CGAL 3D Triangulations in Periodic Spaces

Manuel Caroli, Nico Kruithof, Monique Teillaud

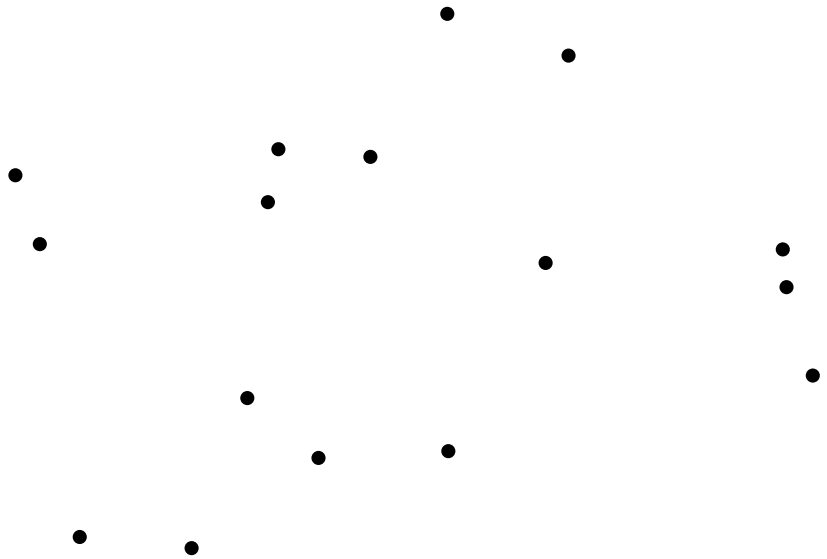
Workshop on Geometric Computing in Periodic Spaces

20th October 2008

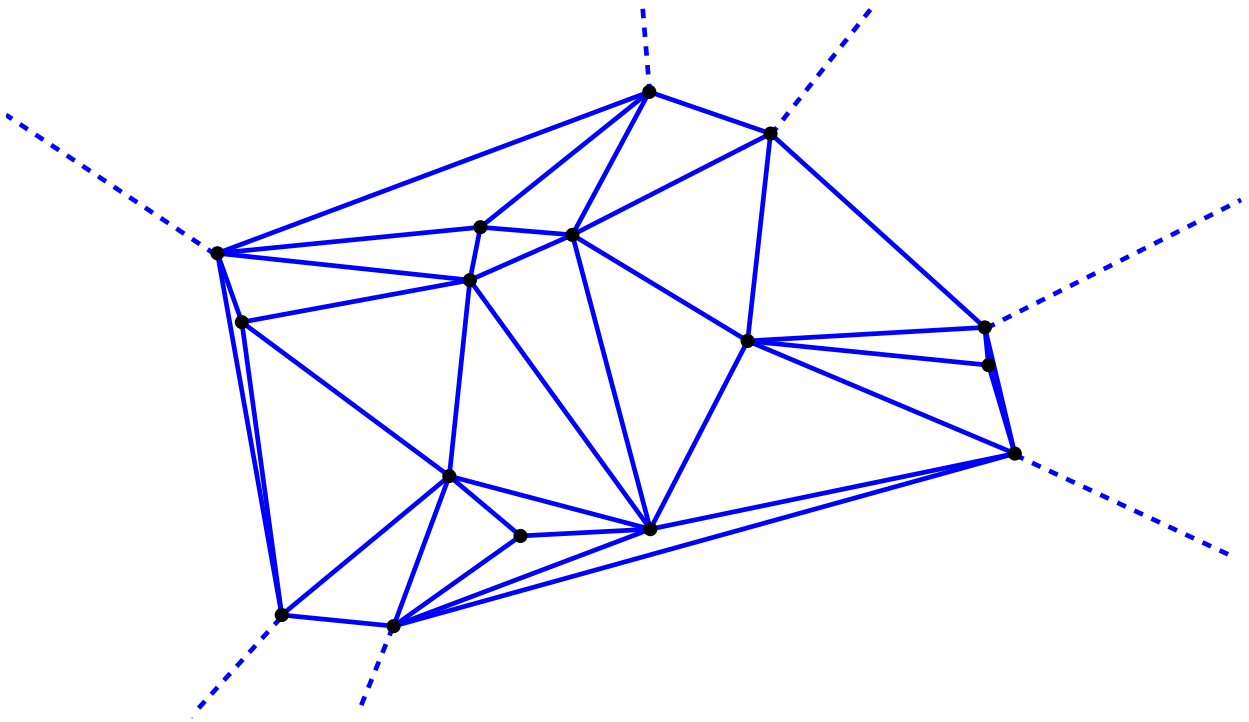
Outline

Outline

Delaunay triangulation



Delaunay triangulation



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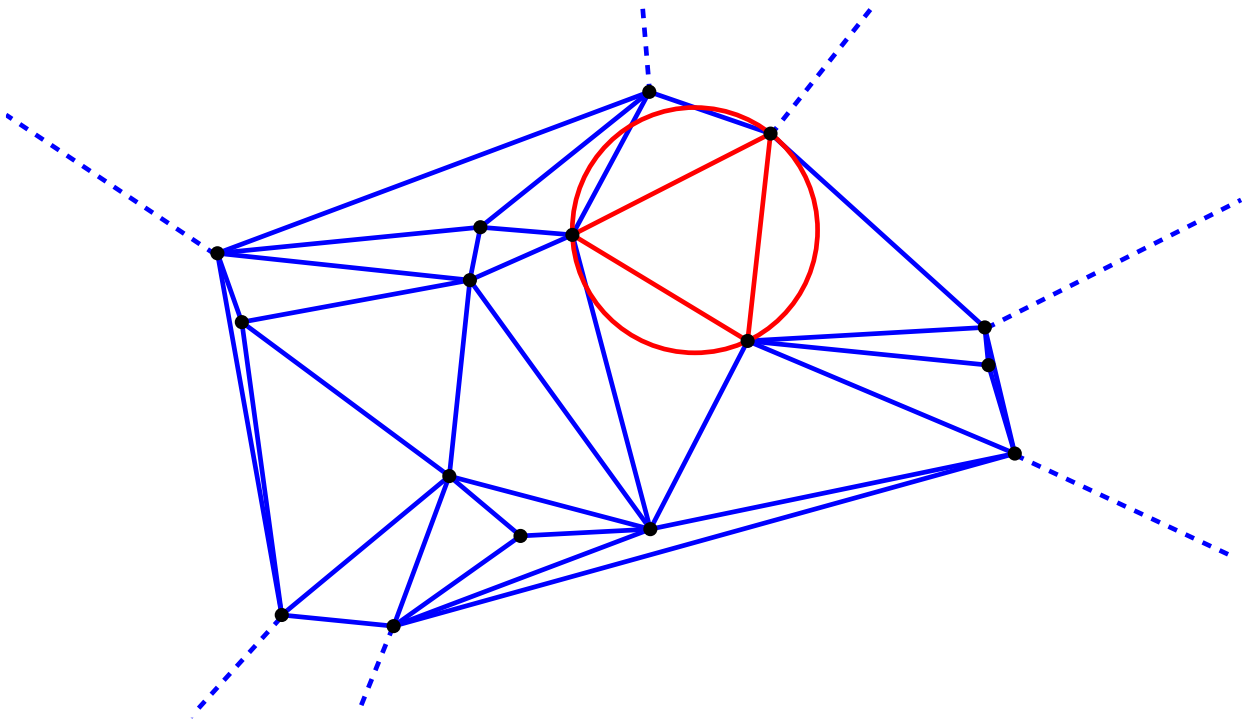

