesmay, 2014].

Storing graphs on surfaces

- Let M be a fixed graph (cellularly) embedded on $\mathscr{S}$.
- The graphs G_1 and G_2 are in general position with respect to M.
- We store the combinatorial map of the overlay of M and G_1, and similarly the overlay of M and G_2.



Data structures for storing graphs on surfaces

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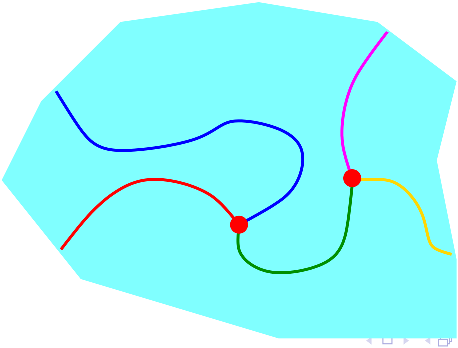

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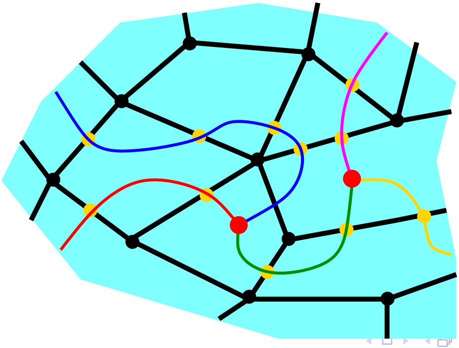
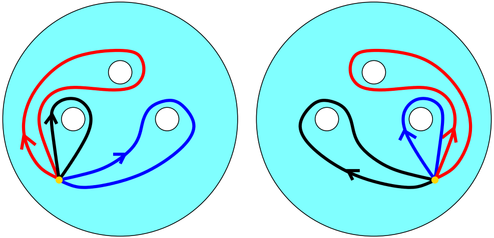


Proof sketch



Some clearly necessary conditions that turn out to be sufficient [Ladegaillerie, 1984]

1. Oriented homeomorphism of $\mathscr{S}$ mapping G_1 to G_2;
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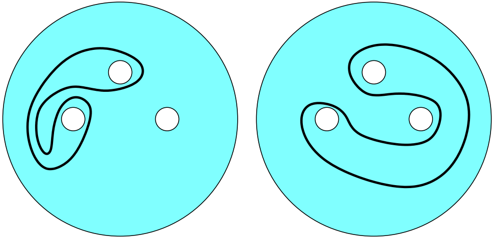

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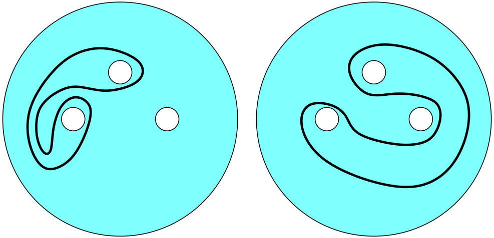

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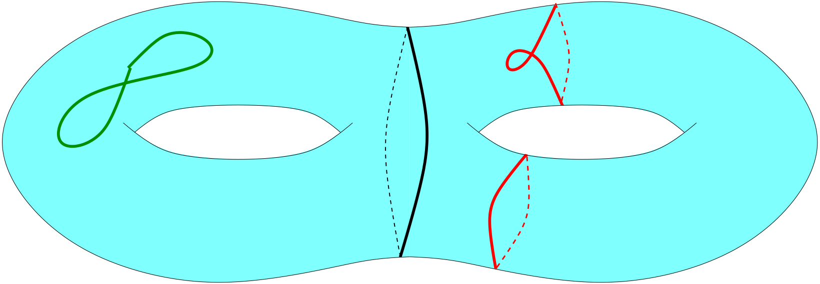
- algorithmically: 1 easy, 2 as before;
- difficulty: small family of cycles for 2;
- tools: universal cover, hyperbolic geometry, Reidemeister moves, [Ringel, 1955], [de Graaf and Schrijver, 1997], ...

3. Shortest non-contractible closed curves

Goal

Problem

Compute a shortest non-contractible closed curve (a.k.a. systole, a.k.a. edge-width).



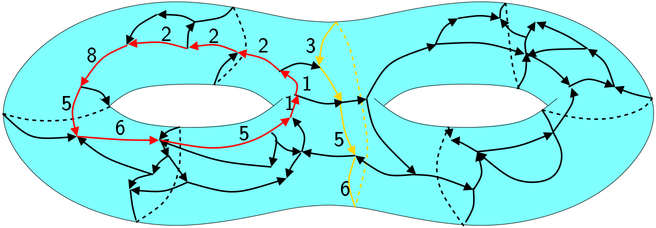
Remark

Similar algorithms for shortest non-separating closed curve.

In a graph cellularly embedded on $\mathscr{S}$: many results!

n: complexity of G g: genus k: output size

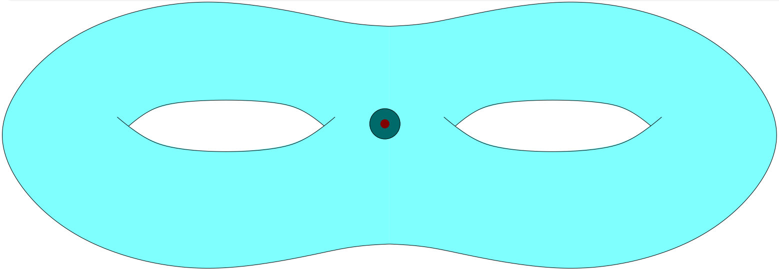
	directed	directed
weighted	$\boldsymbol{O(n^2 \log n)}$ [Erickson–Har-Peled'04] $O(g^{3/2} n^{3/2} \log n)$ [Cabello–Mohar'07] $g^{O(g)} n \log n$ [Kutz'06] $O(g^3 n \log n)$ [Cabello–Chambers'07] $\boldsymbol{O(g^2 n \log n)}$ [Cabello–Chambers–Erickson'13] $\boldsymbol{2^{O(g)} n \log \log n}$ [Fox'13] $\boldsymbol{O(gn \log n)}$ for 2-approx [Erickson–Har-Peled'04]	$O(n^2 \log n)$ [Cab–CdV–Laz'10] $\boldsymbol{O(g^{1/2} n^{3/2} \log n)}$ [Cab–CdV–Laz'10] $g^{O(g)} n \log n$ [Erickson'11] $\boldsymbol{O(g^3 n \log n)}$ [Fox'13]
unweighted	$O(n^3)$ [Thomassen'90] $\boldsymbol{O(n^2)}$ [Cab–CdV–Laz'10] $\boldsymbol{O(gnk)}$ [Cab–CdV–Laz'10] $\boldsymbol{O(gn/\varepsilon)}$ for $(1+\varepsilon)$-approx [Cab–CdV–Laz'16]	$\boldsymbol{O(n^2)}$ [Cab–CdV–Laz'10] $\boldsymbol{O(gnk)}$ [Cab–CdV–Laz'16]



Cut locus

Intermediate step

Let us compute a shortest closed curve passing through a fixed basepoint b.



