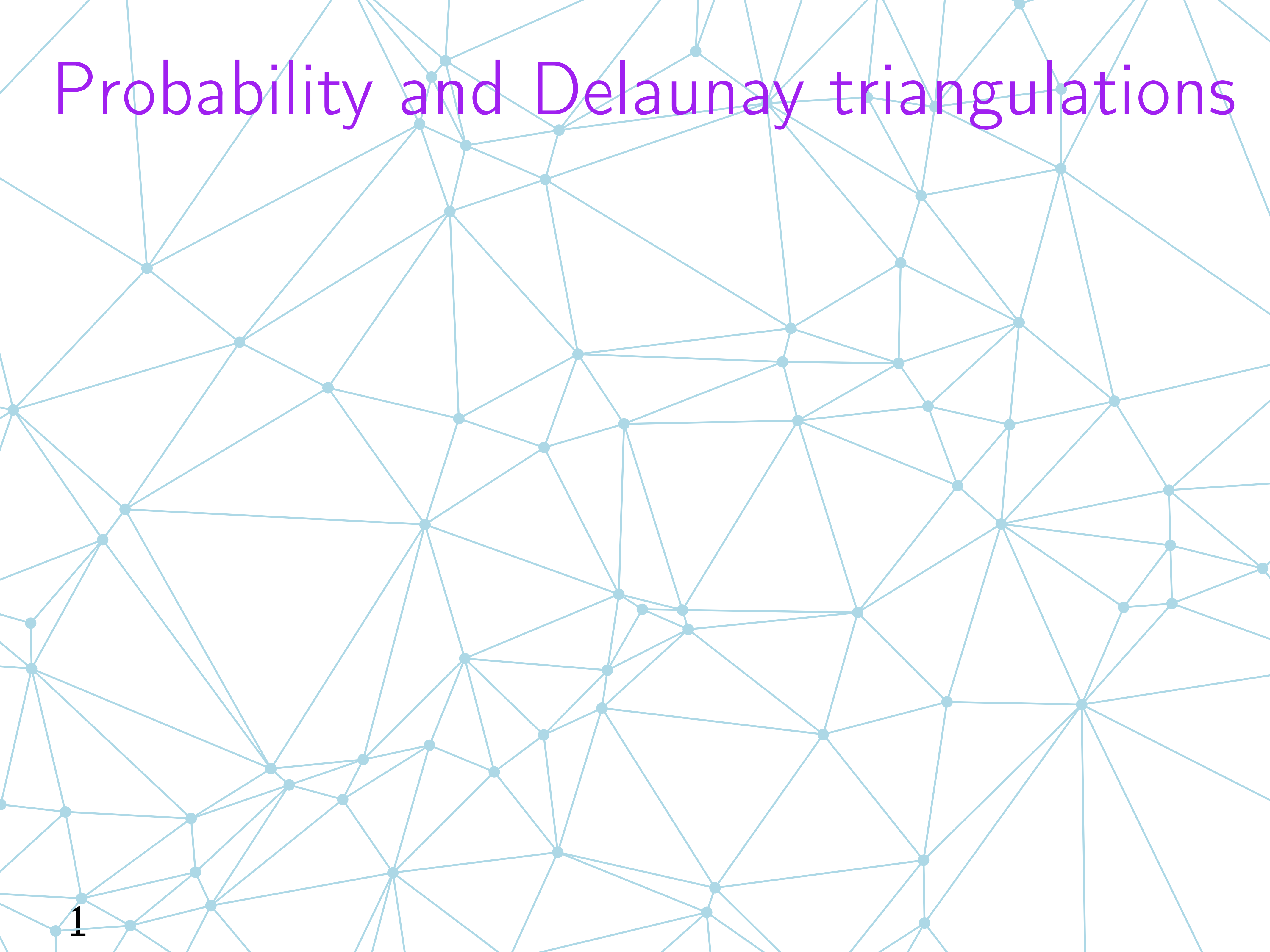
Probability and Delaunay triangulations



Randomized algorithms for Delaunay triangulations

Poisson Delaunay triangulation

Randomized algorithms for Delaunay triangulations

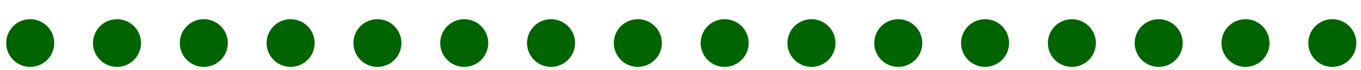
- Randomized backward analysis of binary trees
- Randomized incremental construction of Delaunay
- Jump and walk
- The Delaunay hierarchy
- Biased randomized incremental order
- Chew algorithm for convex polygon

Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties

Sorting

$-\infty$



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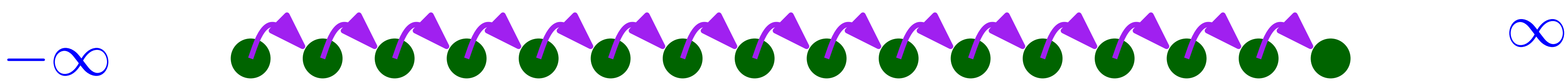
Sorting

