Generalised Eden growth model and random planar trees

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Notation

- Given a finite subset C of $\mathbb{Z}^2$ which we call a crystal, its (external) boundary ∂C are these nodes of $\mathbb{Z}^2 \setminus C$ which have at least one neighbour in C:

$$\partial C = \{y \in \mathbb{Z}^2 \setminus C : \ \exists x \in C \text{ such that } \|x - y\| = 1\}.$$

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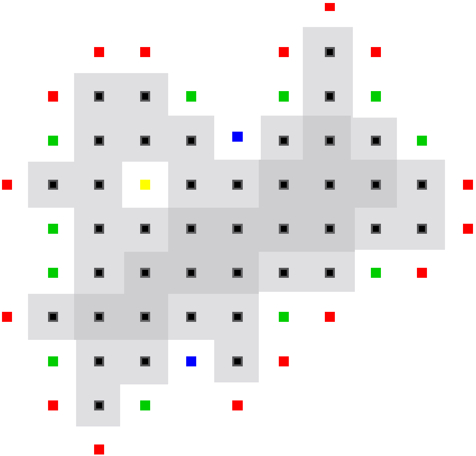
- Four types of nodes: $\partial C = \partial_1 C \cup \partial_2 C \cup \partial_3 C \cup \partial_4 C$, where

$$\partial_i C = \{y \in \mathbb{Z}^2 \setminus C : \text{ exactly } i \text{ neighbours of } y \text{ lie in } C\}, \quad i = 1, 2, 3, 4.$$

Crystal and its boundary



Growth model

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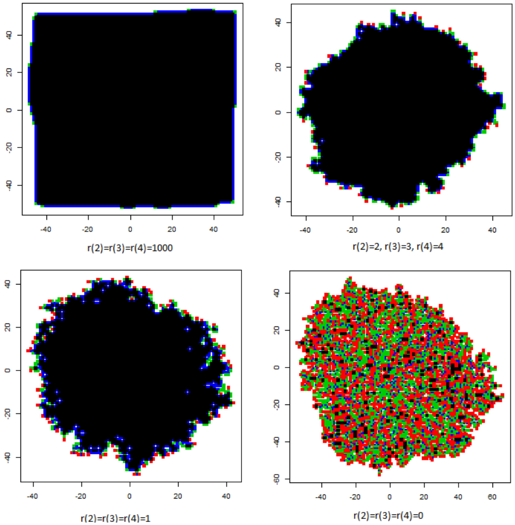
- Let C_n is the crystal at time $t = n$. At time $t = n + 1$ one of the external boundary nodes $z \in \partial C_n$ will become crystallised, i.e. a new crystal is $C_{n+1} = C_n \cup \{z\}$, where z is chosen randomly with probability depending on the number of neighbouring crystallised nodes, i.e. their type.

Generalised Eden model

We consider the following Generalised Eden model: given 4 non-negative parameters $r_1, \dots, r_4$ not all equal 0, the probability that $z \in \partial_i C_n,\ i = 1, 2, 3, 4$ is crystallised at time $n+1$ is given by

$$\frac{r_i}{\sum_{i=1}^{4} r_i |\partial_i C_n|}$$

Once crystallised, nodes stay crystallised forever.



r(2)=r(3)=r(4)=1000

r(2)=2, r(3)=3, r(4)=4

r(2)=r(3)=r(4)=1

r(2)=r(3)=r(4)=0

Continuous time version

- At time $t = 0$, each boundary node $z \in \partial_i C_0$ is given independently an exponentially $\mathrm{Exp}(r_i)$ distributed clock and the one z_1 with the minimal time t_1 is crystallised. Neighbours of z_1 have their clocks reset depending on their new type.

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- Classical Eden model is the one with parameters $r_i = i$. Equivalently, every node retains its Exp(1) clock.
  It is equivalent to first-passage percolation model: the crystal C_t at time t are the nodes which are "wet" at time t when the water source is C_0 and the water speed along each edge is independent $1/$Exp(1) r.v.'s.