Construction

- Grow a disk around b; the cut locus C is the set of points where the disk self-collides.
- Formally, it is the (closure of the) set of points with several shortest paths to b.
- $\mathscr{S} \setminus C$ is (homeomorphic to) an open disk.

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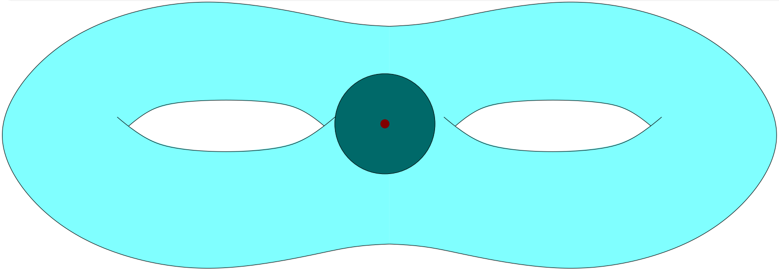

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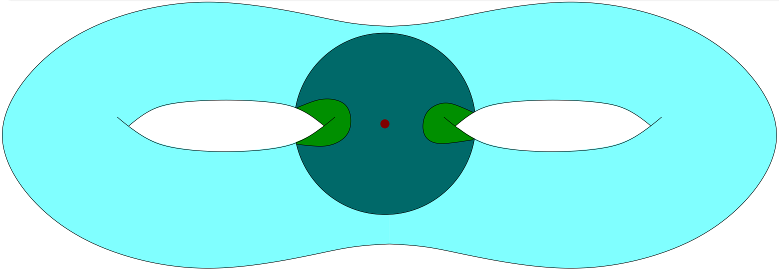

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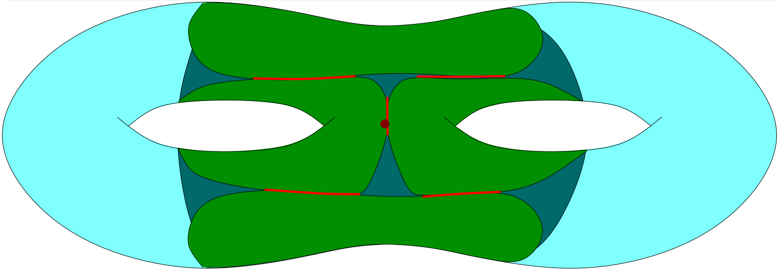

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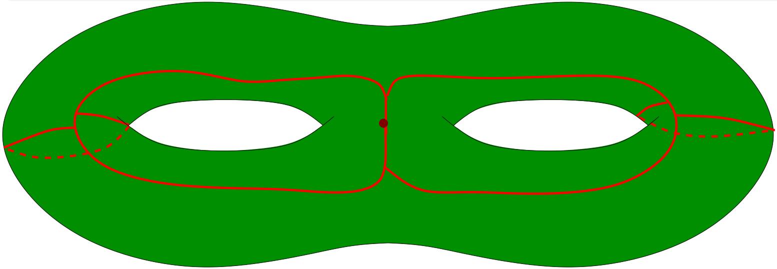

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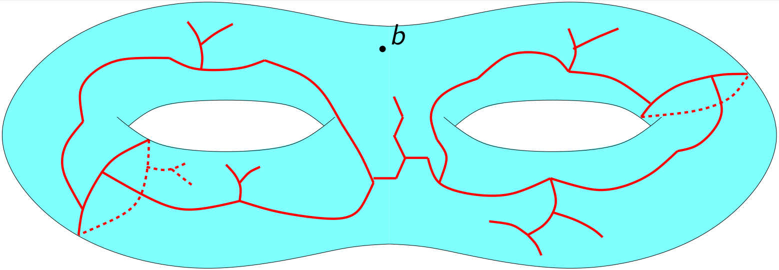

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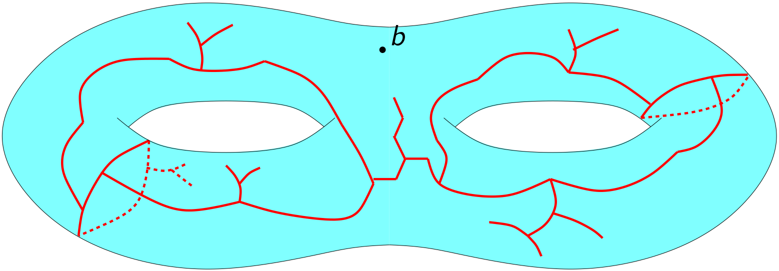

Lemma

- A shortest non-contractible loop based at b crosses the cut locus C exactly once;
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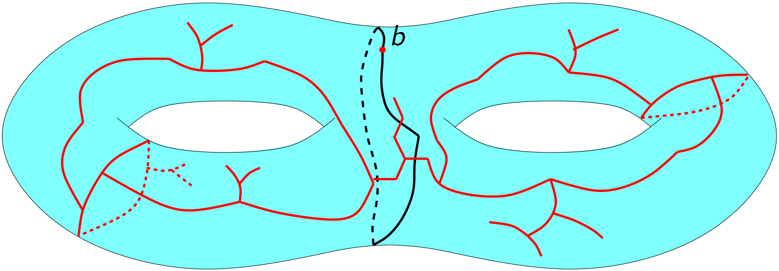

