From Admissibility to a New Hierarchy of Unification Types

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Motivation

Does unification type reflect the connection between unification and admissible rules?
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("Counter") Examples:
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Motivation
Framework

Let us fix:

- $\mathcal{L} :=$ algebraic language;
- $\mathcal{V} :=$ class of $\mathcal{L}$-algebras.
Motivation
Framework

Let us fix:

- \( \mathcal{L} := \) algebraic language;
- \( \mathcal{V} := \) class of \( \mathcal{L} \)-algebras.

Let \( \text{Fm}_\mathcal{L}(X) \) denote the \textbf{formula algebra} (also known as term algebra or absolutely free algebra) of \( \mathcal{L} \) over a set of variables \( X \).
A substitution (homomorphism)

$$\sigma : \text{Fm}_L(X) \to \text{Fm}_L(Y)$$

is called a $\mathcal{V}$-unifier (over $X$) of a set of $L$-identities $\Sigma$ with variables in $X$ if

$$\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi) \text{ for all } \varphi \approx \psi \in \Sigma.$$
Motivation

Unifiers

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is called a \( \mathcal{V} \)-unifier (over \( X \)) of a set of \( \mathcal{L} \)-identities \( \Sigma \) with variables in \( X \) if

\[ \mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi) \text{ for all } \varphi \approx \psi \in \Sigma. \]

Let \( U_\mathcal{V}(\Sigma, X) \) denote the set of \( \mathcal{V} \)-unifiers of \( \Sigma \) over \( X \).
Motivation

Unifiers

If \( \sigma_1, \sigma_2 \in U_{\mathcal{V}}(\Sigma, X) \), we say that \( \sigma_1 \) is more general than \( \sigma_2 \)

\[ \sigma_2 \preceq \sigma_1 \]

if there exists a substitution \( \lambda \) defined on the variables of \( \sigma_1(X) \) such that \( \sigma_2 \simeq_{\mathcal{V}} \lambda \circ \sigma_1 \).
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if there exists a substitution \( \lambda \) defined on the variables of \( \sigma_1(X) \) such that \( \sigma_2 \cong \mathcal{V} \lambda \circ \sigma_1 \).

A complete set for \((U_{\mathcal{V}}(\Sigma, X), \preceq)\) is a subset \( M \subseteq U_{\mathcal{V}}(\Sigma, X) \) such that for every \( \sigma \in U_{\mathcal{V}}(\Sigma, X) \), there exists \( \sigma' \in M \) such that \( \sigma \preceq \sigma' \).
If $\sigma_1, \sigma_2 \in U_V(\Sigma, X)$, we say that $\sigma_1$ is more general than $\sigma_2$

$$\sigma_2 \preceq \sigma_1$$

if there exists a substitution $\lambda$ defined on the variables of $\sigma_1(X)$ such that $\sigma_2 \simeq_V \lambda \circ \sigma_1$.

A complete set for $(U_V(\Sigma, X), \preceq)$ is a subset $M \subseteq U_V(\Sigma, X)$ such that for every $\sigma \in U_V(\Sigma, X)$, there exists $\sigma' \in M$ such that $\sigma \preceq \sigma'$.

$M$ is called a $\mu$-set for $(U_V(\Sigma, X), \preceq)$ if $\sigma_1 \not\preceq \sigma_2$ and $\sigma_2 \not\preceq \sigma_1$ for all distinct $\sigma_1, \sigma_2 \in M$. 
Motivation
Unifiers and Admissibility

If $\Sigma, \Delta$ are finite sets of $\mathcal{L}$-identities, the clause $\Sigma \Rightarrow \Delta$ is $\mathcal{V}$-admissible if for every $\mathcal{V}$-unifier $\sigma$ of $\Sigma$ there exists $\varphi \approx \psi \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$. 
If $\Sigma, \Delta$ are finite sets of $\mathcal{L}$-identities, the clause $\Sigma \Rightarrow \Delta$ is $\mathcal{V}$-admissible if for every $\mathcal{V}$-unifier $\sigma$ of $\Sigma$ there exists $\varphi \approx \psi \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$.

