Partitions and Clifford algebras

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Abstract

Given the set $[n] = \{1, \ldots, n\}$ for positive integer n, combinatorial properties of Clifford algebras are exploited to count partitions and non-overlapping partitions of $[n]$. The result is recovery of Stirling numbers of the second kind, Bell numbers, and Bessel numbers.

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1 Introduction

For positive integer n, the number of ways of partitioning an n-set into k nonempty equivalence classes is $S(n,k)$, the *Stirling number of the second kind.* For fixed $n > 0$, summing $S(n,k)$ over k from 1 to n gives the total number of ways of partitioning the n-set into equivalence classes, defined as the n^{th} *Bell number*, B_n.

Any n-set can be identified with the integer interval $[n] = \{1, 2, \ldots, n\}$. With the implied order structure, two blocks (equivalence classes) $\underline{i}$, $\underline{j}$ are said to *overlap* if

$$\min(\underline{i}) < \min(\underline{j}) < \max(\underline{i}) < \max(\underline{j}).$$

Given blocks $\underline{i}$ and $\underline{j}$, define the notation $\underline{i} \pitchfork \underline{j}$ to indicate that $\underline{i}$ and $\underline{j}$ are non-overlapping. A partition $\mathcal{P}$ of the n-set is said to be *non-overlapping* if $\underline{i} \pitchfork \underline{j}$ whenever $\underline{i} \neq \underline{j} \in \mathcal{P}$.

Let B_n^* denote the number of non-overlapping partitions of the n-set. For $n > 0$, the numbers B_n^* are called the *Bessel numbers* [2].

We show that combinatorial properties of Clifford algebras can be employed to generate Stirling numbers of the second kind, Bell numbers, and Bessel numbers. This is accomplished by considering Clifford exponentials, linear functionals and canonical projections on nilpotent-generated abelian subalgebras of Clifford algebras of particular signature.

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Formal power series expansions in unipotent-generated abelian subalgebras of Clifford algebras have been used previously to study random walks on the hypercube [5]. Combinatorial properties of Clifford algebras have also been used by the authors to enumerate cycles and self-avoiding walks in random graphs [6].

Definition 1.1. For fixed $n \geq 0$, let V be an n-dimensional vector space having orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The 2^n-dimensional *Clifford algebra* of signature (p, q), where $p + q = n$, is defined as the associative algebra generated by the collection $\{\mathbf{e}_i\}$ along with the scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:

$$\mathbf{e}_i\, \mathbf{e}_j + \mathbf{e}_j\, \mathbf{e}_i = 0 \text{ for } i \neq j, \text{ and} \tag{1.1}$$

$${\mathbf{e}_i}^2 = \begin{cases} 1, & \text{if } 1 \leq i \leq p \\ -1, & \text{if } p+1 \leq i \leq p+q = n. \end{cases} \tag{1.2}$$

The Clifford algebra of signature (p, q) is denoted $\mathcal{C}\ell_{p,q}$.

Generally the vectors generating the algebra do not have to be orthogonal. When they are orthogonal, as in the definition above, the resulting multivectors are called *blades*. Clifford algebras have well-known geometric properties and have connections with mathematical physics. Much more can be found in works such as [3] and [4].

The current work requires the construction of a nilpotent-generated abelian sub-algebra of a Clifford algebra (cf. [6]). This sub-algebra will be denoted by ${\mathcal{C}\ell_n}^{\text{nil}}$. Generating functions and functionals for Stirling numbers of the second kind, Bell numbers, and Bessel numbers will be defined on this algebra.

For any $n > 0$, let $\mathcal{G}_n$ denote the associative algebra generated by the elements $g_i = \mathbf{e}_i + \mathbf{e}_{n+i} \in \mathcal{C}\ell_{n,n}$. It is not difficult to see that $\mathcal{G}_n$ is spanned by basis elements of the form

$$\begin{cases} \text{scalars: } g_0 = 1 \in \mathbb{R} \\ \text{vectors: } g_1, \ldots, g_n \\ \text{bivectors: } g_i\, g_j = g_{ij} \;\text{ where } 0 < i < j \leq n \\ \vdots \\ \text{n-vector: } g_1\, g_2\, \cdots\, g_n \end{cases} \tag{1.3}$$

subject to the multiplication rules

$$\begin{cases} g_i\, g_j = -g_j\, g_i \\ g_1\, g_1 = g_1^2 = g_2^2 = \cdots = g_n^2 = 0. \end{cases} \tag{1.4}$$

Let $N = 2n$, and let $\mathbb{G} \subset \mathcal{G}_N$ be any collection of pairwise disjoint bivectors. In other words, $\mathbb{G}$ is a collection of bivectors $\{g_{ij}\}$ such that

$$g_{ij}, g_{k\ell} \in \mathbb{G} \Rightarrow \{i, j\} \cap \{k, \ell\} = \emptyset. \tag{1.5}$$

Clearly the maximal order of such a collection is $\frac{N}{2} = n$. Denote by $\mathbb{G}_{\max}$ the unique (up to isomorphism) collection of maximal order. Since the bivectors are disjoint, $\mathbb{G}_{\max}$ constitutes an abelian group.