Infinite growth

- If $r_1 > 0$, the crystal cannot stop growing. Let $z(C_n)$ be the leftmost among the lowest nodes of C_n and $|C_0| = n_0$. Then $|\partial C_n| \leq 4(n + n_0)$, probability that the node $f(C_n) \in \partial C_n$ just below $z(C_n)$ crystallise is at least $1/(4(n + n_0))$ and by the Borel-Cantelli lemma, this would happen infinitely often.

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- If $r_1 = 0$, the crystal can got stuck (e.g., when $r_2 = 1$ and $C_0 = \{0, 1\}^2$).

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Shape result

Assume $C_0 = 0$. One speaks of a Shape result if there exist a compact set D containing the origin, such that

$$\lim_{n\to\infty} \mathrm{dist}_H(n^{-1/2}C_n, D) = 0 \; a.s.,$$

where $\mathrm{dist}_H(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|$ is the Hausdorff distance between sets.

Non-decreasing rates

For the case $r_1 \le r_2 \le r_3 \le r_4$ the main tool is Kingman's subadditivity theorem for time $t(x,y)$ when y crystallises from initial crystal $C_0 = \{x\}$:

- Show that for co-linear $0, x, y$ along each rational direction $\theta \in [0, 2\pi)$

$$t(0,y) \le t(0,x) + t(x,y). \tag{1}$$

  This is proved by coupling two crystallisation processes, starting from $\{0\}$ and from $\{x\}$. Eq. (1) implies existence of an a.s. limit

$$\lim_{\|y\|\to\infty} \|y\|^{-1} t(0,y) = \rho(\theta)$$

- Then show continuity of $\rho(\theta)$ using subadditivity again:

$$t(0,y) \le t(0,x) + t(x,y) \text{ and } t(0,x) \le t(0,y) + t(x,y)$$

for $\|x\| = \|y\| = n$ and $\|x - y\| = n\varepsilon$.

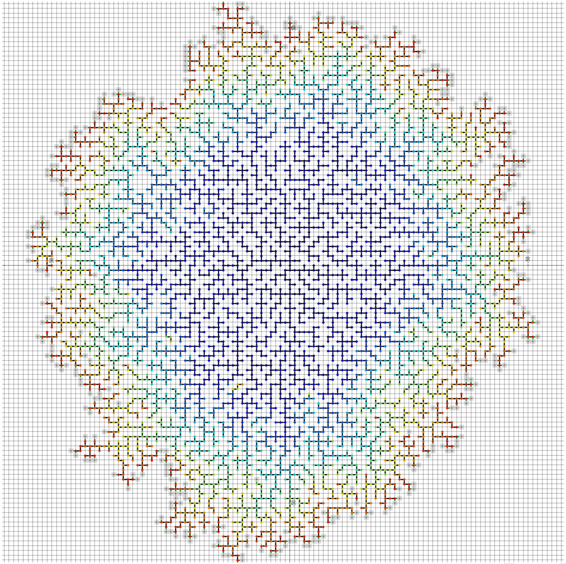
Theorem

When $r_1 \le r_2 \le r_3 \le r_4$ the Shape result holds.

Flake model

Consider now an extreme case $r_1 = 1,\ r_2 = r_3 = r_4 = 0$: a node can crystallise if only one of its neighbour is crystallised.

The crystal is a tree: a node which would close a cycle has at least two crystallised neighbours and so will never crystallise.