Let $\Sigma$, and $\Delta$ be finite sets of $\mathcal{L}$-identities, $X = \text{Var}(\Sigma \cup \Delta)$ and $M$ be a complete (or $\mu$-set) for $U_{\mathcal{V}}(\Sigma, X)$.

The clause $\Sigma \Rightarrow \Delta$ is $\mathcal{V}$-admissible if for every $\mathcal{V}$-unifier $\sigma \in M$

there exists $\varphi \approx \psi \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$. 
Motivation

Given a clause $\Sigma \Rightarrow \Delta$ is there any procedure to obtain a “small” set $M$ of unifiers of $\Sigma$ such that:

$$\Sigma \Rightarrow \Delta \text{ is } \forall\text{-admissible}$$

if $\forall \sigma \in M, \exists \varphi \approx \psi \in \Delta$ such that $\forall |\sigma(\varphi) \approx \sigma(\psi)$?
Main Definition
What should we change?

If $\sigma_1, \sigma_2 \in U_V(\Sigma, X)$, we say that $\sigma_1$ is more general than $\sigma_2$

$$\sigma_2 \preceq \sigma_1$$

if there exists a substitution $\lambda$ defined on the variables of $\sigma_1(X)$ such that $\sigma_2 \cong \lambda \circ \sigma_1$.

A complete set for $(U_V(\Sigma, X), \preceq)$ is a subset $M \subseteq U_V(\Sigma, X)$ such that for every $\sigma \in U_V(\Sigma, X)$, there exists $\sigma' \in M$ such that $\sigma \preceq \sigma'$.

$M$ is called a $\mu$-set for $(U_V(\Sigma, X), \preceq)$ if $\sigma_1 \not\preceq \sigma_2$ and $\sigma_2 \not\preceq \sigma_1$ for all distinct $\sigma_1, \sigma_2 \in M$. 
Main Definition

More Exact Unifiers

Let $\Sigma$ be a finite set of $\mathcal{L}$-equations and $\sigma_1, \sigma_2$ be $\mathcal{V}$-unifiers of $\Sigma$. We say that $\sigma_1$ is more exact than $\sigma_2$ (in symbols $\sigma_2 \sqsubseteq \sigma_1$) if $\sigma_1$ unifies fewer identities than $\sigma_2$. 
Main Definition
More Exact Unifiers

Let \( \Sigma \) be a finite set of \( \mathcal{L} \)-equations and \( \sigma_1, \sigma_2 \) be \( \mathcal{V} \)-unifiers of \( \Sigma \). We say that \( \sigma_1 \) is more exact than \( \sigma_2 \) (in symbols \( \sigma_2 \sqsubseteq \sigma_1 \)) if \( \sigma_1 \) unifies fewer identities than \( \sigma_2 \).

More precisely:

\[
\sigma_2 \sqsubseteq \sigma_1
\]

if

\[
\mathcal{V} \models \sigma_2(\varphi) \approx \sigma_2(\psi) \text{ whenever } \mathcal{V} \models \sigma_1(\varphi) \approx \sigma_1(\psi).
\]
Main Definition

Exact Type

Immediately,

\[ \subseteq \text{ determines a preorder on the } \forall \text{-unifiers of } \Sigma. \]
Main Definition

Exact Type

Immediately,

$\sqsubseteq$ determines a preorder on the $\forall$-unifiers of $\Sigma$.

Lemma

For each $X \supseteq \text{Var}(\Sigma)$,

$$\text{type}(U_\forall(\Sigma, \text{Var}(\Sigma)), \sqsubseteq) = \text{type}(U_\forall(\Sigma, X), \sqsubseteq).$$
Main Definition

Exact Type

Immediately,

$\sqsubseteq$ determines a preorder on the $\forall$-unifiers of $\Sigma$.