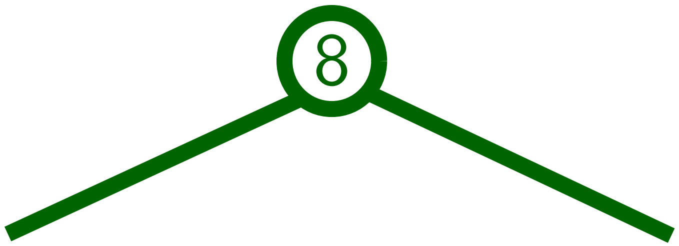


Sorting

Binary tree

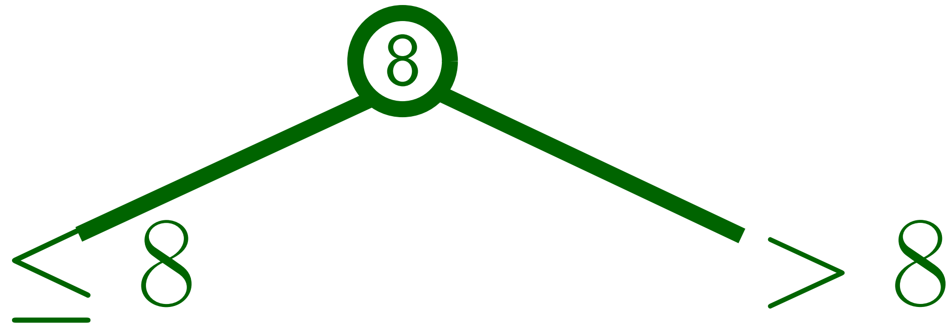
Sorting

Binary tree



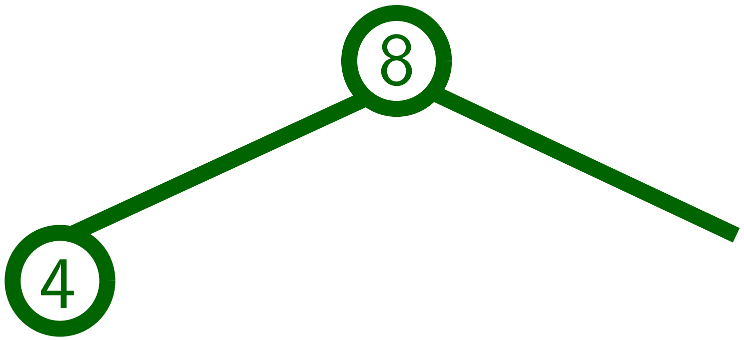
Sorting

Binary tree



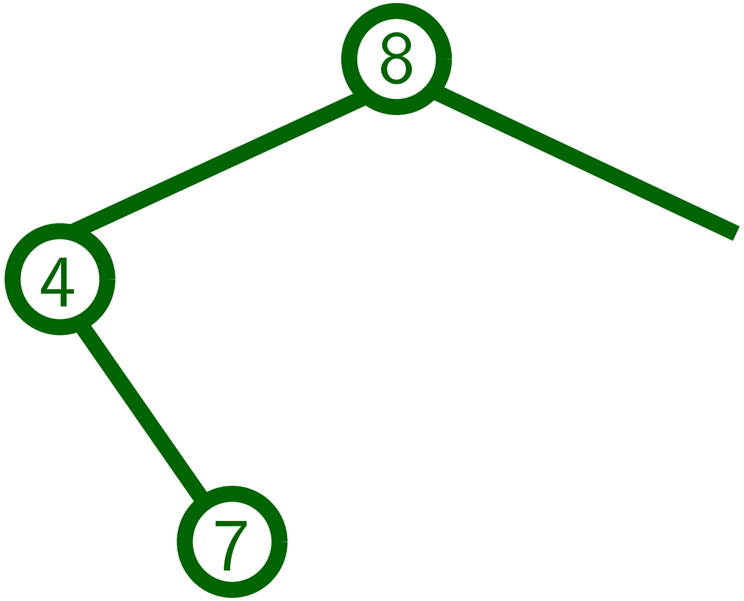
Sorting

Binary tree



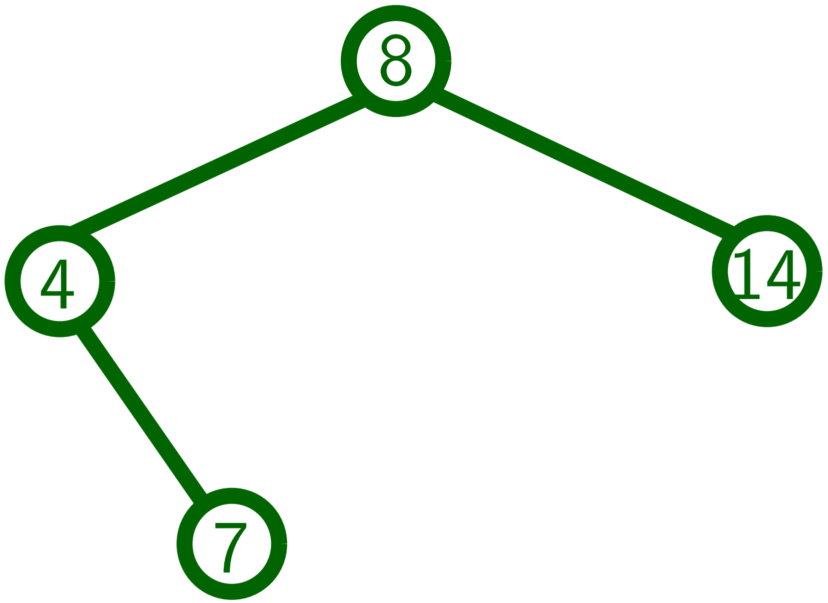
Sorting

Binary tree



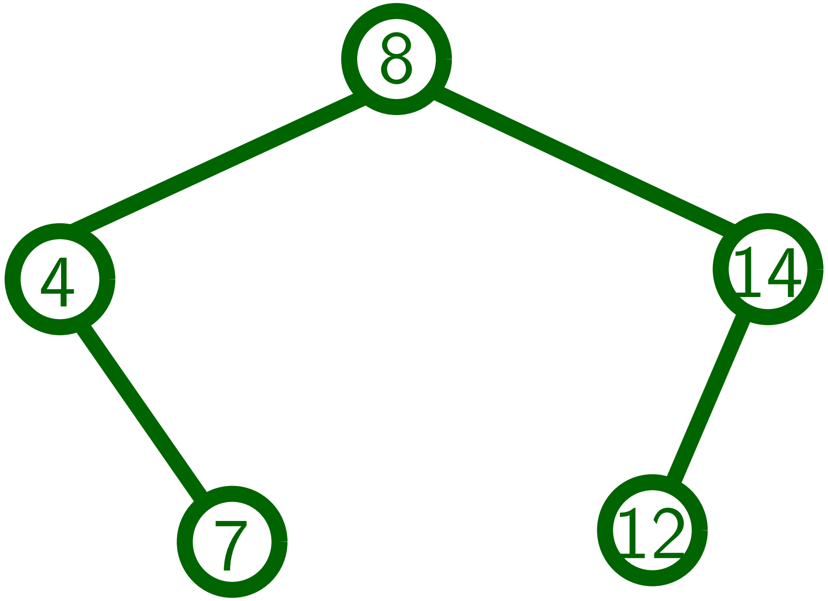
Sorting

Binary tree



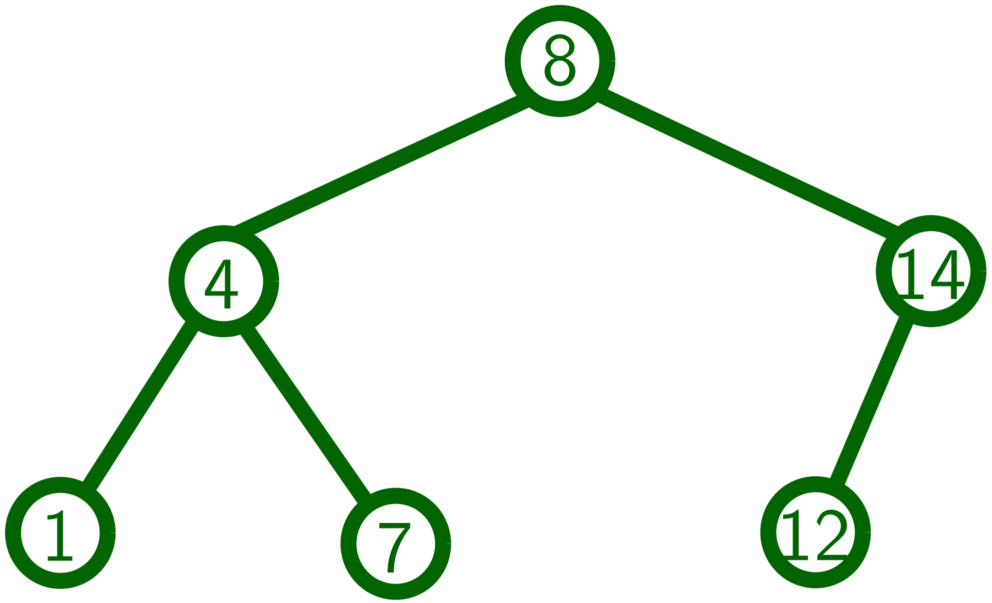
Sorting

Binary tree



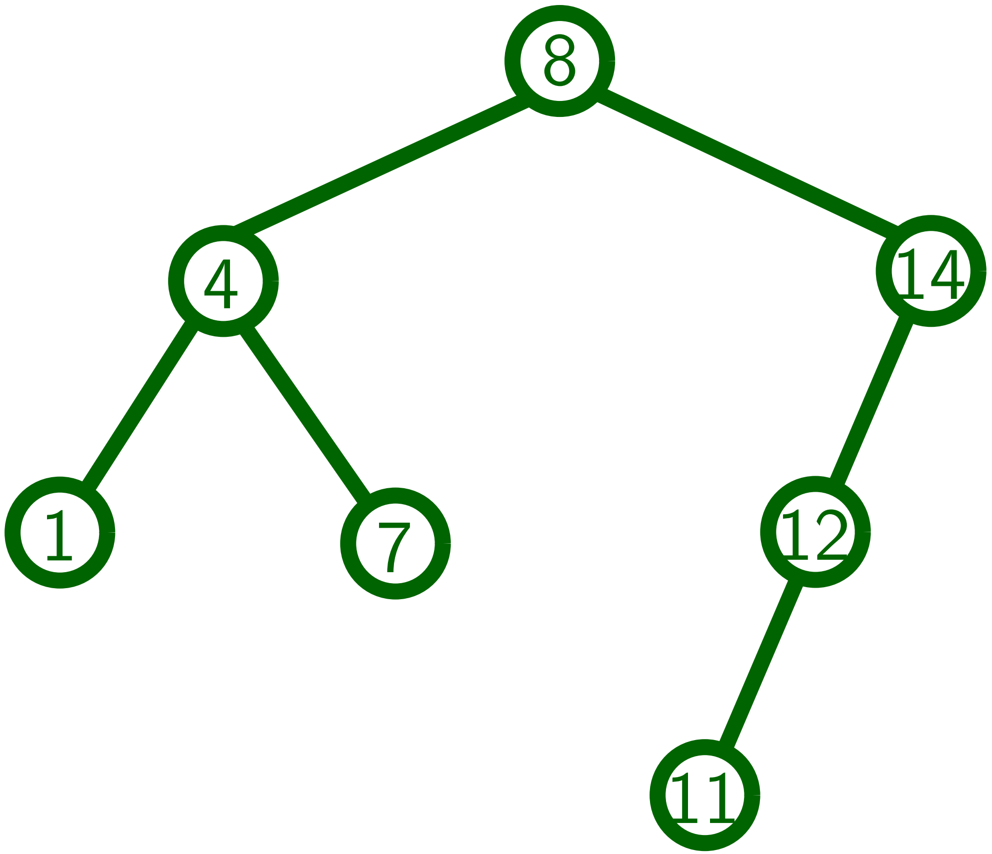
Sorting

Binary tree



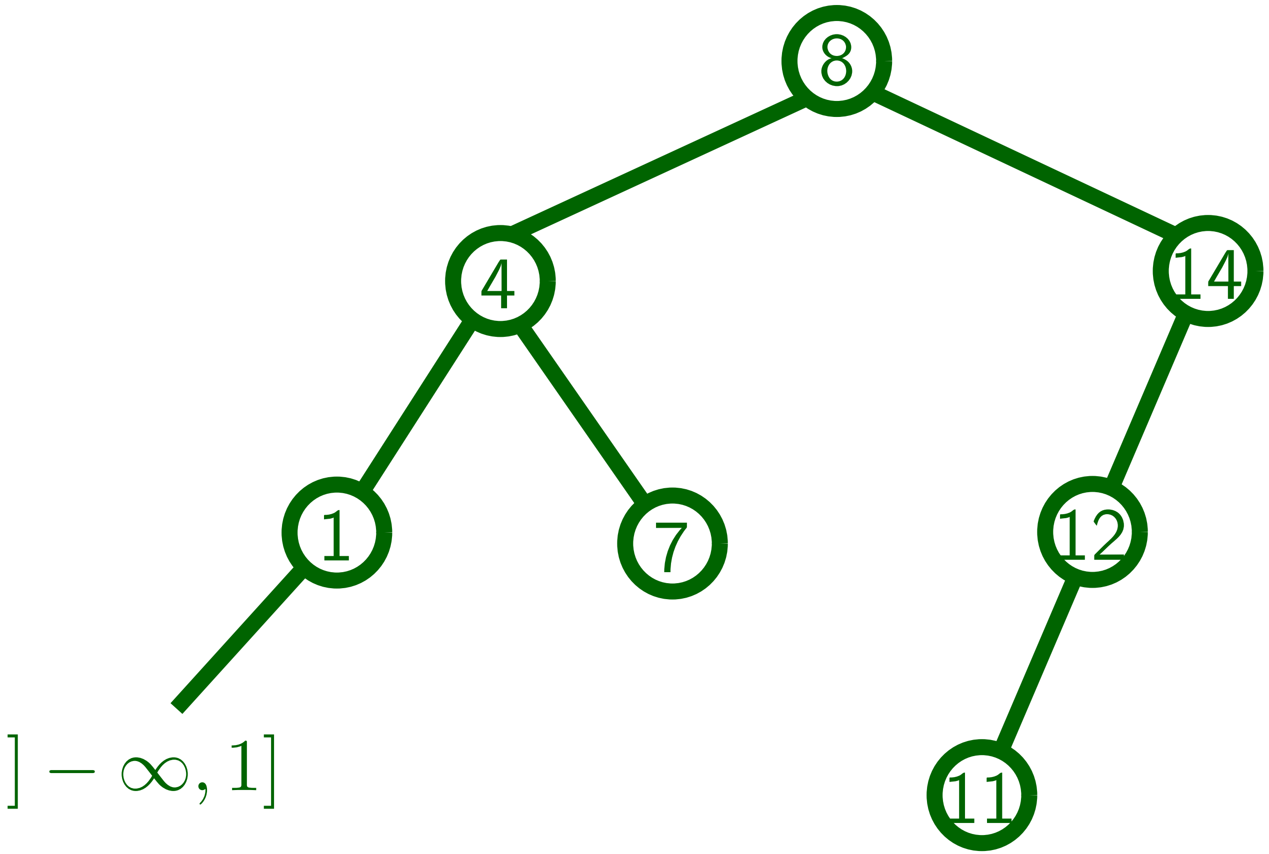
Sorting

Binary tree



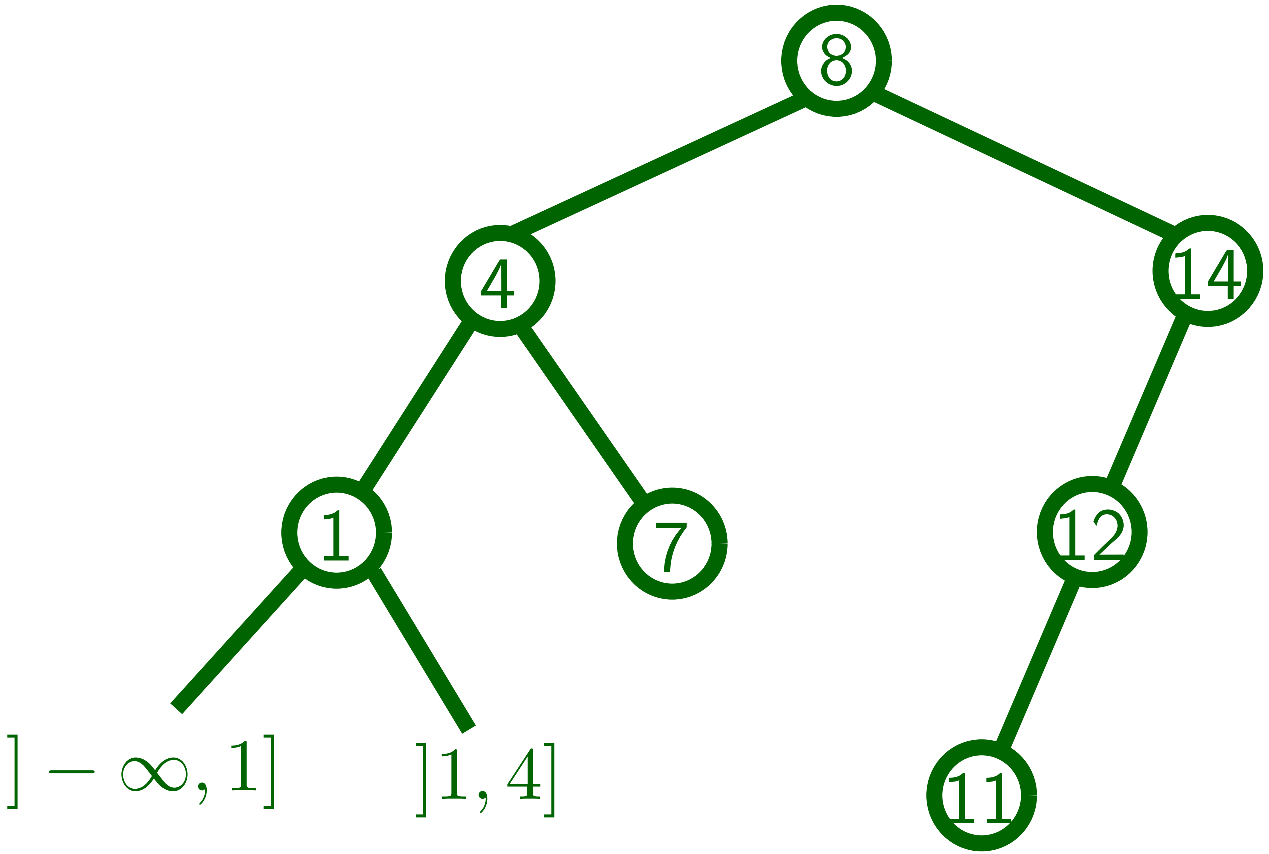
Sorting

Binary tree



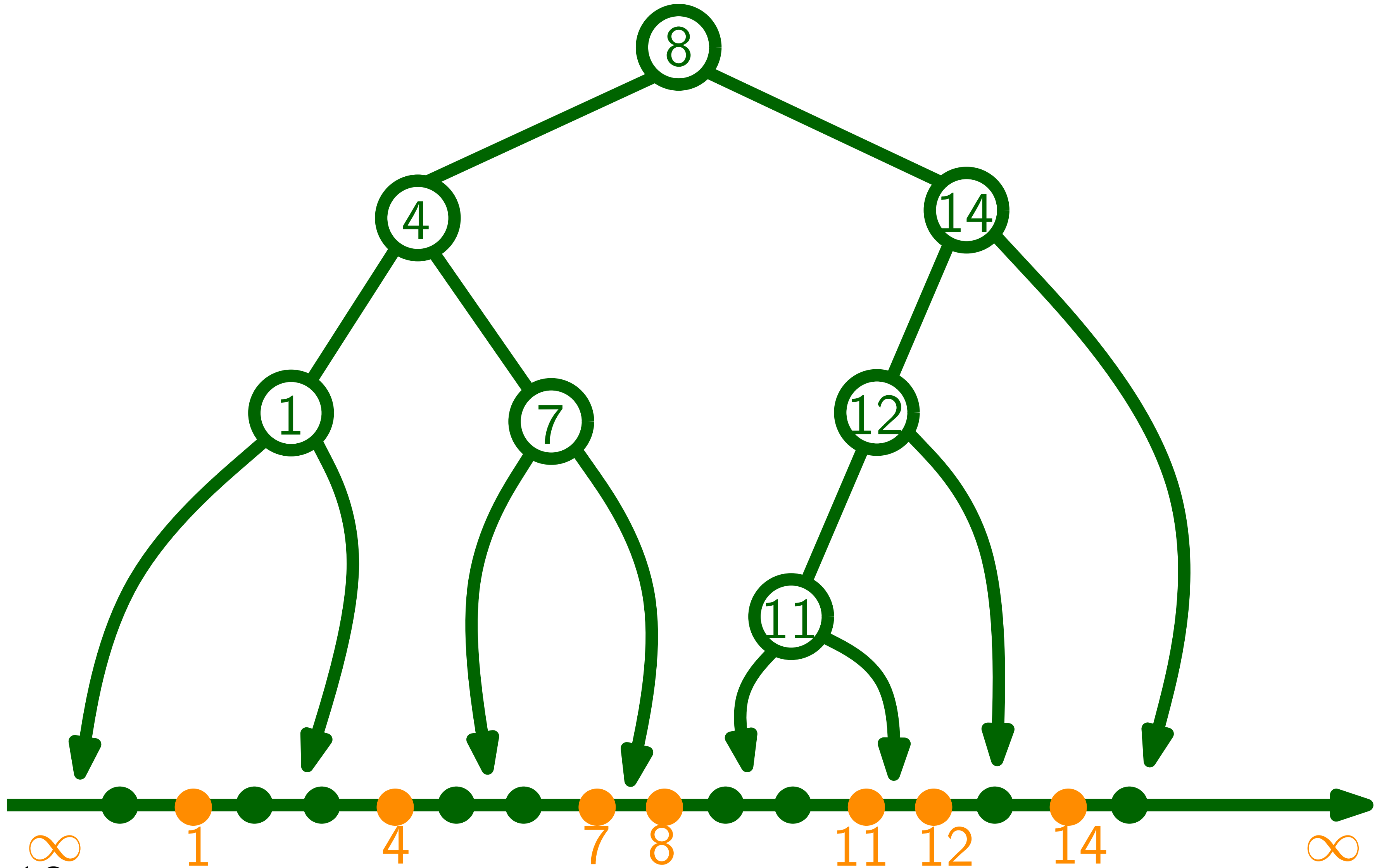
Sorting

Binary tree

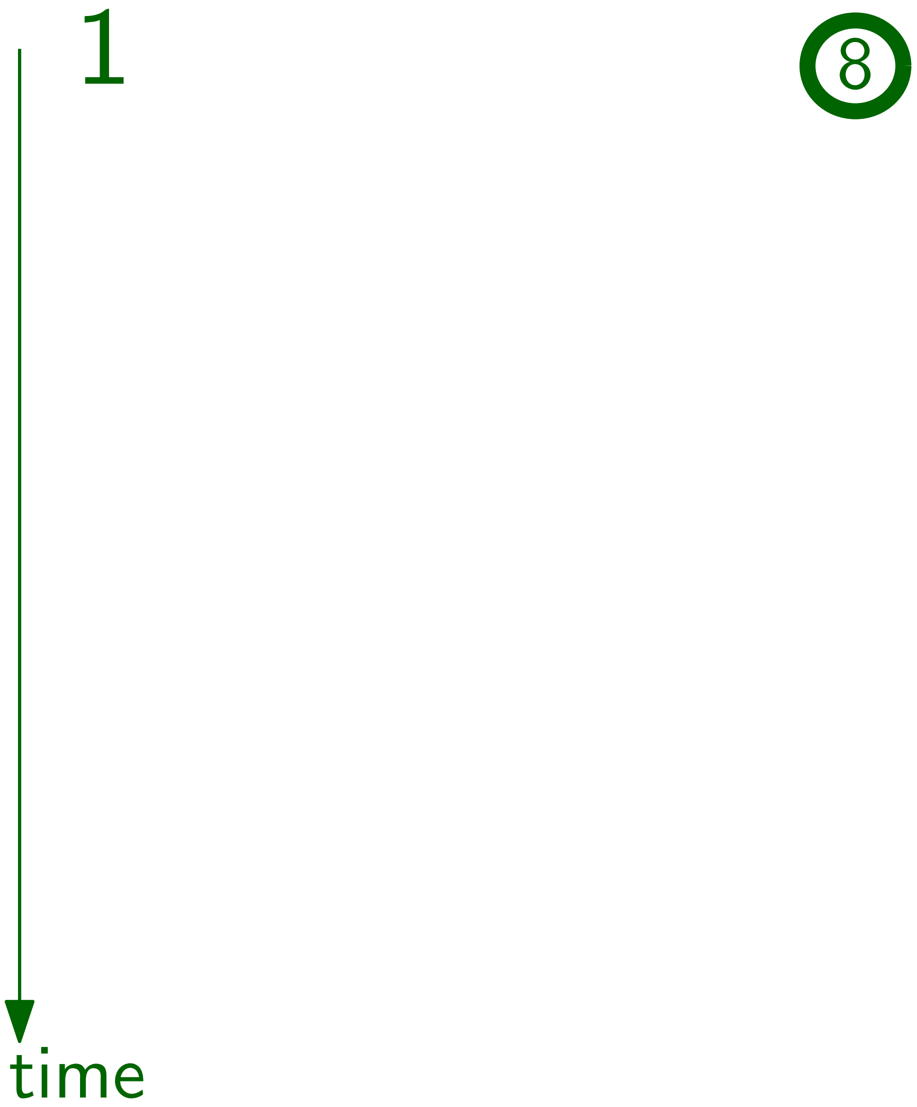


Sorting

Binary tree

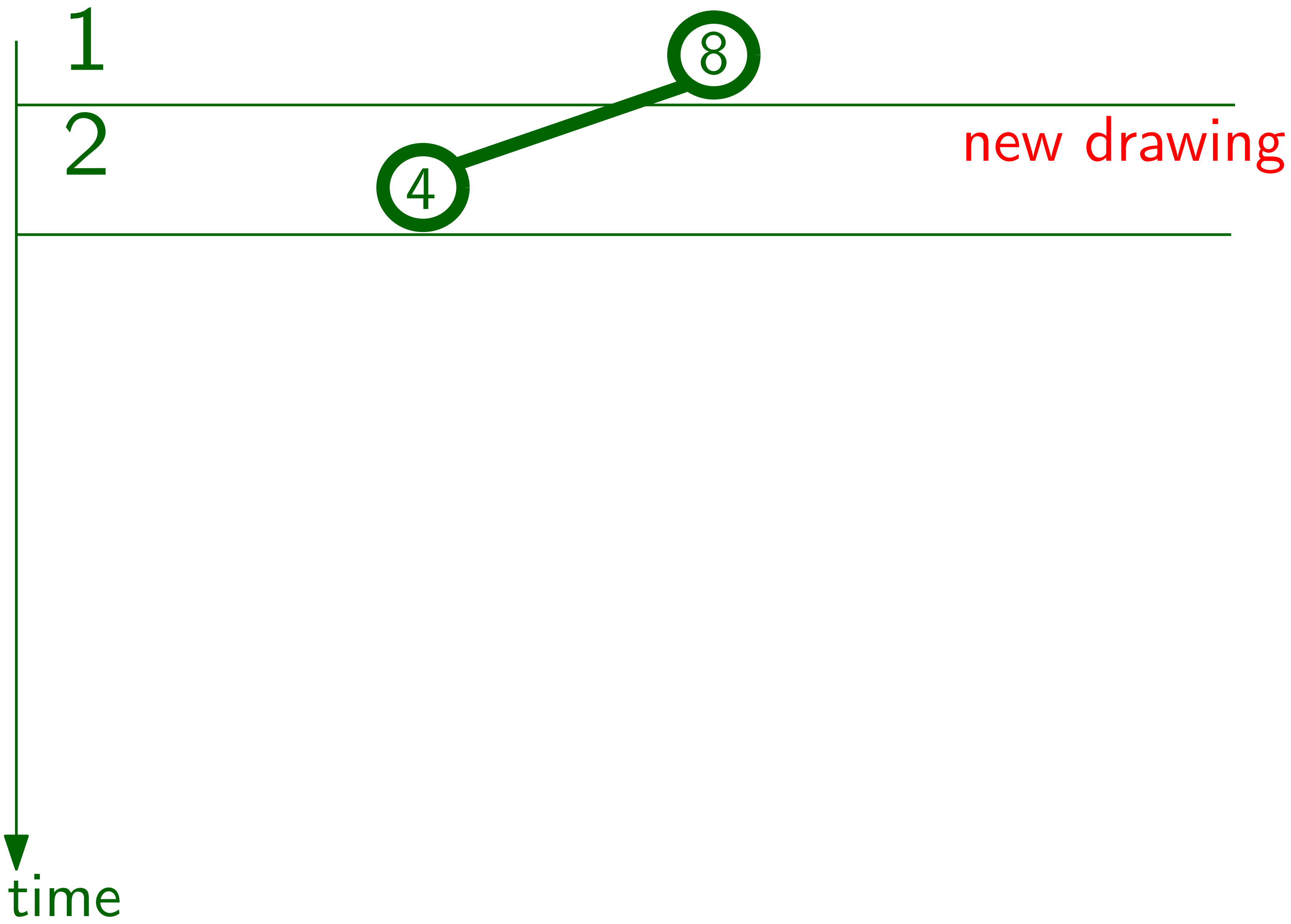


Sorting

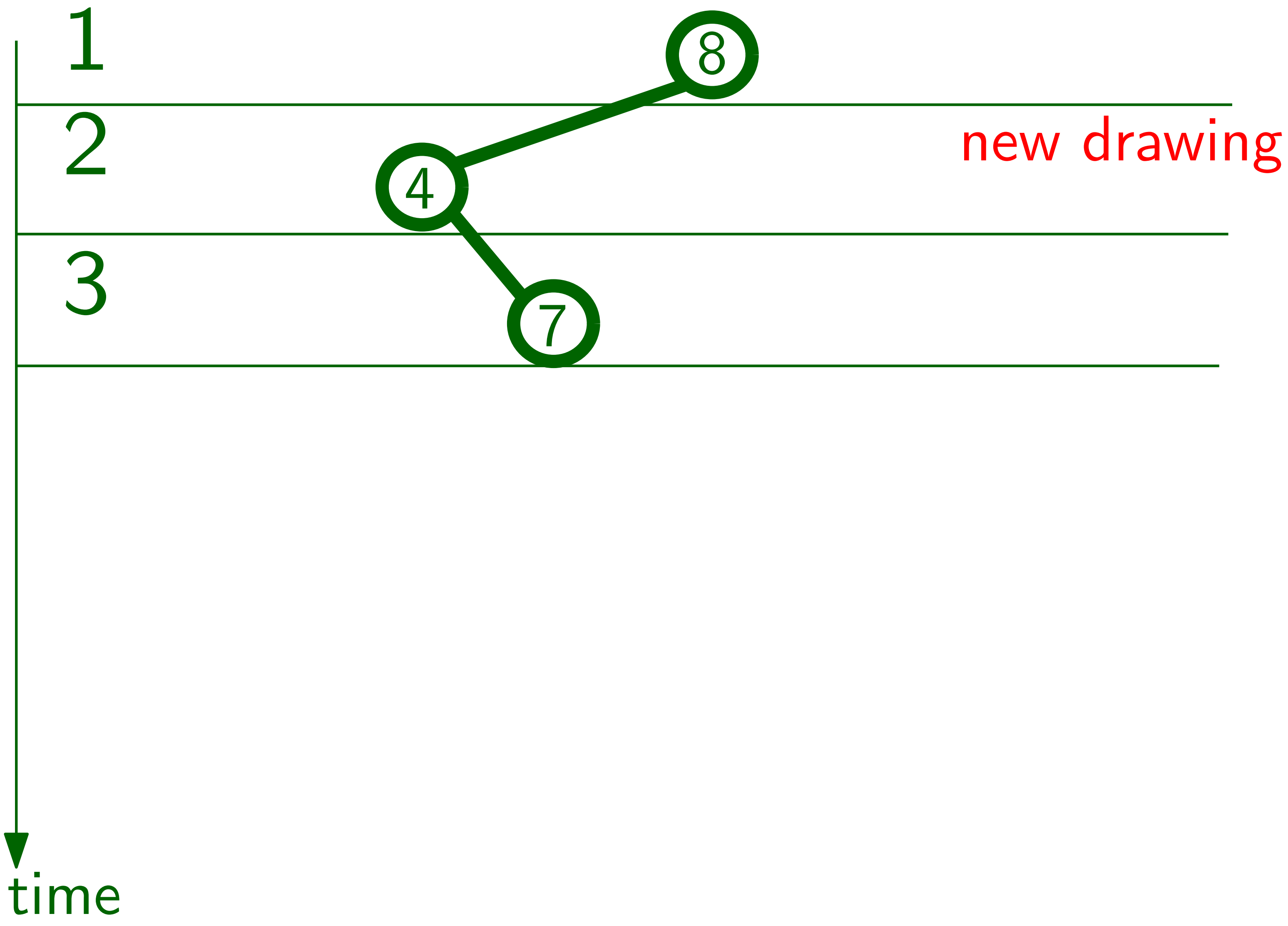


new drawing

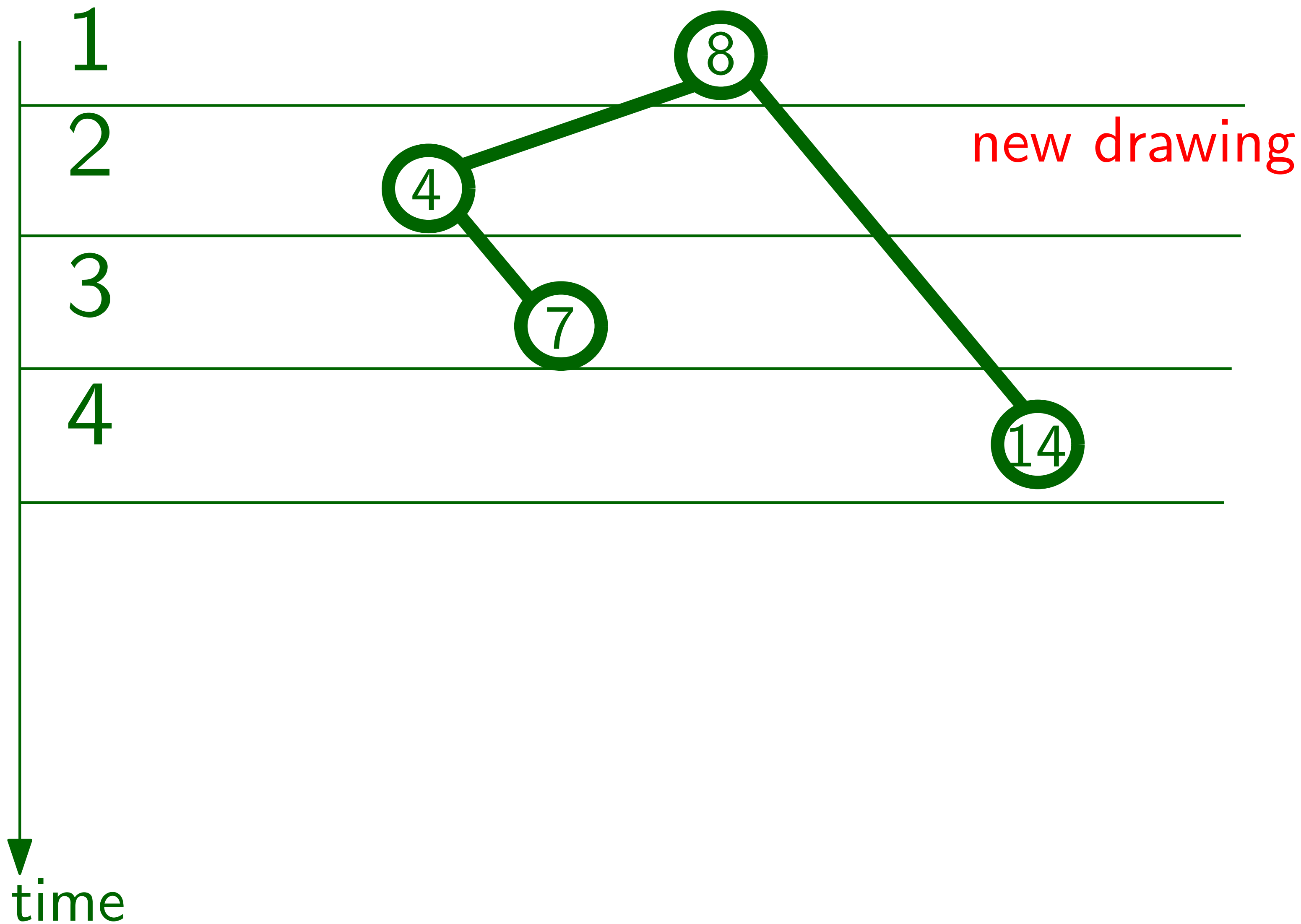
Sorting



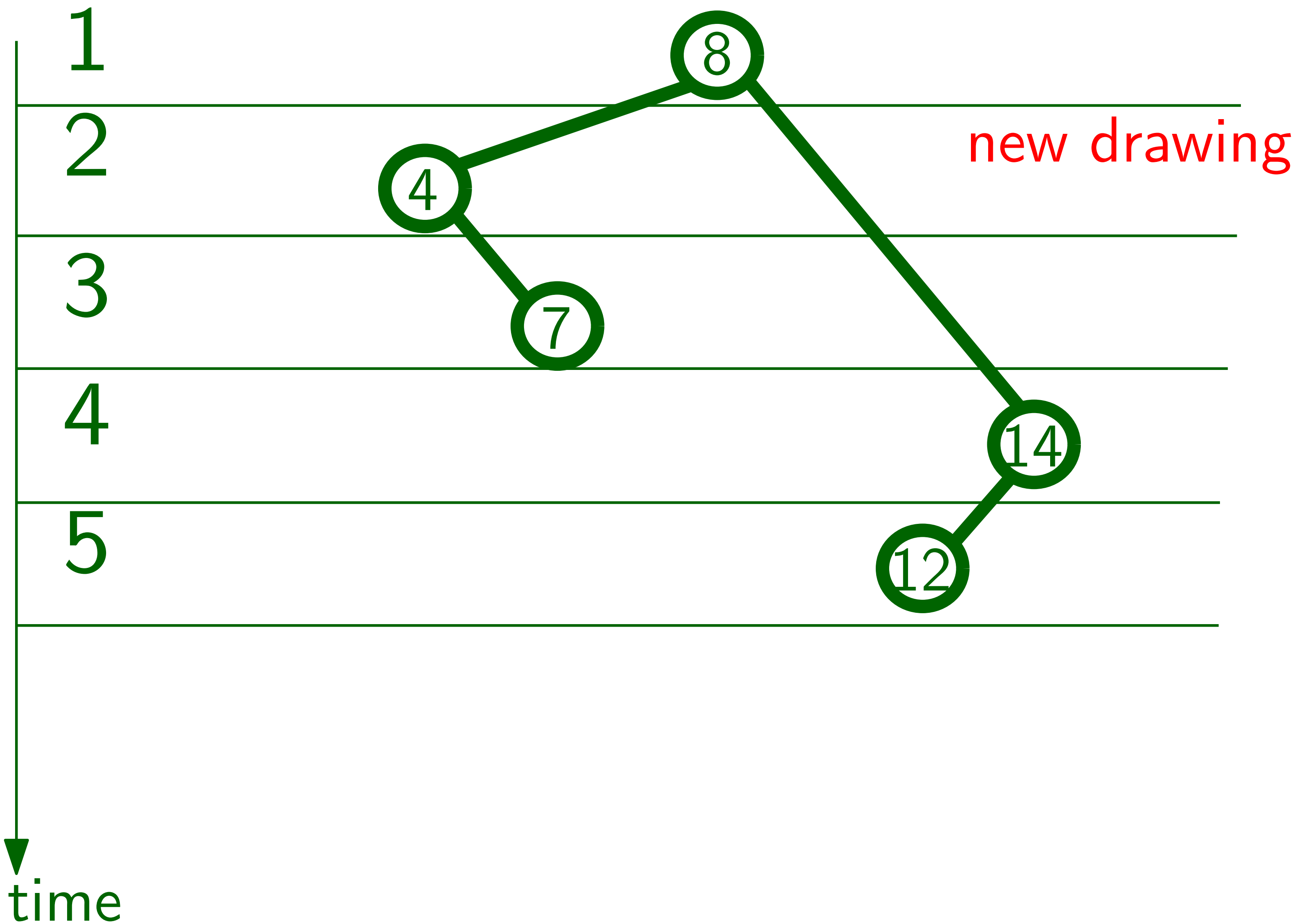
Sorting



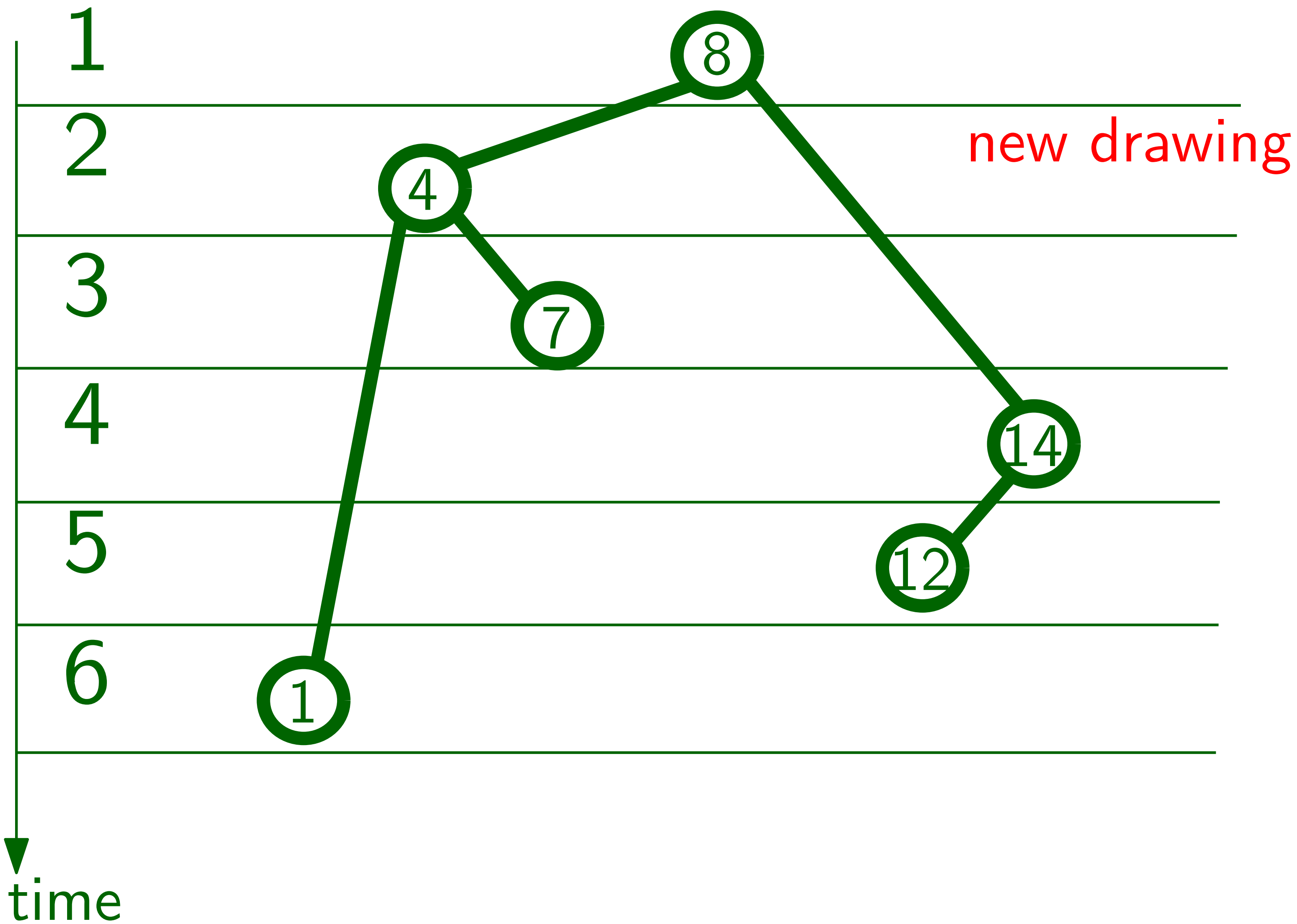
Sorting



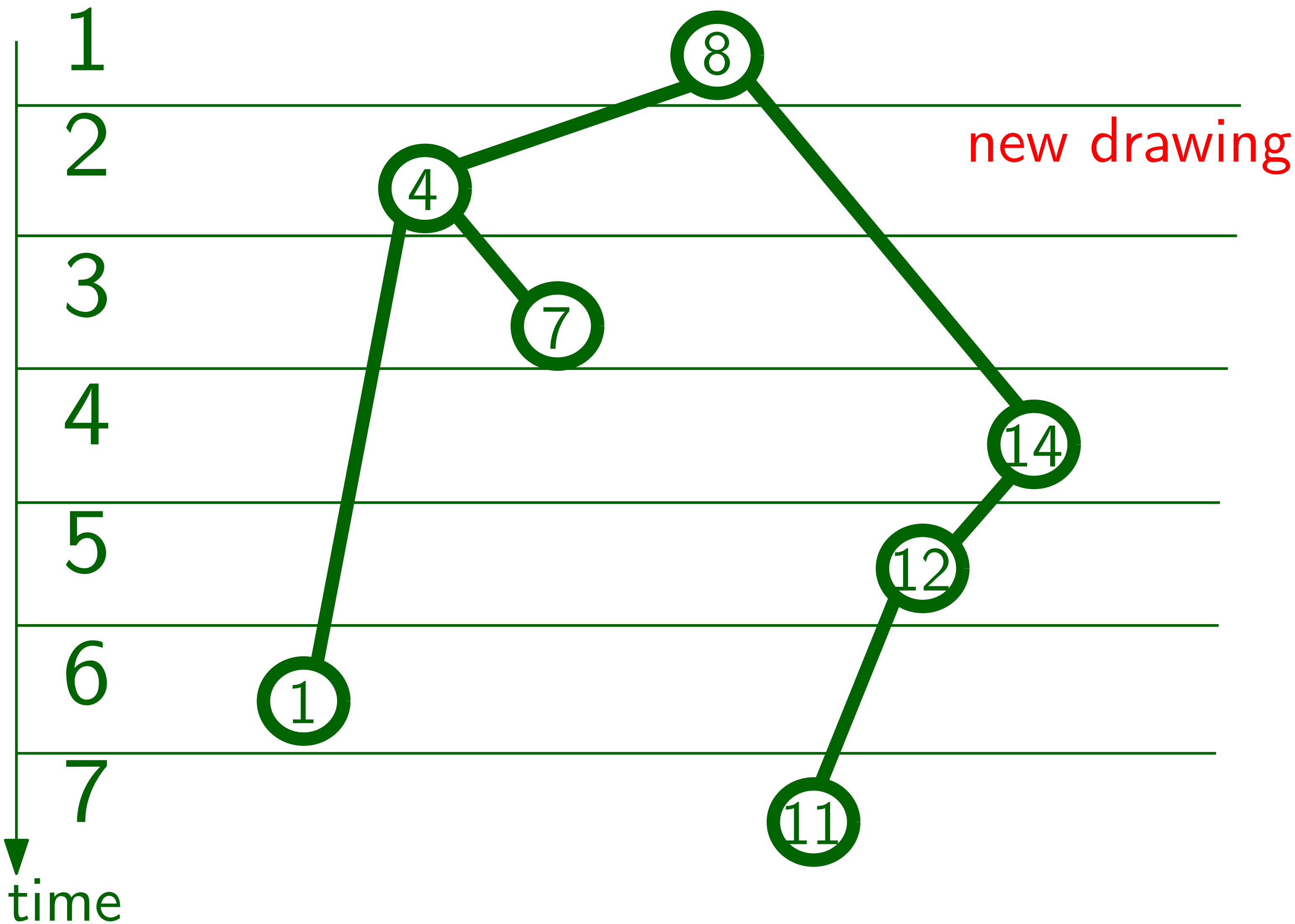
Sorting



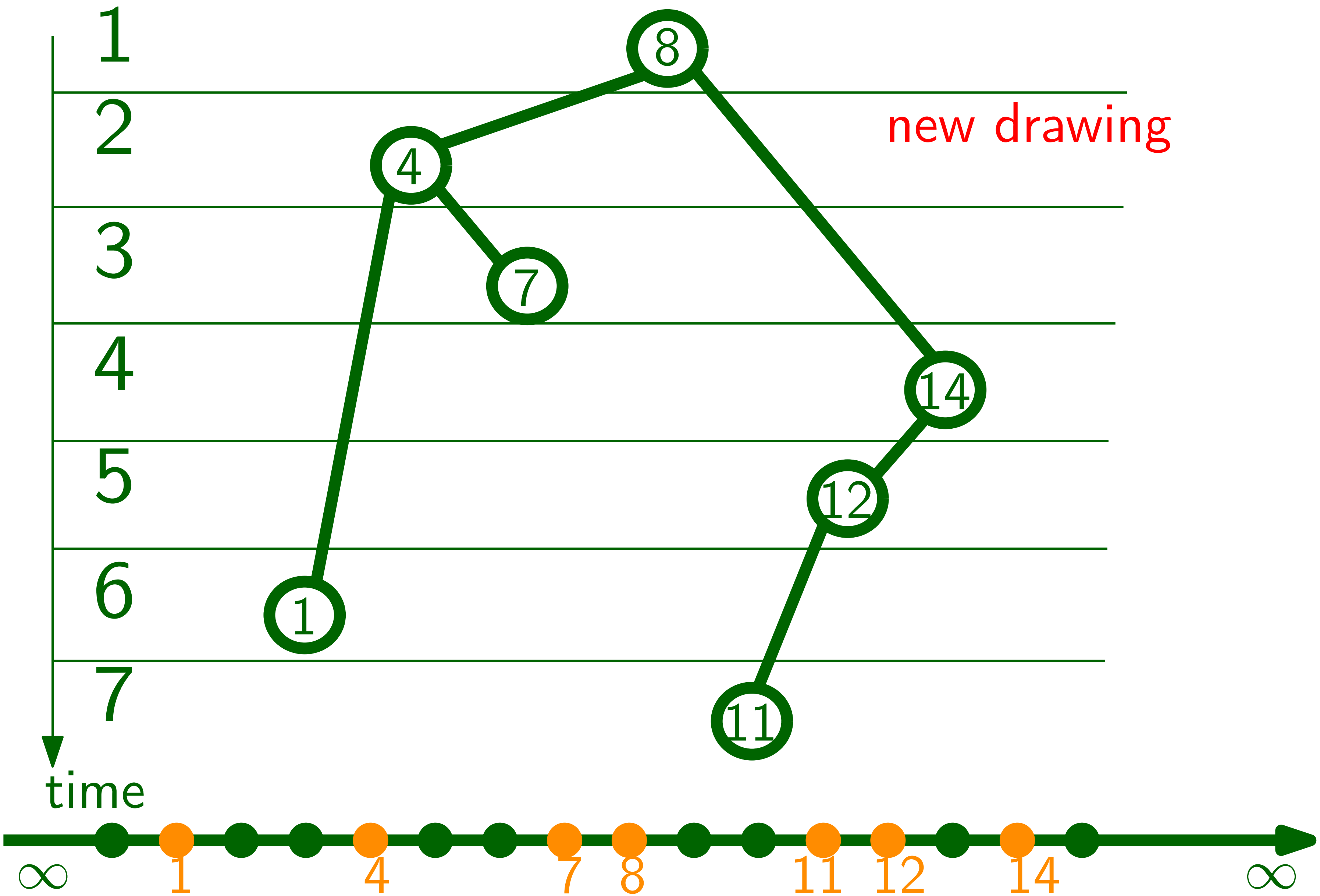
Sorting



Sorting



Sorting



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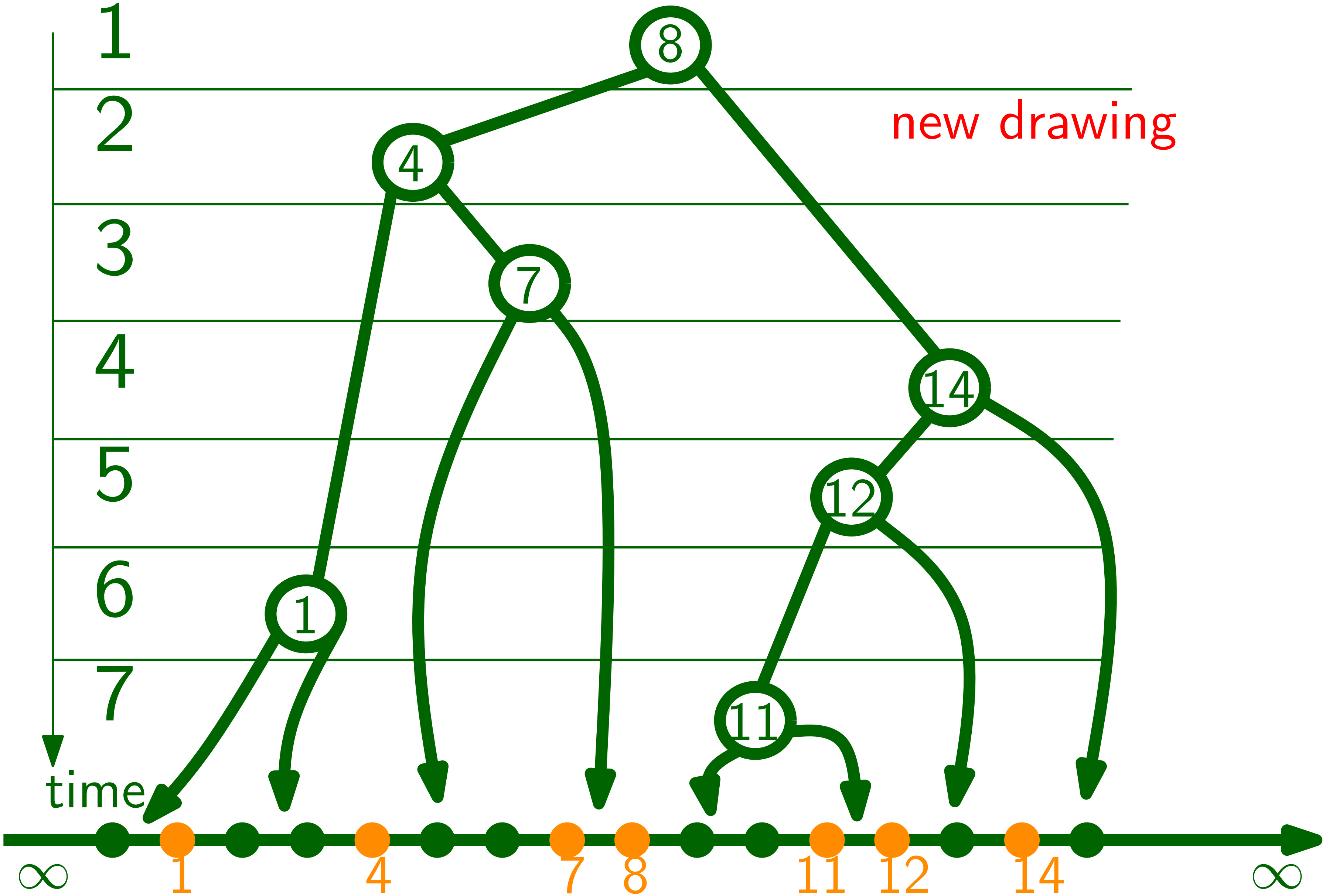

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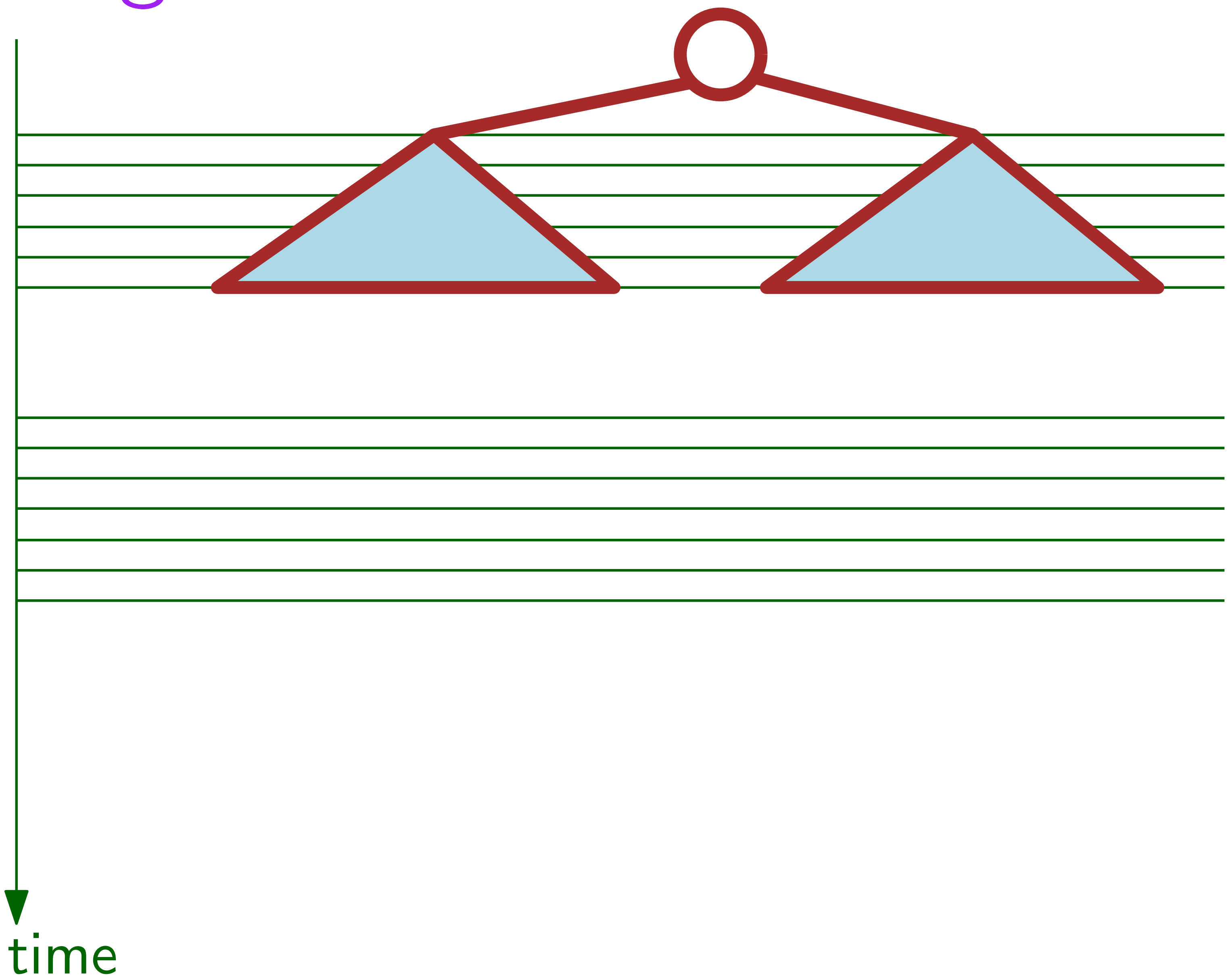
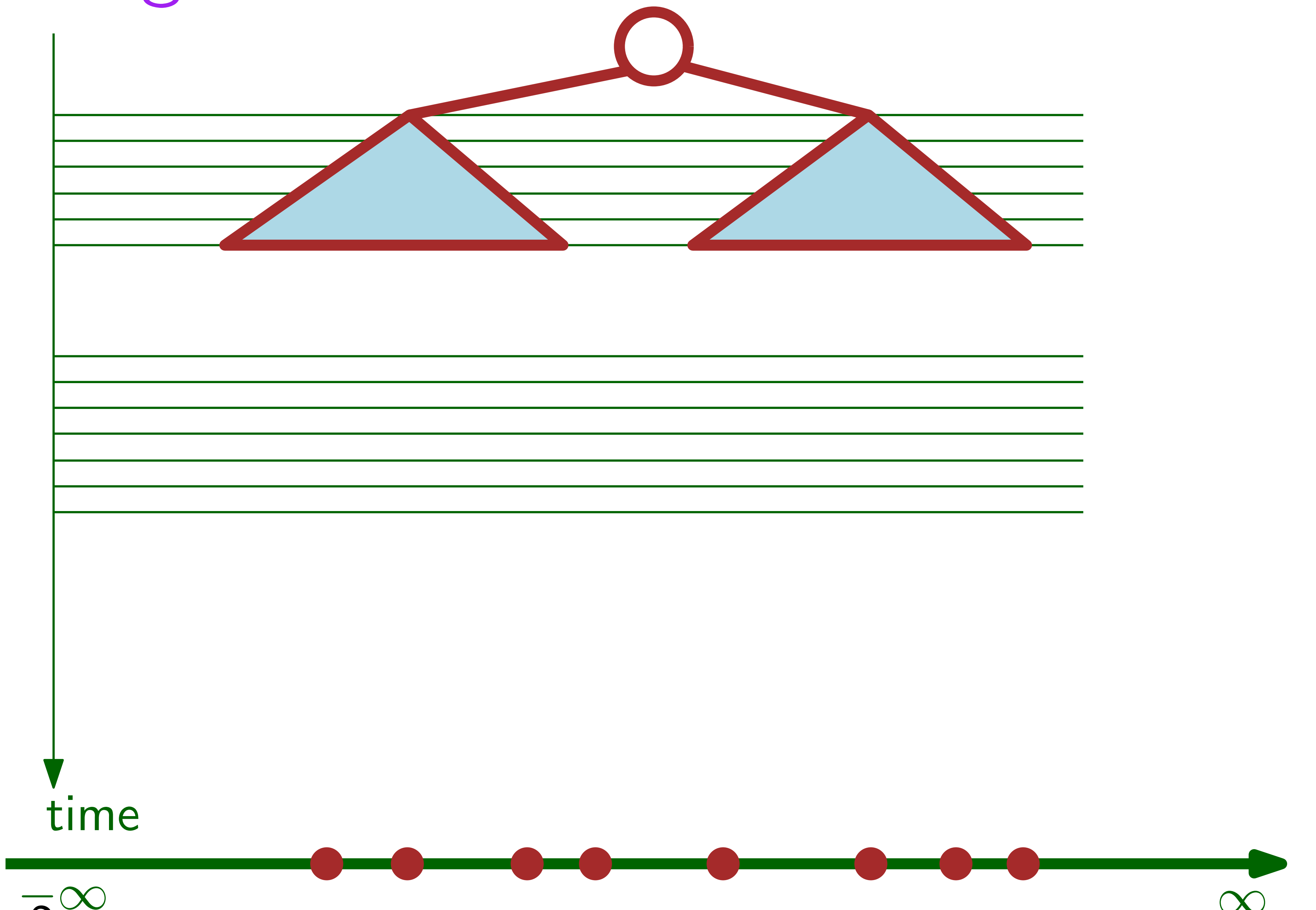