Types of nodes

One may distinguish

1. The crystallised nodes: C_n – the crystal
2. The nodes $\partial_1 C_n$ which can be crystallised at the next step (their clocks are set)

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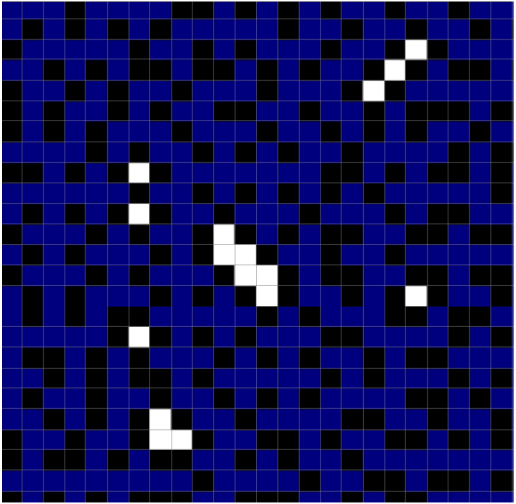
3. forbidden nodes: $F_n = \partial_2 C_n \cup \partial_3 C_n \cup \partial_4 C_n$
4. All the rest: $\mathbb{Z}^2 \setminus (C_n \cup \partial C_n)$ among which are the nodes which will never get crystallised since they belong to holes.

Definition

A hole is a finite connected set $H \subset \mathbb{Z}^2 \setminus (C_n \cup \partial C_n)$ such that $\partial H \subset F_n$.

Holes



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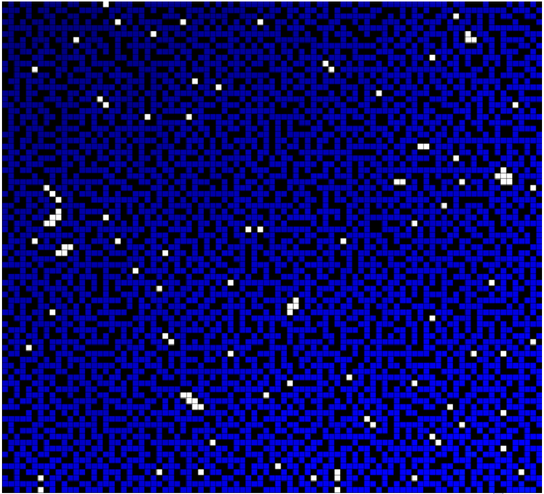
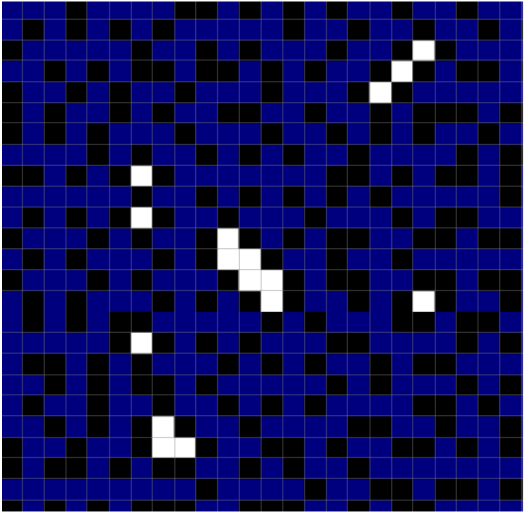


Geometry of a hole



Consider $F = \partial H$ of a hole H and let $f(H)$ be the leftmost of its lowest nodes.

- F contains no more than 2 neighbouring horizontally or vertically aligned nodes. If there are 3, the central one cannot be forbidden, since its neighbours are 2 forbidden and 1 node from the hole.