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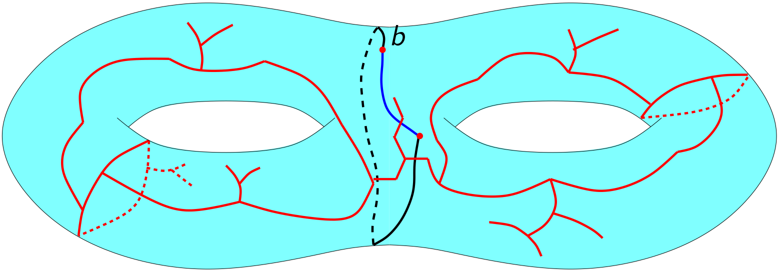

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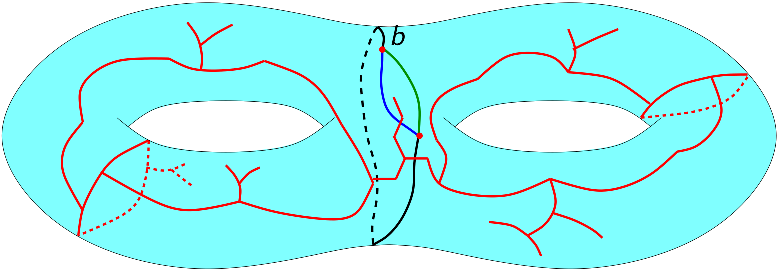

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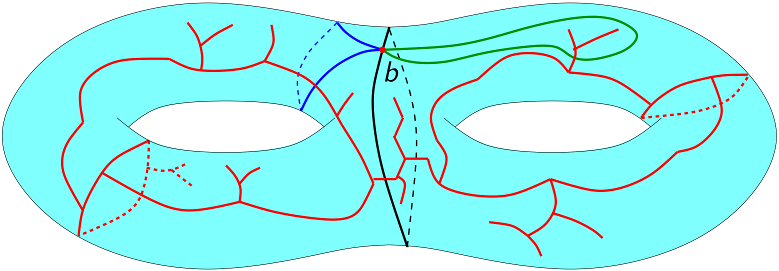

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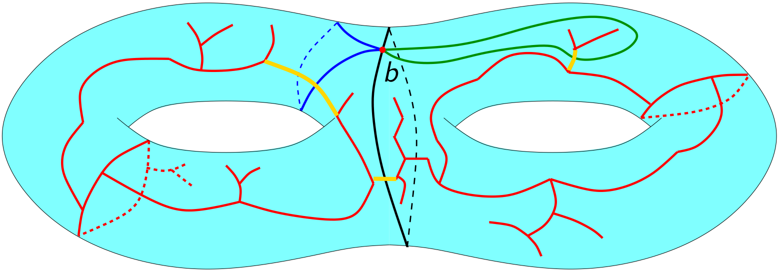

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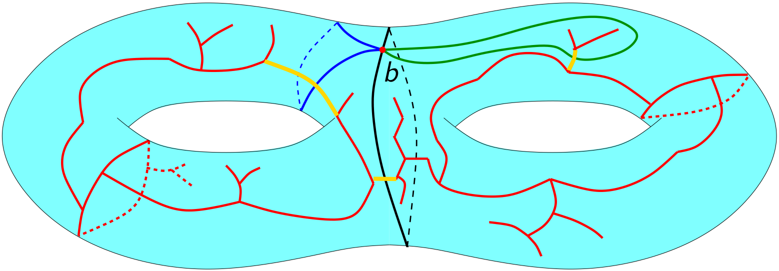

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Generic algorithm [Erickson and Whittlesey, 2005] [CdV, 2009]

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- return the shortest loop from b crossing C' exactly once.

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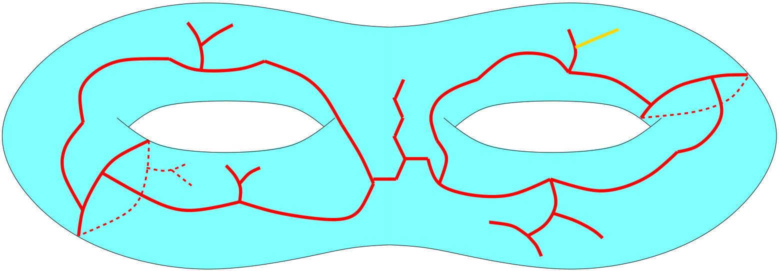

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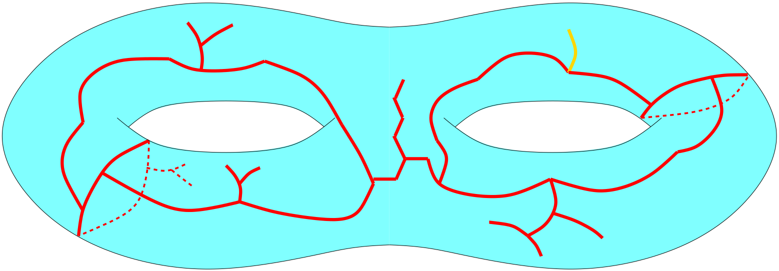

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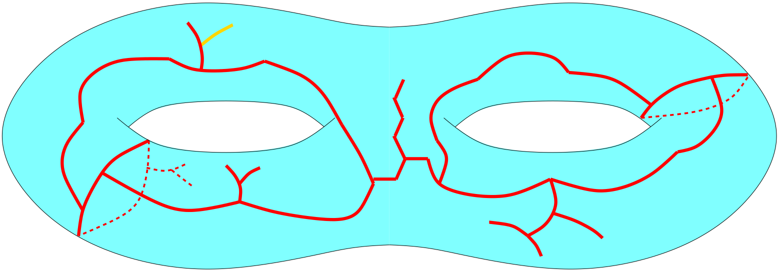

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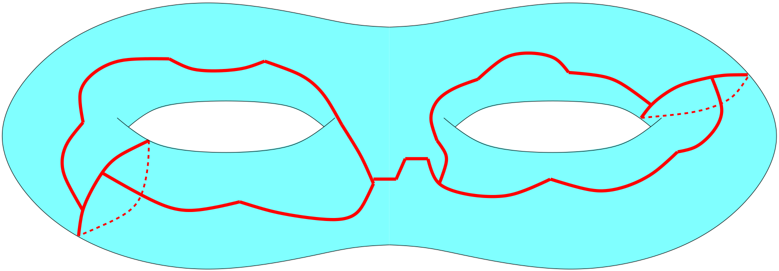