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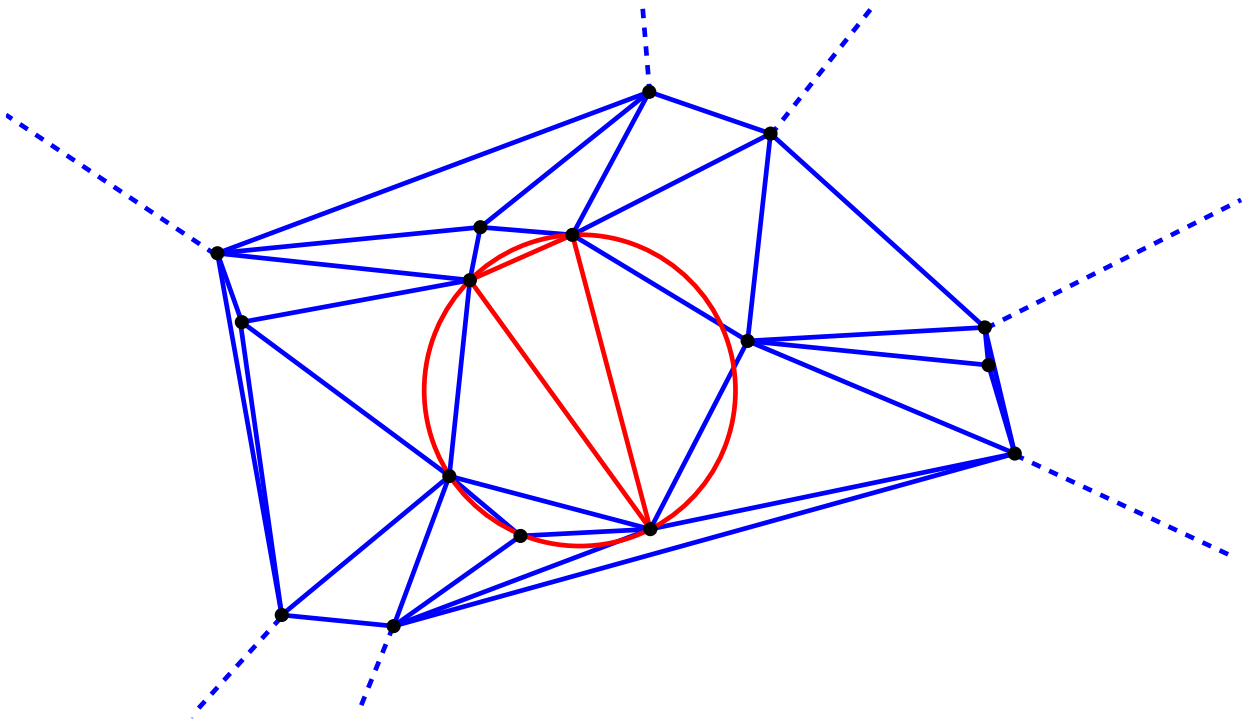
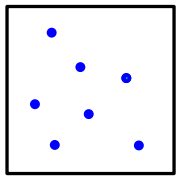


Periodic space $\mathbb{T}^2$

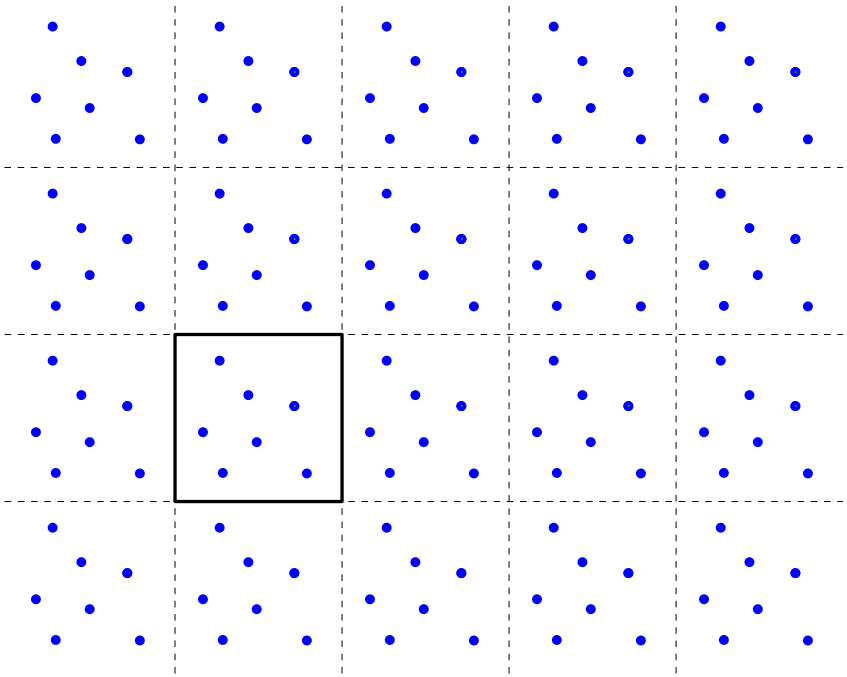
- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\pi := \mathbb{R}^2 \rightarrow \mathbb{T}^2$



$\mathcal{S}$

Periodic space $\mathbb{T}^2$

- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\pi := \mathbb{R}^2 \to \mathbb{T}^2$
- $\pi^{-1}(p)$ maps p onto a regular point lattice



$\pi^{-1}(\mathcal{S})$

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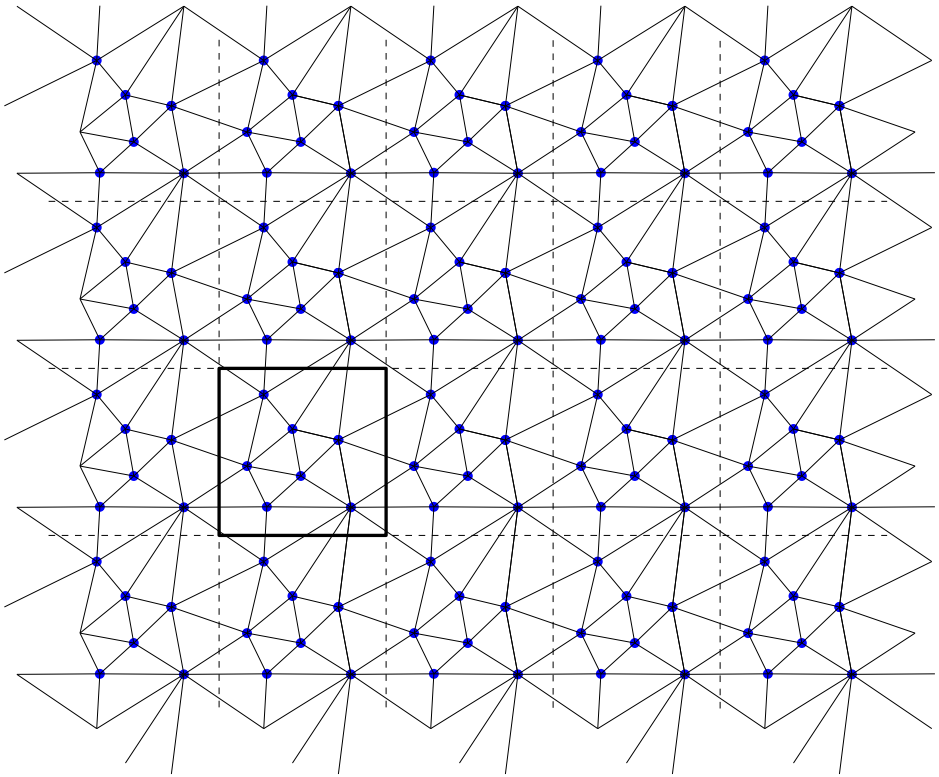

$DT(\pi^{-1}(\mathcal{S}))$

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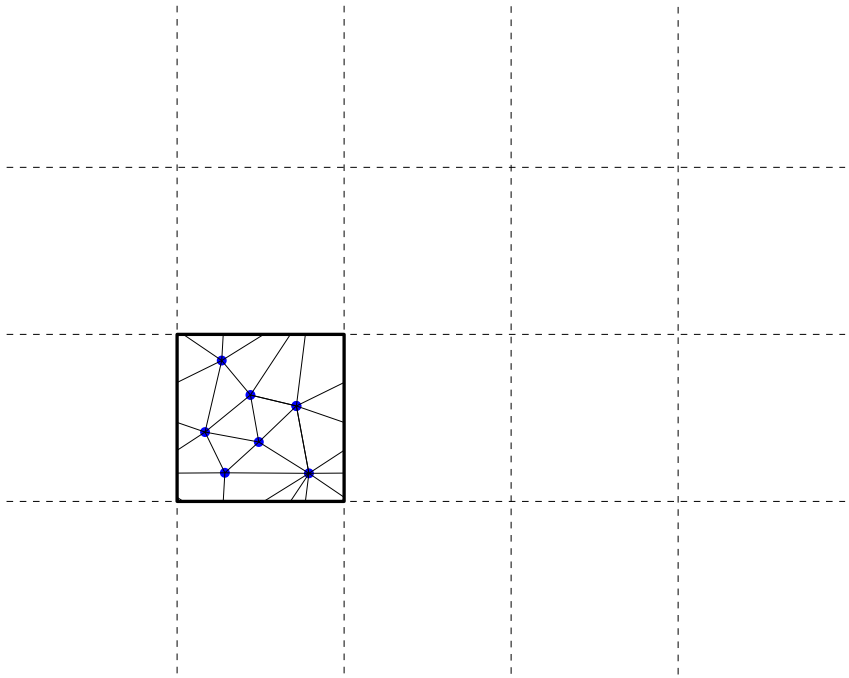


$$\pi(DT(\pi^{-1}(\mathcal{S})))$$

Triangulation

Definition

A *Triangulation of a point set* $\mathcal{S}$ is
a simplicial complex K in
a topological space $\mathbb{X}$, if

1. each point in $\mathcal{S}$ is a vertex of K
2. $\bigcup_{\sigma \in K} \sigma$ is homeomorphic to $\mathbb{X}$.