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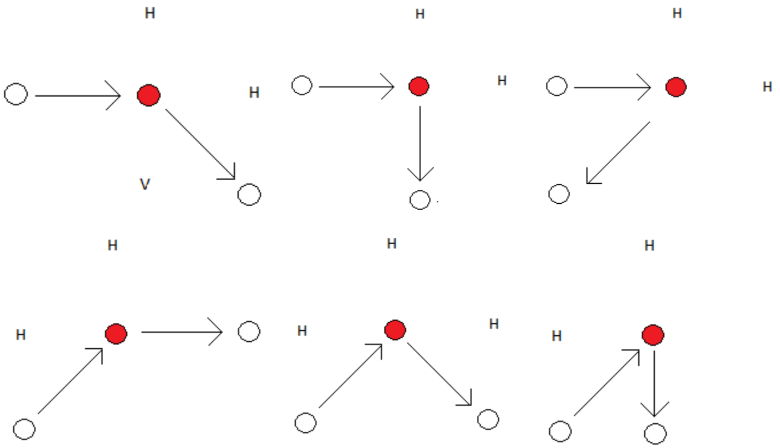
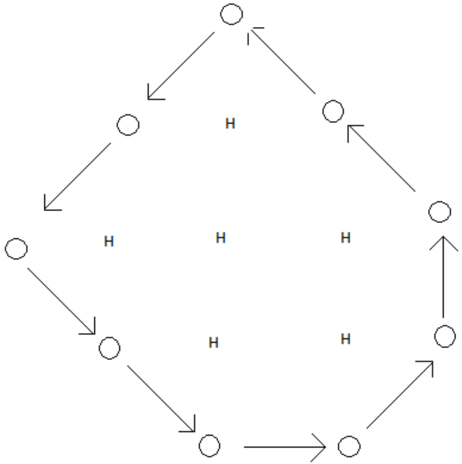
- F contains no more than 2 neighbouring horizontally or vertically aligned nodes. If there are 3, the central one cannot be forbidden, since its neighbours are 2 forbidden and 1 node from the hole.
- Connect the consecutive nodes from F by arrows going counter-clockwise starting from $f(H)$ so that the hole stays “on the left”. The angle these arrows form with the abscissa cannot decrease and can increase only by $\pi/4$ or $\pi/2$.

Impossible turns



Geometry of hole's boundary



Big holes

There are $O(n)$ possible configurations of holes with perimeter $|\partial H| = n$ with a fixed $f(H)$.

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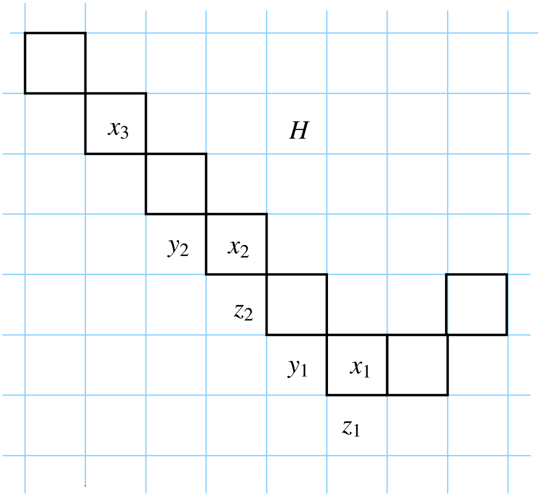
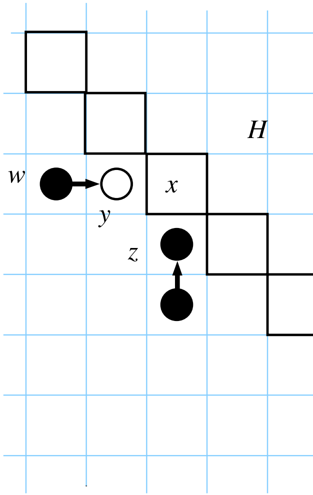
Probability to observe a hole with diameter at least n with a fixed $f(H)$ is at most $\exp\{-\beta n\}$ for some $0 < \beta < \log 2$.