Sorting



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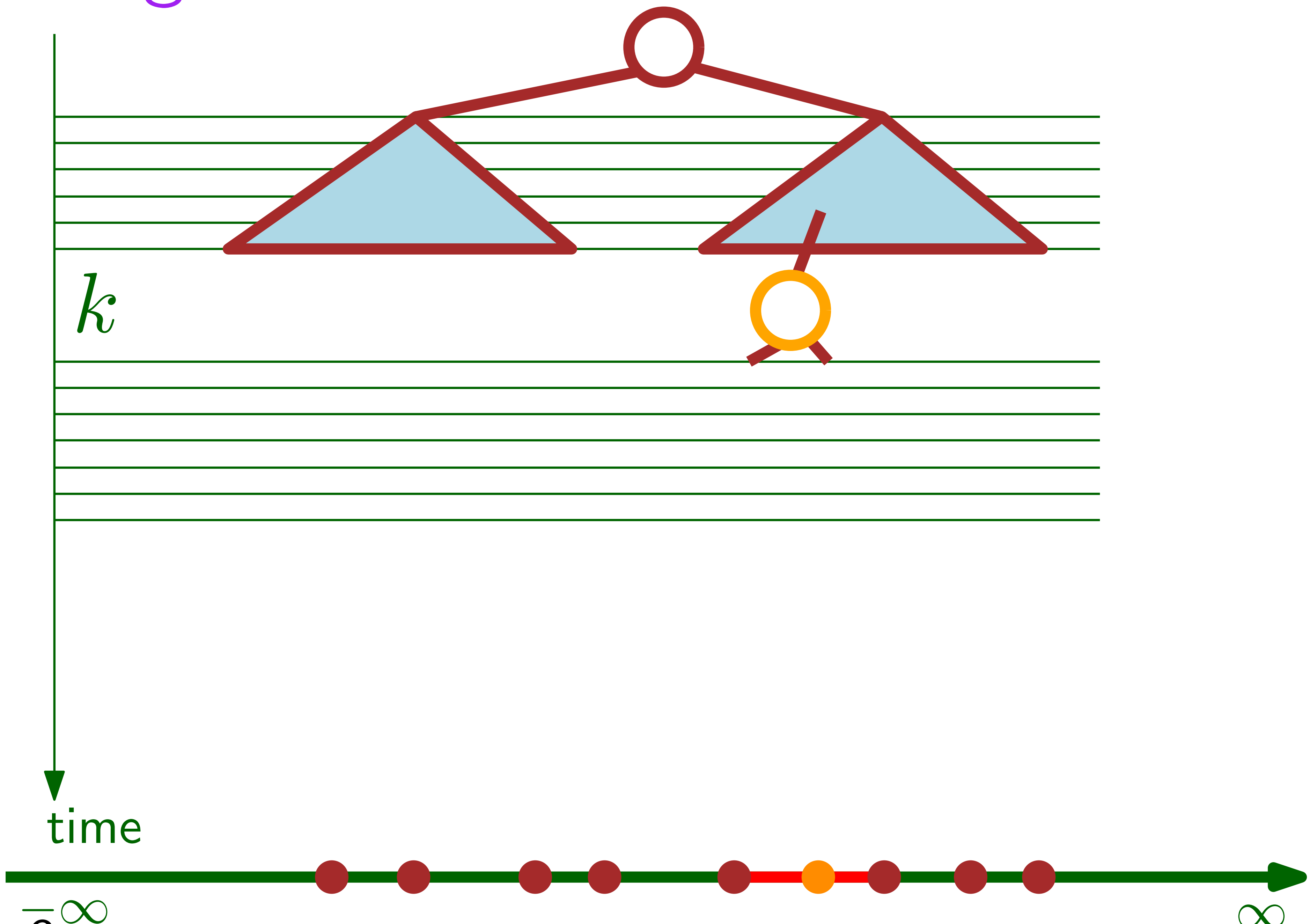
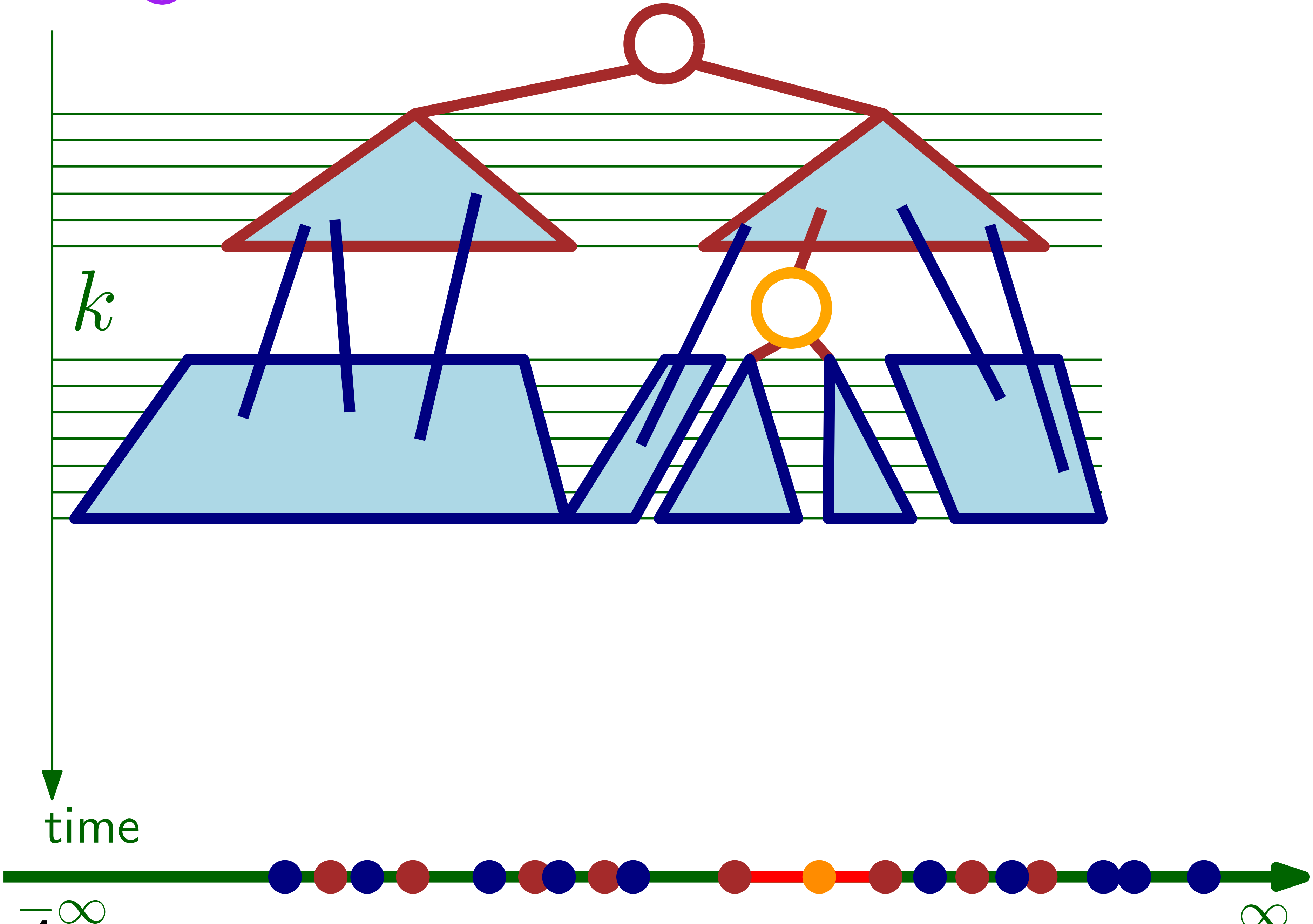
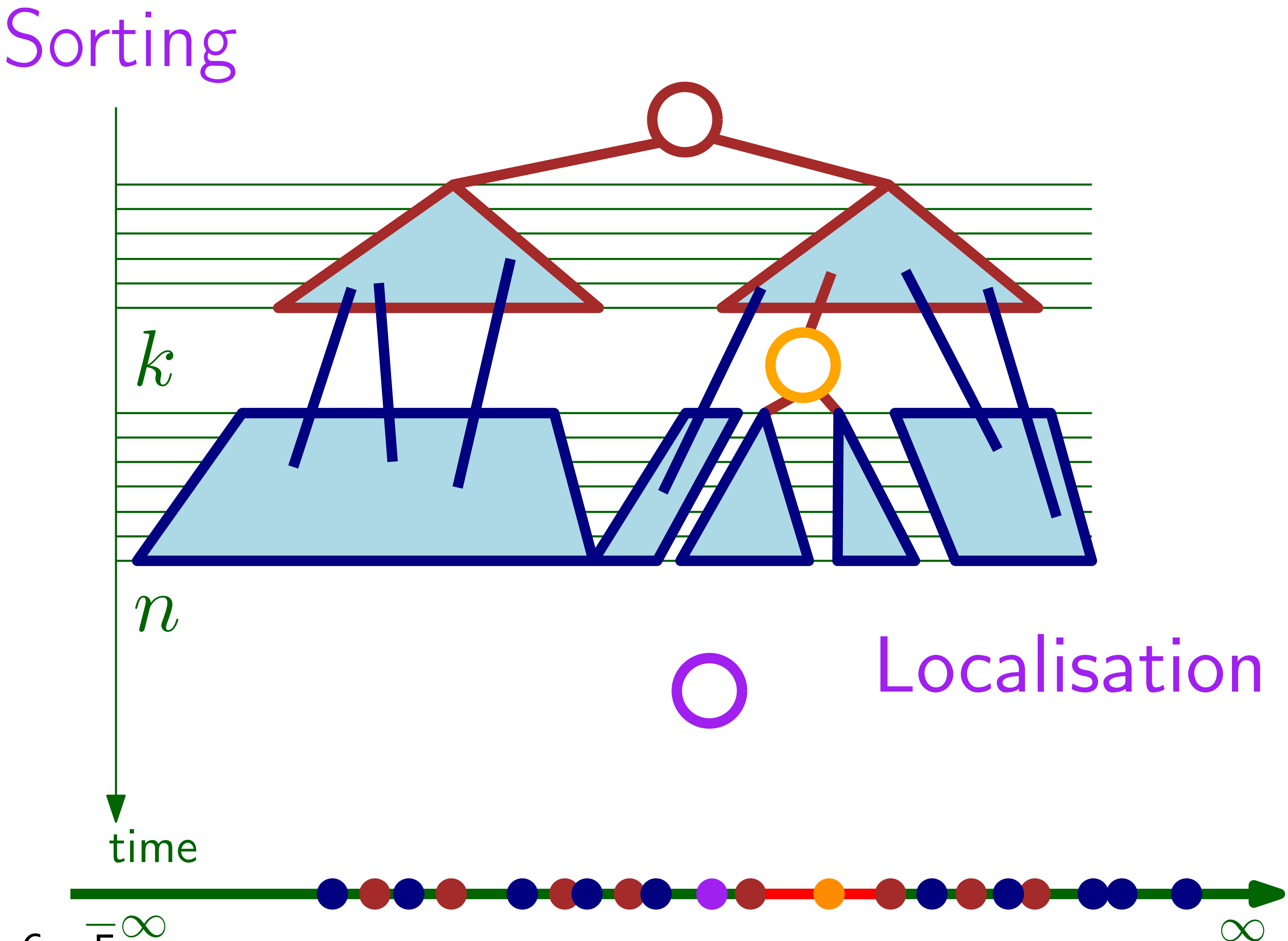


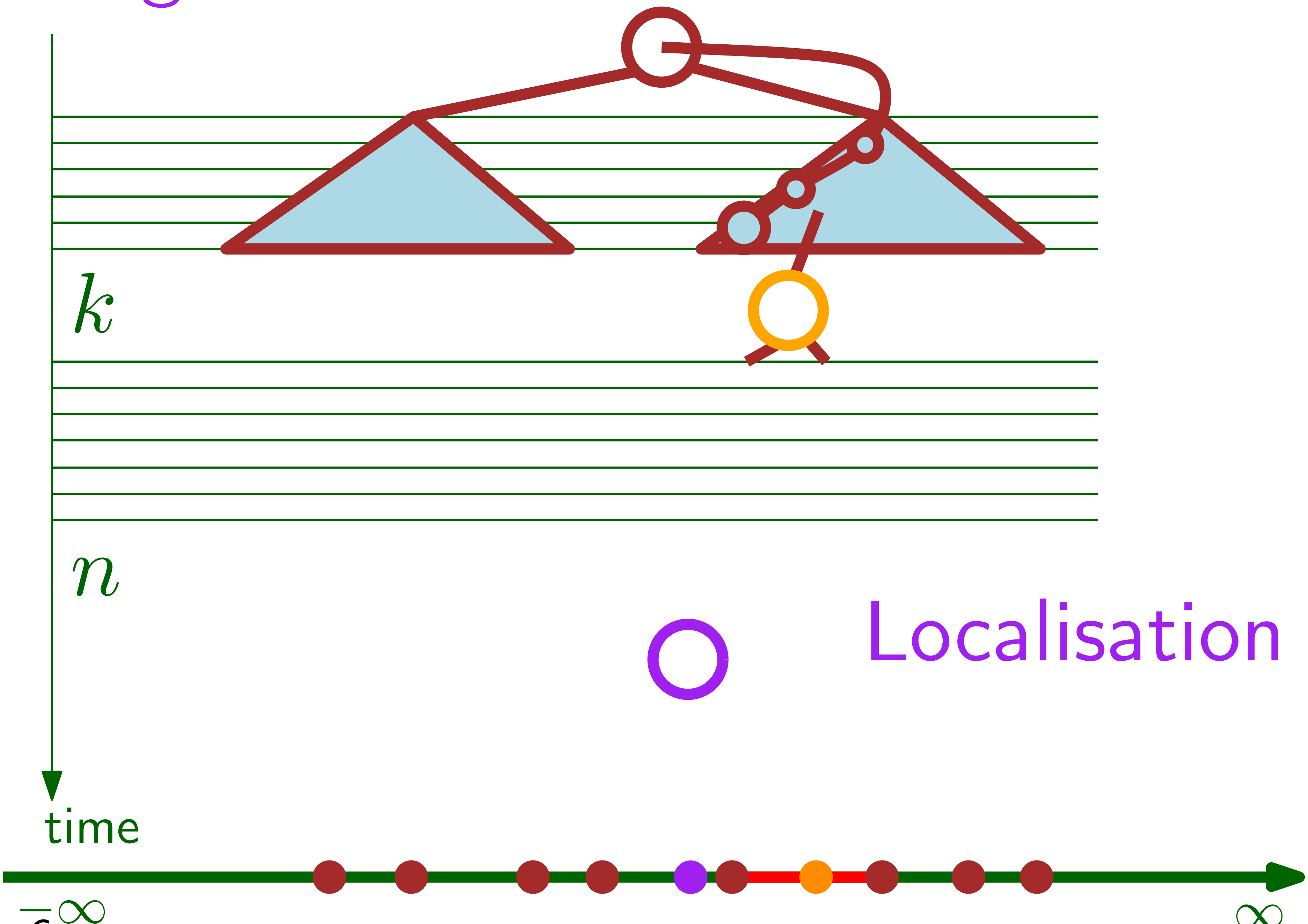
Sorting



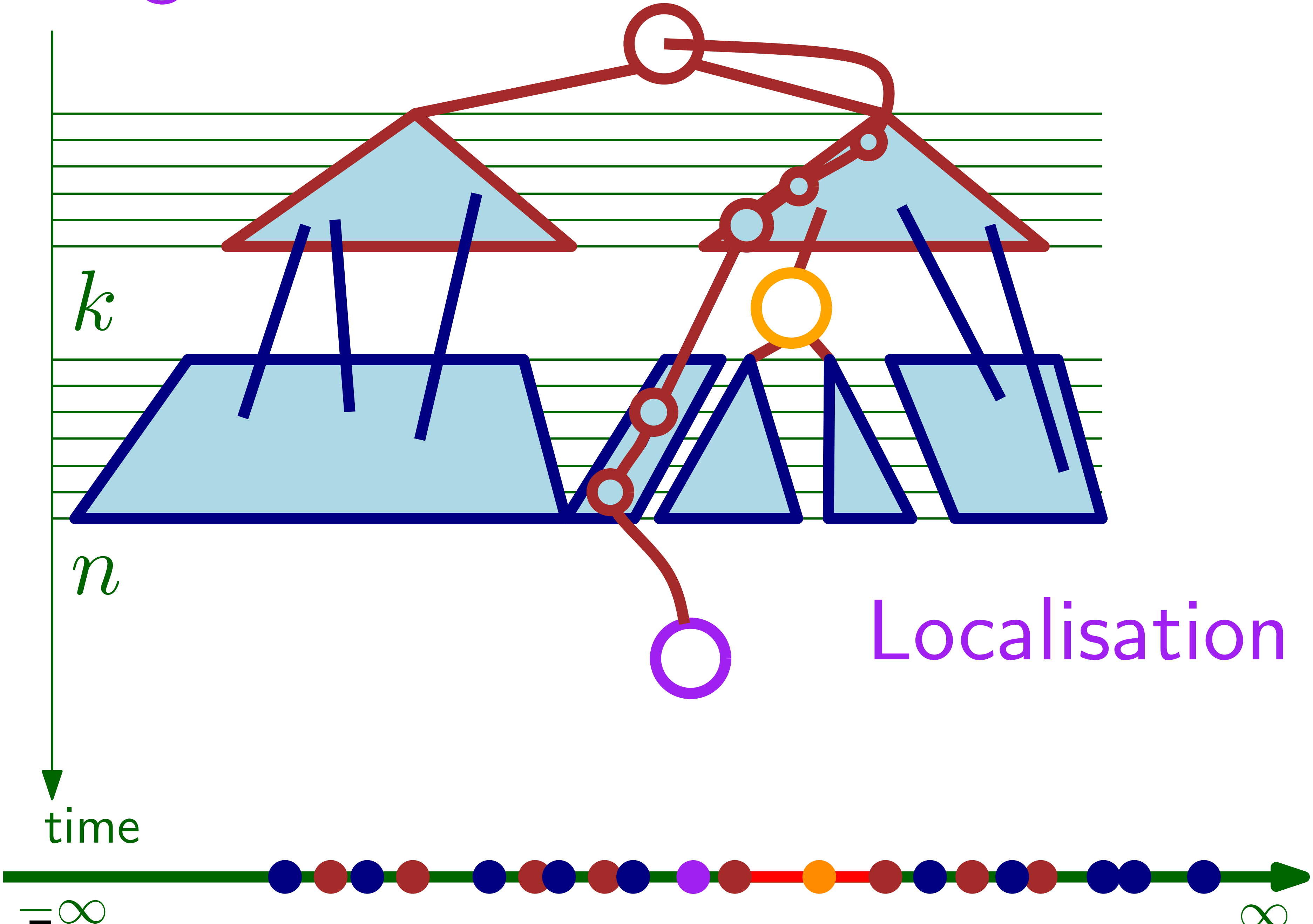


Sorting
k
n
Localisation
time
$-\infty$
∞

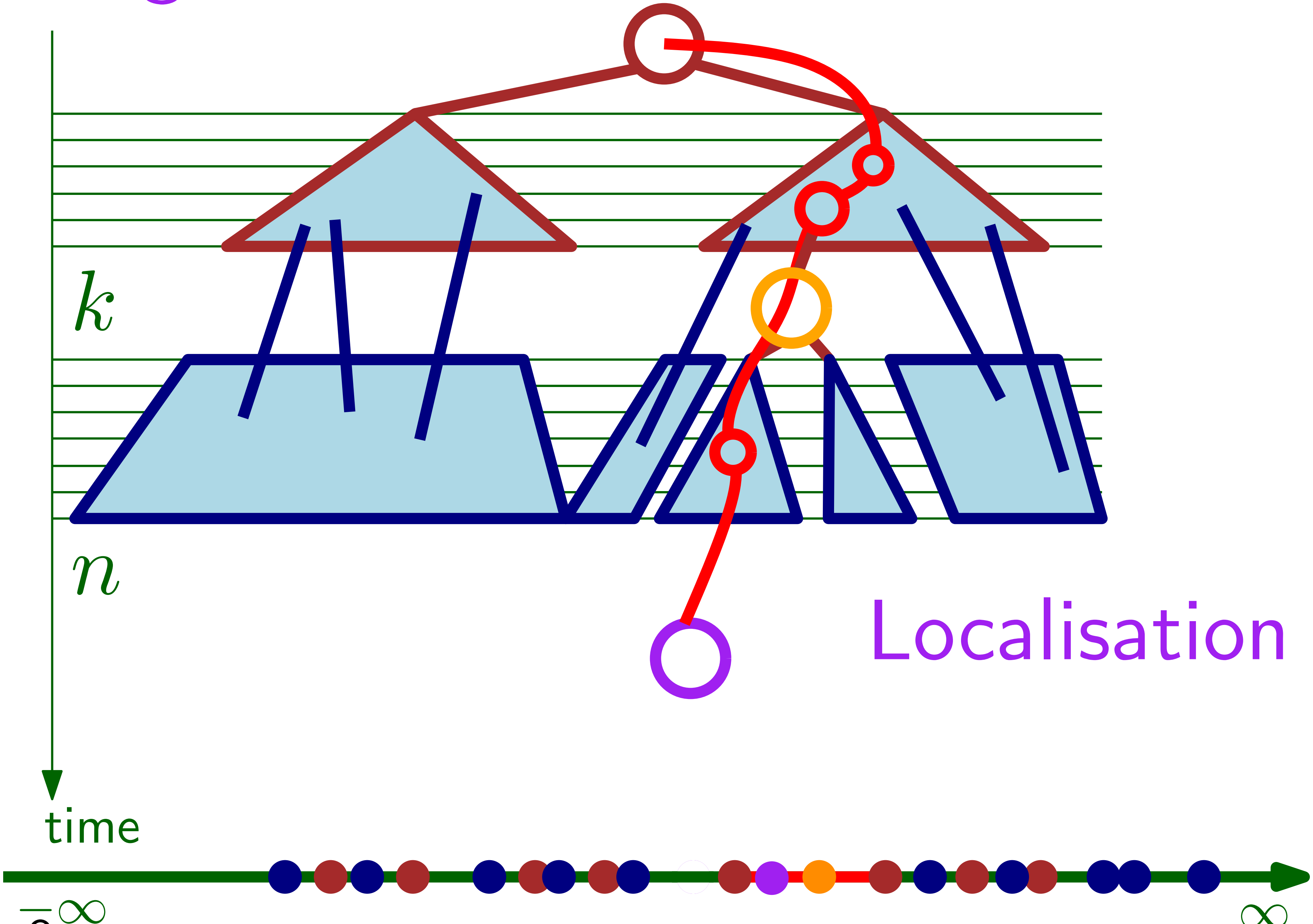
Sorting



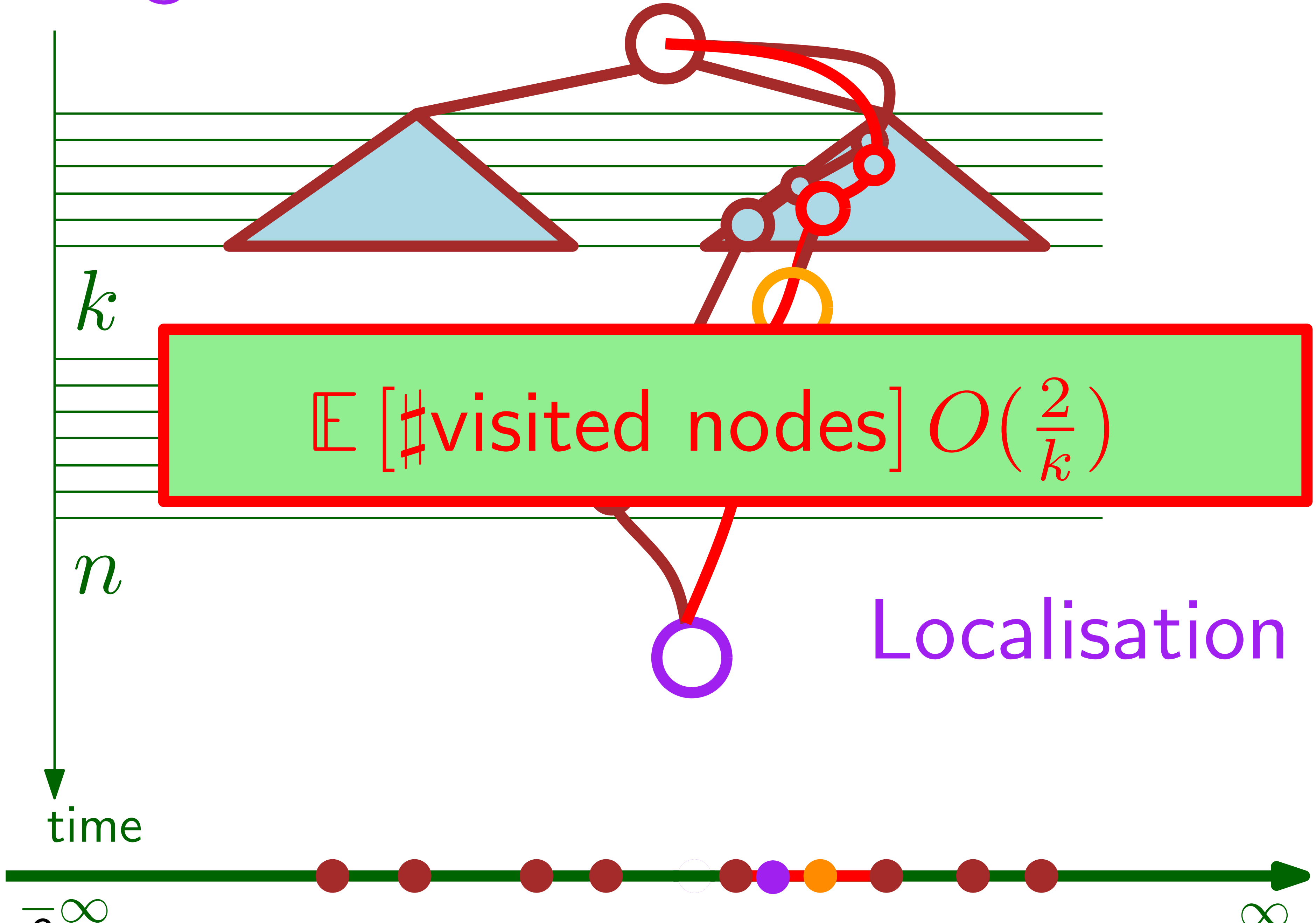
Sorting



Sorting

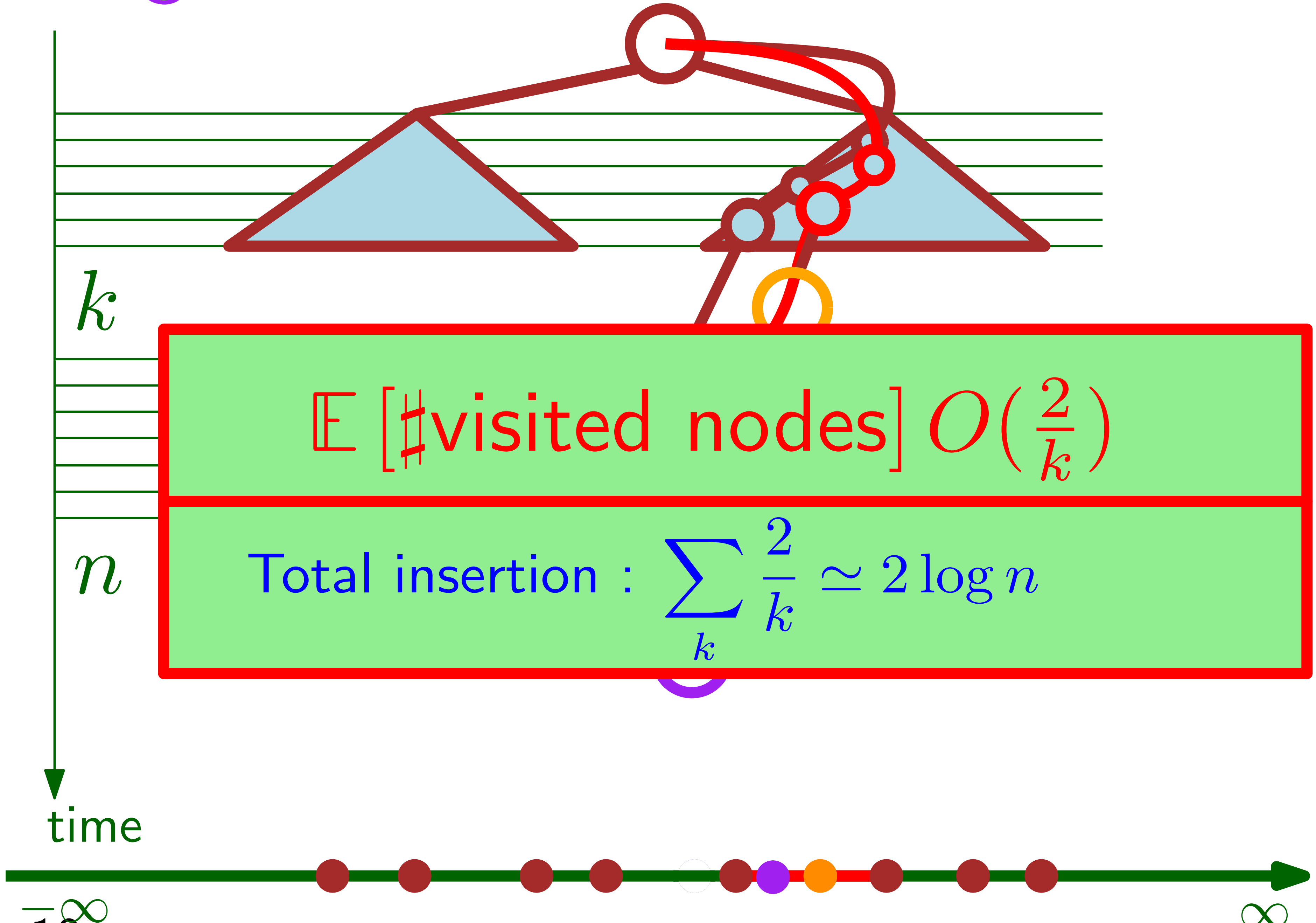


Sorting



k
$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$
n
Localisation
time
$-\infty$
∞