Definition 1.2. Let ${\mathcal{C}\ell_n}^{\text{nil}}$ denote the associative algebra generated by the disjoint bivectors $\{\varepsilon_i\}_{1\leq i\leq n} = \mathbb{G}_{\max}$ along with the scalar $\varepsilon_\emptyset = 1 \in \mathbb{R}$. Observe that

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i, \text{ for } 1 \leq i, j \leq n, \text{and} \tag{1.6}$$

$$\varepsilon_i^2 = 0, \text{ for all } 1 \leq i \leq n. \tag{1.7}$$

As shorthand, denote the product $\varepsilon_i\,\varepsilon_j$ as ε_{ij}. Further, allow $\underline{i}$ to represent a canonically ordered multi-index consisting of some subset of $[n] = \{1, 2, \ldots, n\}$. Thus arbitrary elements of ${\mathcal{C}\ell_n}^{\text{nil}}$ have the form

$$u = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\,\varepsilon_{\underline{i}}\,, \tag{1.8}$$

where $u_{\underline{i}} \in \mathbb{R}$ for all $\underline{i} \in 2^{[n]}$ and $\varepsilon_{\underline{i}} = \prod_{k\in\underline{i}} \varepsilon_k$. The *degree* of a term $u_{\underline{i}}\,\varepsilon_{\underline{i}}$ is defined as the cardinality of the index $\underline{i}$.

The following definition is based on Berezin's definition dealing with second quantization [1].

Definition 1.3. Let $u \in {\mathcal{C}\ell_n}^{\text{nil}}$. Then the *Berezin integral* of u is defined by

$$\int u\,d\varepsilon_n \cdots d\varepsilon_1 = \int \left(\sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\,\varepsilon_{\underline{i}}\right) d\varepsilon_n \cdots d\varepsilon_1 = u_{\{1,2,\ldots n\}}. \tag{1.9}$$

In other words, $\int u\,d\varepsilon_n \cdots d\varepsilon_1$ is the "top-form" coefficient in the canonical expansion of u.

2 Results

Let $n \in \mathbb{N}$ be fixed, and let ${\mathcal{C}\ell_n}^{\text{nil}}$ denote the abelian algebra with nilpotent generators $\{\varepsilon_1, \ldots, \varepsilon_n\}$ along with the unit scalar $\varepsilon_\emptyset = 1$.

Proposition 2.1. *Let*

$$A = \sum_{\emptyset\neq\underline{j}\in 2^{[n]}} \varepsilon_{\underline{j}} \in {\mathcal{C}\ell_n}^{\text{nil}}. \tag{2.1}$$

Then, for $1 \leq k \leq n$,

$$\frac{1}{k!}\int A^k\,d\varepsilon_n \cdots d\varepsilon_1 = S(n, k), \tag{2.2}$$

where $S(n, k)$ *denotes the* Stirling number of the second kind, *defined as the number of partitions of* $[n]$ *into* k *nonempty subsets.*

Proof. For $1 \le k \le n$,

$$\left(\sum_{\emptyset \ne \underline{j} \in 2^{[n]}} \varepsilon_{\underline{j}}\right)^k = \sum_{\substack{(\underline{j}_1,\ldots,\underline{j}_k)\\ \emptyset \ne \underline{j}_i \in 2^{[n]}, 1 \le i \le k}} \varepsilon_{\underline{j}_1} \cdots \varepsilon_{\underline{j}_k}. \tag{2.3}$$

By the commutative and nilpotent properties of the generators, this reduces to

$$\sum_{\substack{\{\underline{j}_1,\ldots,\underline{j}_k\} \subset 2^{[n]}\\ \{\underline{j}_\ell\} \text{ pairwise disjoint}}} k!\, \varepsilon_{\underline{j}_1} \cdots \varepsilon_{\underline{j}_k}.$$

Taking the Berezin integral of this term further reduces to only those k-subsets of the power set whose union is $\{1, \ldots, n\}$. Dividing by $k!$ cancels the summation over all permutations and yields the number of k-block partitions of the n-set. □

Example 2.2. Stirling numbers of the second kind $\{S(6, k)\}$, $1 \le k \le 6$, generated using Clifford-algebraic methods and Mathematica.

```
(* Compute Stirling numbers S(n,k) *)
n = 6;
B = ClBasis[n];
A = Sum[B[[i]], {i, 2, 2^n}];
Table[1 / k ! * ClBerInt[ClPwr[A, k, n, "nil"], n], {k, 1, n}]

{1., 31., 90., 65., 15., 1.}
```

Proposition 2.3. *Let*

$$A = \sum_{\emptyset \ne \underline{j} \in 2^{[n]}} \varepsilon_{\underline{j}} \in \mathcal{C}\ell_n{}^{\text{nil}}. \tag{2.4}$$

Then,

$$\int e^A \, d\varepsilon_n \cdots d\varepsilon_1 = B_n, \tag{2.5}$$

where B_n denotes the n^{th} Bell number.