Idea of the proof

- Consider continuous time version and let $F = \cup_{t \geq 0} F(C_t)$ – all the forbidden nodes.
- Consider a hole H with breadth along $(1, 1)$ and $(1, -1)$ directions n and centroid $f(H)$ at a fixed node x_1. For definitiveness, let the longest boundary is along $(1, -1)$ direction.
- Enumerate each second node going clockwise from x_1: $x_1, x_2, \dots, x_{[n/2]}$ and on the opposite side $x_{[n/2]+1}, \dots, x_n$. Let y_i, z_i be their boundary nodes at the left and below.


x_3
H
y_2
x_2
z_2
y_1
x_1
z_1


H
w
x
y
z

Let $\tau_i = \min\{t(y_i), t(z_i)\}$ be the time the first neighbour of x_i crystallises, $\tau_{(1)} \le \tau_{(2)} \dots$ and $x_{(i)}$ the i-th among x's whose neighbour crystallises.

$$\begin{aligned}
&\mathbf{P}\{\partial H \in F \;\Big|\; x_1 = f(H)\} \\
&\qquad \le \mathbf{P}\{x_1, x_2, \dots, x_n \in F \;\Big|\; x_1 = f(H)\} \\
&\qquad = \mathbf{E}\,\mathbf{P}\{x_{(1)} \in F \;\Big|\; \tau_{(1)}, x_1 = f(H)\} \\
&\qquad \times \mathbf{P}\{x_{(2)} \in F \;\Big|\; \tau_{(1)}, \tau_{(2)}, x_{(1)} \in F, x_1 = f(H)\} \\
&\qquad \dots \\
&\qquad \times \mathbf{P}\{x_{(n)} \in F \;\Big|\; \tau_{(1)}, \dots, \tau_{(n)}, x_{(1)}, \dots, x_{(n-1)} \in F, x_1 = f(H)\}
\end{aligned}$$

By the strong Markov property, since $\{\tau_{(i)}\}$ are stopping times,

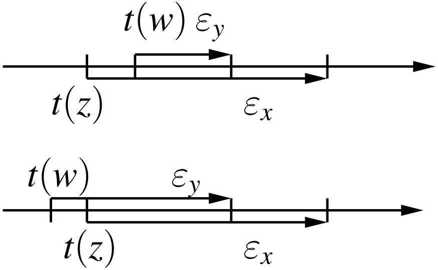
$$\mathbf{P}\{x_{(i)} \in F \;\Big|\; x_{(1)}, \dots, x_{(i-1)} \in F, \tau_{(1)}, \dots, \tau_{(i)}, x_1 = f(H)\}$$
$$= \mathbf{P}\{x_{(i)} \in F \;\Big|\; x_{(1)}, \dots, x_{(i-1)} \in F, \tau_{(i)}, x_1 = f(H)\}$$

Moreover, $x_{(i)} \in F$ depends on the clocks at nodes $y_{(i)},\ z_{(i)}$ which are reset at time $\tau_{(i)}$ by memoryless of the Exponential distribution. Thus

$$= \mathbf{P}\{x_{(i)} \in F \;\Big|\; \tau_{(i)}\}$$

Omitting index (i), let ε_x be the clock started at node x at time $t(z)$ so that x is set to crystallise at time $t(z) + \varepsilon_x$. Let ε_y is the clock started at node y when the first of its neighbours, say w crystallised. We have two cases:

1. at time $t(z)$, both neighbours of y was not yet crystallised so y did not have clock set yet: $t(z) < t(w)$;
2. at time $t(z)$, y was not crystallised, but had a clock ε_y already ticking: $t(w) < t(z)$.



$x \in F$ if $t(z) + \varepsilon_x > t(w) + \varepsilon_y$. By memoryless of the exponential r.v.'s, this is equivalent $\varepsilon_x > \varepsilon_y$ so that

$$\mathbf{P}\{x_{(i)} \in F \;\big|\; \tau_{(i)}\} = 1/2.$$

Thus for a given configuration of H with diameter n,

$$\mathbf{P}\{\partial H \in F \mid x_1 = f(H)\} \le 2^{-n}$$

$$\mathbf{P}\{\text{there is a hole } H \text{ with } f(H) = x_1 \text{ with diameter } \ge n\} \le C \sum_{m=n}^{\infty} \frac{m}{2^m}$$

so by Borel-Cantelli, the probability that there is always a hole of diameter $\alpha\sqrt{N}$ in C_N is 0. Thus