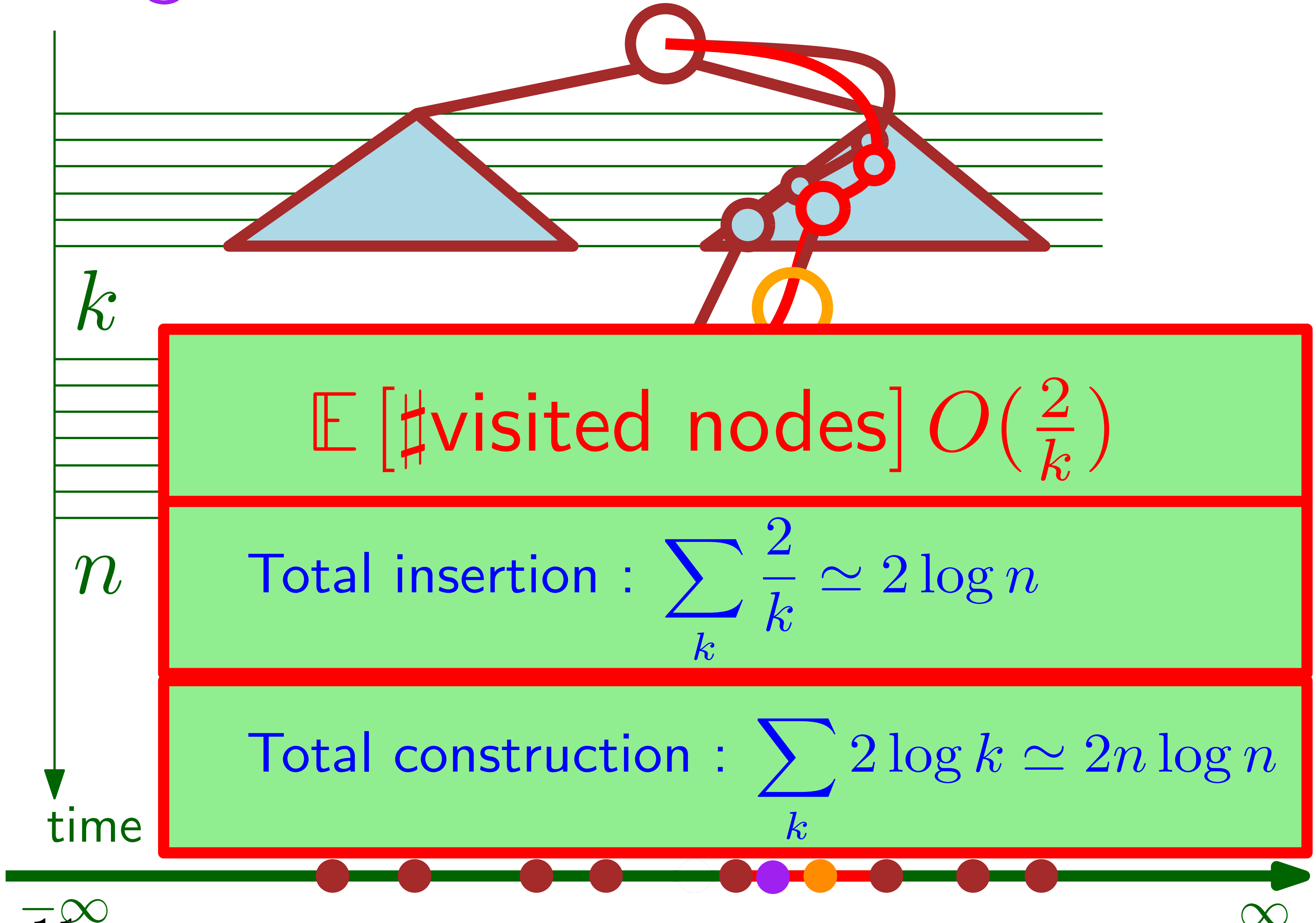
Sorting



$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

Total insertion : $\sum_{k} \frac{2}{k} \simeq 2 \log n$

Sorting

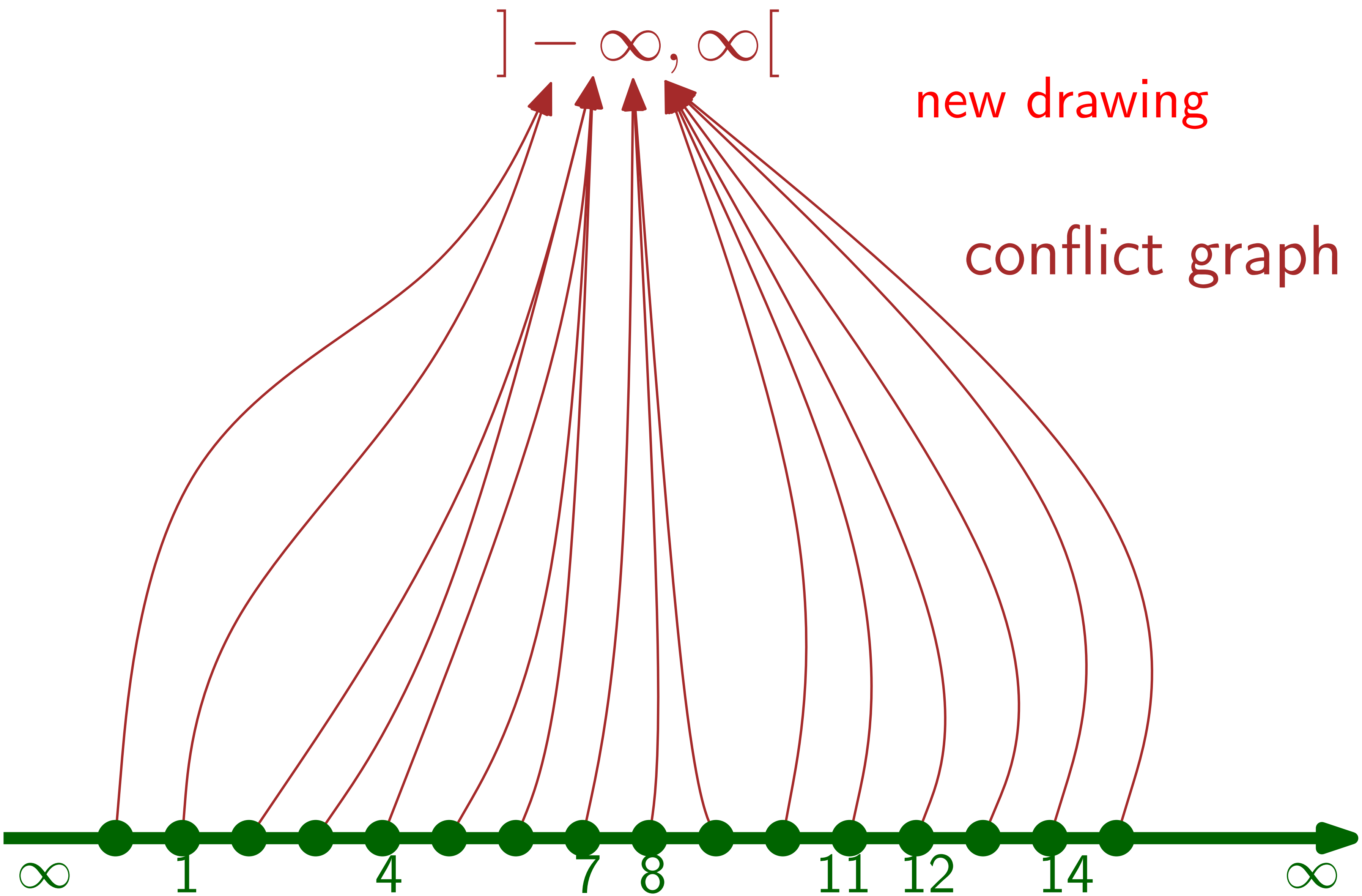


$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

Total insertion : $\sum_k \frac{2}{k} \simeq 2\log n$

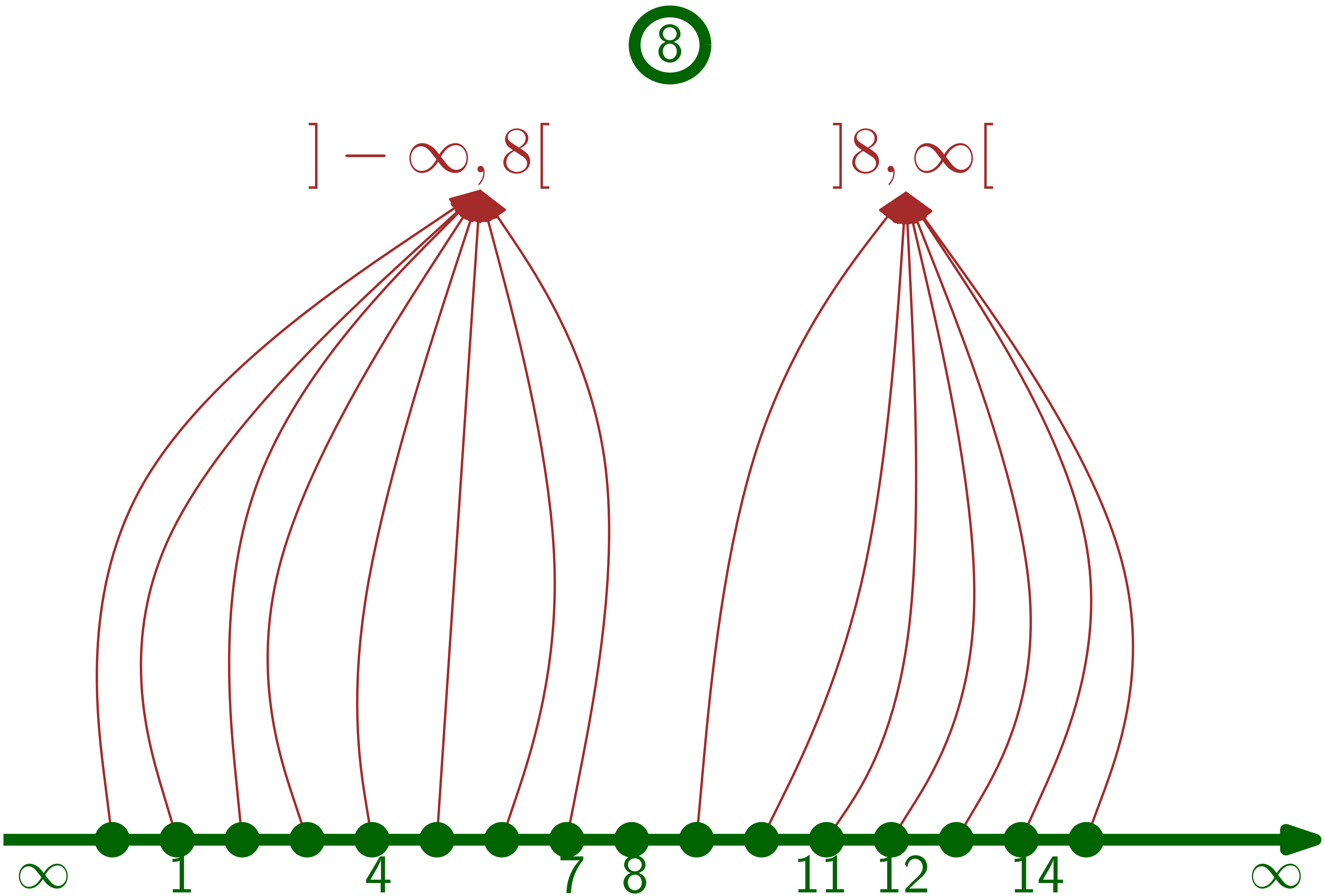
Total construction : $\sum_k 2\log k \simeq 2n\log n$

Sorting

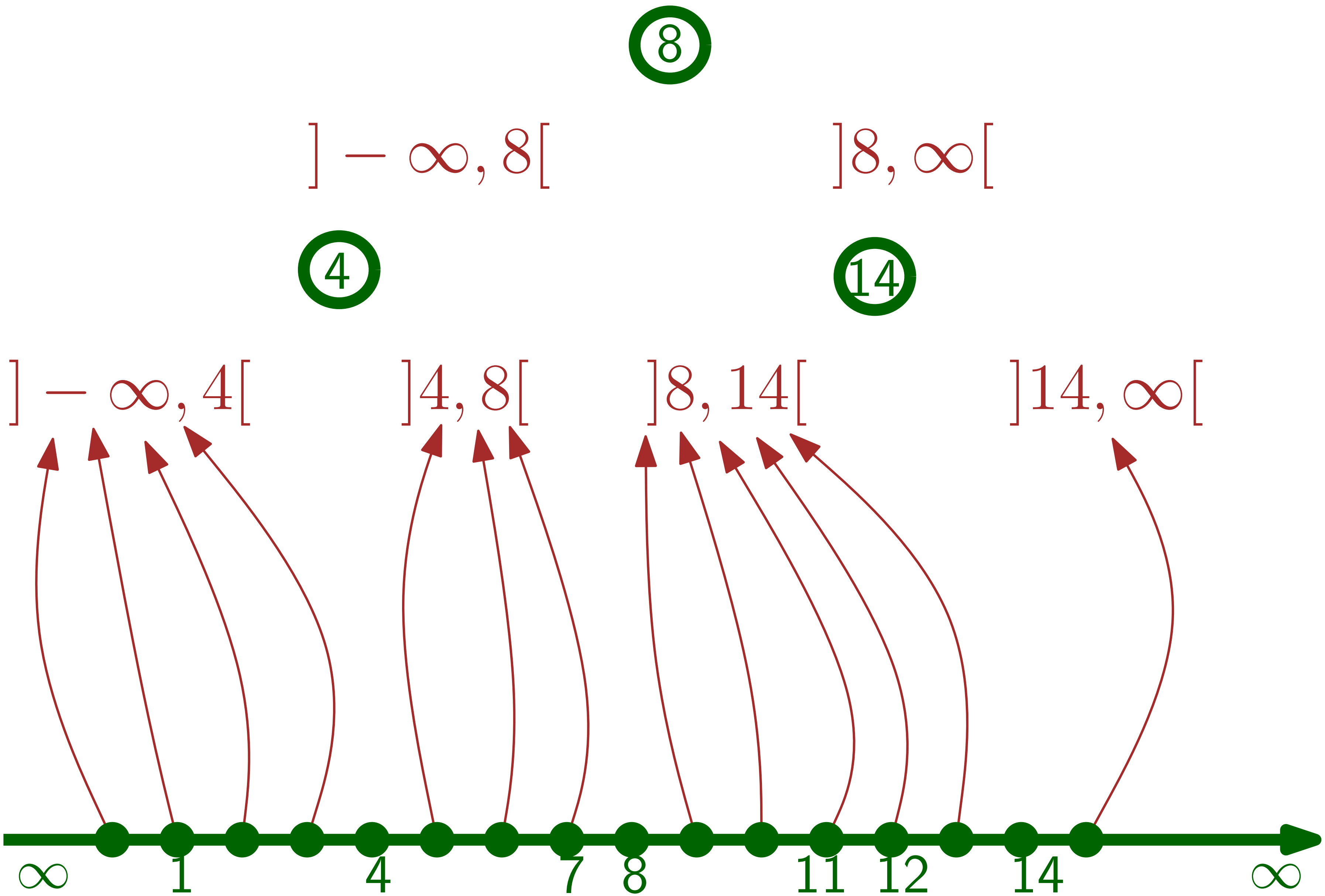


$]-\infty,\infty[$
new drawing
conflict graph
∞
1
4
7
8
11
12
14
∞

Sorting



Sorting



8
$]-\infty, 8[$
$]8, \infty[$
4
14
$]-\infty, 4[$
$]4, 8[$
$]8, 14[$
$]14, \infty[$
∞
1
4
7
8
11
12
14
∞

Sorting

Unbalanced binary tree

Quicksort

$O(n \log n)$

History graph

Conflict graph

Same analysis

Backwards analysis

Analyse last insertion and sum

Last object is a random object

Randomization

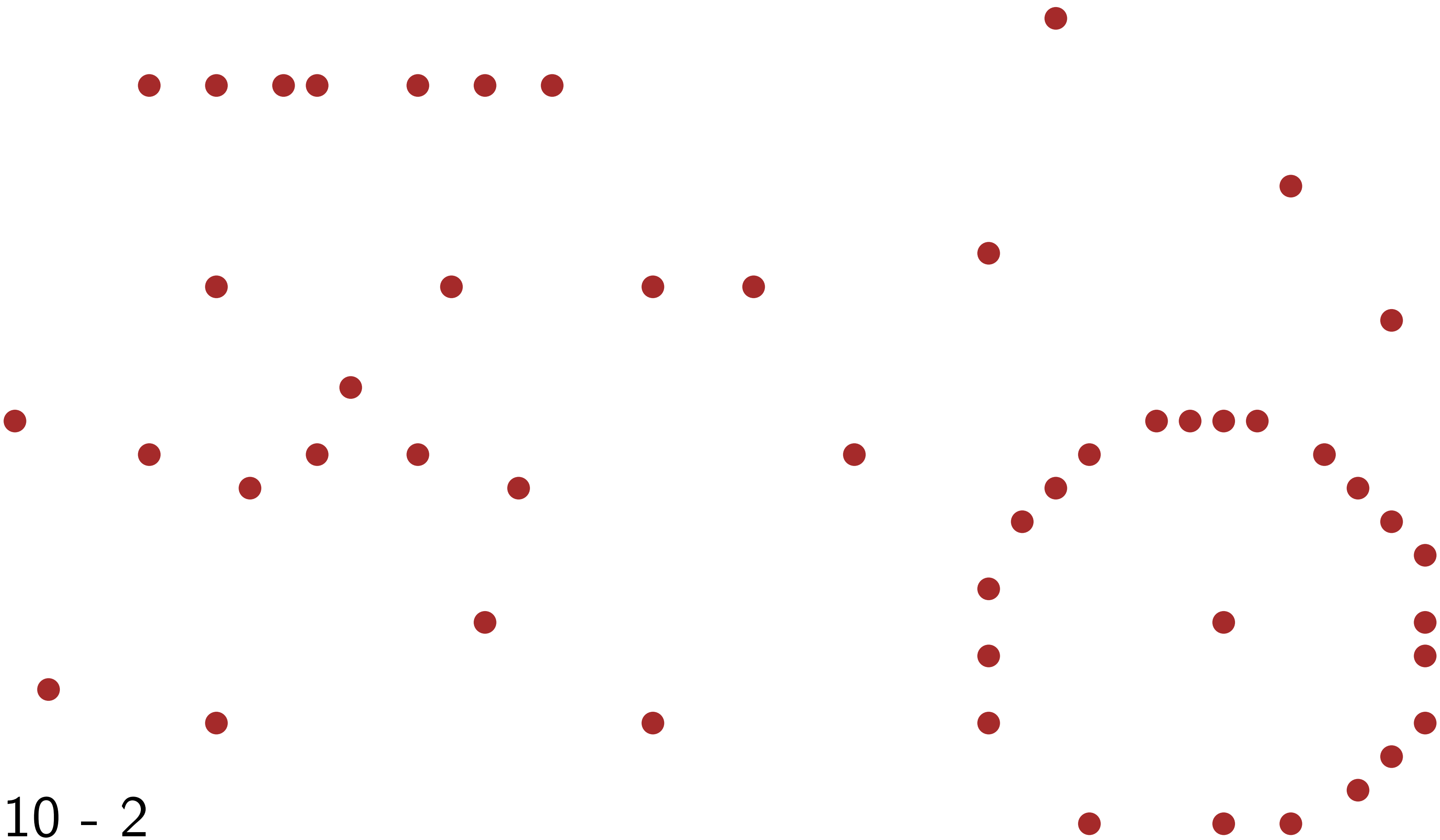
Backwards analysis for Delaunay triangulation

Delaunay triangulation

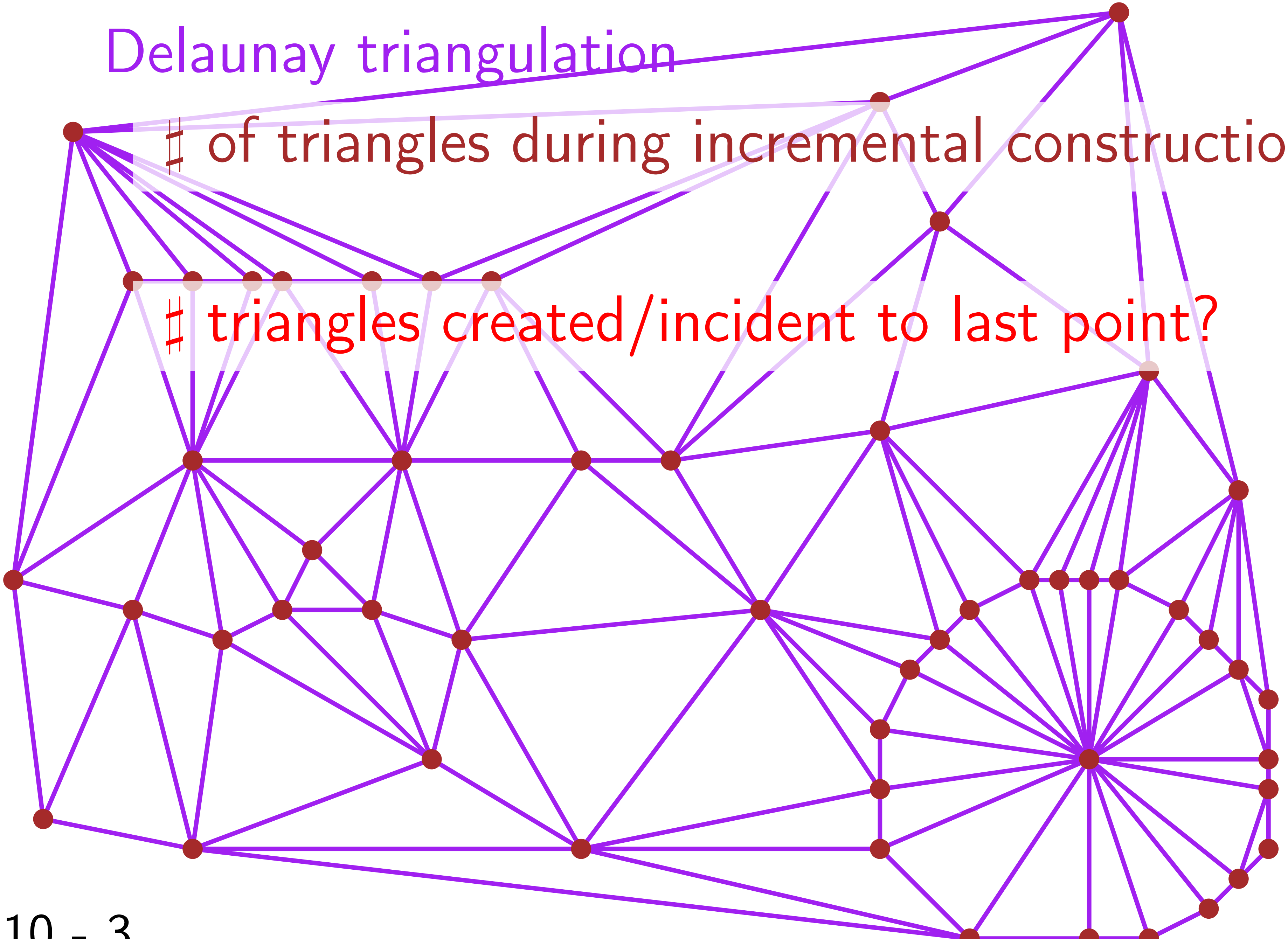
$\sharp$ of triangles during incremental construction?

Delaunay triangulation

- $\sharp$ of triangles during incremental construction?



Delaunay triangulation



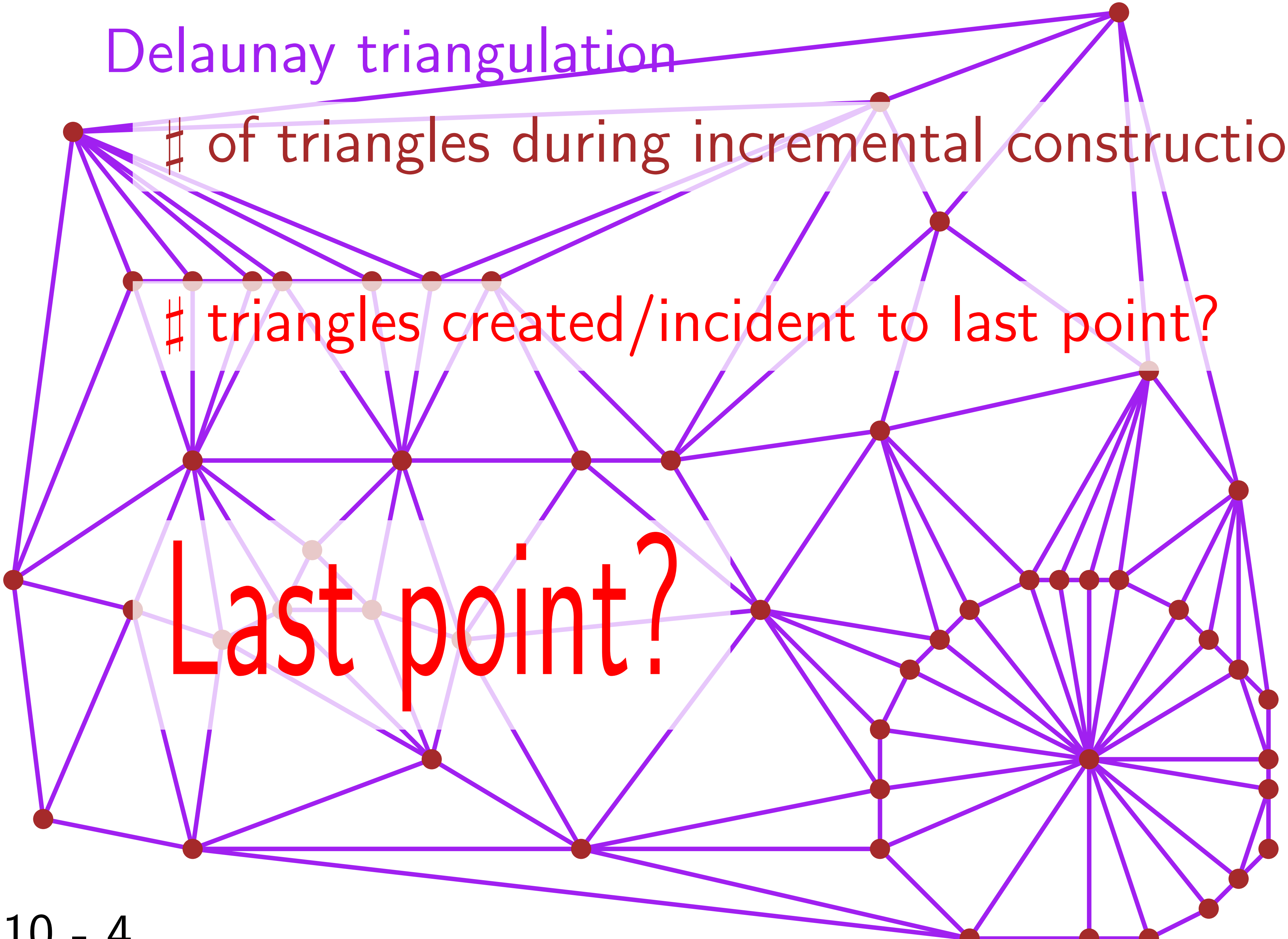
$\sharp$ of triangles during incremental construction?

$\sharp$ triangles created/incident to last point?

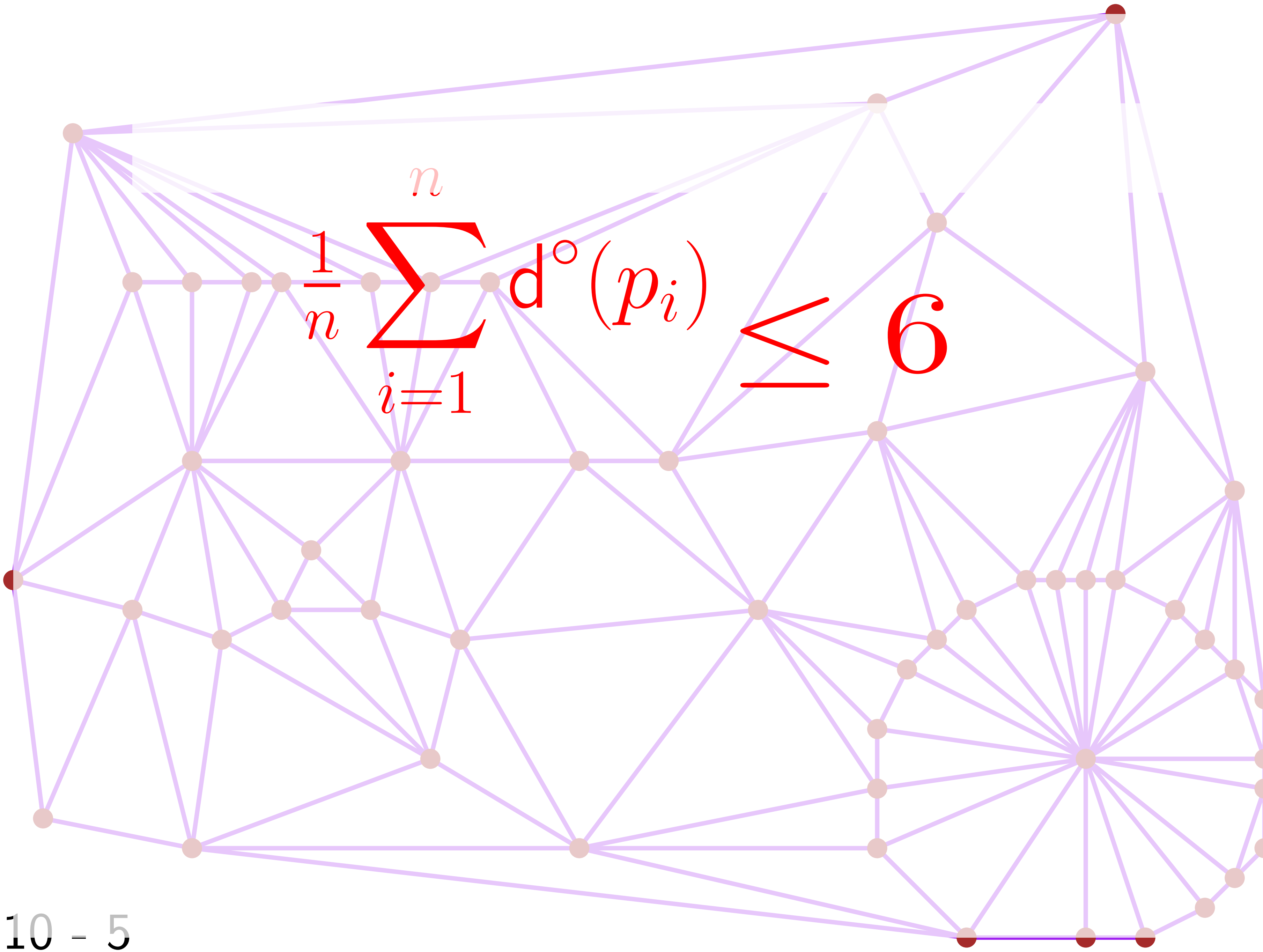
Delaunay triangulation

♯ of triangles during incremental construction?

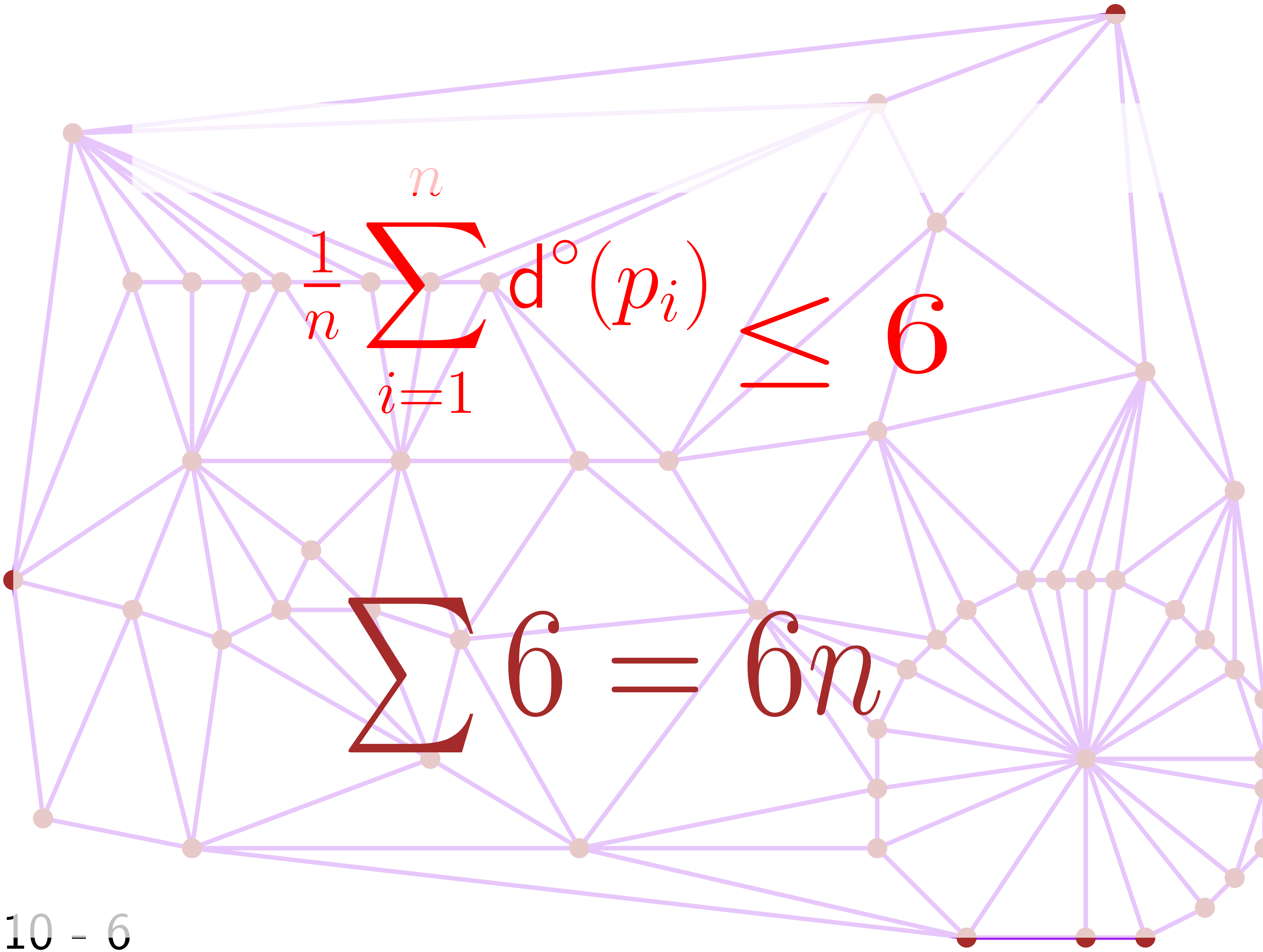
♯ triangles created/incident to last point?



Last point?



$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{d}^\circ(p_i) \leq 6$$

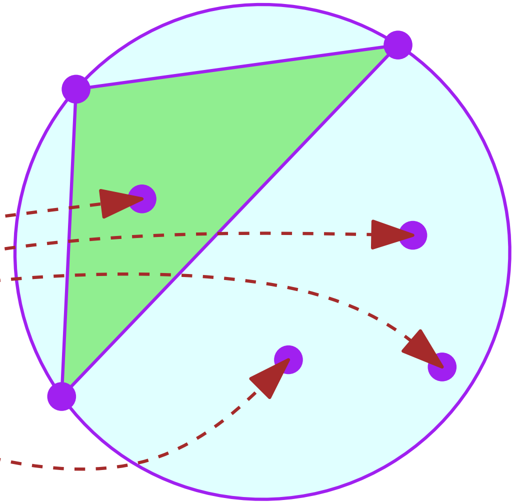


$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{d}^\circ(p_i) \leq 6$$

$$\sum 6 = 6n$$

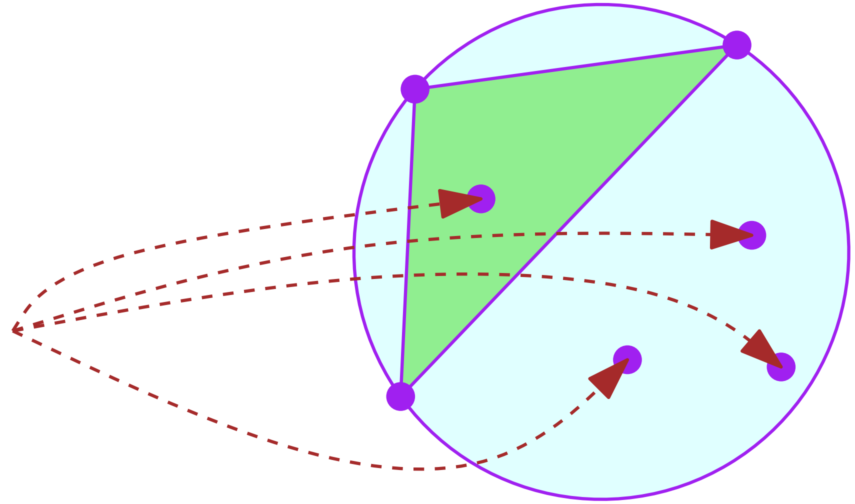
Alternative analysis

Triangle Δ with j stoppers



Alternative analysis

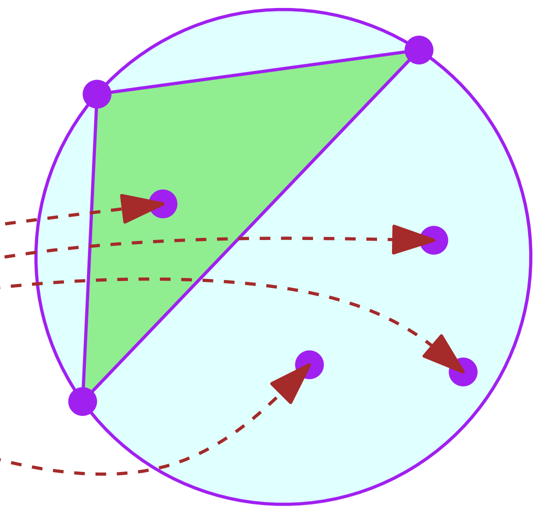
Triangle Δ with j stoppers



Probability that it exists in the triangulation of a sample of size αn

$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4}\alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Alternative analysis



Triangle Δ with j stoppers

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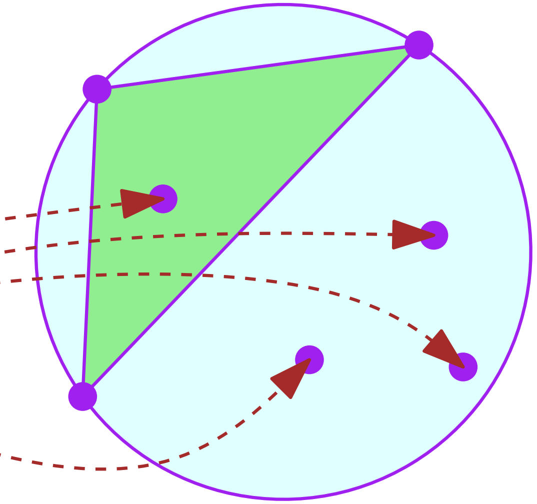
Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



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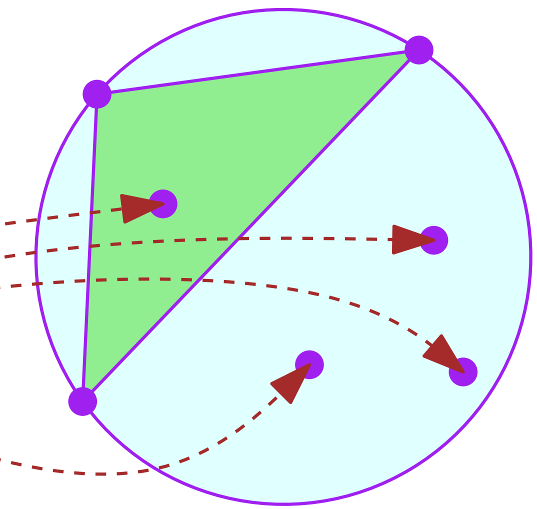
$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis