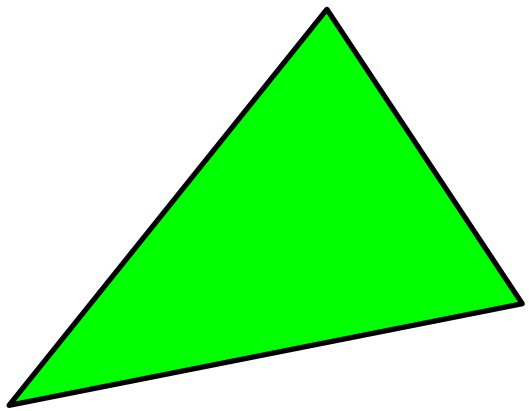
Simplicial complex

Definition

A *simplicial complex* K is

a finite set of simplices such that

1. $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$
2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$



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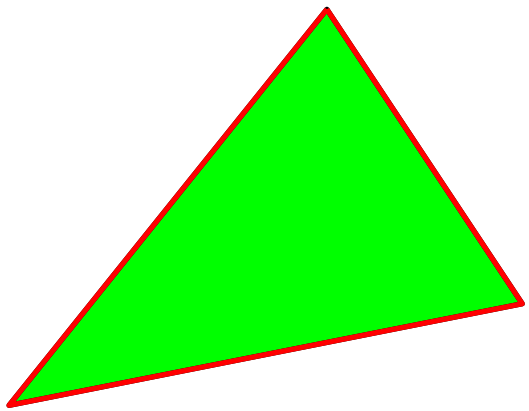

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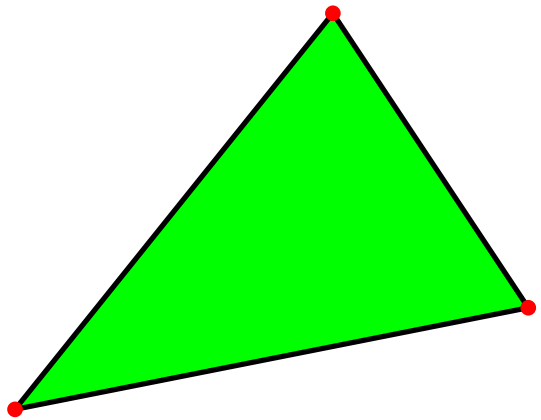

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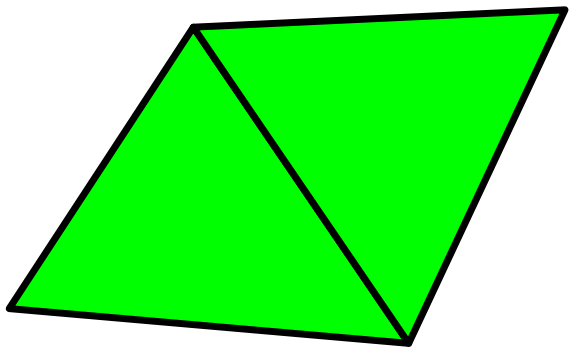

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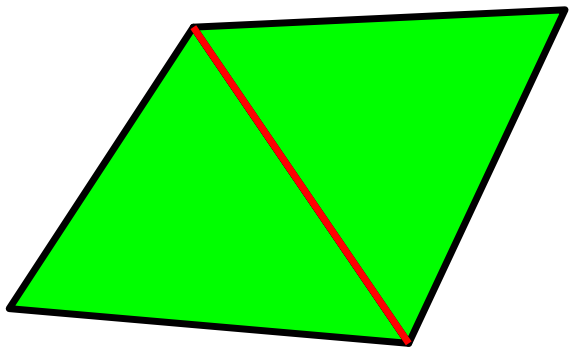
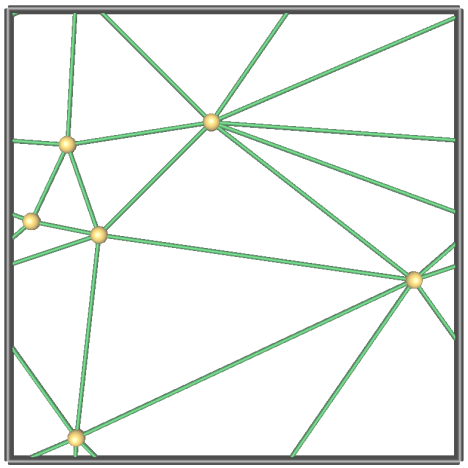


Delaunay triangulation in $\mathbb{T}^3$.

Definition

We call $DT_{\mathbb{T}}(\mathcal{S}) := \pi(DT(\pi^{-1}(\mathcal{S})))$ a *Delaunay tessellation*.

We call $DT_{\mathbb{T}}(\mathcal{S})$ a Delaunay *triangulation* if it is a simplicial complex.



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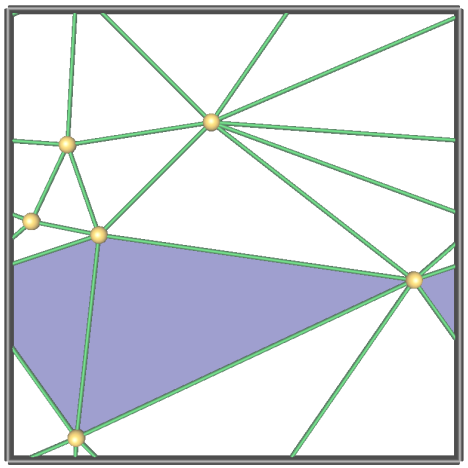
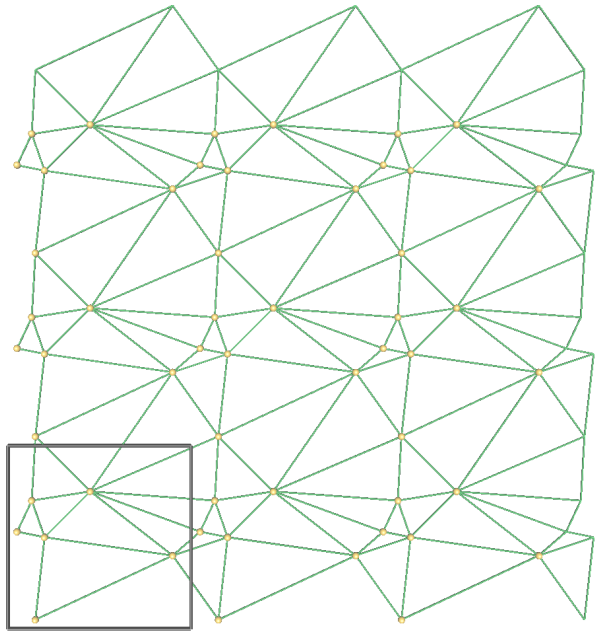
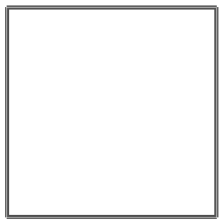


Covering spaces

Solution: Computing in $\mathbb{T}^2_3 := \mathbb{R}^2/3\mathbb{Z}^3$
with point set $\pi_3(\pi^{-1}(\mathcal{S}))$



Building up a triangulation



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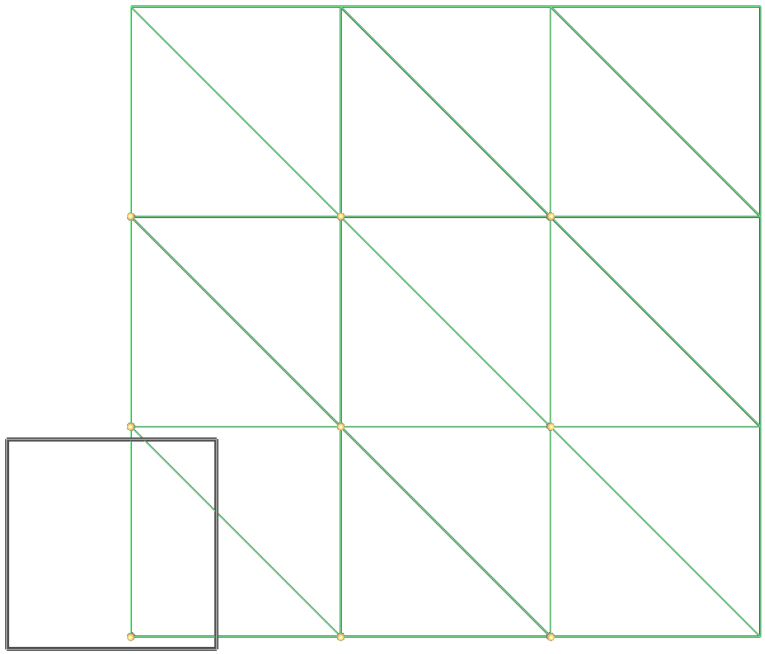