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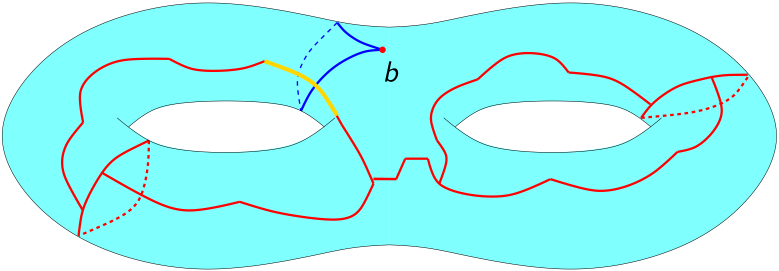

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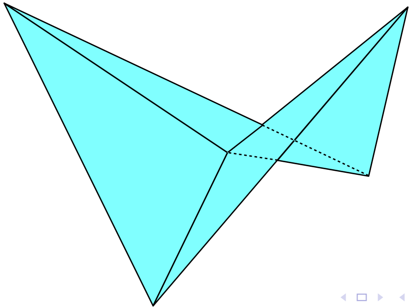
Polyhedral surfaces

Everything relies on the computation of the cut locus!

- [Chen and Han, 1996]: $O(n^2)$, where n is the number of triangles (or the total complexity of the polygons);
- Thus, algorithm with running-time $O(n^2)$ when b is fixed. But the loop is not necessarily simple, it may "run along itself".



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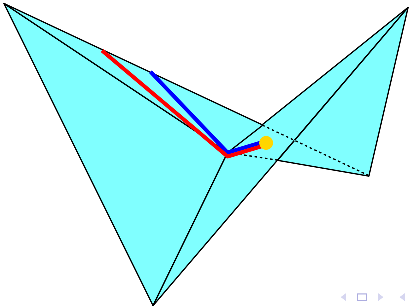

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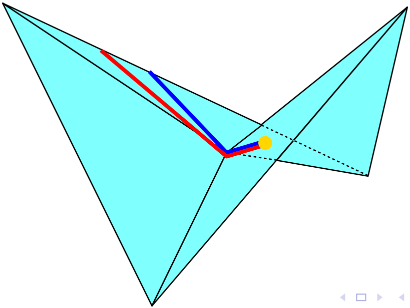
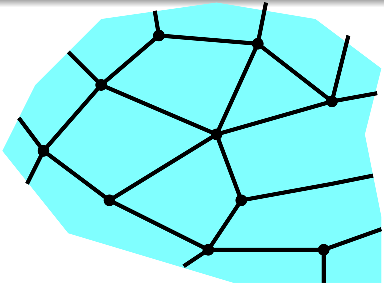
- Observation: A shortest non-contractible closed curve passes through a vertex. Thus, a shortest non-contractible loop (without fixing b) can be computed in $O(n^3)$.



Cross-metric surfaces [CdV, Erickson, 2006]

A discretization of metric surfaces, suitable for many purposes.



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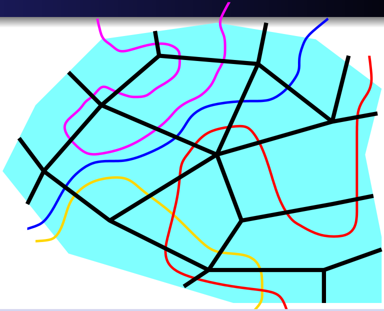
- Let M be an edge-weighted graph (cellularly) embedded on $\mathscr{S}$.

Cross-metric surfaces



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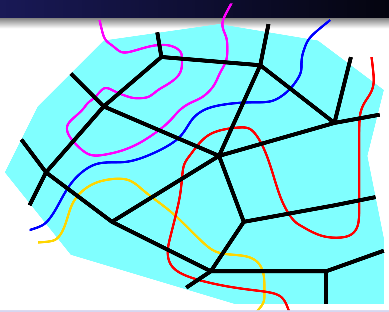

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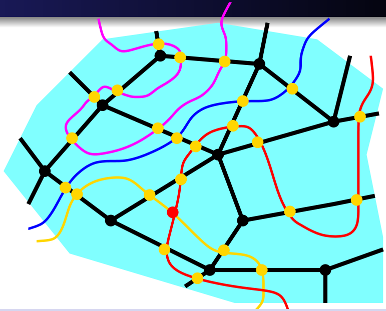
- The length of a curve is, by definition, the sum of the weights of the edges of M crossed by that curve.

Cross-metric surfaces



Cross-metric surfaces [CdV, Erickson, 2006]

A discretization of metric surfaces, suitable for many purposes.

Storing curves on surfaces

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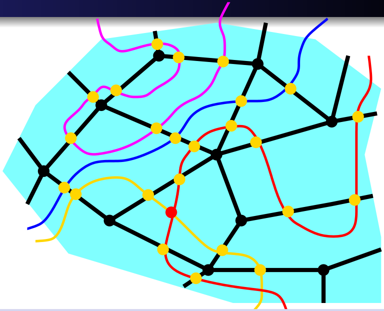

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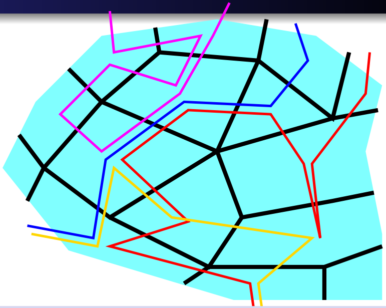