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Size of the triangulation of the sample

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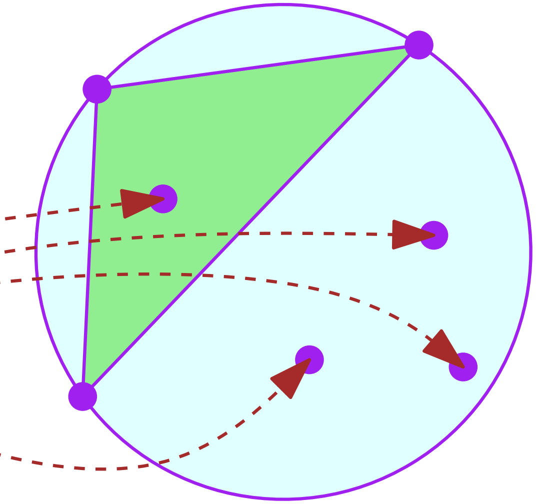
$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



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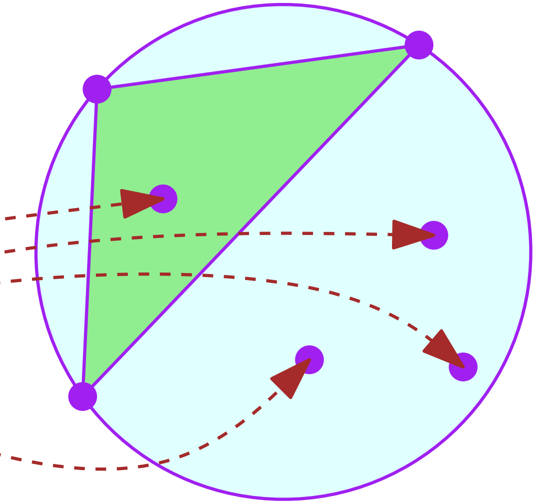
Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers} \qquad = O(\alpha n)$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



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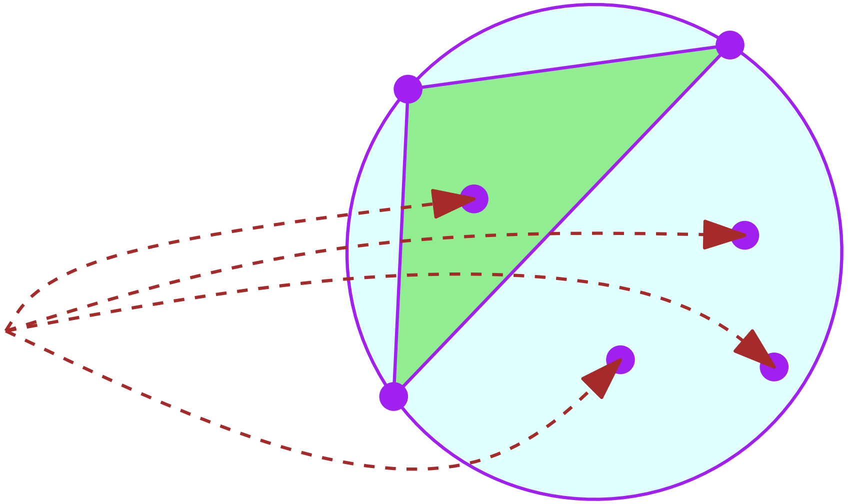
$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$

Alternative analysis

Triangle Δ with j stoppers

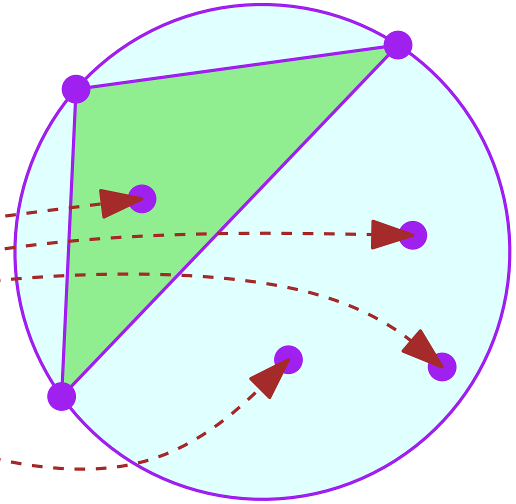


Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

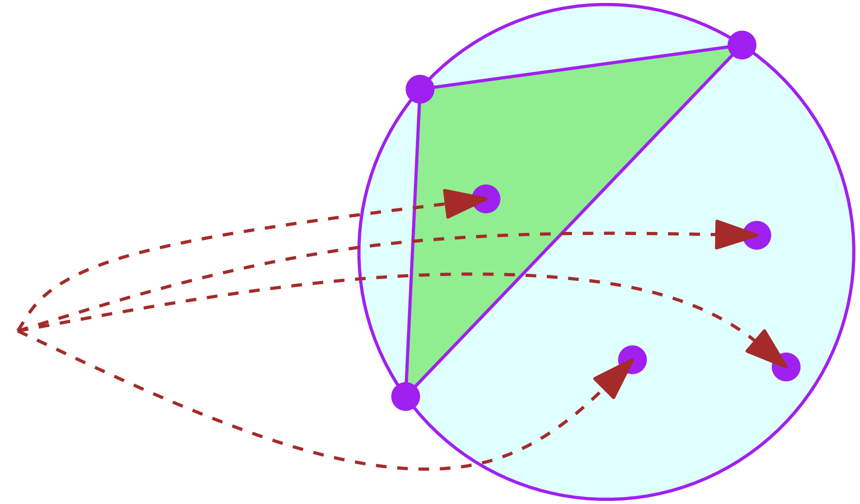
$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

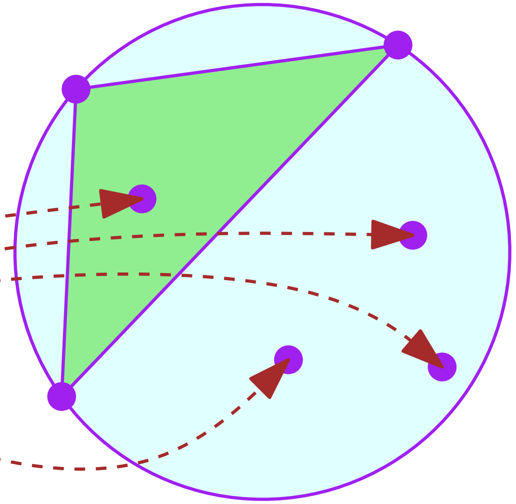
$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

Alternative analysis



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$\sharp$ of created triangles

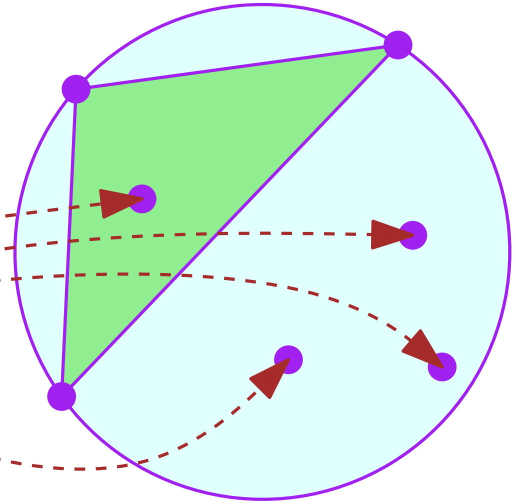
$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

$$\simeq \sum_{j=0}^{n} \frac{18}{j^4} \times nj^2 = O(n \sum \frac{1}{j^2}) = O(n)$$

Alternative analysis

Triangle Δ with j stoppers

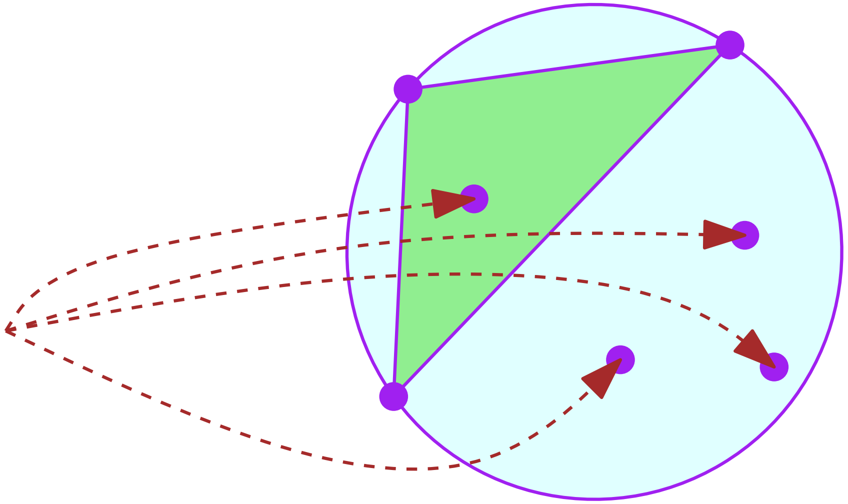


Conflict graph / History graph

It remains to analyze conflict location

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

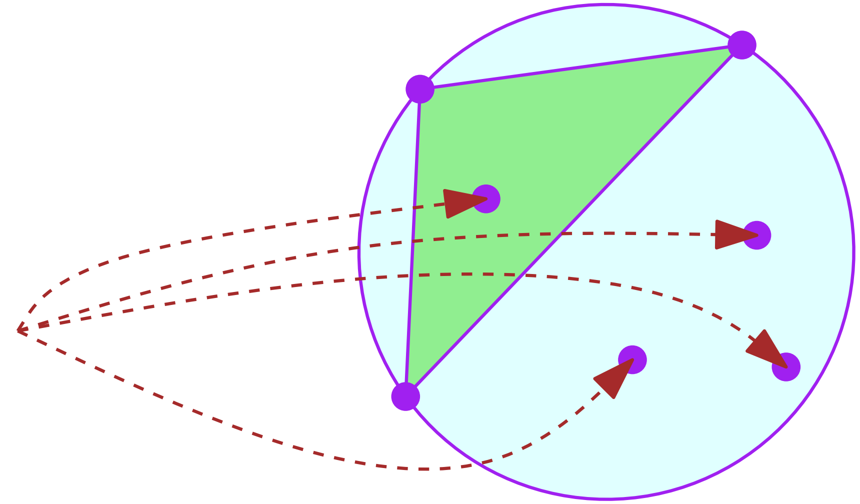
$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

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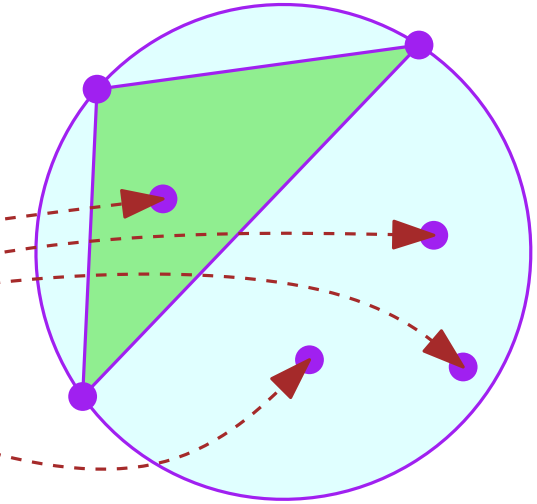
Alternative analysis



Triangle Δ with j stoppers

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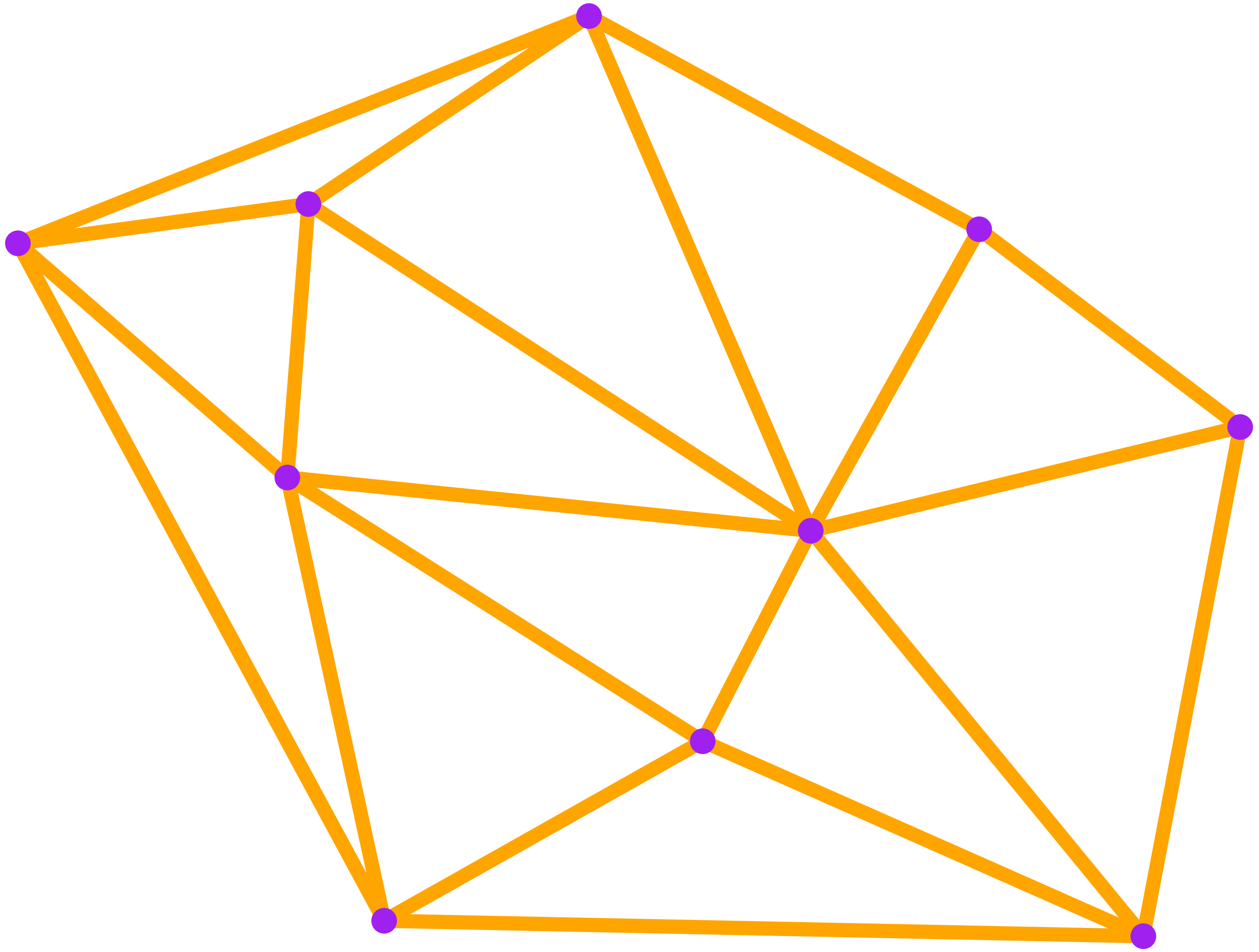
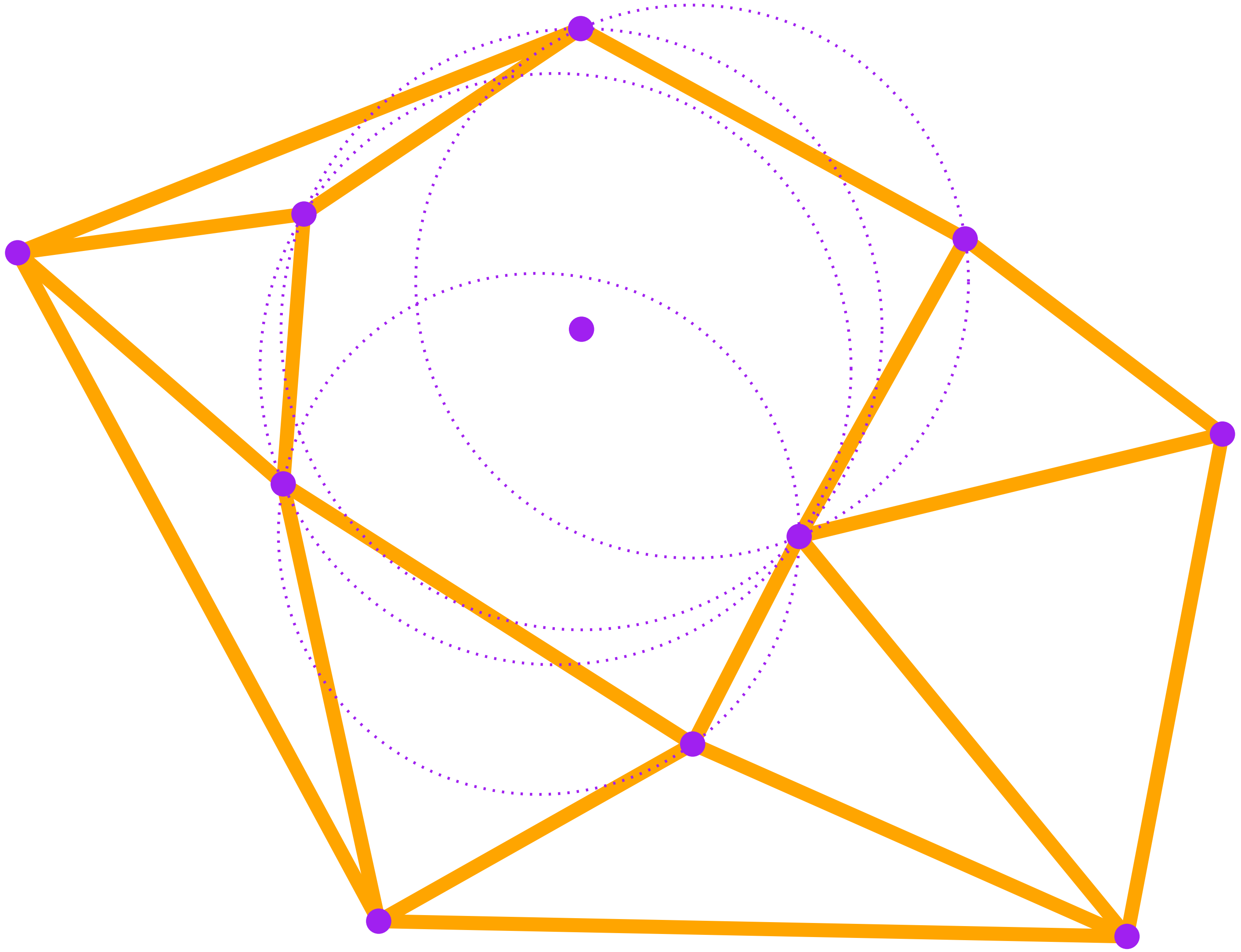
$$= \sum_{j=0}^{n} j \times \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

$$\simeq \sum_{j=0}^{n} j \times \frac{18}{j^4} \times nj^2 = O(n \sum \frac{1}{j}) = O(n \log n)$$

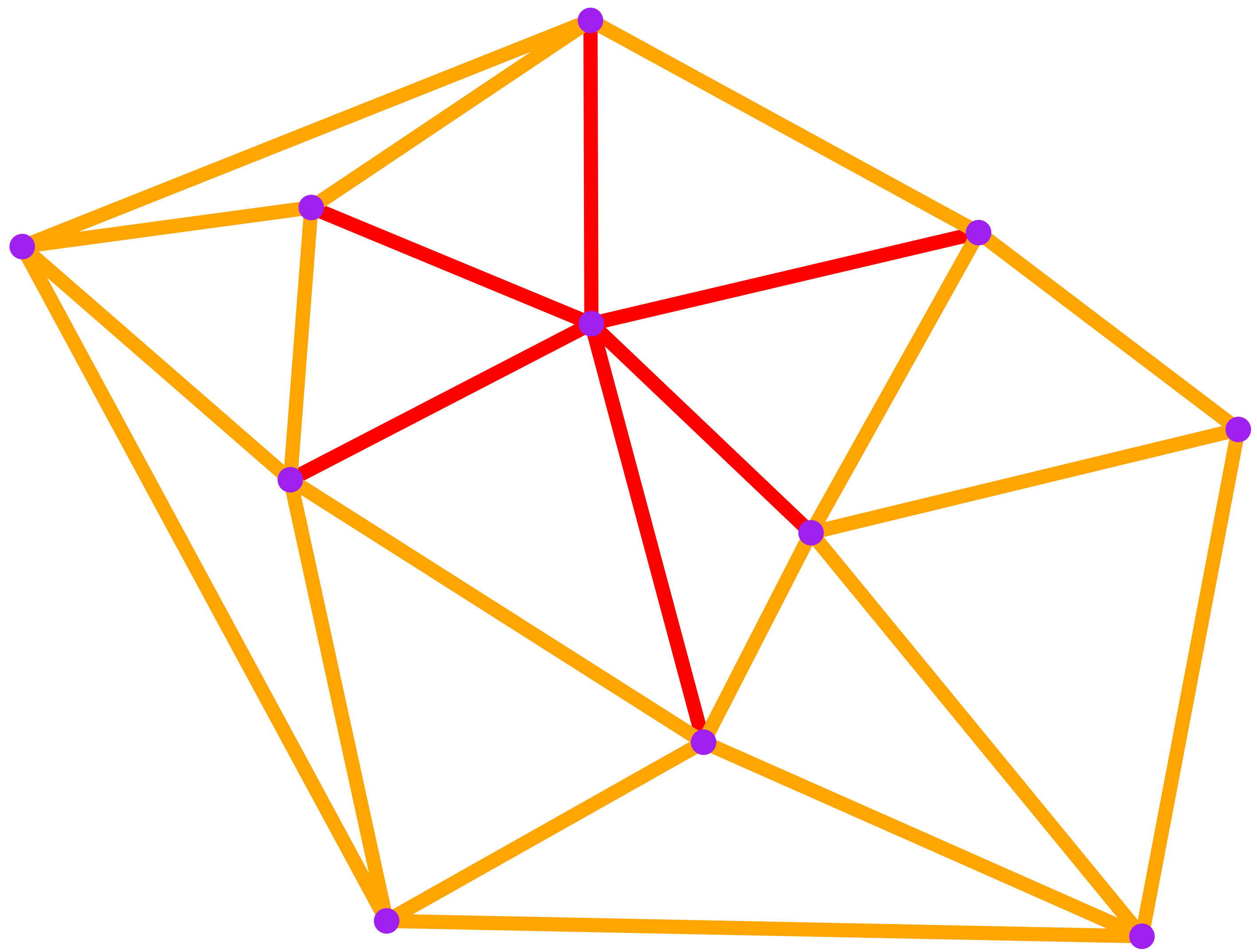
History graph



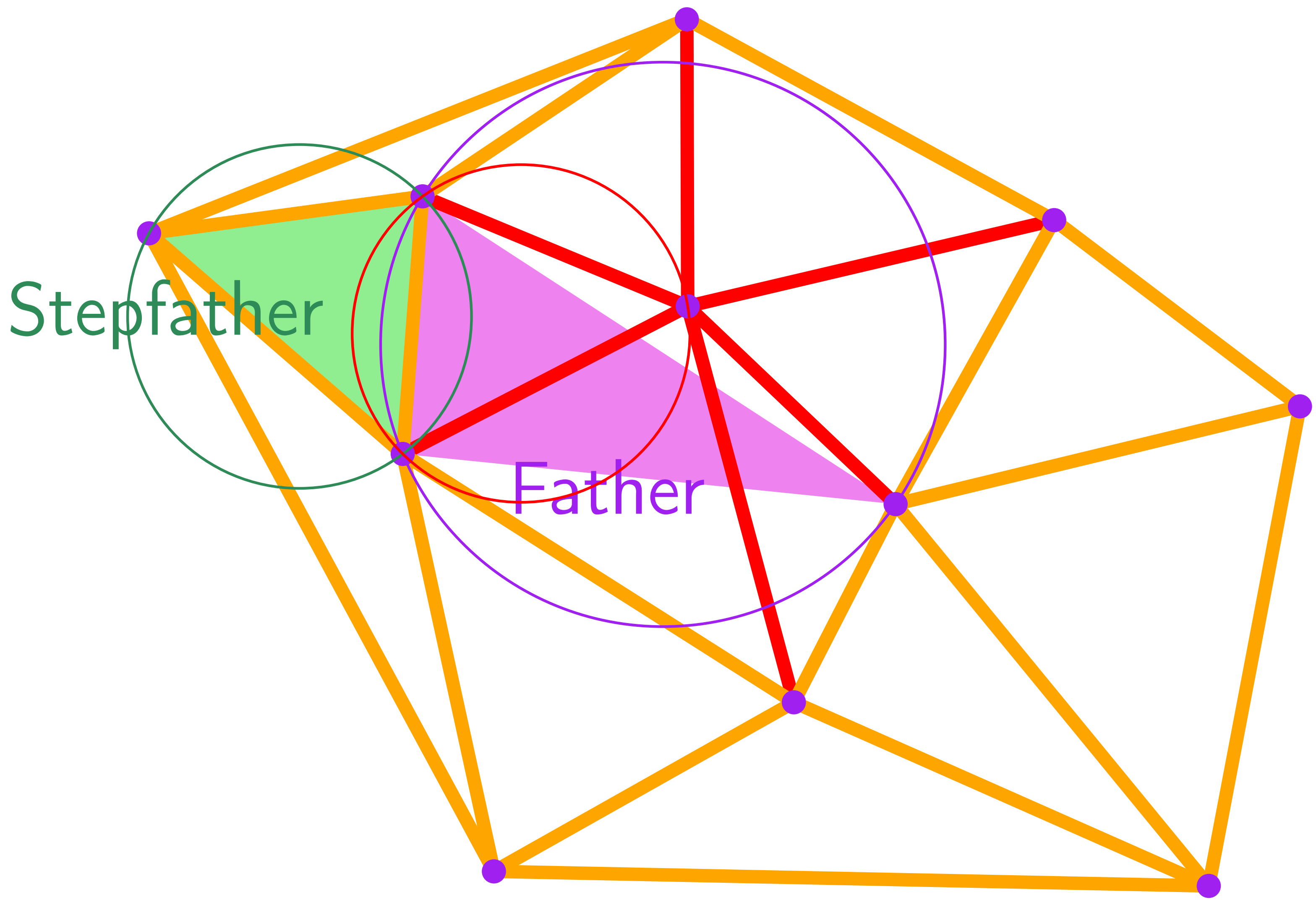
History graph



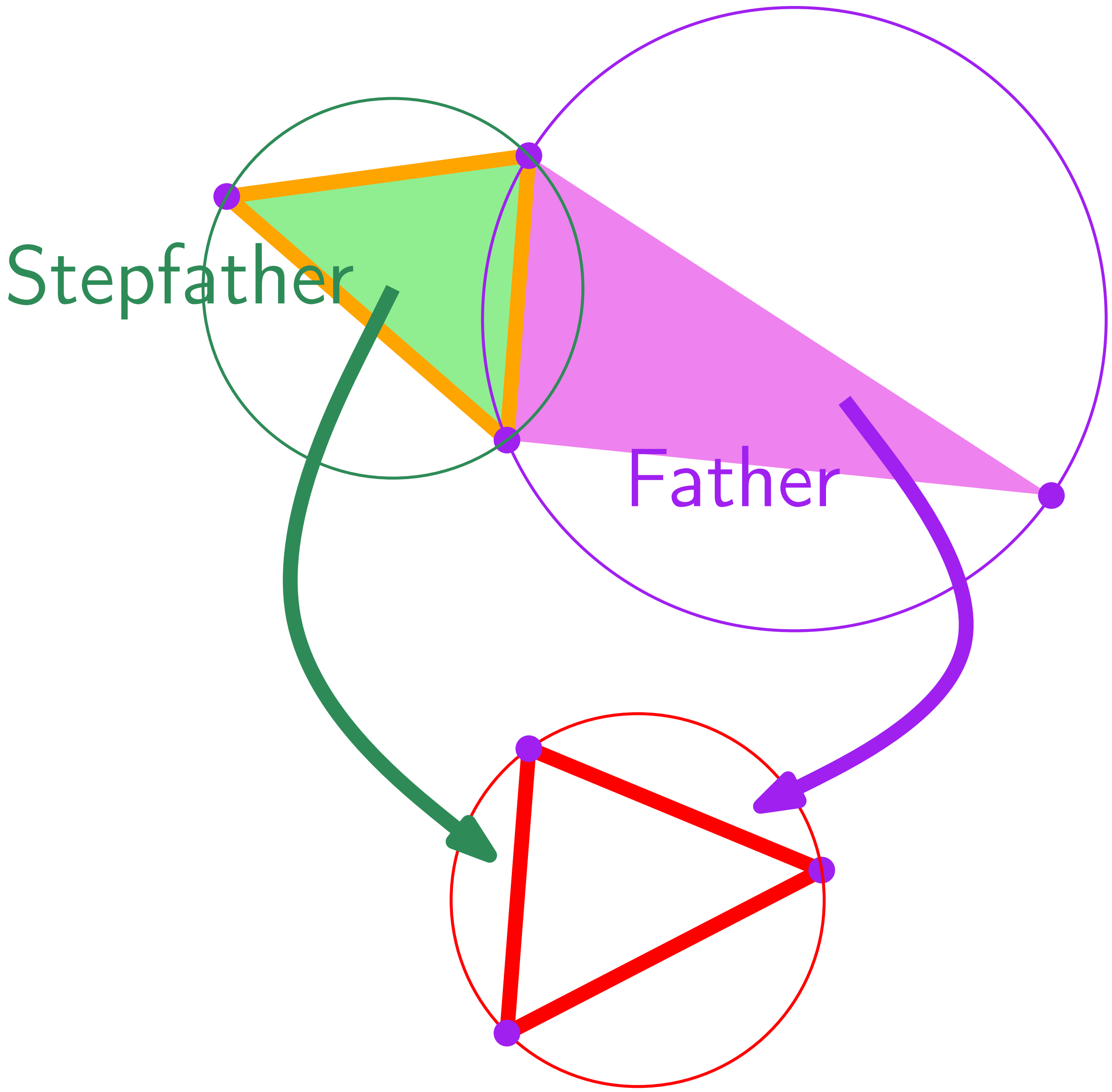
History graph

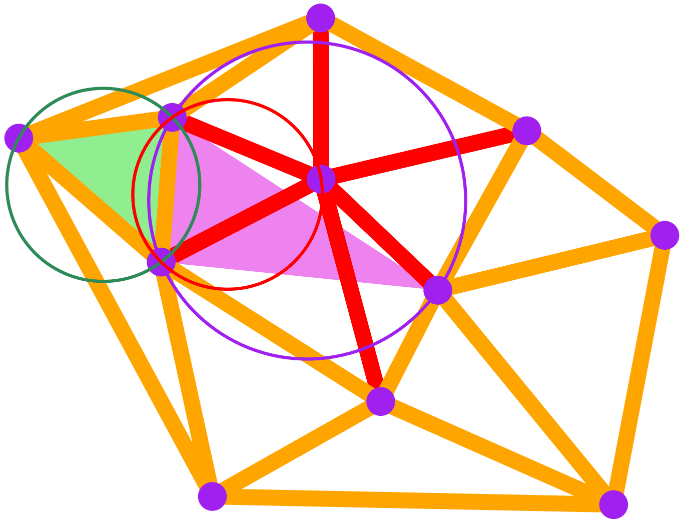


History graph

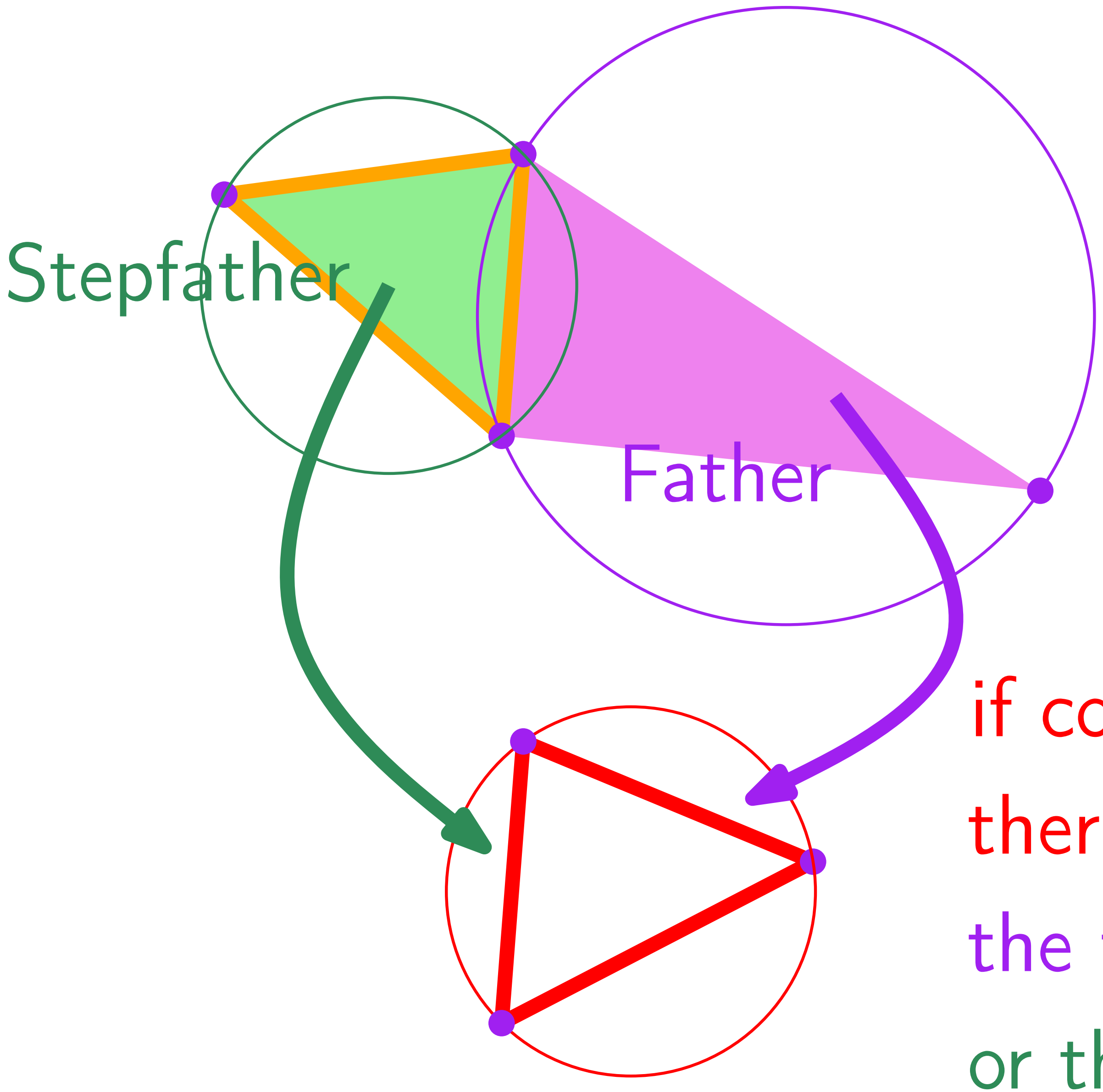


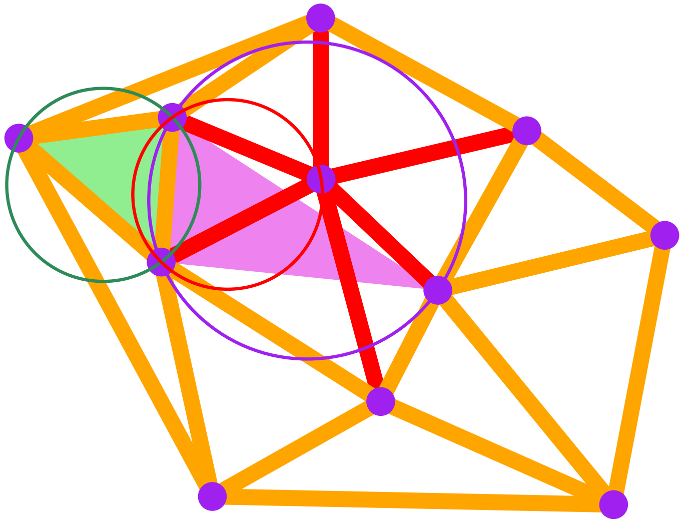
History graph (Delaunay tree)