Proof. By definition, $e^A = \displaystyle\sum_{k=0}^{\infty} \frac{A^k}{k!}$. Further, noting that the nilpotent property of the generators $\{\varepsilon_i\}$ implies $A^k = 0$ for all $k > n$,

$$\int e^A \, d\varepsilon_n \cdots d\varepsilon_1 = \int \left(\sum_{k=0}^{n} \frac{A^k}{k!}\right) d\varepsilon_n \cdots d\varepsilon_1 = \sum_{k=0}^{n} \int \left(\frac{A^k}{k!}\right) d\varepsilon_n \cdots d\varepsilon_1. \tag{2.6}$$

By Proposition 2.1,

$$\int \frac{A^k}{k!} \, d\varepsilon_n \cdots d\varepsilon_1 = S(n, k),$$

so that summing over $k = 1, 2, \ldots, n$ gives the total number of partitions of $\{1, \ldots, n\}$, which is the n^{th} Bell number. $\square$

Example 2.4. The first six Bell numbers generated with Mathematica.

```
In[22]:= (* Generate Bell numbers *)
         L = {};
         For[j = 1, j ≤ 6, j ++,
          B = Sum[ClBasis[j][[i]], {i, 1, 2^j}] - 1;
          L = Append[L, ClBerInt[ClXP[B, j, j, "nil"], j]];]
         Print[L]

{1., 2., 5., 15., 52., 203.}
```

Proposition 2.5. *Let $\chi : 2^{[n]} \times 2^{[n]} \to \{0, 1\}$ be defined by*

$$\chi(\underline{i}, \underline{j}) = \begin{cases} 0 & \text{if } \underline{i} \pitchfork \underline{j} \\ 1 & \text{otherwise.} \end{cases} \tag{2.7}$$

Let

$$A = \sum_{\emptyset \prec \underline{i} \preceq [n]} \varepsilon_{\underline{i}} \otimes \prod_{\emptyset \prec \underline{j} \preceq [n]} \Big(\chi(\underline{i}, \underline{j}) \upsilon_{f(\underline{i}, \underline{j})} + (1 - \chi(\underline{i}, \underline{j})) \upsilon_{\emptyset} \Big) \in \mathcal{C}\ell_n{}^{\text{nil}} \otimes \mathcal{C}\ell_{\binom{2^n}{2}}{}^{\text{nil}} \tag{2.8}$$

where $f : 2^{[n]} \times 2^{[n]} \to [\binom{2^n}{2}]$ is a symmetric integer-labeling of pairs of multi-indices, and $\upsilon_{f(\underline{i}, \underline{j})}$ is a nilpotent generator of $\mathcal{C}\ell_{\binom{2^n}{2}}{}^{\text{nil}}$. Then,

$$\int \vartheta(e^A)\, d\varepsilon_n \cdots d\varepsilon_1 = B_n^*, \tag{2.9}$$

where $\vartheta : \mathcal{C}\ell_n{}^{\text{nil}} \otimes \mathcal{C}\ell_{\binom{2^n}{2}}{}^{\text{nil}} \to \mathcal{C}\ell_n{}^{\text{nil}}$ is canonical projection, and B_n^* denotes the n^{th} Bessel number.

Proof. The proof is based on the observation that non-overlapping partitions are constructed from non-overlapping blocks. By construction of the sum in (2.8), if multi-indices $\underline{i}$ and $\underline{j}$ are overlapping, their associated multi-vectors $\varepsilon_{\underline{i}}$ and $\varepsilon_{\underline{j}}$ are multiplied by the same abelian nilpotent generator of $\mathcal{C}\ell_{\binom{2^n}{2}}{}^{\text{nil}}$. In this way, the products of multi-vectors associated with overlapping blocks are always zero. The remainder of the proof follows those of Propositions 2.1 and 2.3.

$\square$

Example 2.6. A few Bessel numbers generated with Mathematica.

```
In[19]:= (* First 6 Bessel numbers *)
         L = {};
         For[m = 1, m ≤ 6, m++,
           V = Table[ClBasis[m][[i]], {i, 2, Length[ClBasis[m]]}];
           λ = V;
           tk = 1;
           For[j = 1, j ≤ Length[V] - 1, j++,
            For[k = j + 1, k ≤ Length[V], k++,
             If[ClOverlap[V[[k]], V[[j]]] == 1,
              λ[[k]] = λ[[k]] ** υtk;
              λ[[j]] = λ[[j]] ** υtk;
              tk++;
             ]]];
           B* = Sum[λ[[i]], {i, 1, Length[λ]}];
           L = Append[L, ClBerInt[Θ[ClXP[B*, m, m, "nil"]], m]]];
         Print[L];

{1., 2., 5., 14., 43., 143.}
```

References

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