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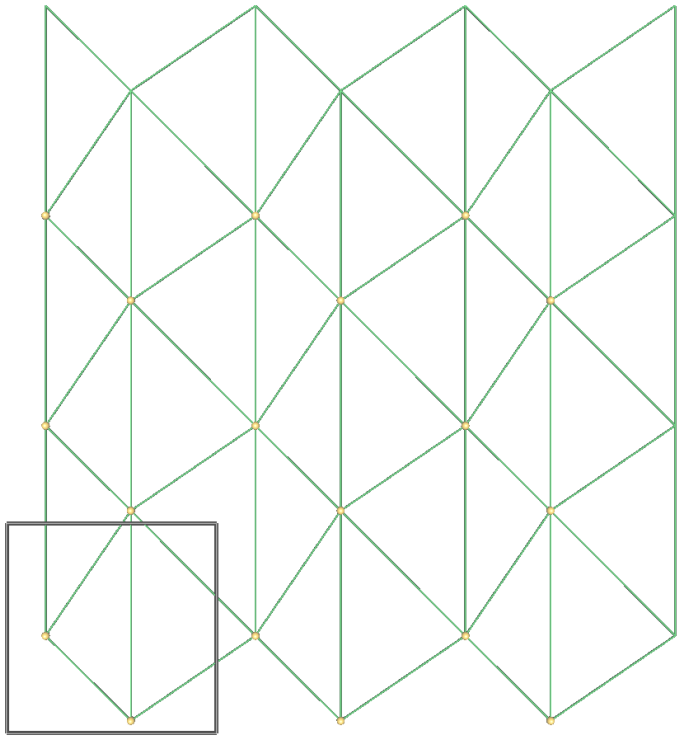

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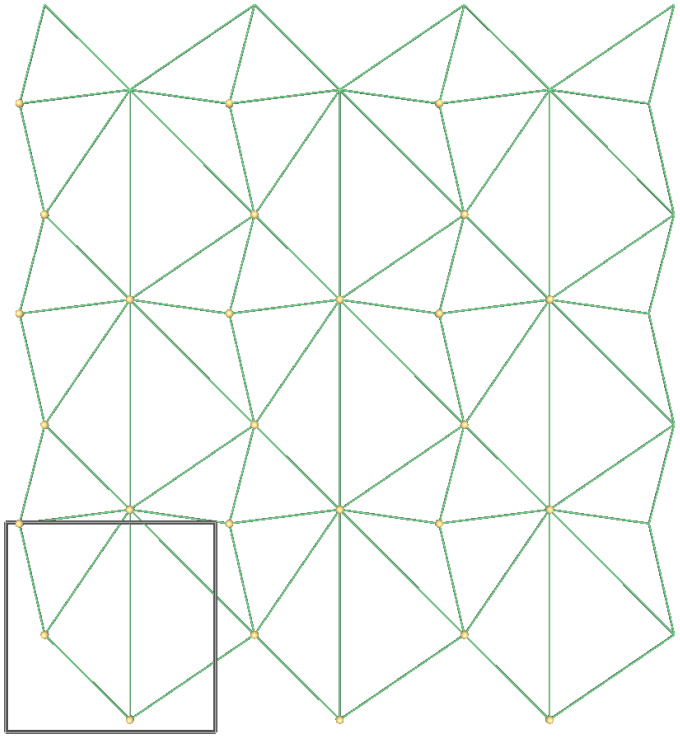

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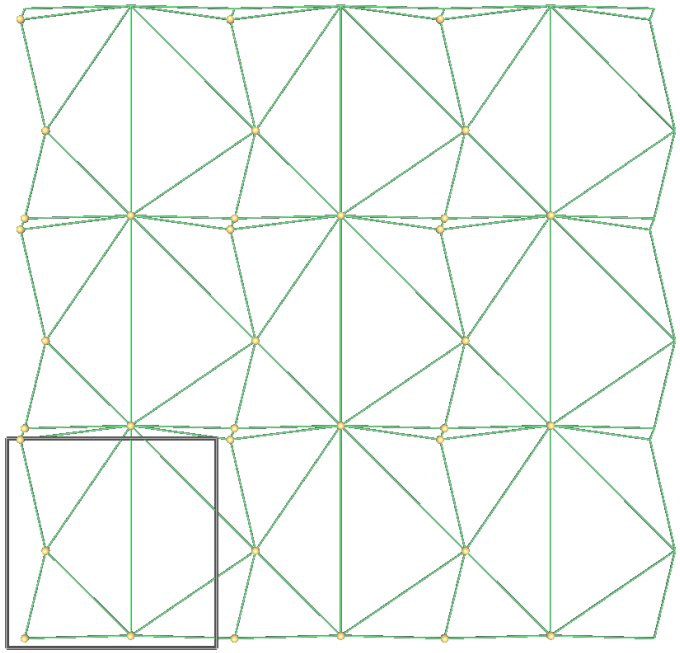

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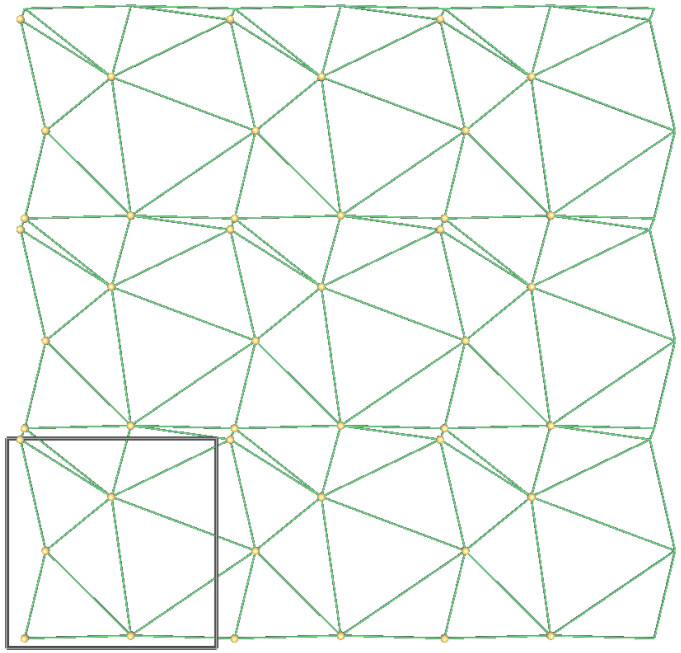

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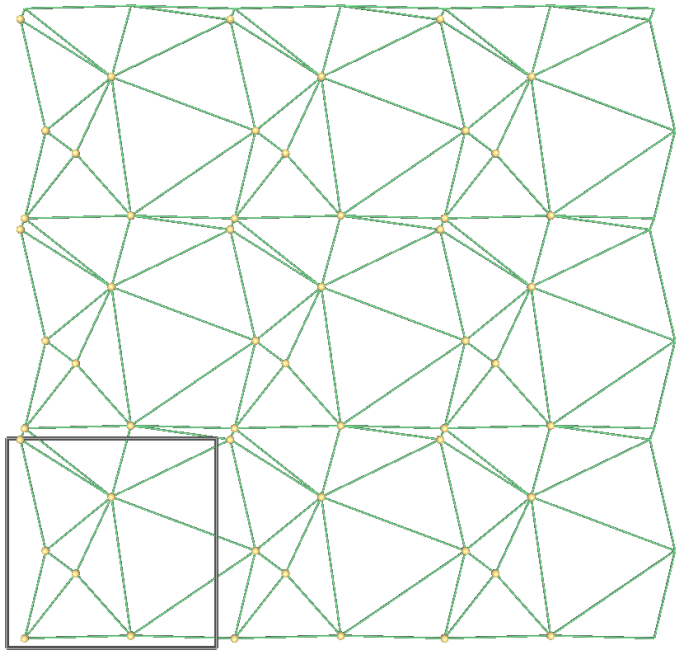