History graph (Delaunay tree)





if conflict

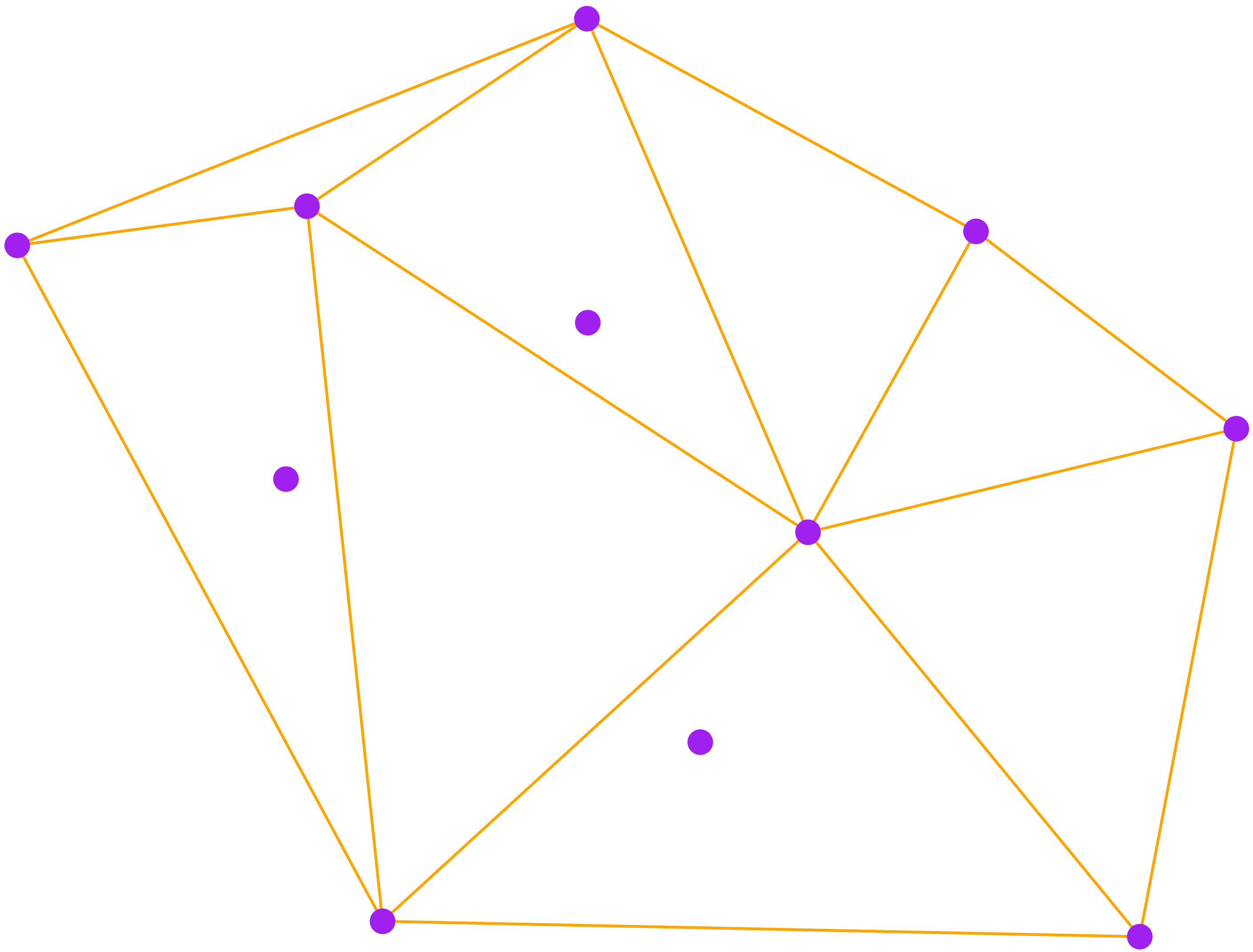
there was a conflict with

the father

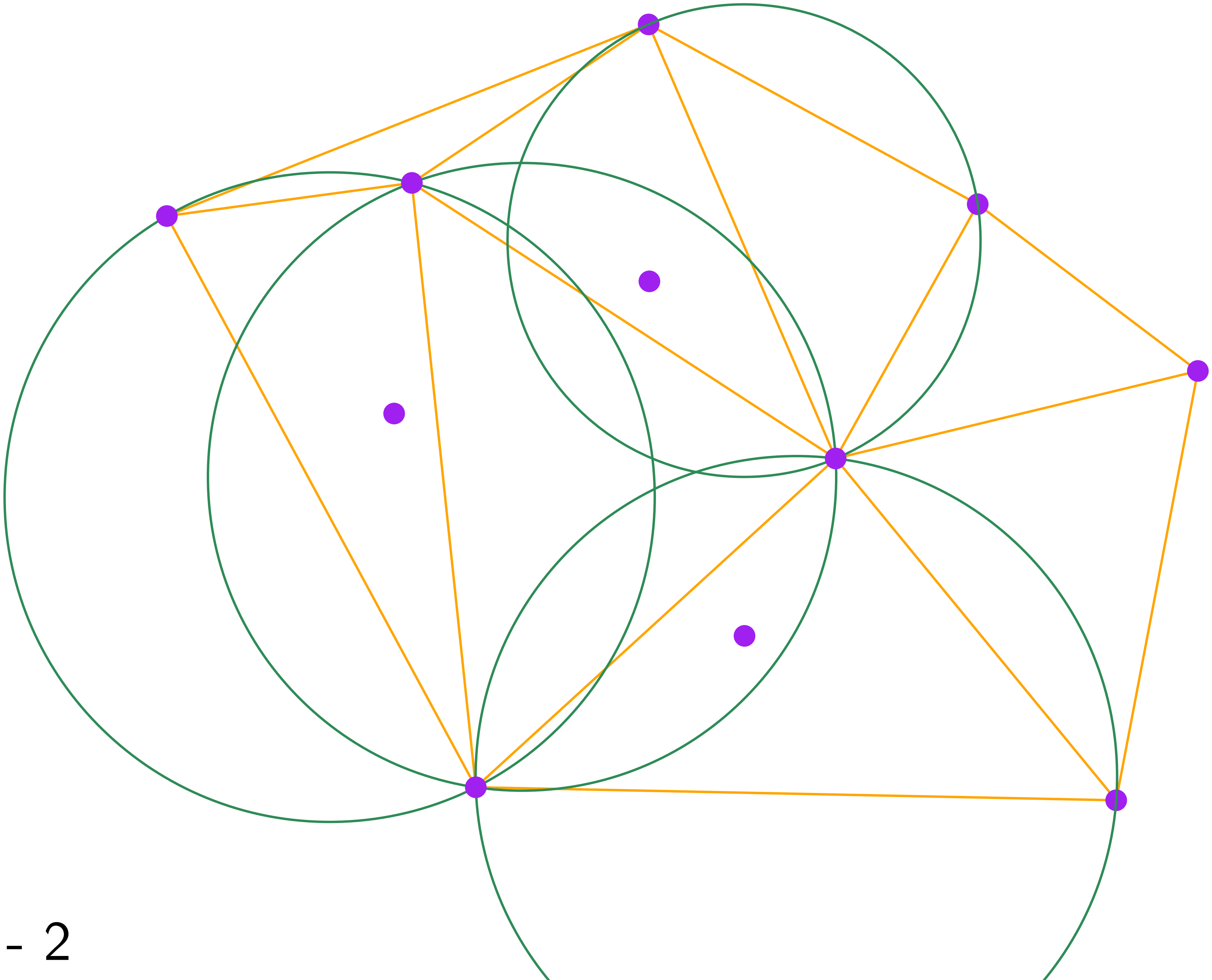
or the stepfather

or both

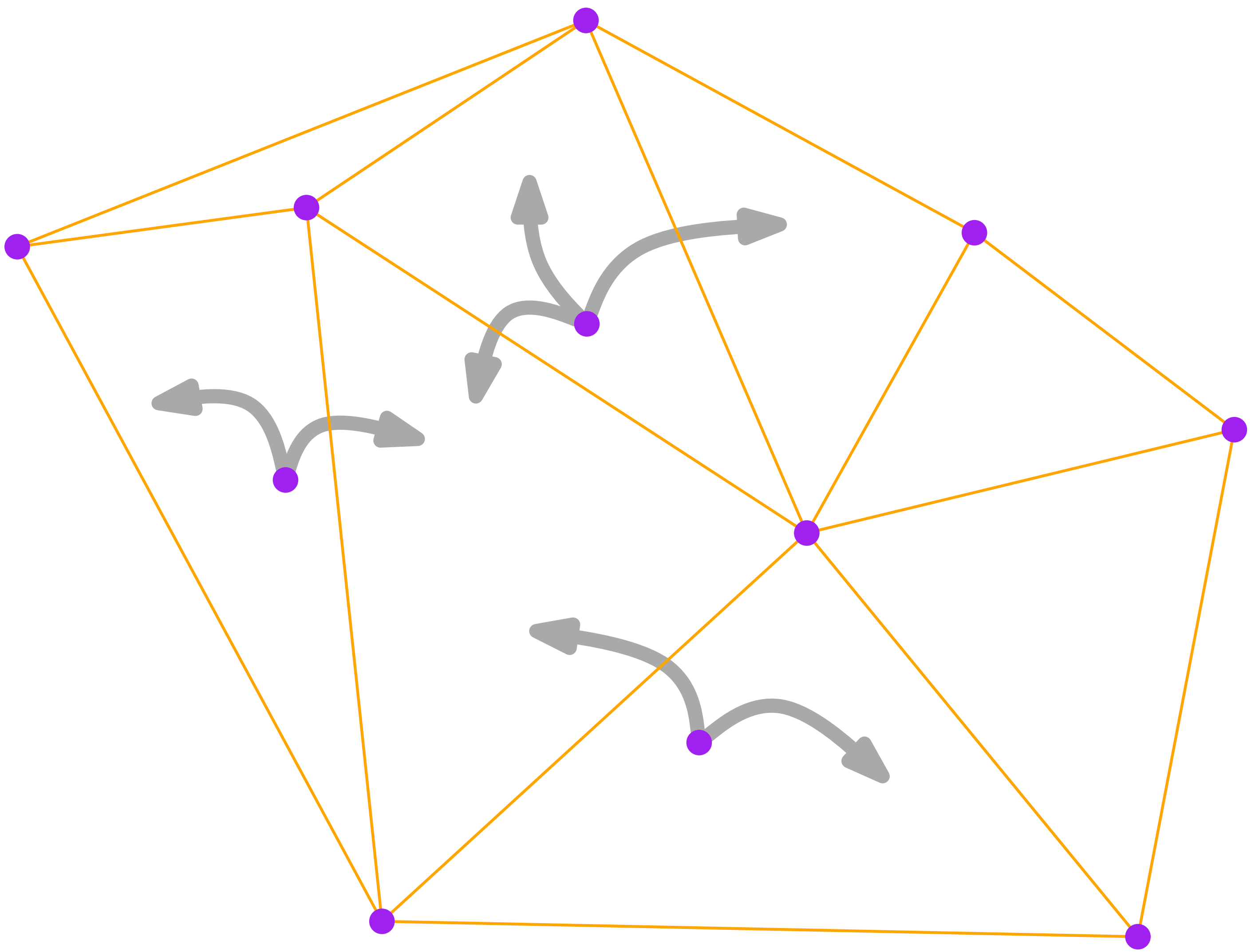
Conflict graph



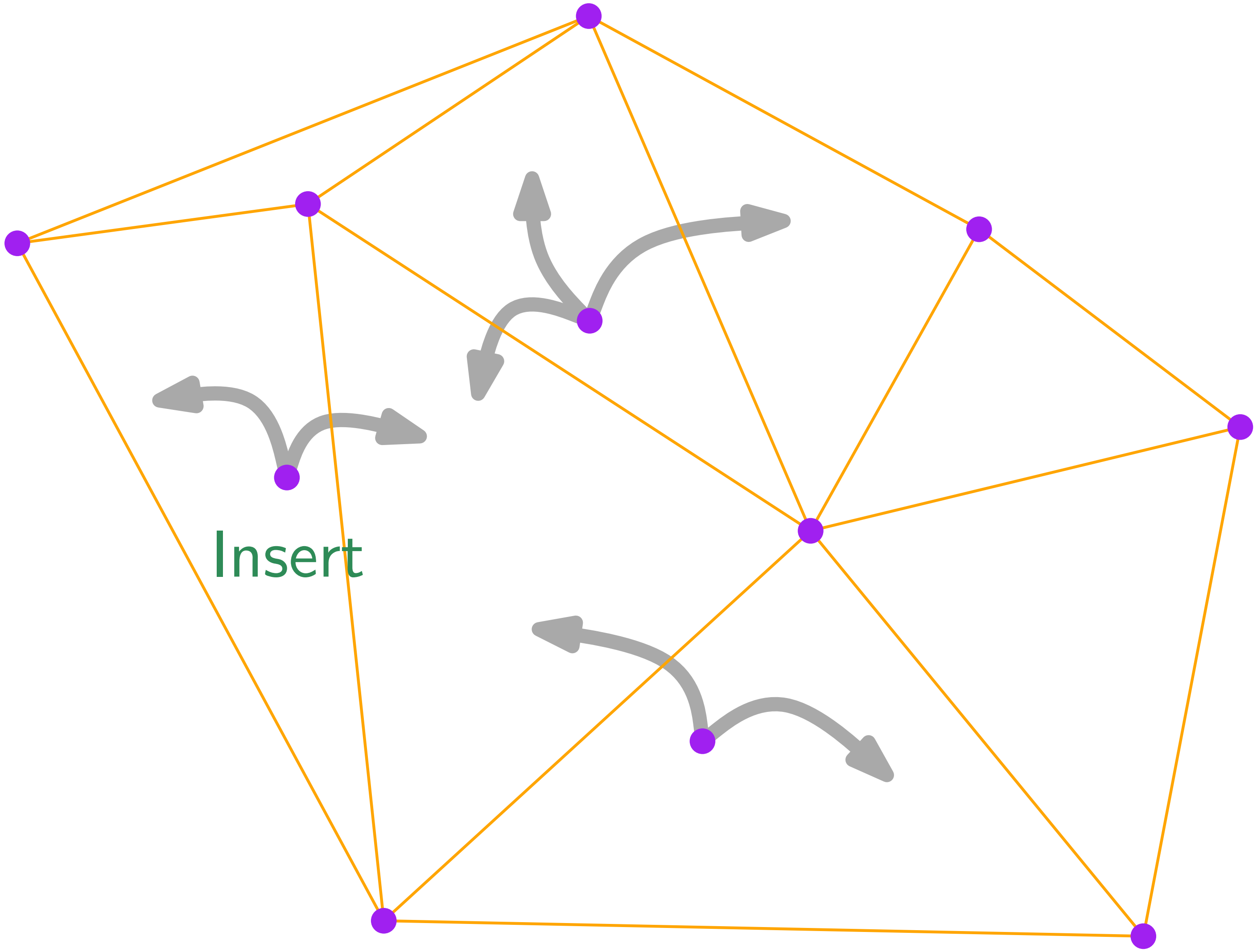
Conflict graph



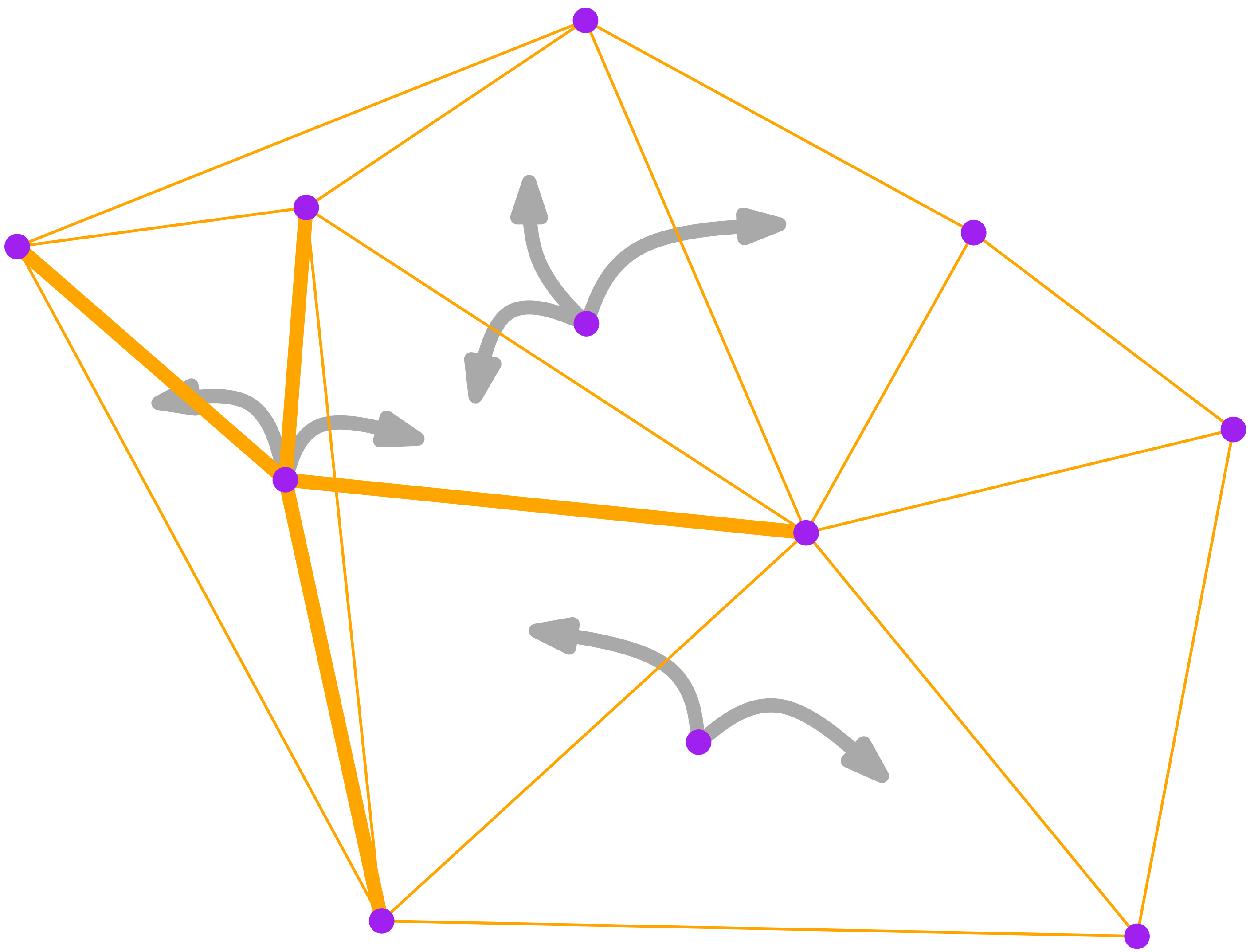
Conflict graph



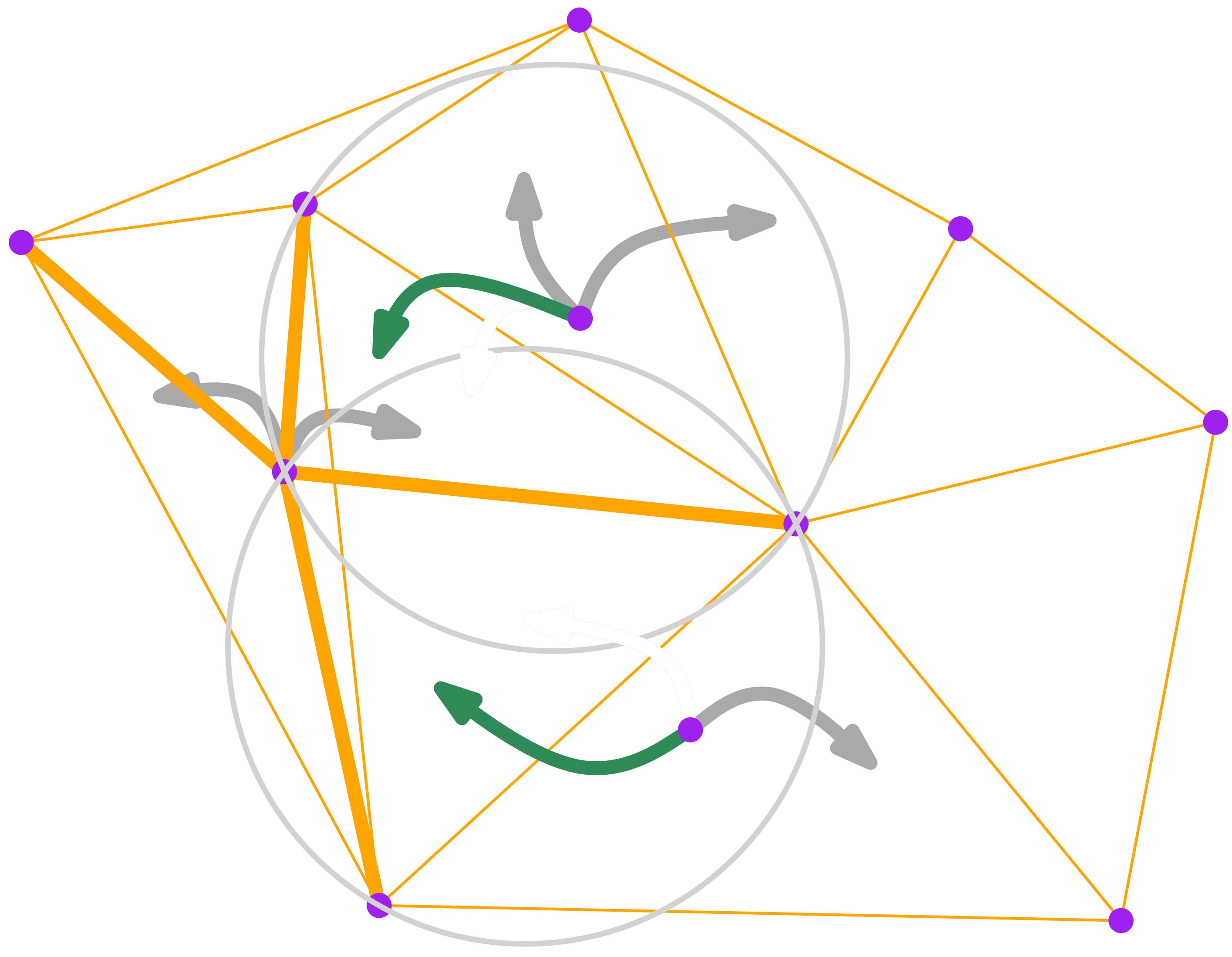
Conflict graph



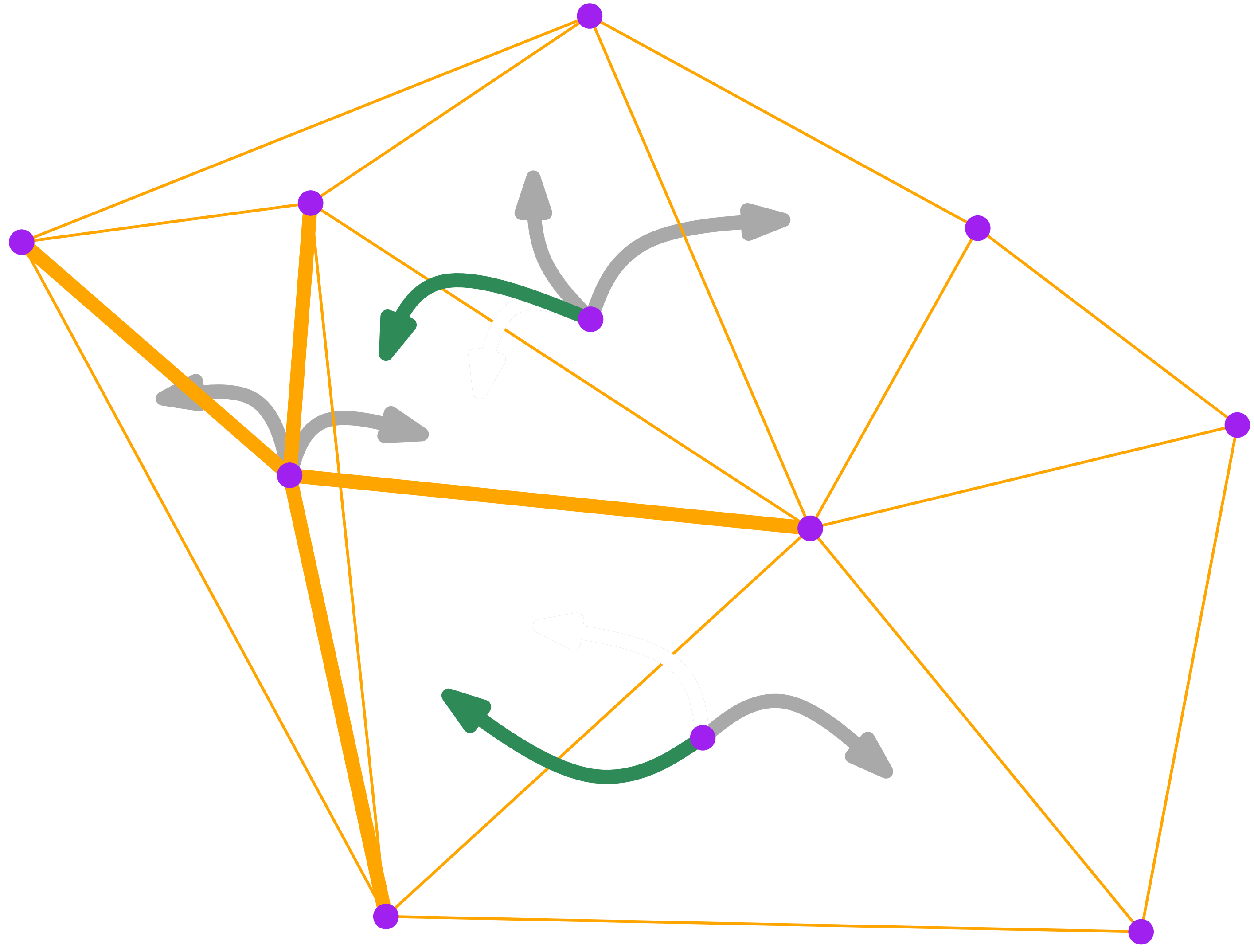
Conflict graph



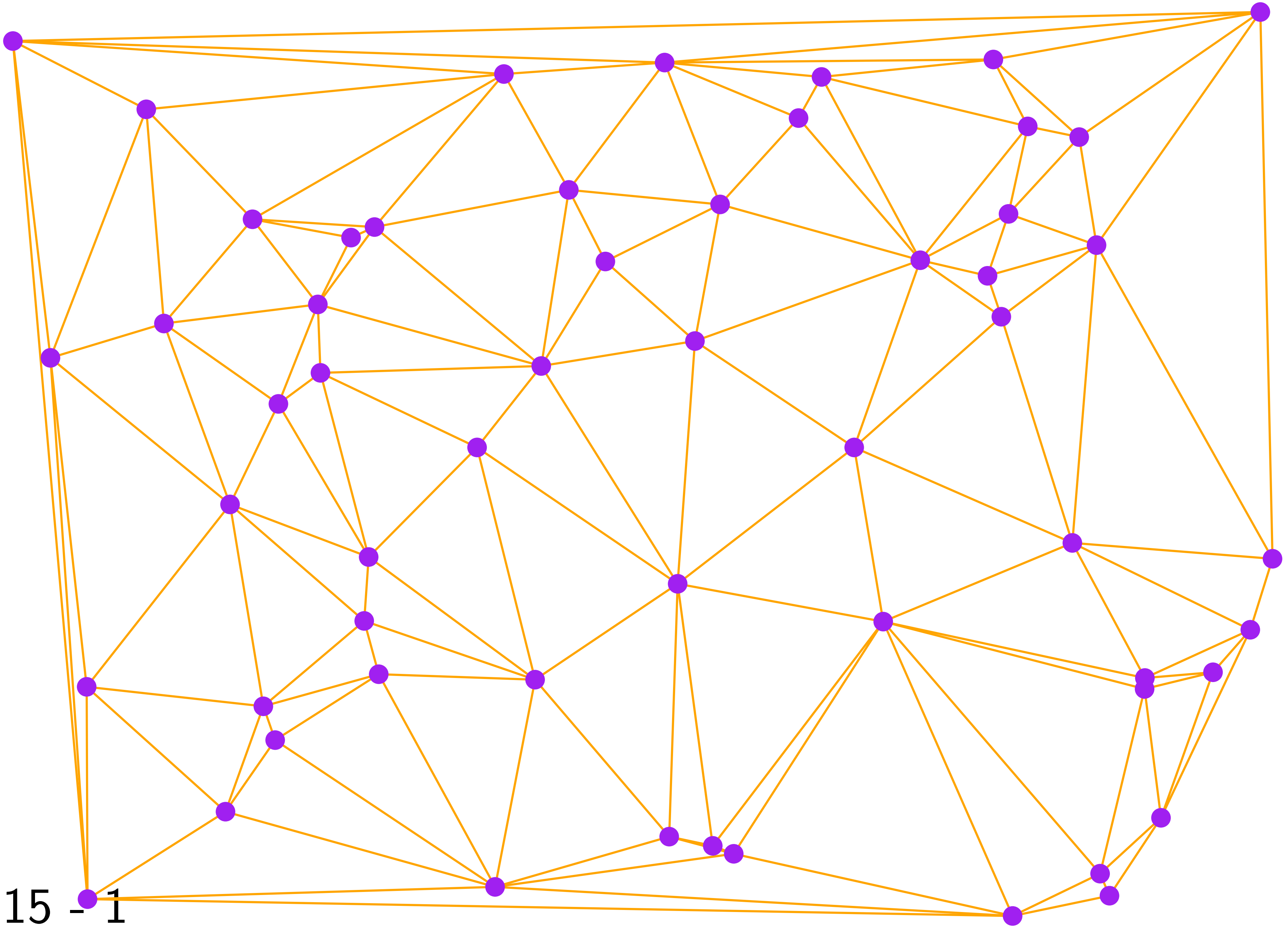
Conflict graph



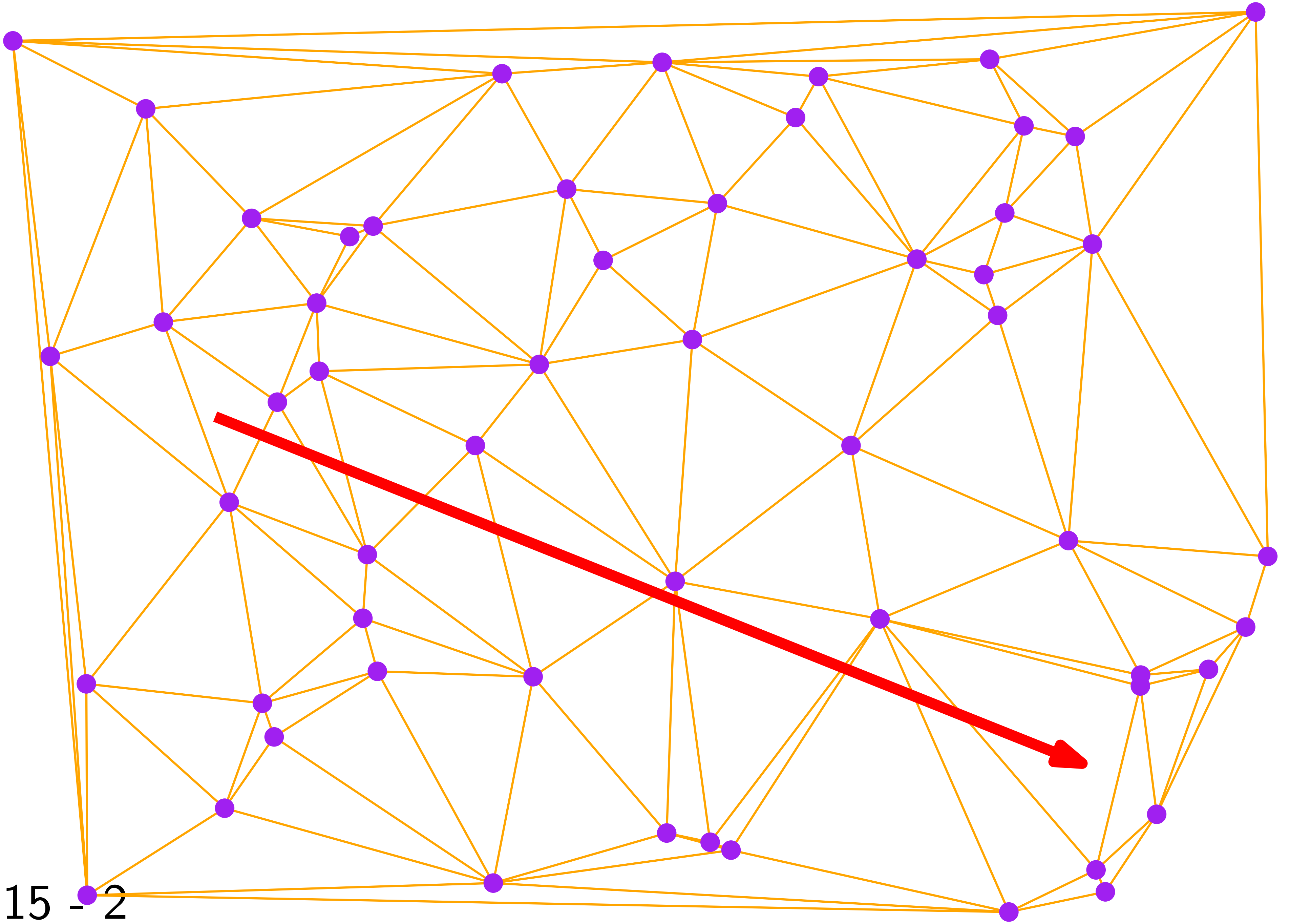
Conflict graph



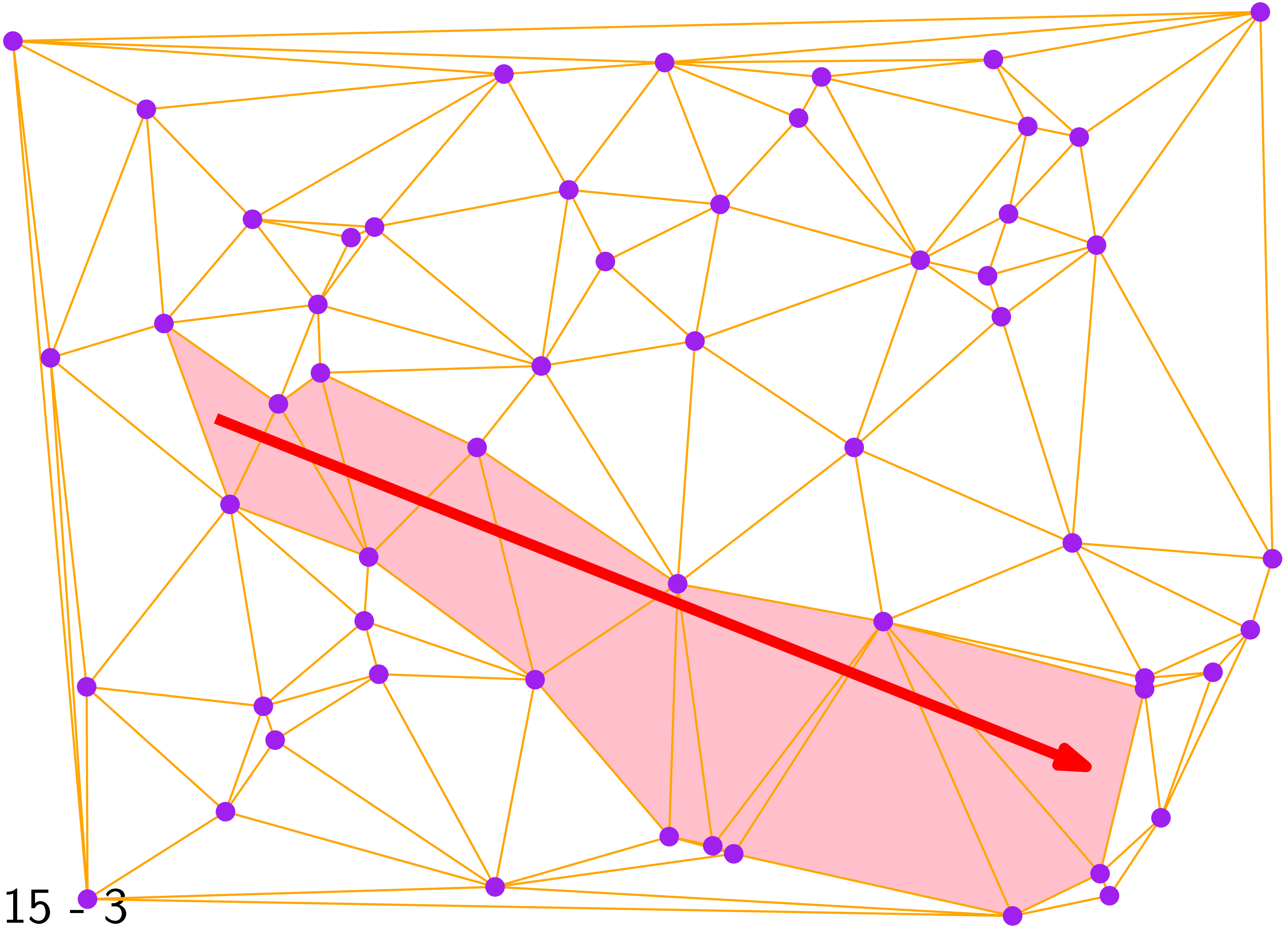