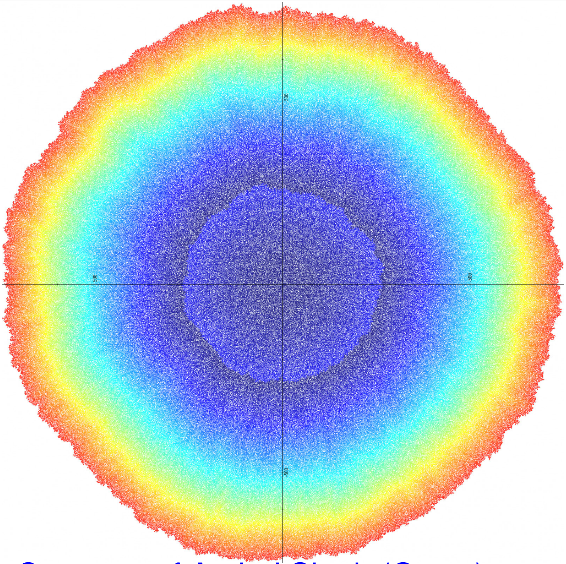
If the shape result still holds, D is 1-connected.

Open problems

Does the shape result still hold for non-monotonely growing rates?

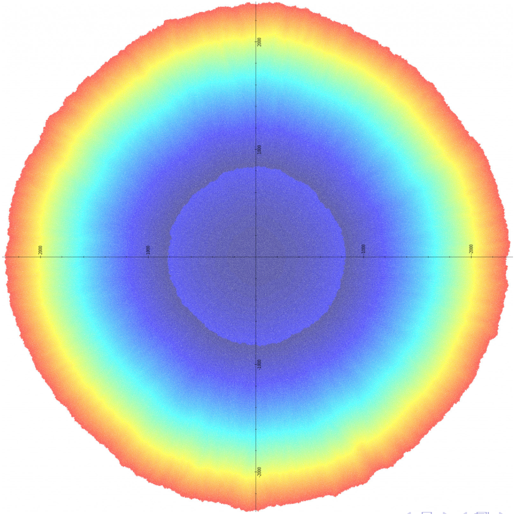
Problem: coupling argument does not work – crystal at 0 inhibits growth of crystal at x!

10^5 steps

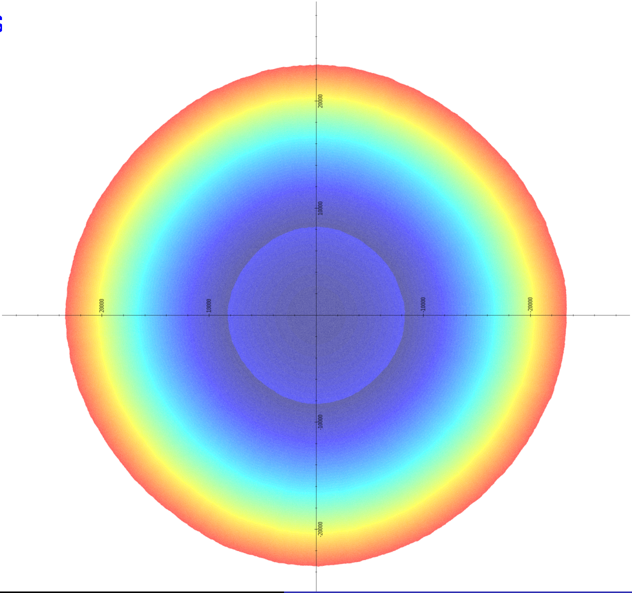


Courtesy of Arvind Singh (Orsay)