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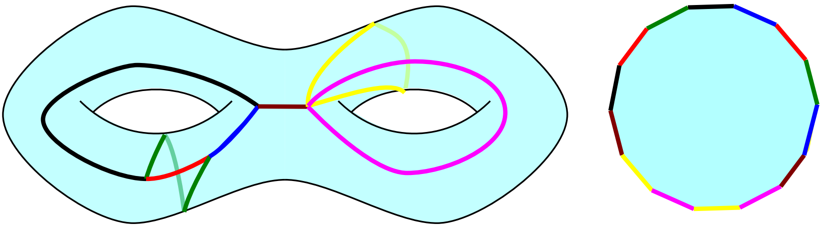
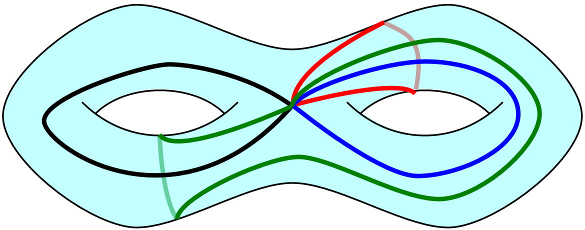
4. Topological decompositions of surfaces

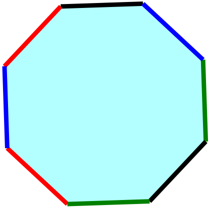
Cut graphs



- cut graph: a graph that cuts $\mathscr{S}$ into a disk.
- system of loops: a one-vertex cut graph.
- canonical system of loops: a one-vertex cut graph in which the loops appear in canonical order.

Cut graphs





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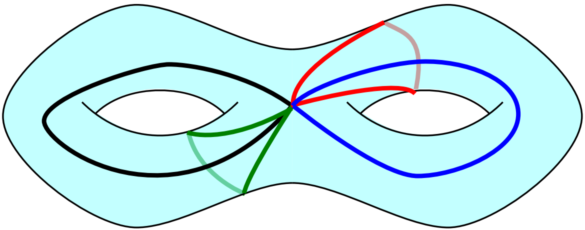

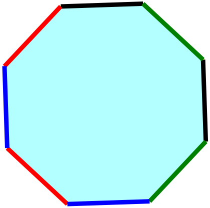

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- Computing a cut graph: easy!
- Shortest cut graph:
  - NP-hard in general [Erickson, Har-Peled, 2004];
  - ε-approximation in $f(g, \varepsilon) \cdot n^3$ [Cohen-Addad and de Mesmay, 2015];
  - easy if one wishes to compute the shortest cut graph with specified vertex set P: doable in $O(n \log n + gn + |P|)$ time [CdV, 2010];
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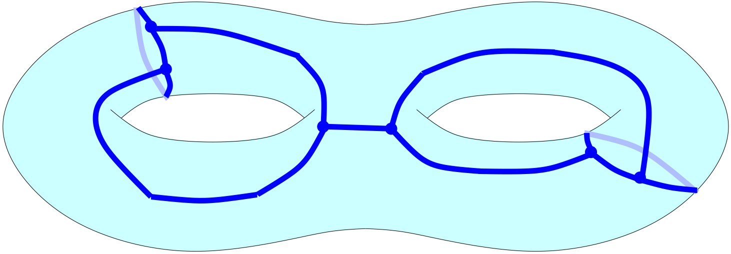

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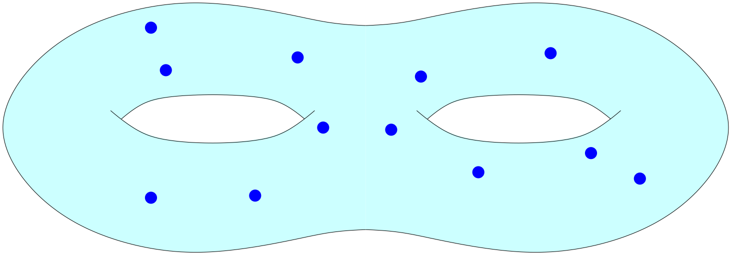

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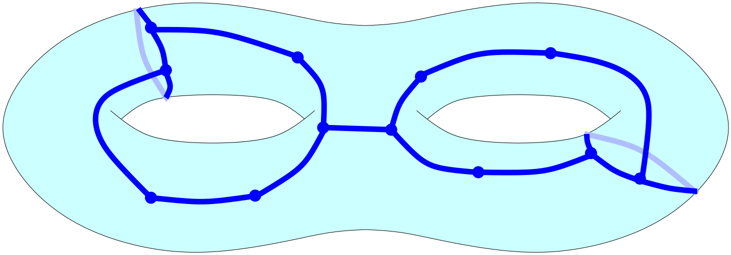
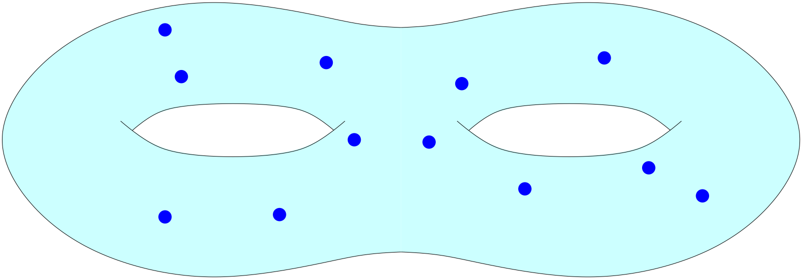


The cut locus w.r.t. P



The cut locus, a.k.a. “Voronoi” diagram

- Grow disks around each point of P simultaneously;

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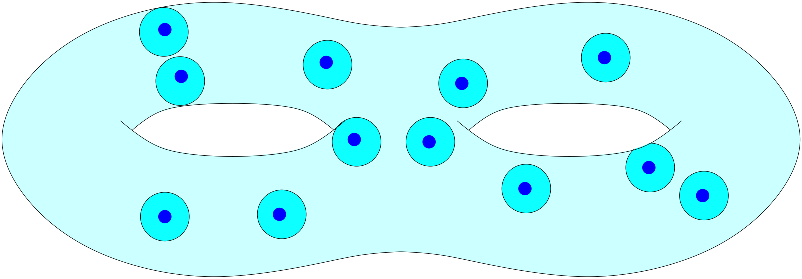