Walk



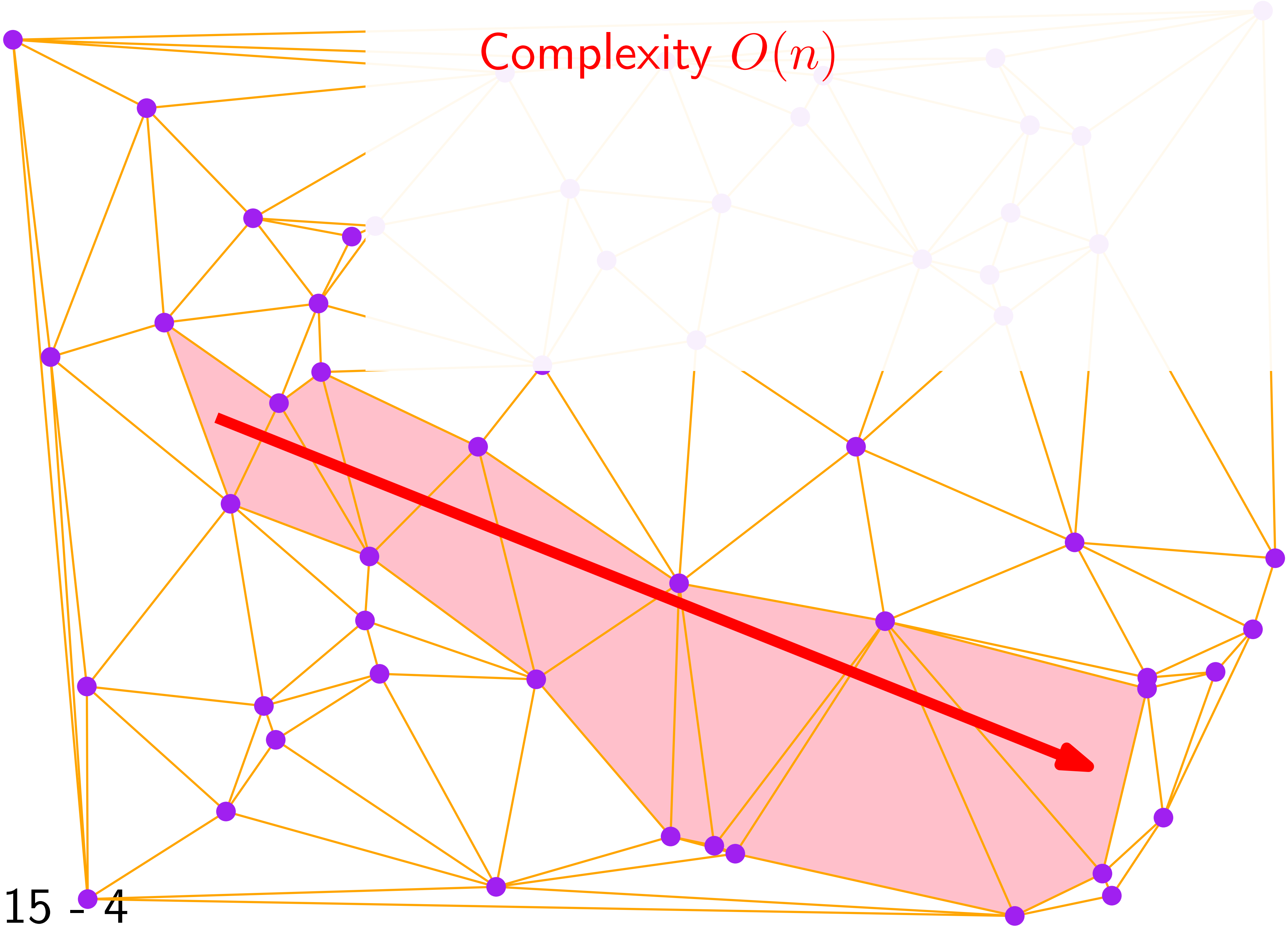
Walk



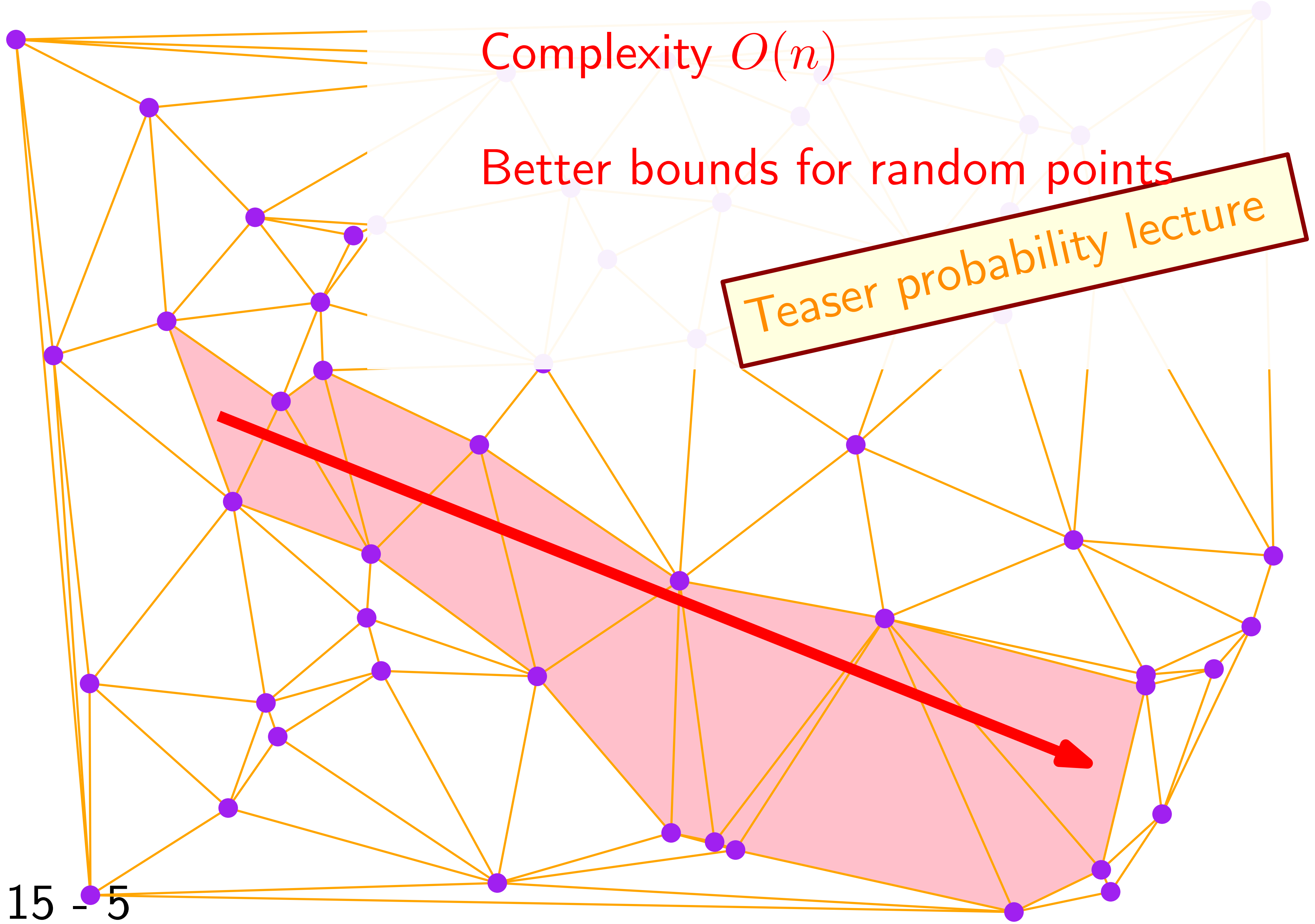
Walk



Walk



Walk

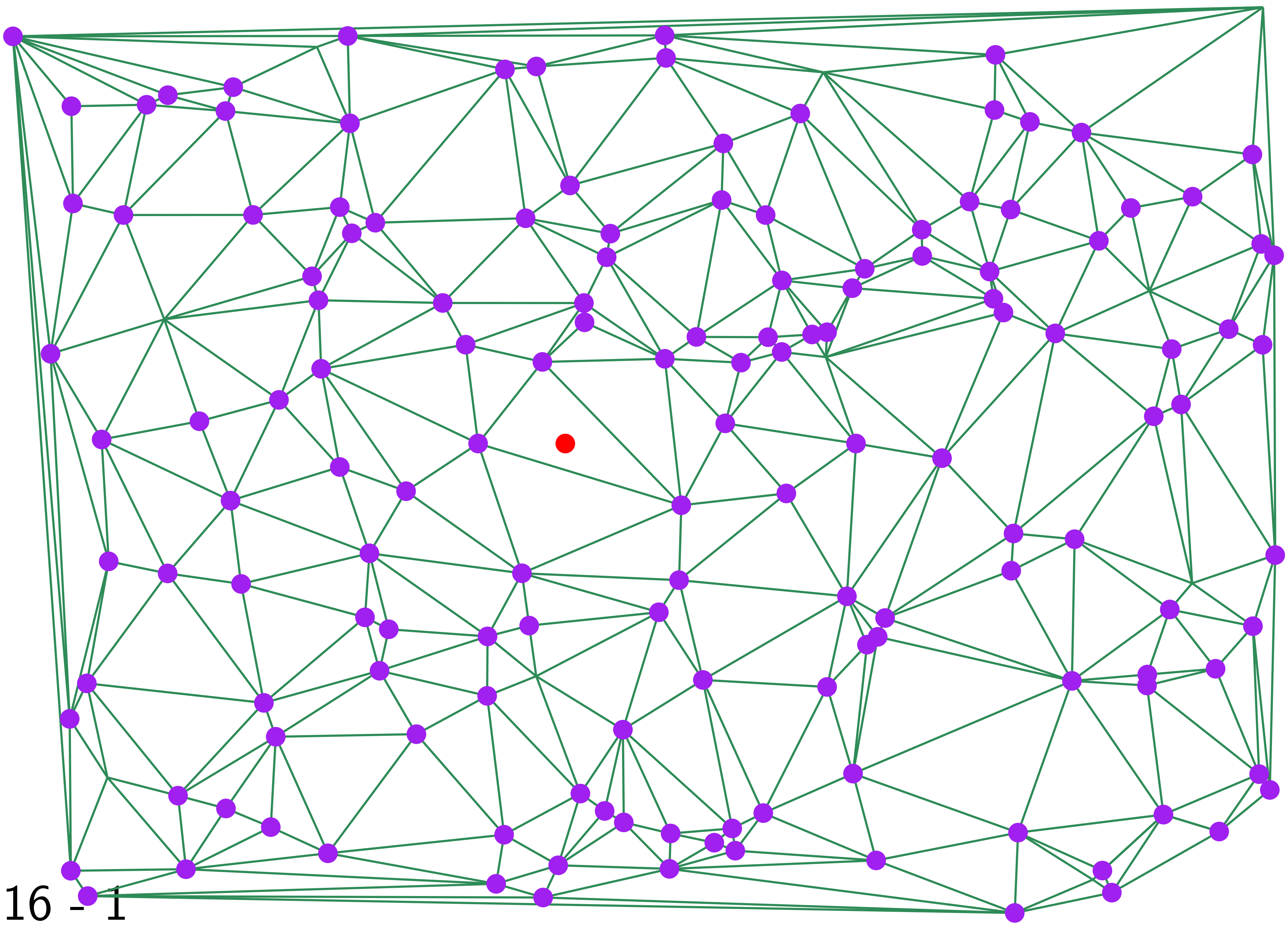


Complexity $O(n)$

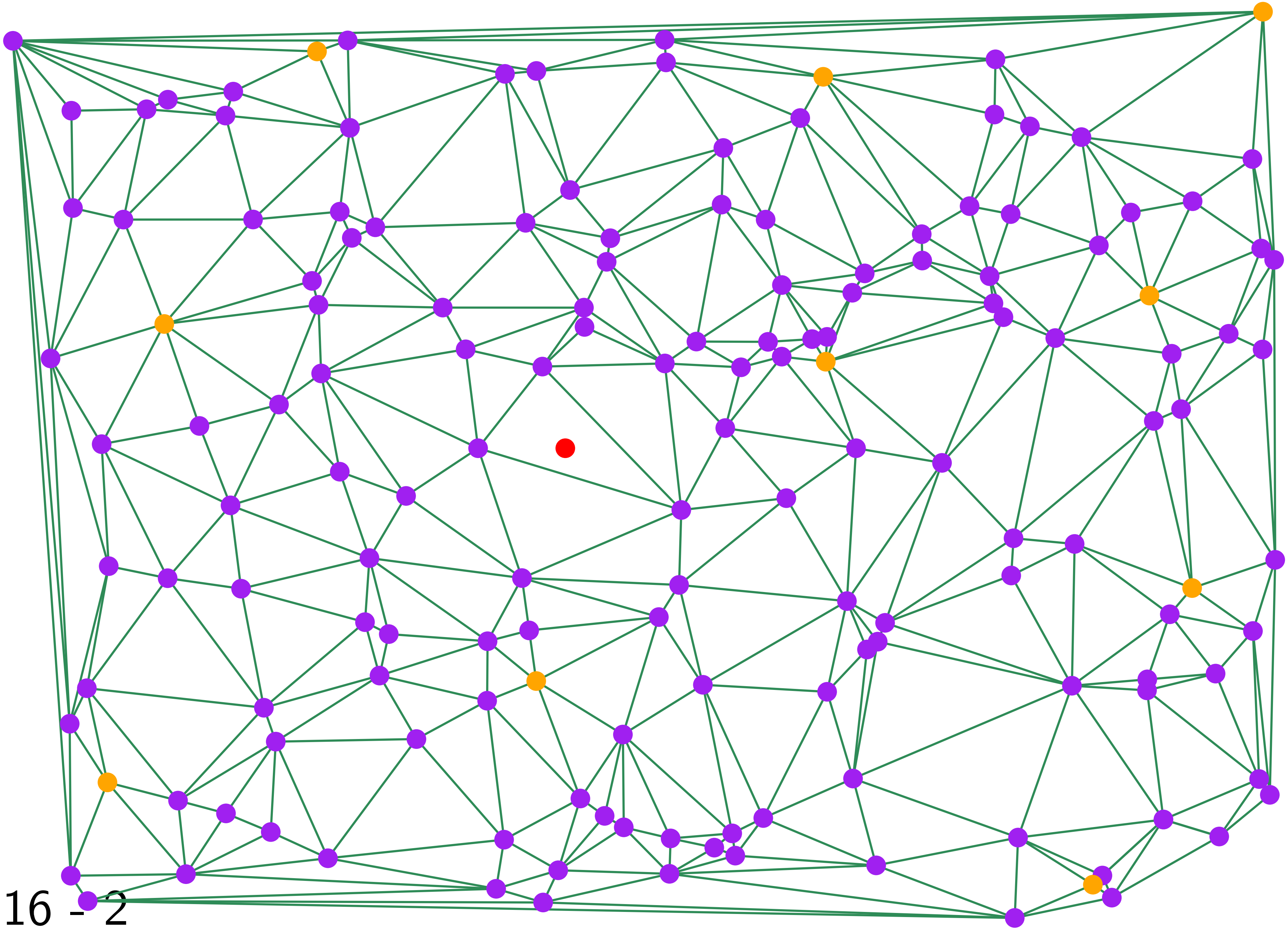
Better bounds for random points

Teaser probability lecture

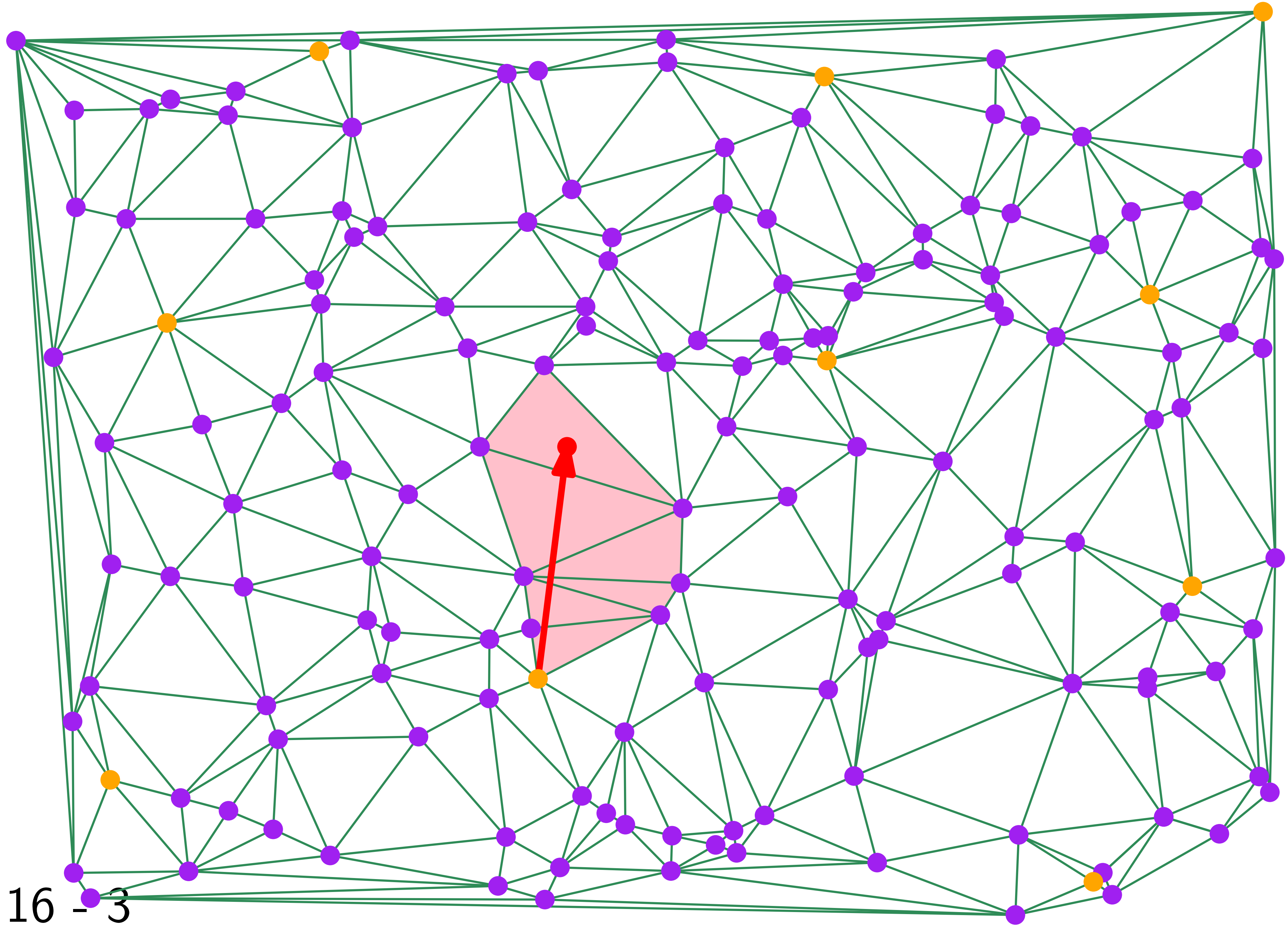
Jump and walk



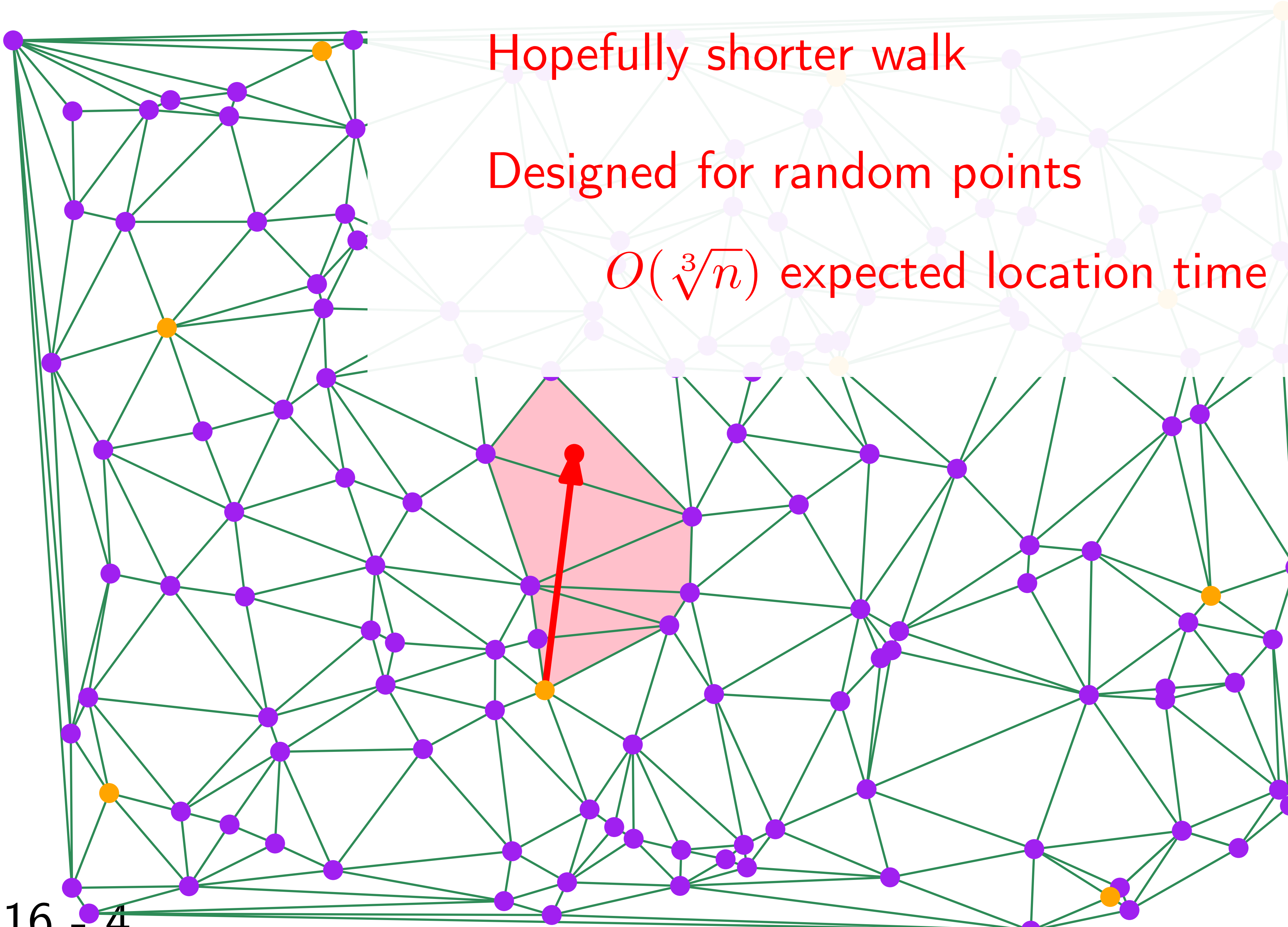
Jump and walk



Jump and walk



Jump and walk

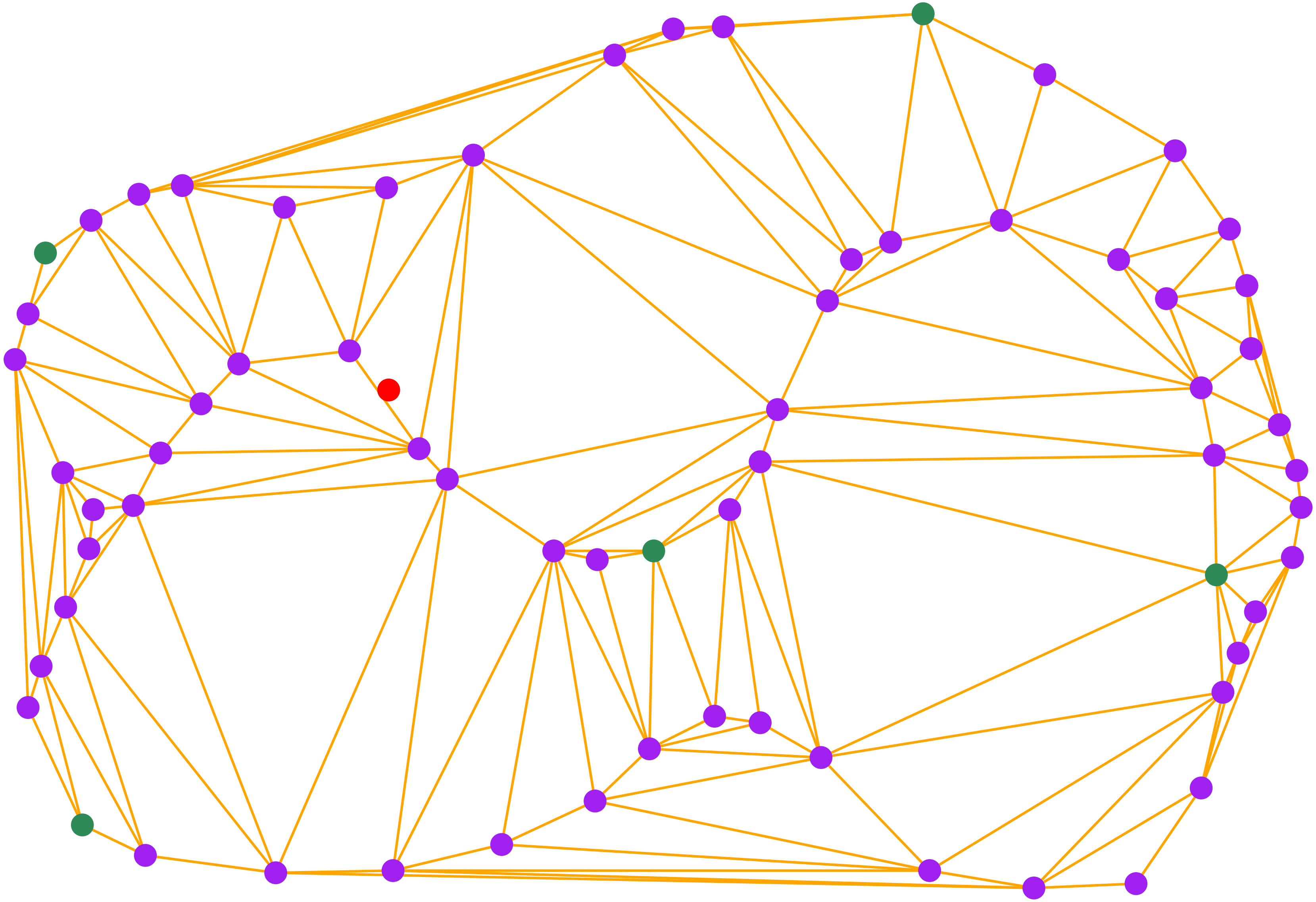


Hopefully shorter walk

Designed for random points

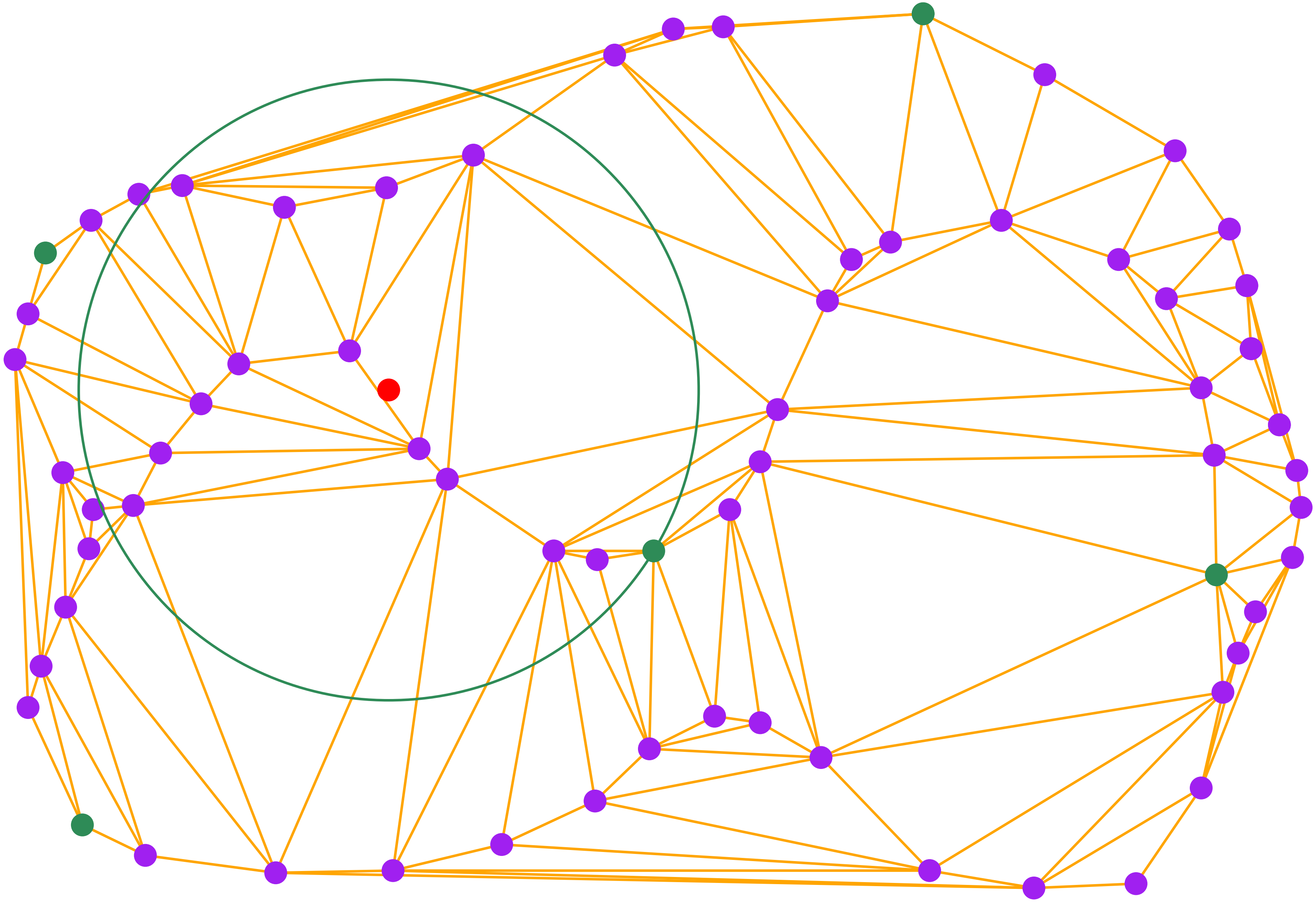
$O(\sqrt[3]{n})$ expected location time

Jump and walk (no distribution hypothesis)