Lemma

For each $X \supseteq \text{Var}(\Sigma)$,

$$\text{type}(U_\forall(\Sigma, \text{Var}(\Sigma)), \sqsubseteq) = \text{type}(U_\forall(\Sigma, X), \sqsubseteq).$$

We define the **exact type of $\Sigma$ in $\forall$** to be

$$\text{type}(U_\forall(\Sigma, \text{Var}(\Sigma)), \sqsubseteq) \quad \text{(for } U_\forall(\Sigma, \text{Var}(\Sigma)) \neq \emptyset).$$
Main Definition

Consequences

Let $\Sigma$, and $\Delta$ be finite sets of $\mathcal{L}$-identities, $X = \text{Var}(\Sigma \cup \Delta)$ and

\[ M \text{ be a complete (or } \mu\text{-set) for } (U_{\mathcal{V}}(\Sigma, X), \sqsubseteq). \]

The clause $\Sigma \Rightarrow \Delta$ is $\mathcal{V}$-admissible if for every $\mathcal{V}$-unifier $\sigma \in M$ there exists $\varphi \approx \psi \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$. 
Main Definition

Consequences

\[ \sigma_2 \preceq \sigma_1 \; \text{implies} \; \sigma_2 \subseteq \sigma_1. \]
Main Definition

Consequences

► \( \sigma_2 \preceq \sigma_1 \) implies \( \sigma_2 \sqsubseteq \sigma_1 \).

► For each \( X \supseteq \text{Var}(\Sigma) \), if \( M \) is a complete set for \((U \vee (\Sigma, X), \preceq)\), then \( M \) is a complete set for \((U \vee (\Sigma, X), \sqsubseteq)\).
Main Definition

Consequences

- $\sigma_2 \preceq \sigma_1$ implies $\sigma_2 \subseteq \sigma_1$.

- For each $X \supseteq \text{Var}(\Sigma)$, if $M$ is a complete set for $(U\forall(\Sigma, X), \preceq)$, then $M$ is a complete set for $(U\forall(\Sigma, X), \subseteq)$.

Proposition

*If we consider the the set of types $\{1, \omega, \infty, 0\}$ preordered as follows $1 \leq \omega \leq \infty \leq 0 \leq \infty,$*
Main Definition

Consequences

- $\sigma_2 \nleq \sigma_1$ implies $\sigma_2 \sqsubseteq \sigma_1$.
- For each $X \supseteq \text{Var}(\Sigma)$, if $M$ is a complete set for $(U_\nu(\Sigma, X), \nleq)$, then $M$ is a complete set for $(U_\nu(\Sigma, X), \sqsubseteq)$.

Proposition

If we consider the set of types $\{1, \omega, \infty, 0\}$ preordered as follows $1 \leq \omega \leq \infty \leq 0 \leq \infty$, then

$$\text{type}(U_\nu(\Sigma), \sqsubseteq) \leq \text{type}(U_\nu(\Sigma), \nleq).$$
### Examples

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<th>Unification Type</th>
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Algebraic Translation
Ghilardi’s Algebraic Translation


\[ \mathsf{Fm}_\mathcal{L}(X) \xrightarrow{\sigma} \mathsf{Fm}_\mathcal{L}(Y) \]

\[
\begin{align*}
\text{Fm}_\mathcal{L}(X) \xrightarrow{\sigma} & \text{Fm}_\mathcal{L}(Y) \\
\downarrow /\nu & \quad \downarrow /\nu \\
\text{F}_\nu(X) \xrightarrow{\sigma\nu} & \text{F}_\nu(Y)
\end{align*}
\]
Algebraic Translation
Ghilardi’s Algebraic Translation


\[
\begin{align*}
\text{Fm}_\mathcal{L}(X) &\xrightarrow{\sigma} \text{Fm}_\mathcal{L}(Y) \\
\downarrow / \mathcal{V} & & \downarrow / \mathcal{V} \\
\text{F}_\mathcal{V}(X) &\xrightarrow{\sigma \mathcal{V}} \text{F}_\mathcal{V}(Y) \\
\downarrow / \Sigma & & \uparrow \\
\text{F}_\mathcal{V}(X)/(\Sigma) &\xrightarrow{h} P
\end{align*}
\]
Algebraic Translation
Ghilardi’s Algebraic Translation