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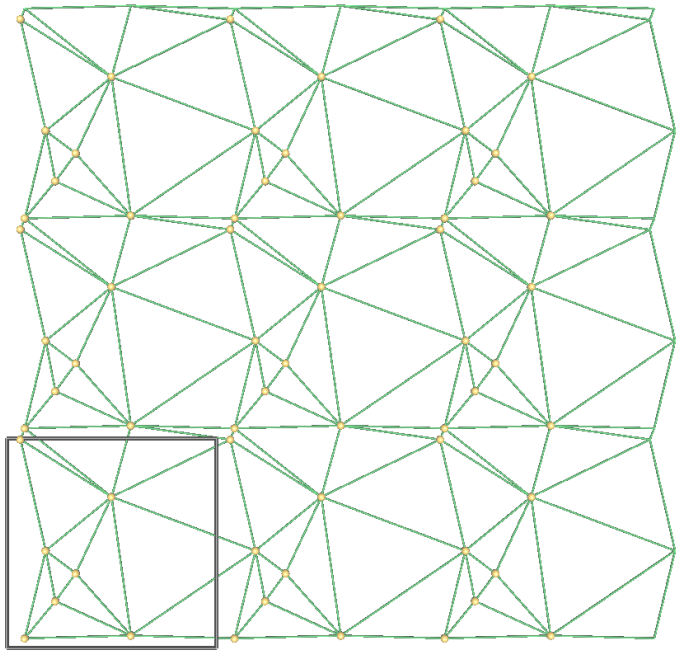

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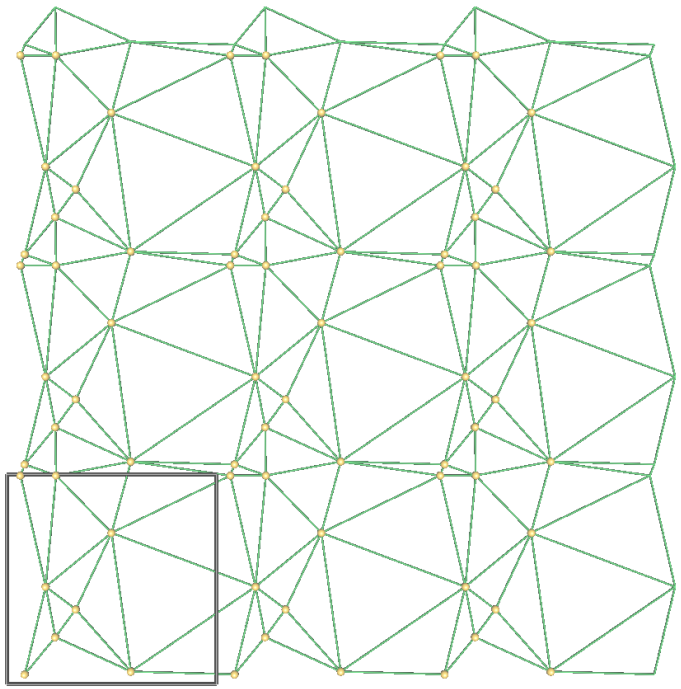

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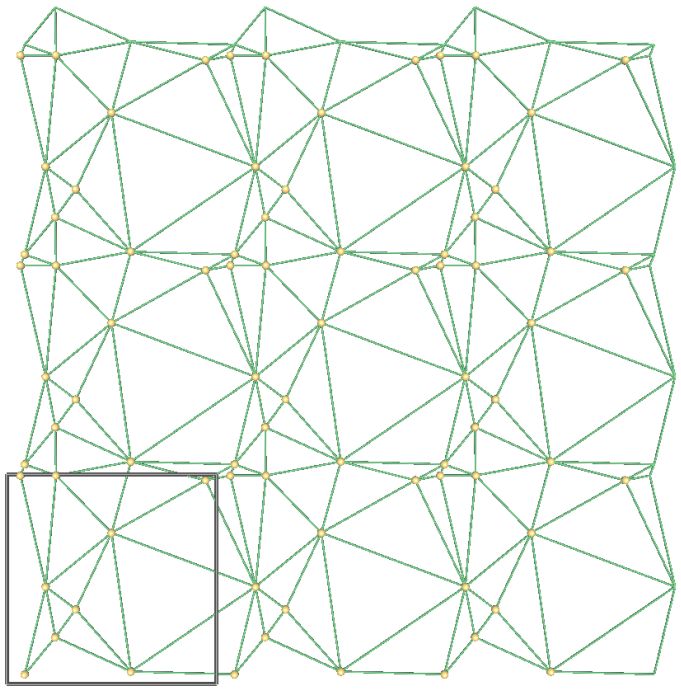

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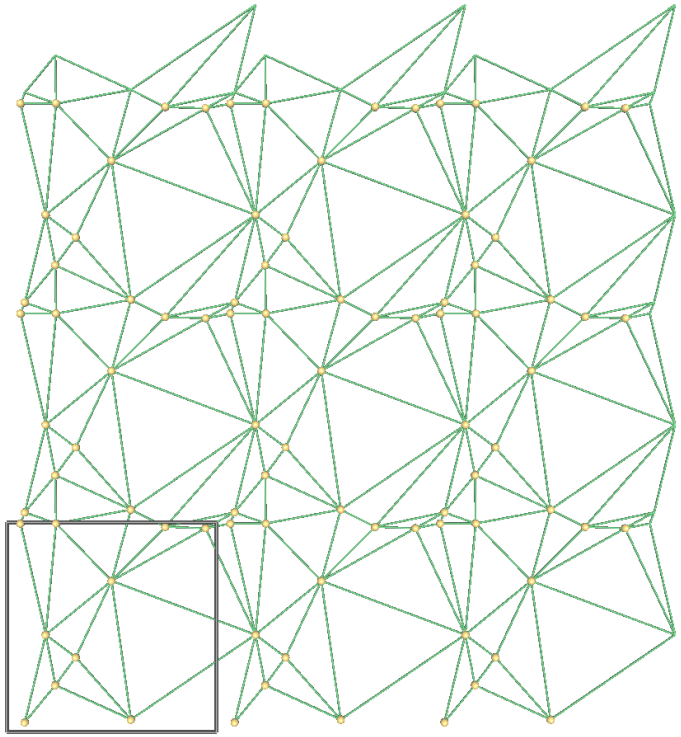

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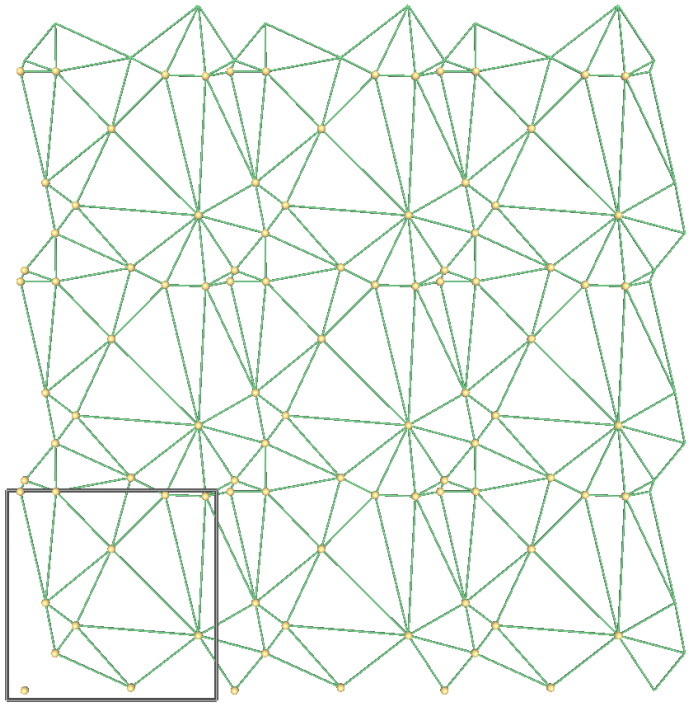

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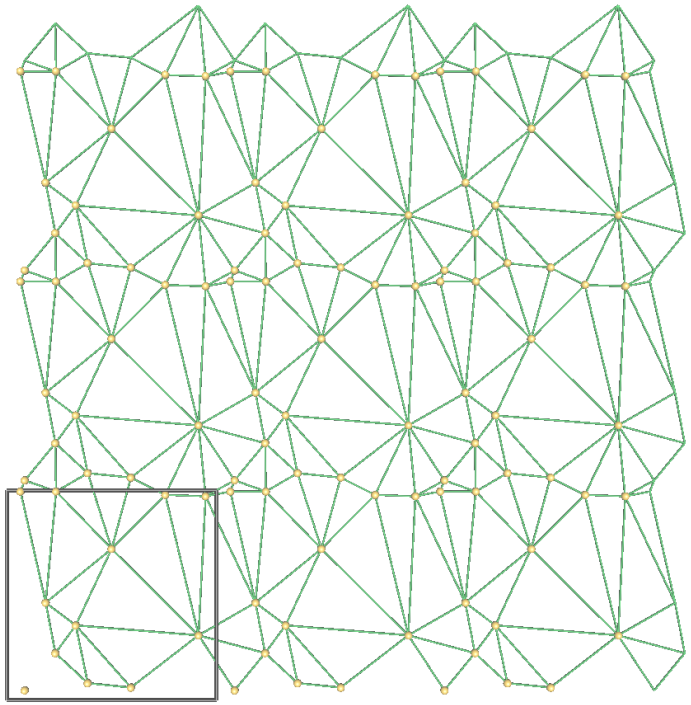