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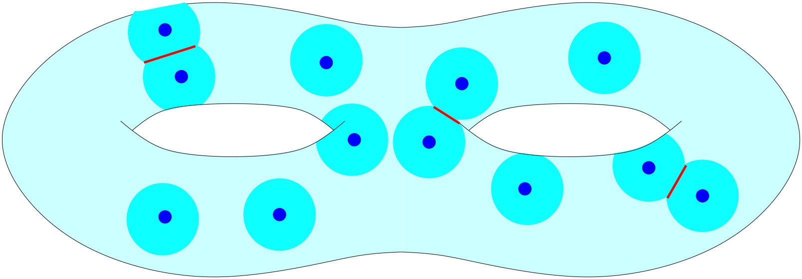

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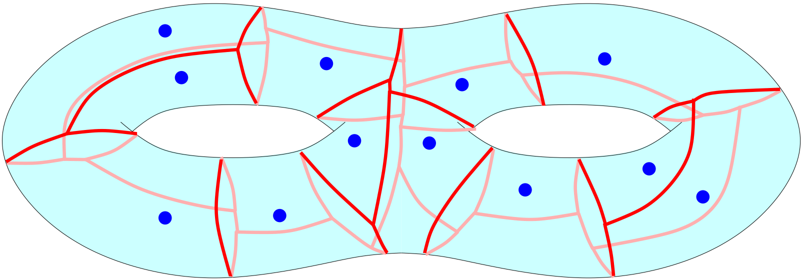

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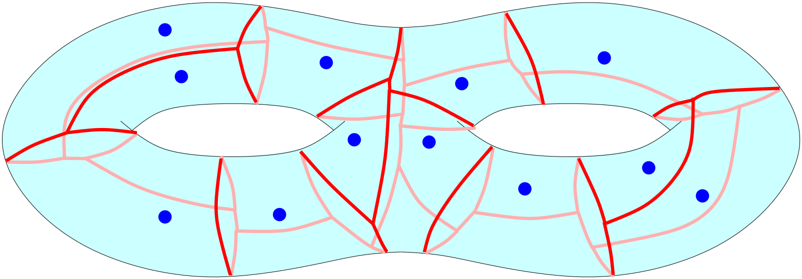

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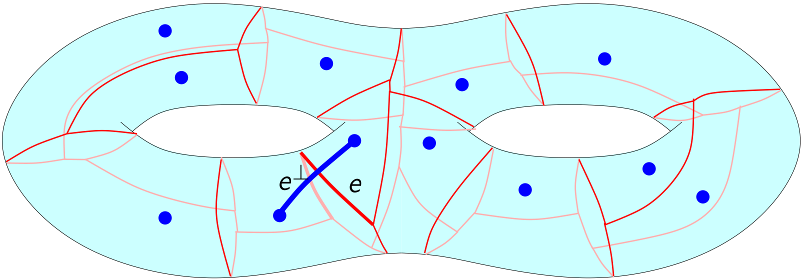

The cut locus, a.k.a. "Voronoi" diagram

- Grow disks around each point of P simultaneously;
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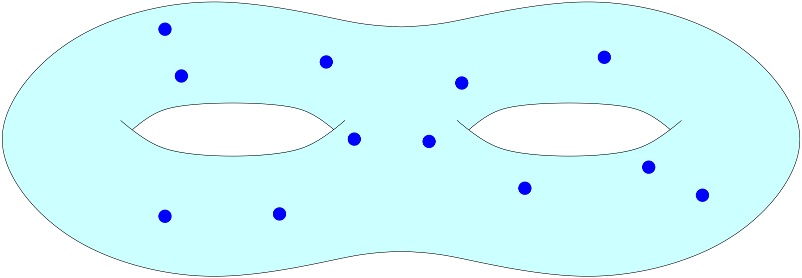
- Given an edge e of C, let $e^\perp$ be a "Delaunay" shortest path with endpoints in P that crosses e and no other edge of C.

Algorithm sketch



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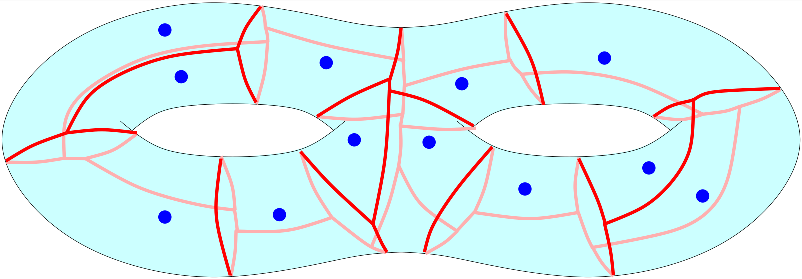

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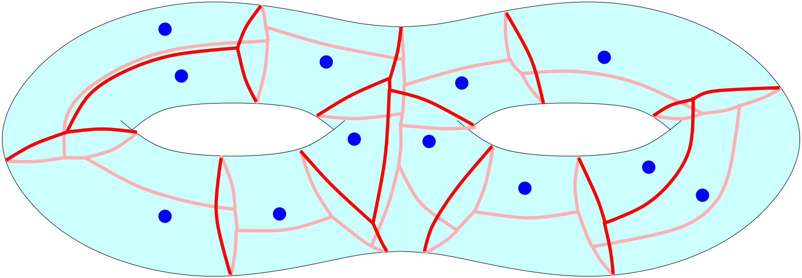

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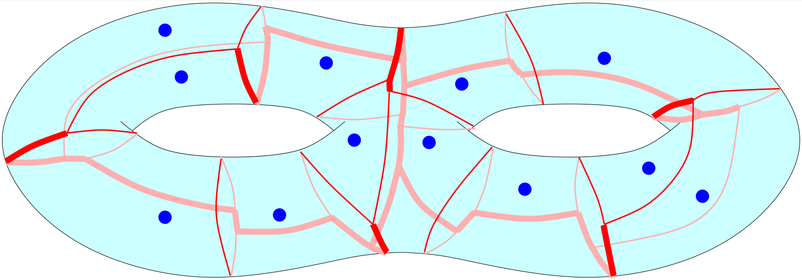