Unification Problem: Finitely presented algebra $A$
Algebraic Translation
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Solution (Unifier): $h: A \rightarrow P$

$P$ is projective
Algebraic Translation
Ghilardi’s Algebraic Translation

Unification Problem: Finitely presented algebra $A$

Solution (Unifier): $h: A \rightarrow P$

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Pre-order:

$A \xymatrix{ & P_1 \ar[dd]^-{f} \\ h_1 \ar[ru] & }$

$A \xymatrix{ & P_2 \\ \ar[lu]^-{h_2} & }$
Theorem (S. Ghilardi)

For each \( \forall \)-unifiable finite set of identities \( \Sigma \),

\[
\text{type}(U_\forall(\Sigma), \preceq) = \text{type}(U_\forall(F_\forall(X)/(\Sigma)), \leq)
\]
We call an algebra $E$ exact in $\mathcal{V}$ if it is finitely generated and embeds into $F_\mathcal{V}(X)$ for some set $X$. 
Algebraic Translation
Algebraic Co-Exact Unifiers

We call an algebra \( E \) **exact in** \( V \) if it is finitely generated and embeds into \( F_V(X) \) for some set \( X \).

Unification Problem: Finitely presented algebra \( A \)
Algebraic Translation
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We call an algebra \( E \) **exact in** \( \mathcal{V} \) if it is finitely generated and embeds into \( F_{\mathcal{V}}(X) \) for some set \( X \).

**Unification Problem:** Finitely presented algebra \( A \)

**Solution (Unifier):** \( h: A \rightarrow E \)

\( E \) is exact in \( \mathcal{V} \)
We call an algebra $E$ **exact in** $\forall$ if it is finitely generated and embeds into $F\forall(X)$ for some set $X$.

**Unification Problem:** Finitely presented algebra $A$

**Solution (Unifier):** $h: A \rightarrow E$

$E$ is exact in $\forall$

**Pre-order:**

$A \xrightarrow{h_1} E_1$

$A \xrightarrow{h_2} E_2$

$f: E_1 \rightarrow E_2$
Theorem

Let $\mathcal{V}$ be an equational class $\Sigma$ a finite set of $\mathcal{V}$-unifiable $\mathcal{L}$-identities and $A$ the algebra finitely presented by $\Sigma$. Let $EU_{\mathcal{V}}(A)$ denote the preorder set of co-exact unifiers of $A$. Then

$$\text{type}(U_{\mathcal{V}}(\Sigma), \sqsubseteq) = \text{type}(EU_{\mathcal{V}}(A)).$$
Corollary

If $A$ the finitely presented algebra by $\Sigma$ has a finitely many congruences, then $\text{type}(U_V(\Sigma), \sqsubseteq)$ is unitary or finitary.
Algebraic Translation
Algebraic Co-Exact Unifiers

Corollary

If $A$ the finitely presented algebra by $\Sigma$ has a finitely many congruences, then $\text{type}(U_{\mathcal{V}}(\Sigma), \sqsubseteq)$ is unitary or finitary.

Corollary

If $\mathcal{V}$ is a locally finite variety, then $\mathcal{V}$ has exact unification type unitary or finitary.
Future Work

- Obtain separating examples.
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- Procedures to determine $\mu$-sets.
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- Procedures to determine $\mu$-sets.
- Applications to admissible rules.
Future Work

- Obtain separating examples.
- Procedures to determine $\mu$-sets.
- Applications to admissible rules.
- Purpose designed types.
Thank you for your attention!

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