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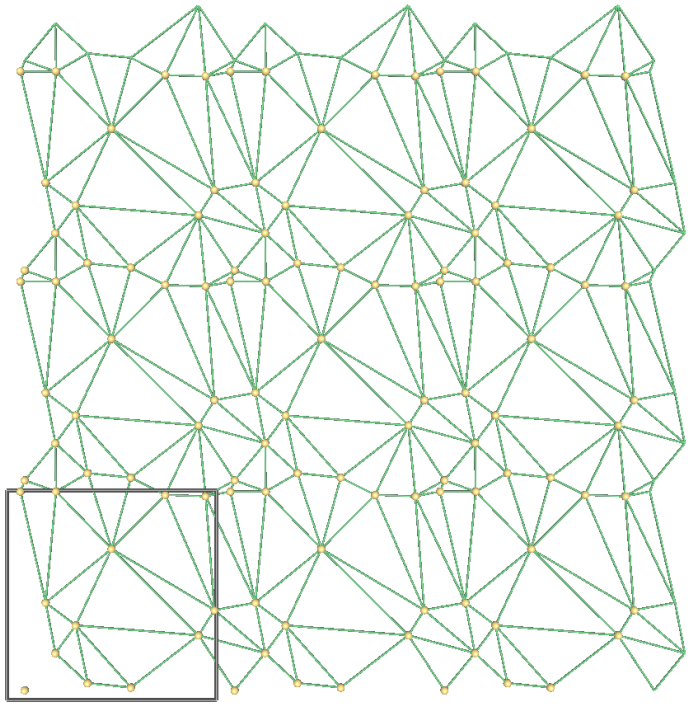

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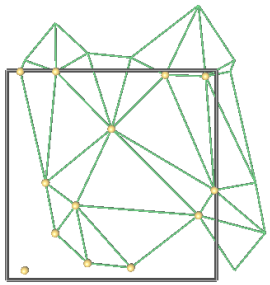

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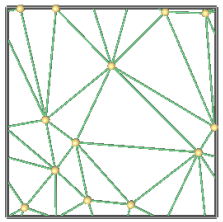
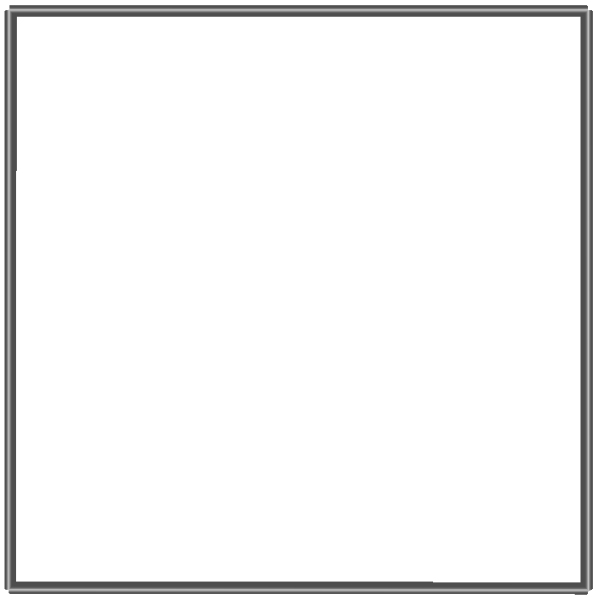


Periodic space $\mathbb{T}^2$



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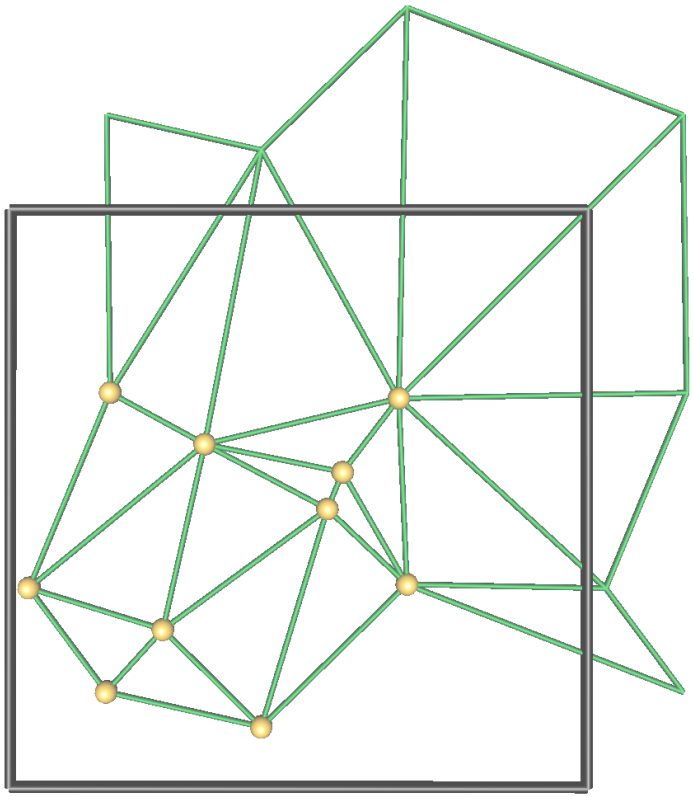

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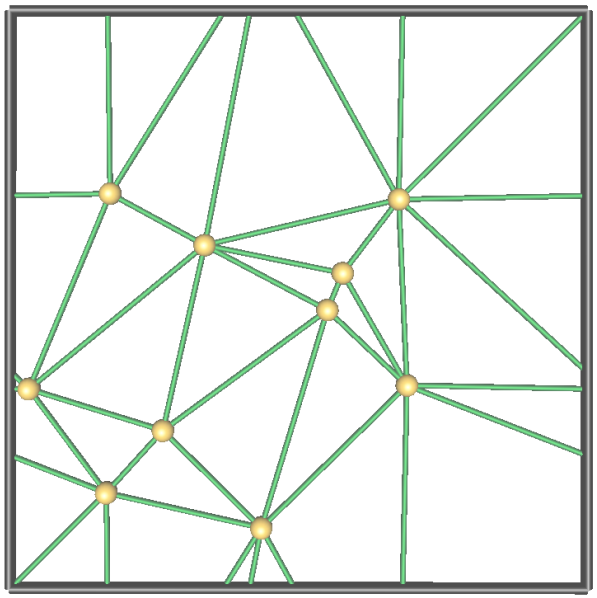

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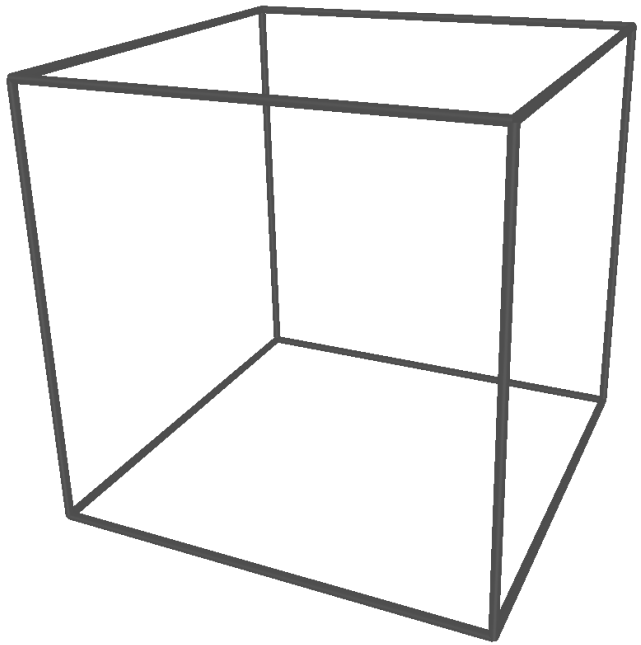