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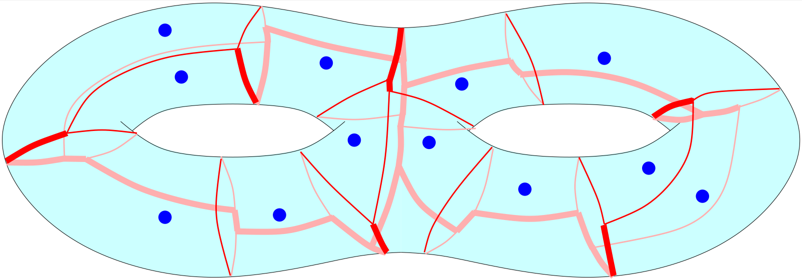

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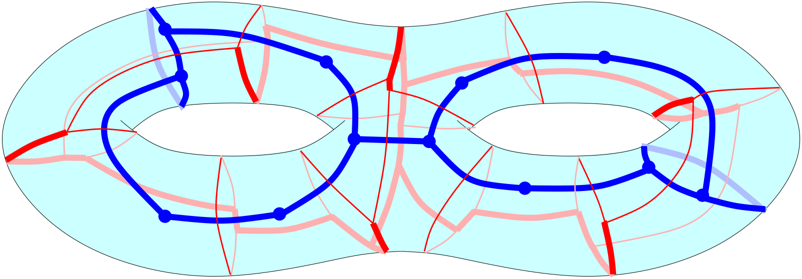

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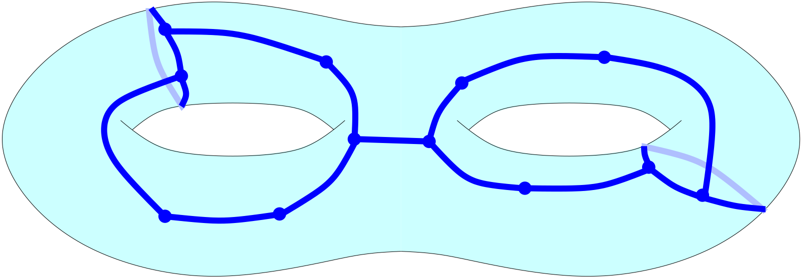

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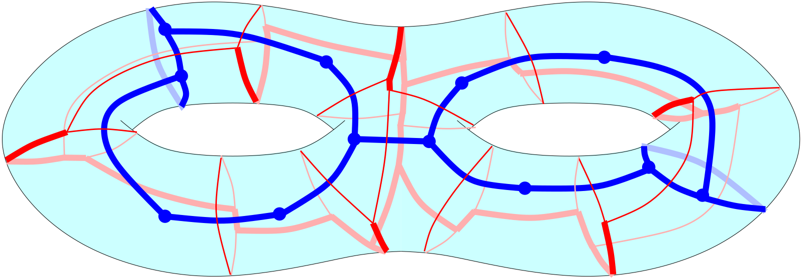

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If $w(e) :=$ length of $e^{\perp}$, and T is a maximum spanning tree w.r.t. w, then K is a shortest cut graph with vertex set P.

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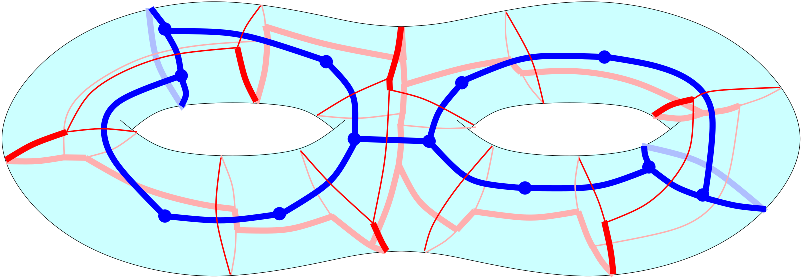

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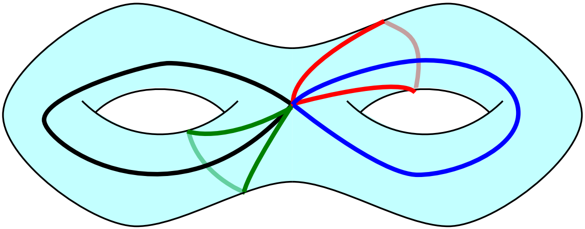
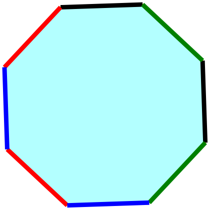
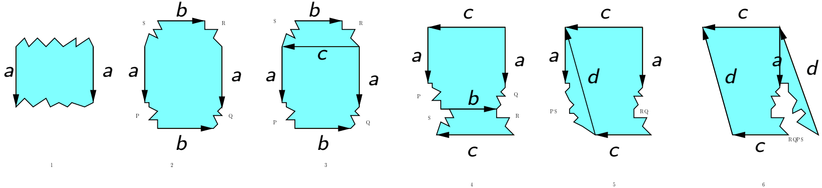
It suffices to prove that each edge of the shortest cut graph is of the form $e^{\perp}$. Proof idea: shortest cut graph = shortest basis of 1-dimensional homology of $\mathscr{S}$ relatively to P.

Canonical system of loops

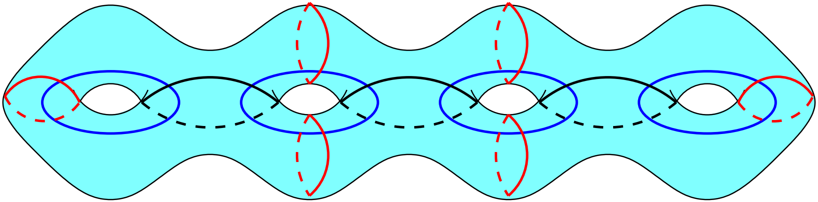




- Shortest: open!
- Some canonical system of loops with $O(gn)$ complexity: doable in $O(gn)$ time [Lazarus, Pocchiola, Vegter, Verroust, 2001].