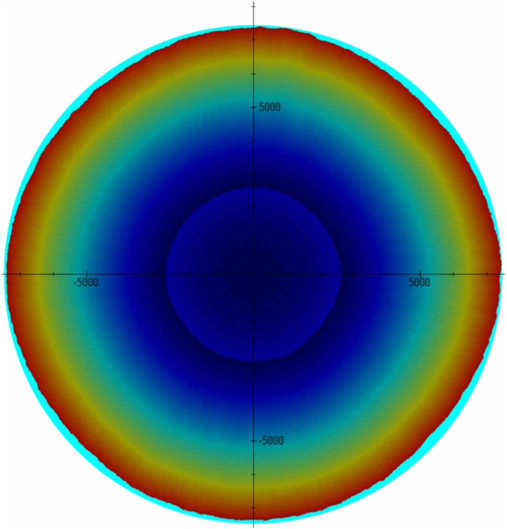
10^7 steps



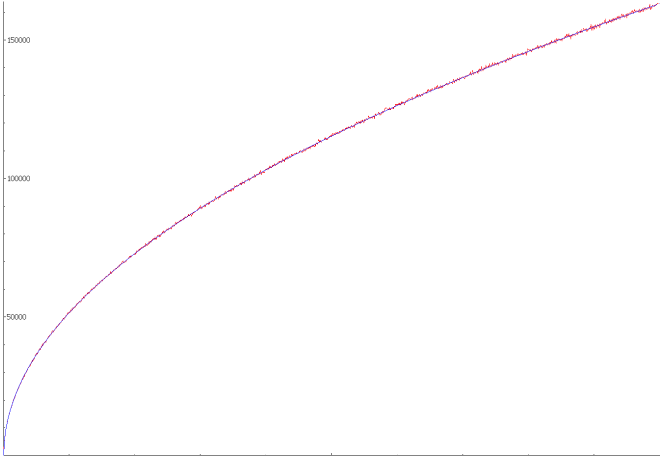
10^9 steps



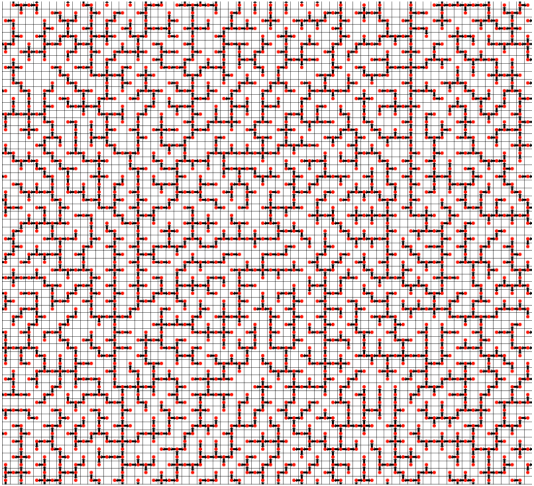
Limiting shape is not a ball



Square root law for boundary size



Long memory



Connctivity

Let $C_0 = \{0\}$ and grow the crystal to infinity. Let $L(x, y)$ be the length of the path from x to y given they are crystallised. Will an a.s. limit exist:

$$\lim_{n\to\infty} (2n)^{-1} L\big((-n, 0), (n, 0)\big) \ ?$$

If yes, will it be different from

$$\lim_{n\to\infty} (2n)^{-1} L\big((-n, N), (n, N)\big) \ ?$$

Random forest

- If $C_0 = \{x, y\}$ with $\|x - y\| > 1$ then the trees connected to x and y are disjoint. How does 'interface' looks like? If the shape result then like a bisector up to $o(r)$ at distance r?

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- Choose nodes to C_0 independently with prob. p. When $p \downarrow 0$, would the trees converge to the Voronoi cells when the grid size is $\sqrt{p}$?

References

1. Klaus Schürger On the asymptotic geometrical behavior of a class of contact interaction process with a monotone infection rate, Z. Wahrsch. verw Gebiete, **48**, 35–48, 1979

Thank you!

Questions?