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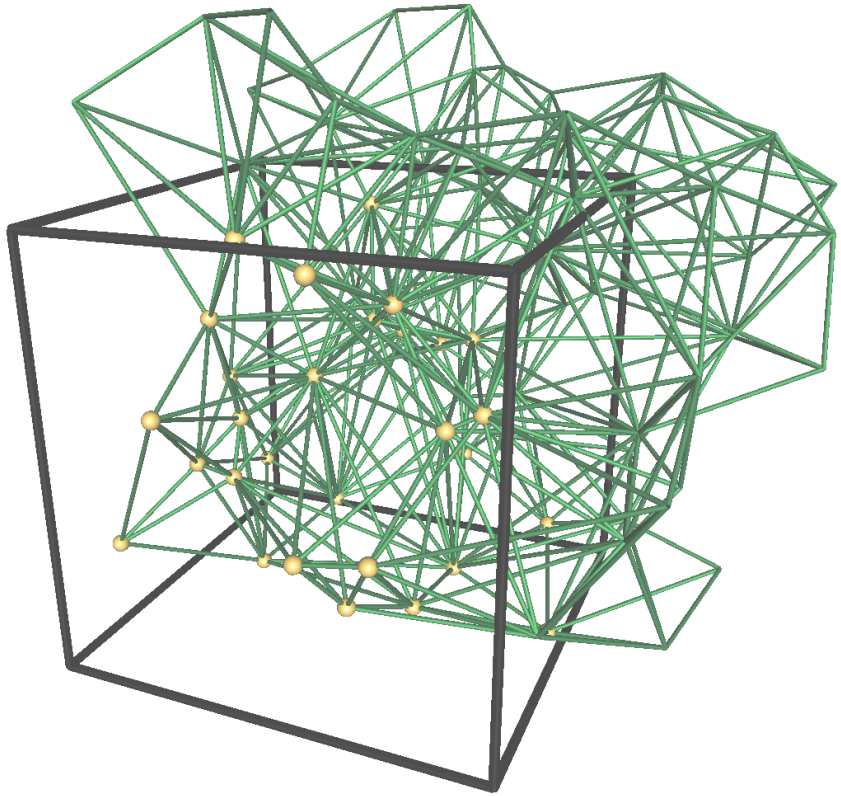

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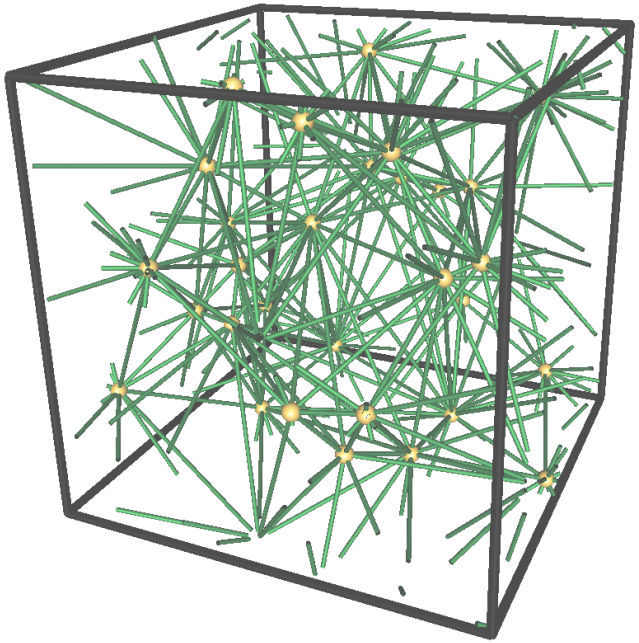

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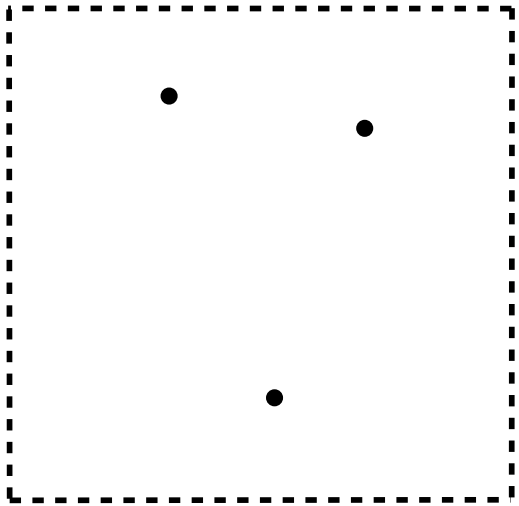
Offsets



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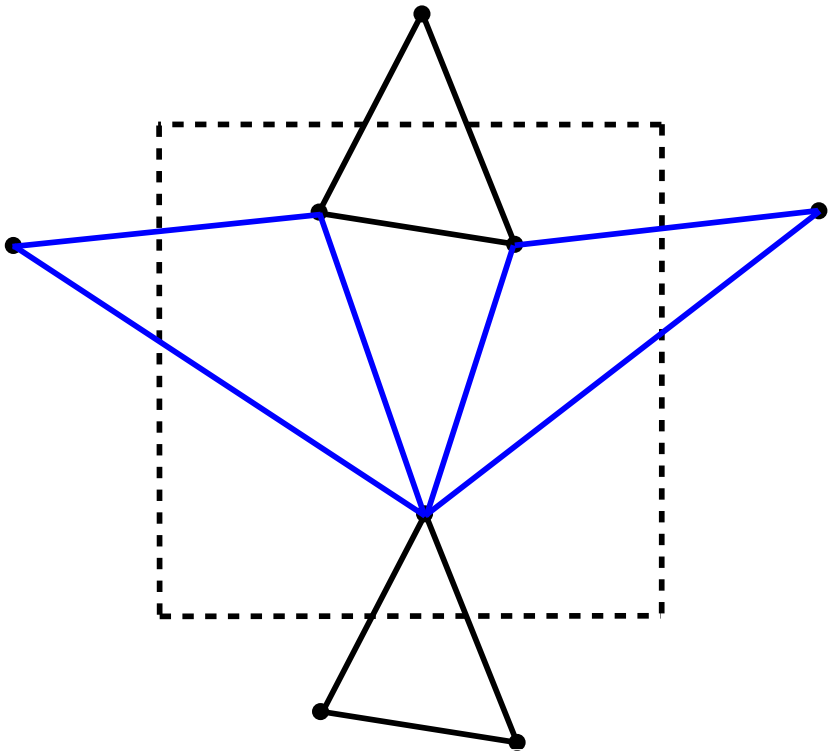

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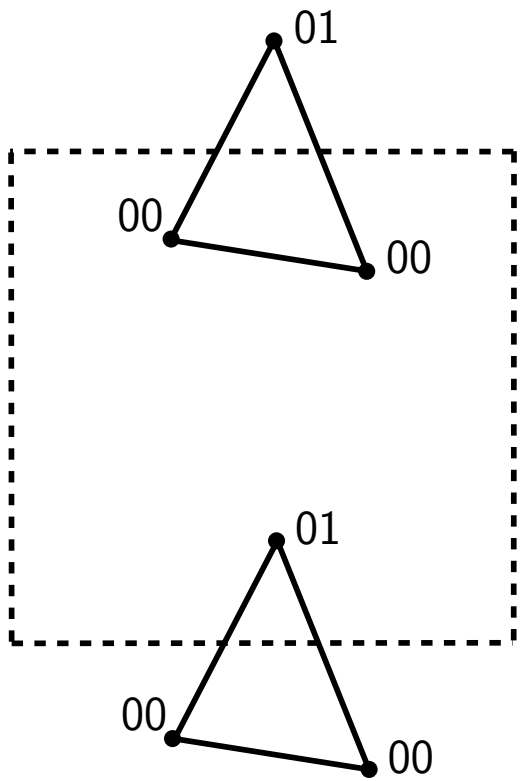

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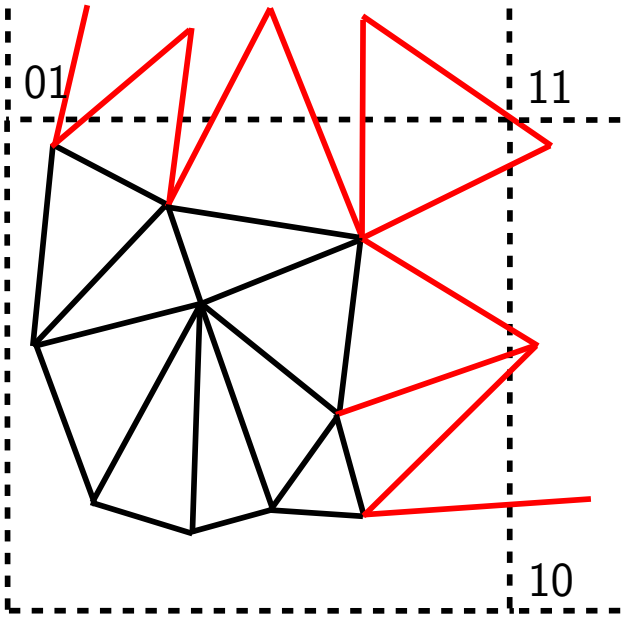

further differences to $\mathbb{R}^3$.