Octagonal decomposition



- Shortest: open!
- Some octagonal decomposition
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  - such that each closed curve is as short as possible in its homotopy class

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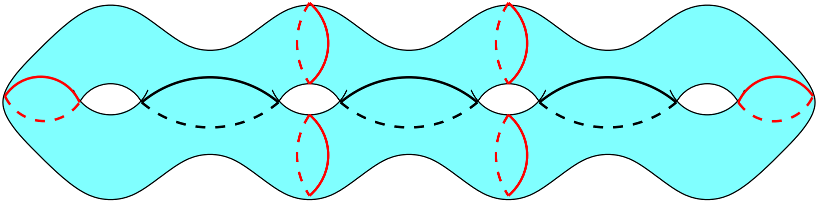

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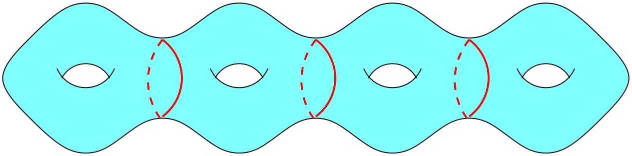
Contains a pants decomposition.

5. Other problems solved

Other curves

- shortest splitting closed curve (separating but non-contractible)

  → *crosses each shortest path $O(g)$ times; each loop of the shortest system of loops is the concatenation of two shortest paths.*



Other curves

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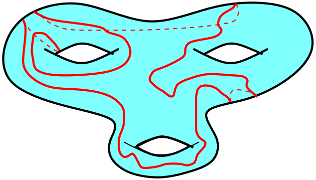
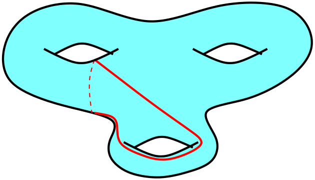
  → *crosses each shortest path $O(g)$ times; each loop of the shortest system of loops is the concatenation of two shortest paths.*

- shortest path homotopic to a given path;
  shortest closed curve freely homotopic to a given closed curve

  → *octagonal decomposition lifts, in the universal cover, to a regular tiling; defines a region of the universal cover to be explored.*





More generally. . .

These building blocks apply to seemingly unrelated problems:

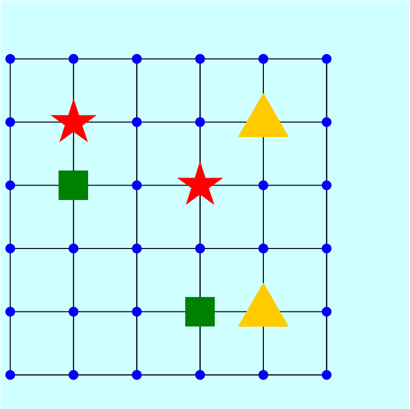
- topological graph theory: crossing number of graphs [Kawarabayashi, Reed, 2007];
- algorithms for planar graphs: maximum flow [Erickson, 2010], shortest non-crossing paths [Erickson, Nayyeri, 2009], multicut [CdV, 2015], [Cohen-Addad, CdV, de Mesmay, 2018?].
- algorithms for surface-embedded graphs: minimum cut [Chambers, Erickson, Nayyeri, 2009], maximum flow [Chambers, Erickson, Nayyeri, 2009].

various models:

- the plane with polygonal obstacles;
- polyhedral surfaces;
- disjoint curves in graphs;
- normal curves.

Example: multicut problem

- Input: $G = (V, E)$: a graph; pairs of vertices, called terminals.
- Output: $E' \subseteq E$ of minimum weight such that: after removing E', there is no path in G connecting the two vertices of a pair.



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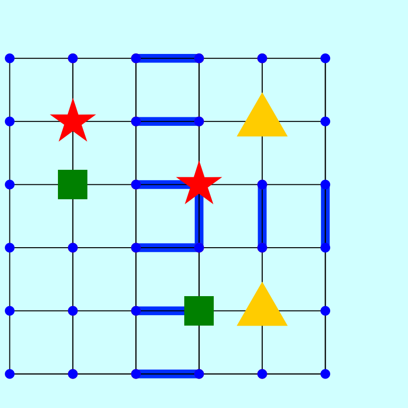

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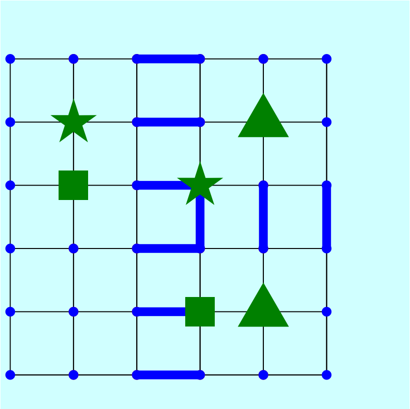
Reformulation: compute a "shortest" (in the cross-metric sense) graph in general position w.r.t. G and separating each pair of terminals.



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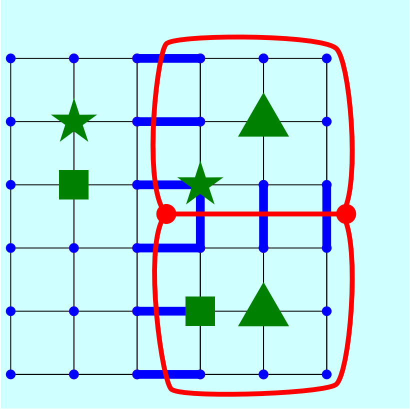

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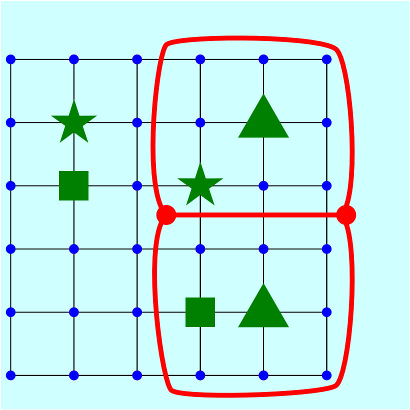

6. Open problems

Open problems

- Shortest decompositions (pants decomposition / octagonal decomposition / canonical system of loops);
- shortest graph embedding (possibly fixing vertices / homotopy / isotopy / combinatorial map);
- a conjecture by Negami: Given two graphs G and H embeddable on a fixed surface, can we embed them so that they cross at most $c \cdot |E(G)| \cdot |E(H)|$ times (for some absolute constant c)?

Thanks for your attention!

Questions?

1. 1. Topological graphs on surfaces in general
2. 2. Decision problems: homotopy and isotopy
3. 3. Shortest non-contractible closed curves
4. 4. Topological decompositions of surfaces
5. 5. Other problems solved
6. 6. Open problems