- Offsets:
  - `Periodic_3_offset_3` ($\mathbb{Z}^3$)
  - `Cell_handle ch`:
    `ch->vertex(i)`, $0 \leq i < 4$
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  - ...
- no degenerate dimensions:
  - `dimension()`
  - no side of circle test

Triangulation traits

`Triangulation_3` parameterized by `Geometric_traits`:

- Geometric primitives: points, segments, triangles ...
- Number types, basic operations
- Geometric predicates: side of sphere, ...
- Geometric constructors: construct triangle, ...

```
Triangulation_3<
    TriangulationTraits_,
    TriangulationDataStructure_3 >
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Periodic_3_Triangulation_3<
    Periodic_3TriangulationTraits_3,
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```

Periodic traits

- side_of_sphere, orientation, etc. with offsets
- Delaunay triangulation traits:
  - orientation
  - coplanar orientation
  - side of sphere
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$$\texttt{pred}(p_0 , p_1 , \ldots, p_k)$$

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- Periodic Delaunay triangulation traits:
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  - ...
- Offsets:

  `periodic_pred`$(p_0, p_1, \ldots, p_k, o_0, o_1, \ldots, o_k)$
  $:=$`pred`$(p_0 + o_0, p_1 + o_1, \ldots, p_k + o_k)$

Access

- Combinatorial iterators:
  - `Vertex_iterator`
  - `Edge_iterator`
  - `Facet_iterator`
  - `Cell_iterator`
- Return all primitives from the triangulation.
- Expectation: no periodic copies
- Inconsistencies if not simplicial complex

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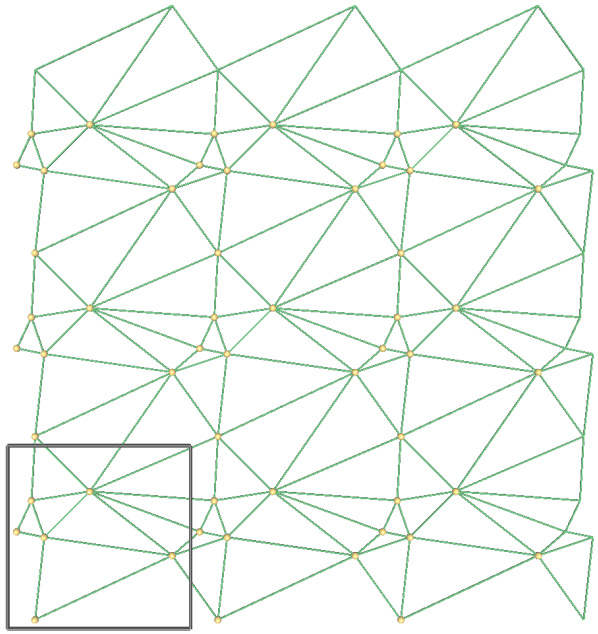

Access II

- Geometric iterators:
  - `Point_iterator`
  - `Segment_iterator`
  - `Triangle_iterator`
  - `Tetrahedron_iterator`

Access II

- Geometric iterators

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- Parameter `Iterator_type it`

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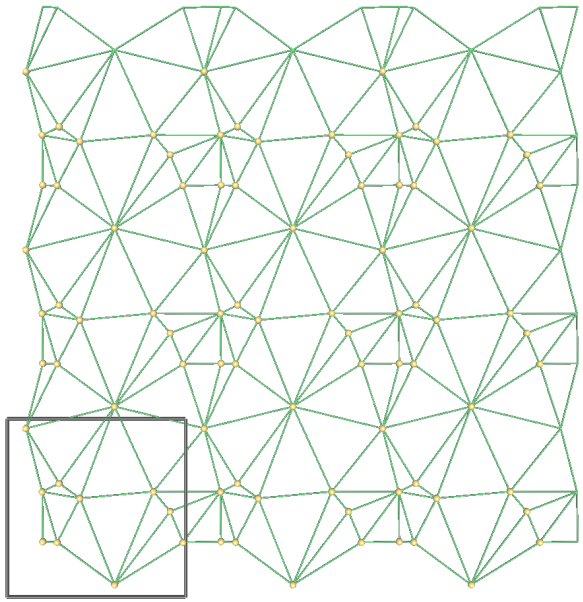
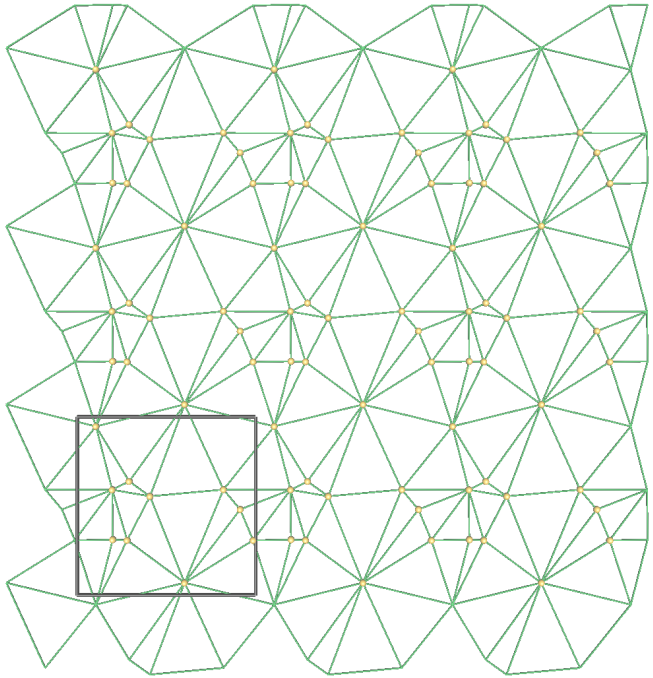


`it == AS_STORED`

Access II

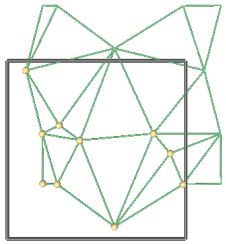
- Geometric iterators
- Parameter `Iterator_type it`



`it == COVER_DOMAIN`

Access II

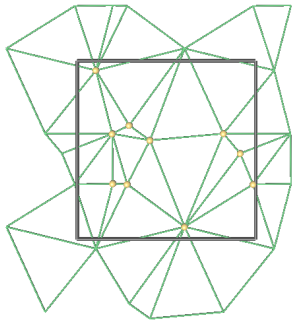
- Geometric iterators
- Parameter `Iterator_type it`



`it == UNIQUE`

Access II

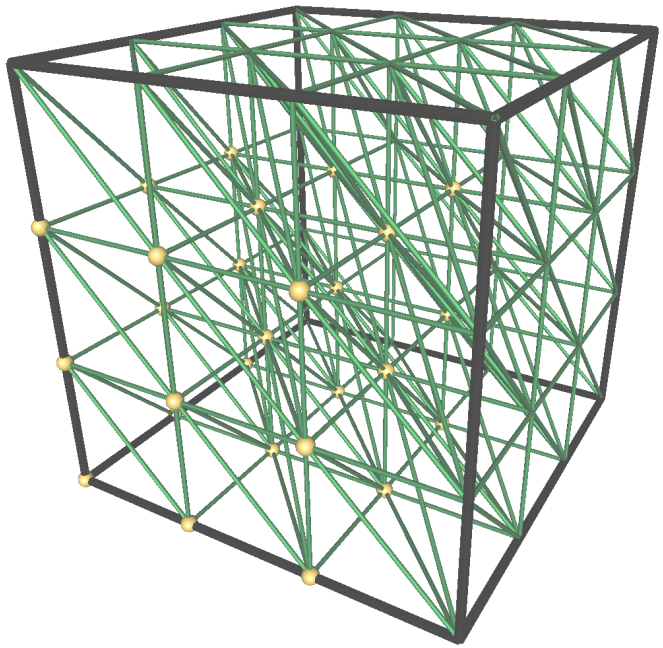
- Geometric iterators
- Parameter `Iterator_type it`



`it == COVER_DOMAIN_UNIQUE`

Heuristic for large point sets

- insert dummy point set
- compute triangulation in 1-sheeted covering
- remove dummy points



$\mathbb{R}^3$ vs. $\mathbb{T}^3$

Delaunay triangulation ($\mathbb{R}^3$ vs. $\mathbb{T}^3$):

No. of points	$\mathbb{R}^3$	$\mathbb{T}^3$	factor	old
1000	0.015	0.022	1.47	1.62
10000	0.12	0.21	1.75	1.53
100000	1.3	2.1	1.61	1.42
1000000	15	23	1.53	1.31

all times in seconds

Computing in 1-sheeted covering in $\mathbb{T}^3$.

Machine: MacBook Pro
Processor: 2.33 GHz Intel Core 2 Duo

Outline

Demo

Conclusion and Outlook

- Delaunay triangulation in $\mathbb{T}^3$
- Implementation in CGAL.
- Demo
- Improve implementation
- Non-cubic fundamental domain
- More features?

Conclusion and Outlook

- Delaunay triangulation in $\mathbb{T}^3$
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Thank you
for your attention!