Jump and walk (no distribution hypothesis)

$\mathbb{E}\,[\sharp$ of in $] = \frac{n}{k}$

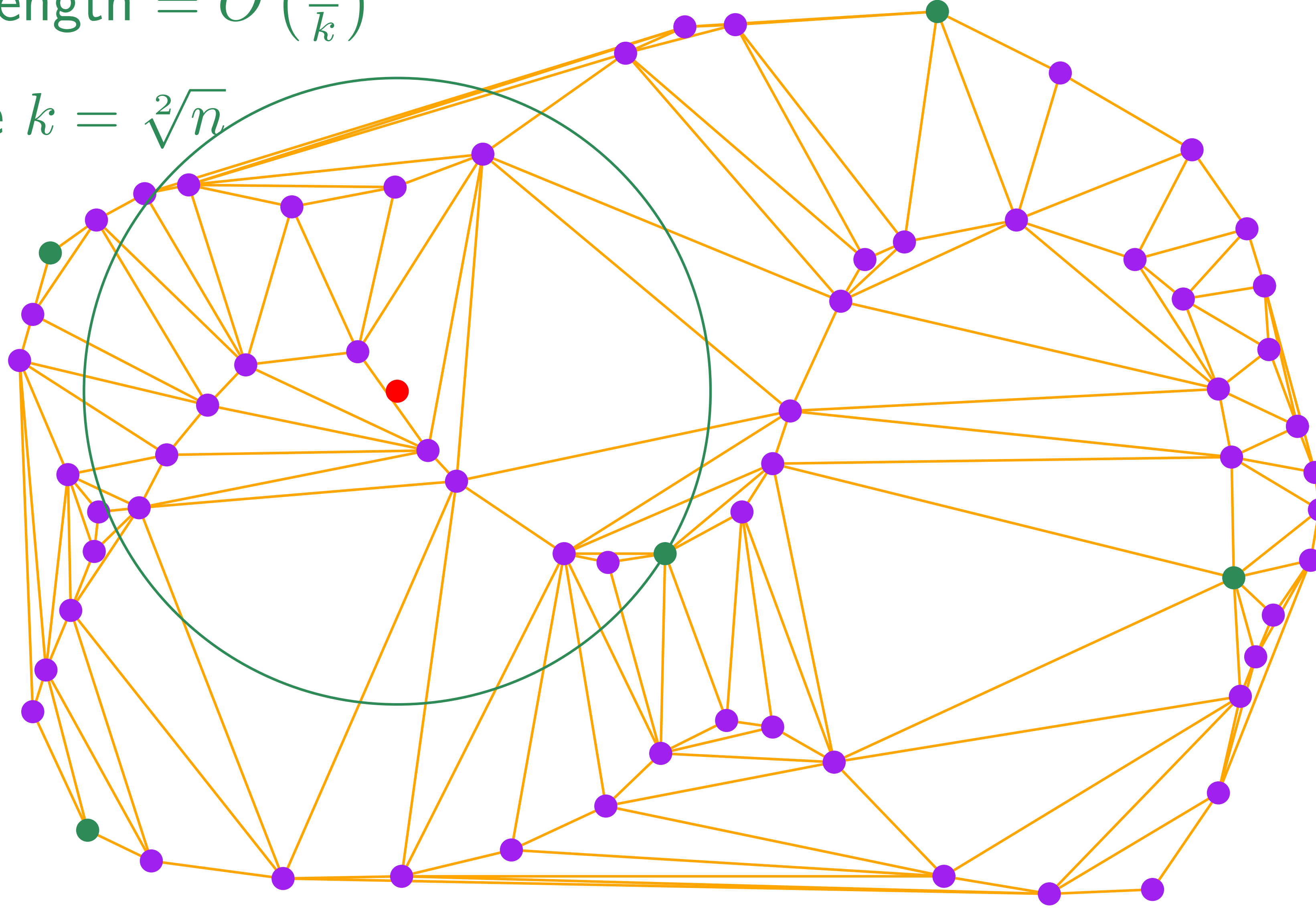


Jump and walk (no distribution hypothesis)

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

Walk length $= O\left(\frac{n}{k}\right)$

choose $k = \sqrt[2]{n}$

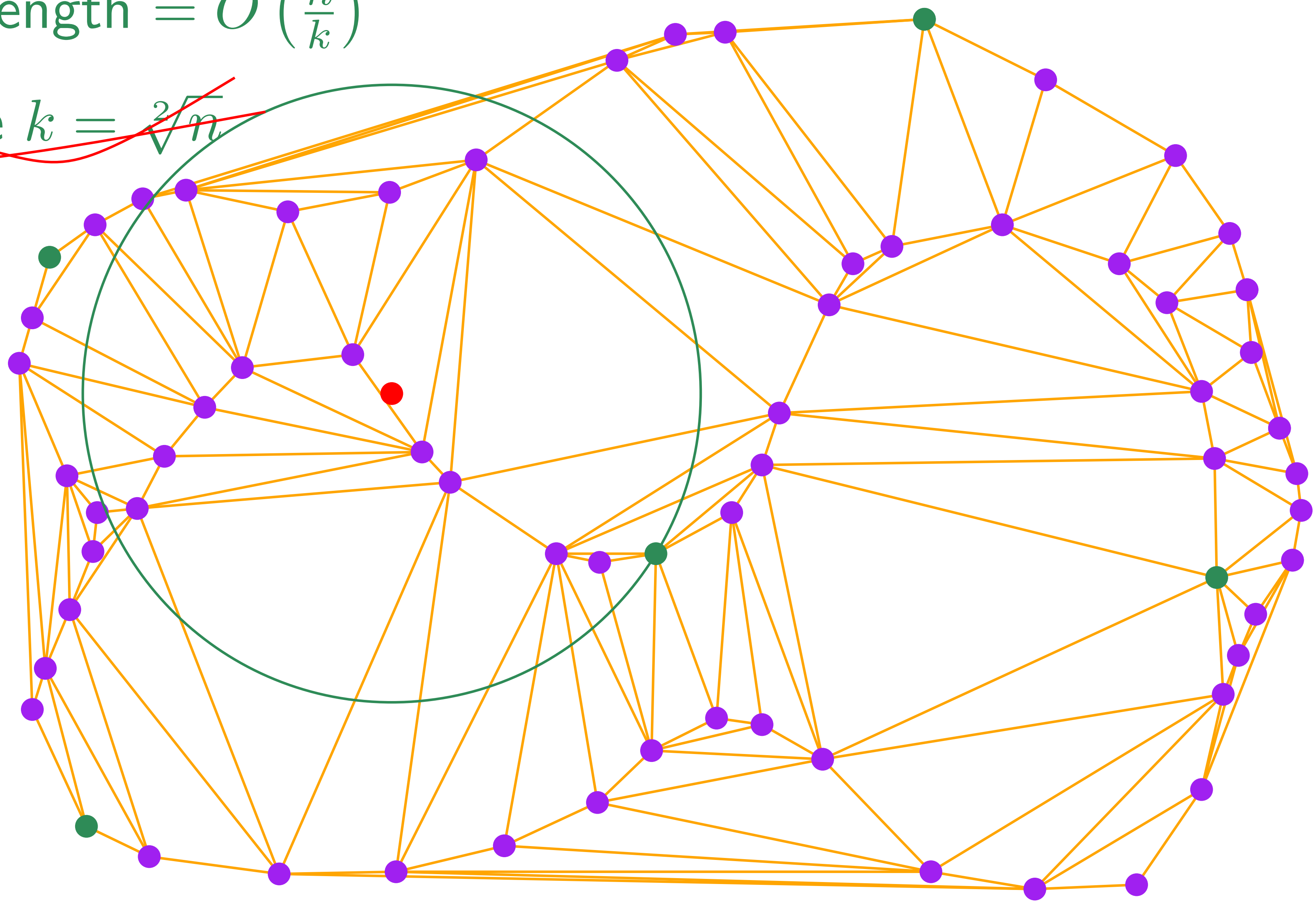


Jump and walk (no distribution hypothesis) Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



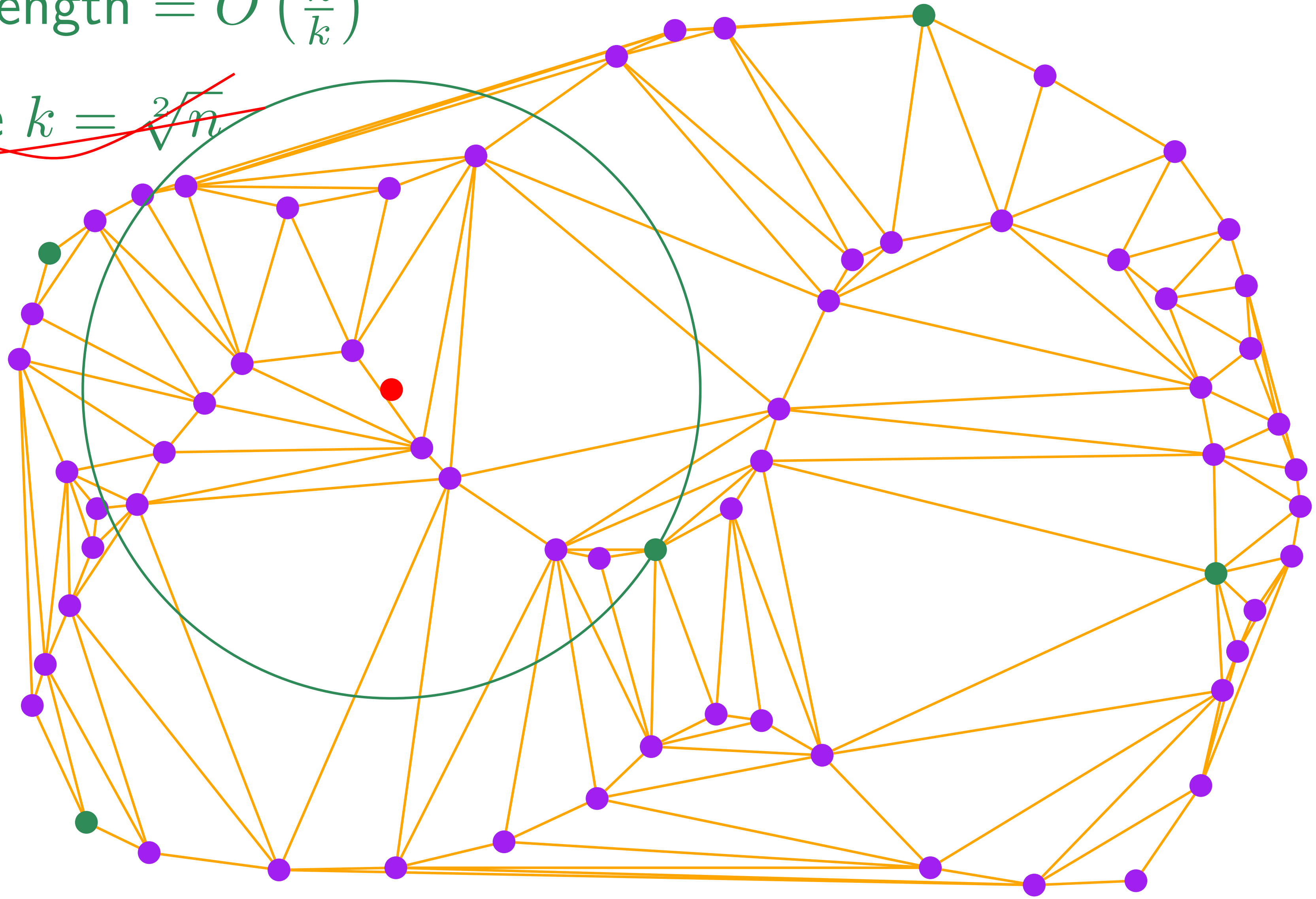
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$\frac{n}{k_1}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



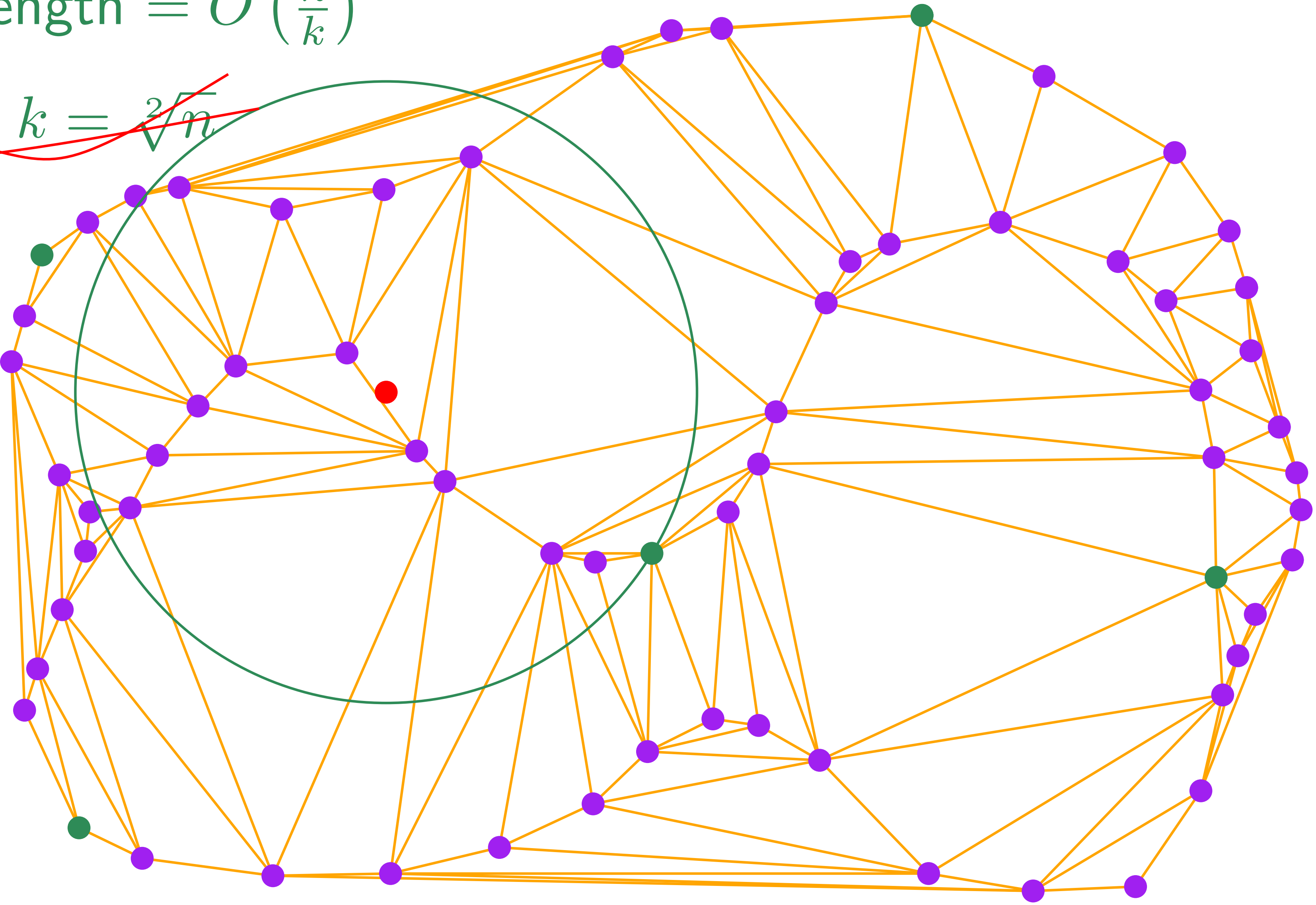
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



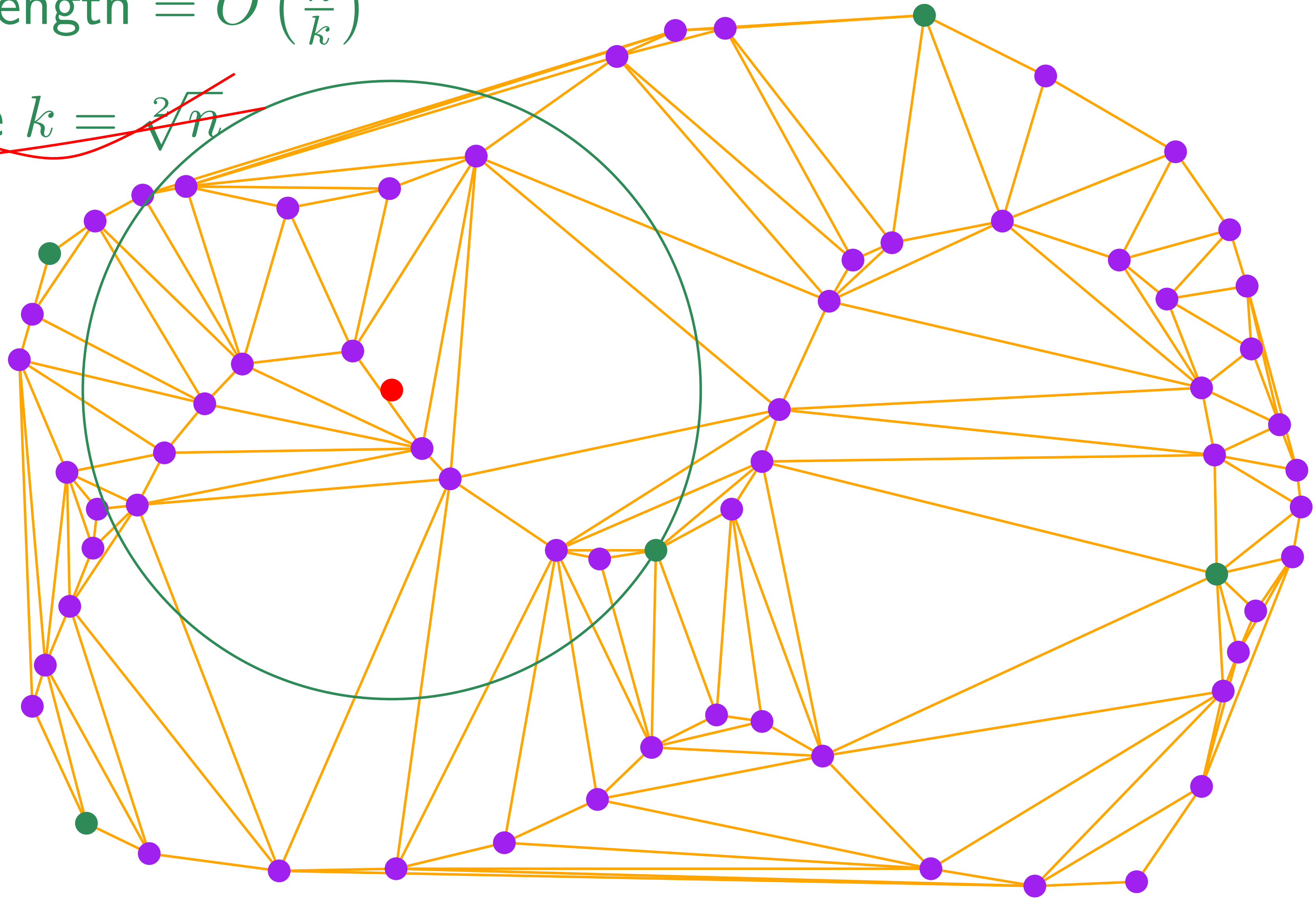
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose~~ $\cancel{k = \sqrt[2]{n}}$



Jump and walk (no distribution hypothesis)Delaunay hierarchy

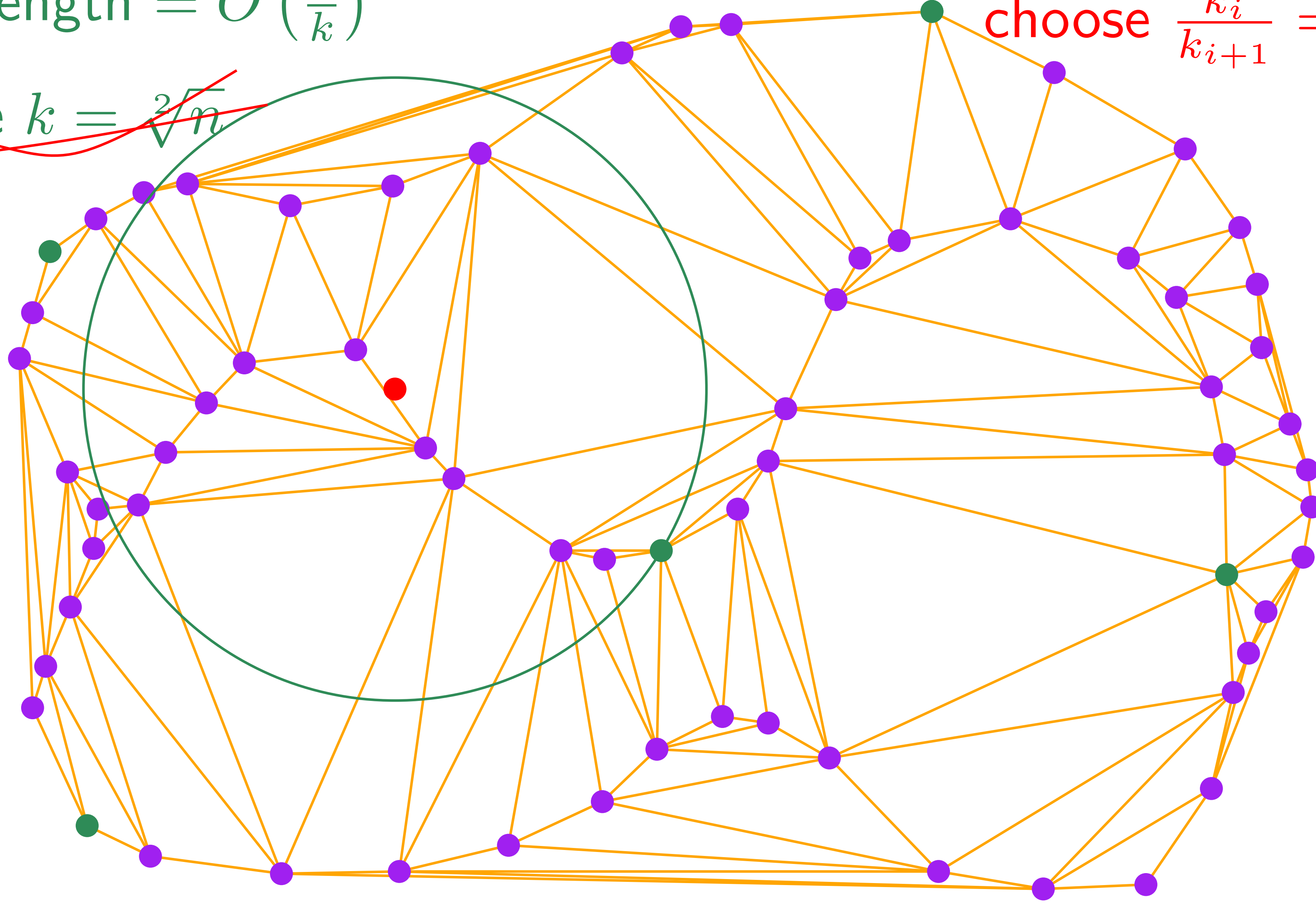
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \dots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

~~choose $k = \sqrt[2]{n}$~~



Jump and walk (no distribution hypothesis) Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

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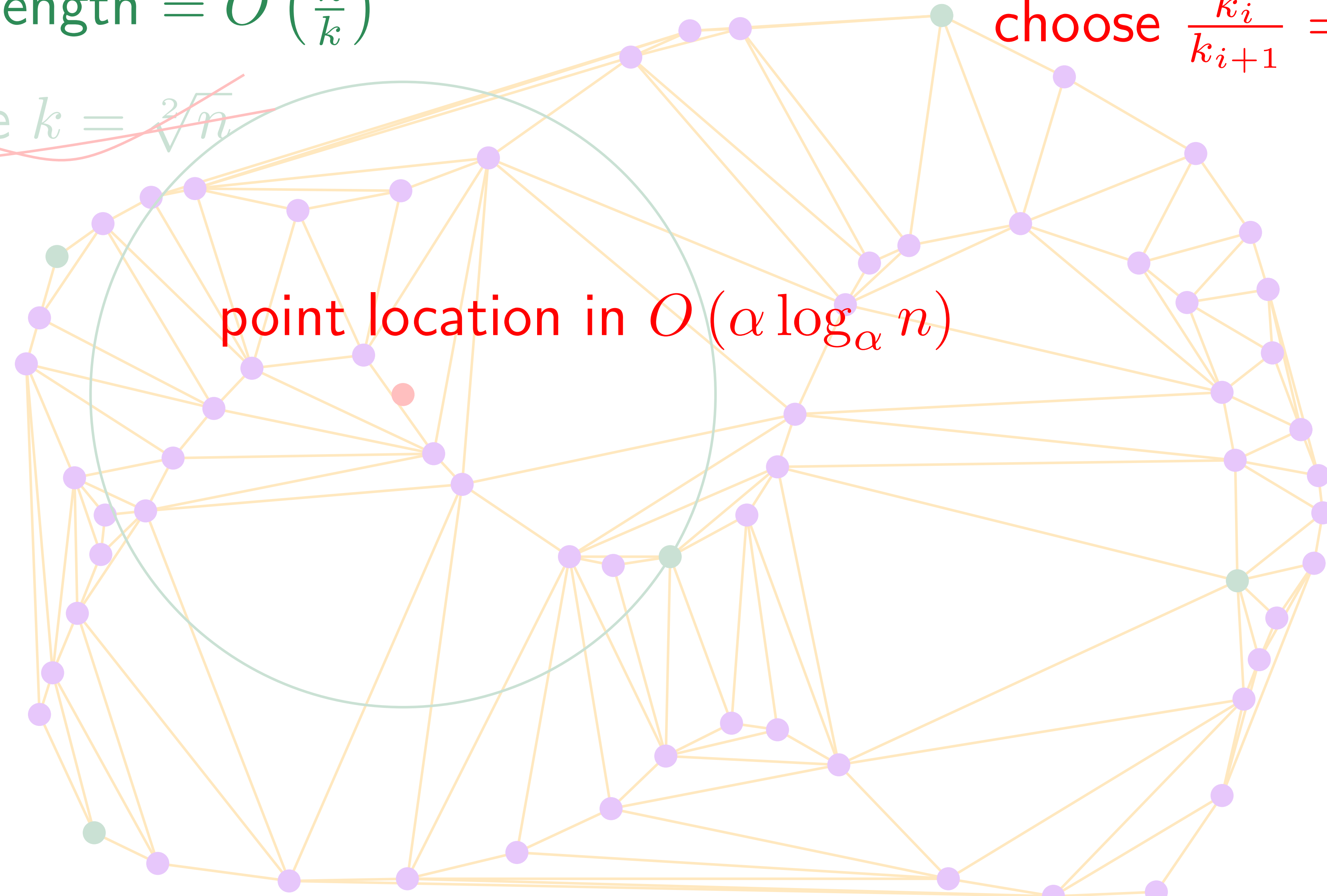


point location in $O\left(\alpha \log_\alpha n\right)$

Jump and walk (no distribution hypothesis) Delaunay hierarchy

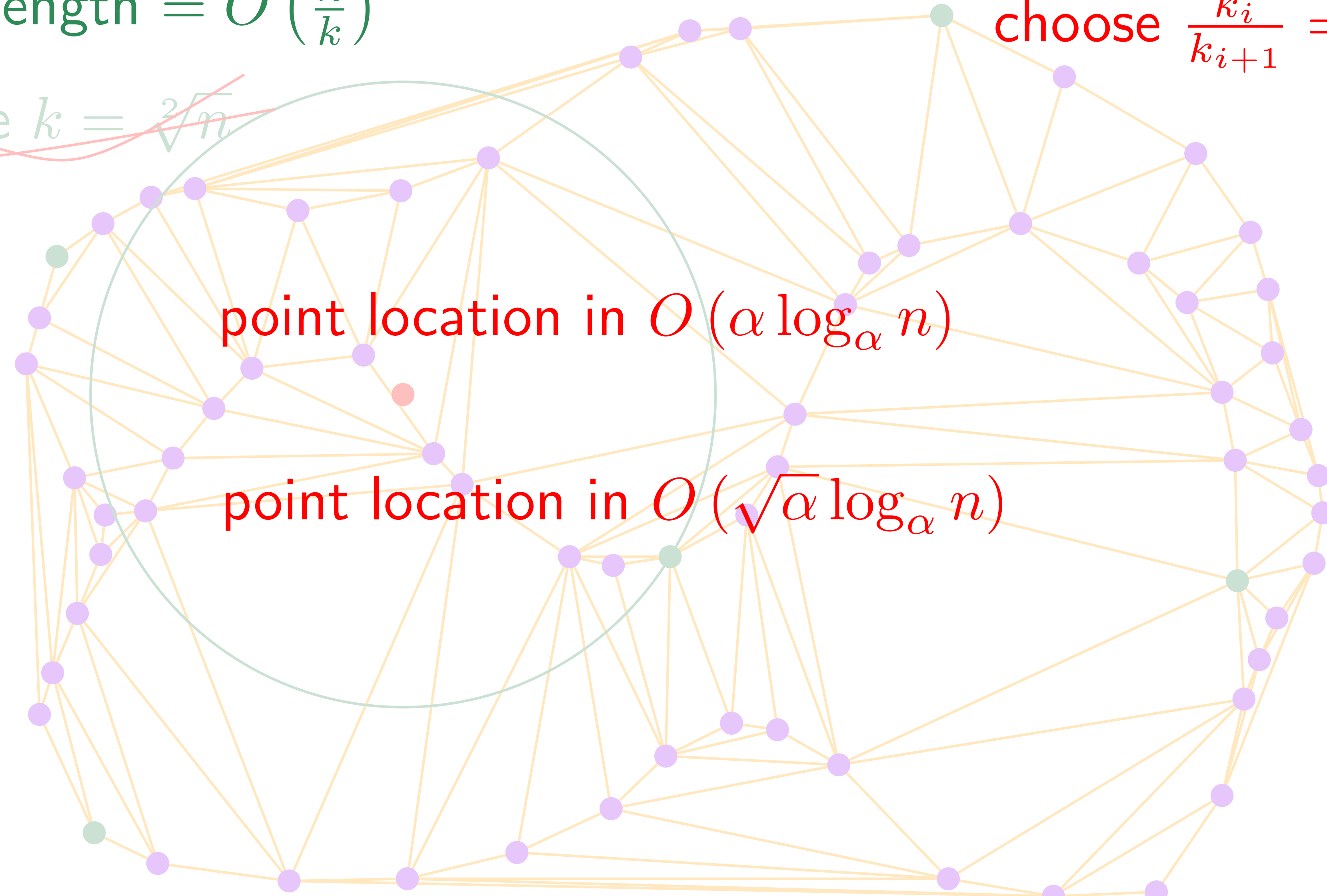
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

~~choose $k = \sqrt[2]{n}$~~



point location in $O\left(\alpha \log_\alpha n\right)$

point location in $O\left(\sqrt{\alpha} \log_\alpha n\right)$

Randomization

How many randomness is necessary?

If the data are not known in advance

shuffle locally

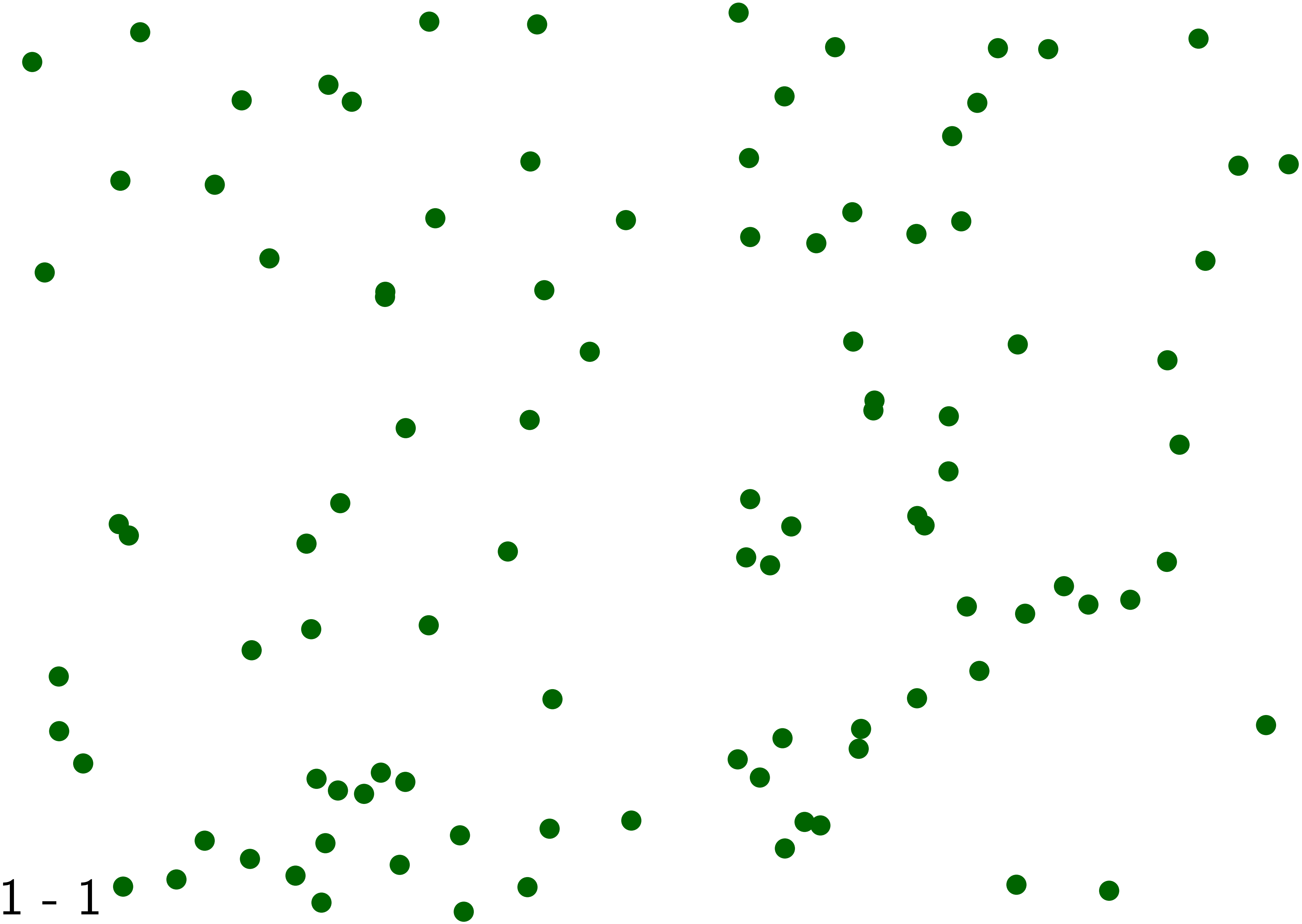
Randomization

Drawbacks of random order

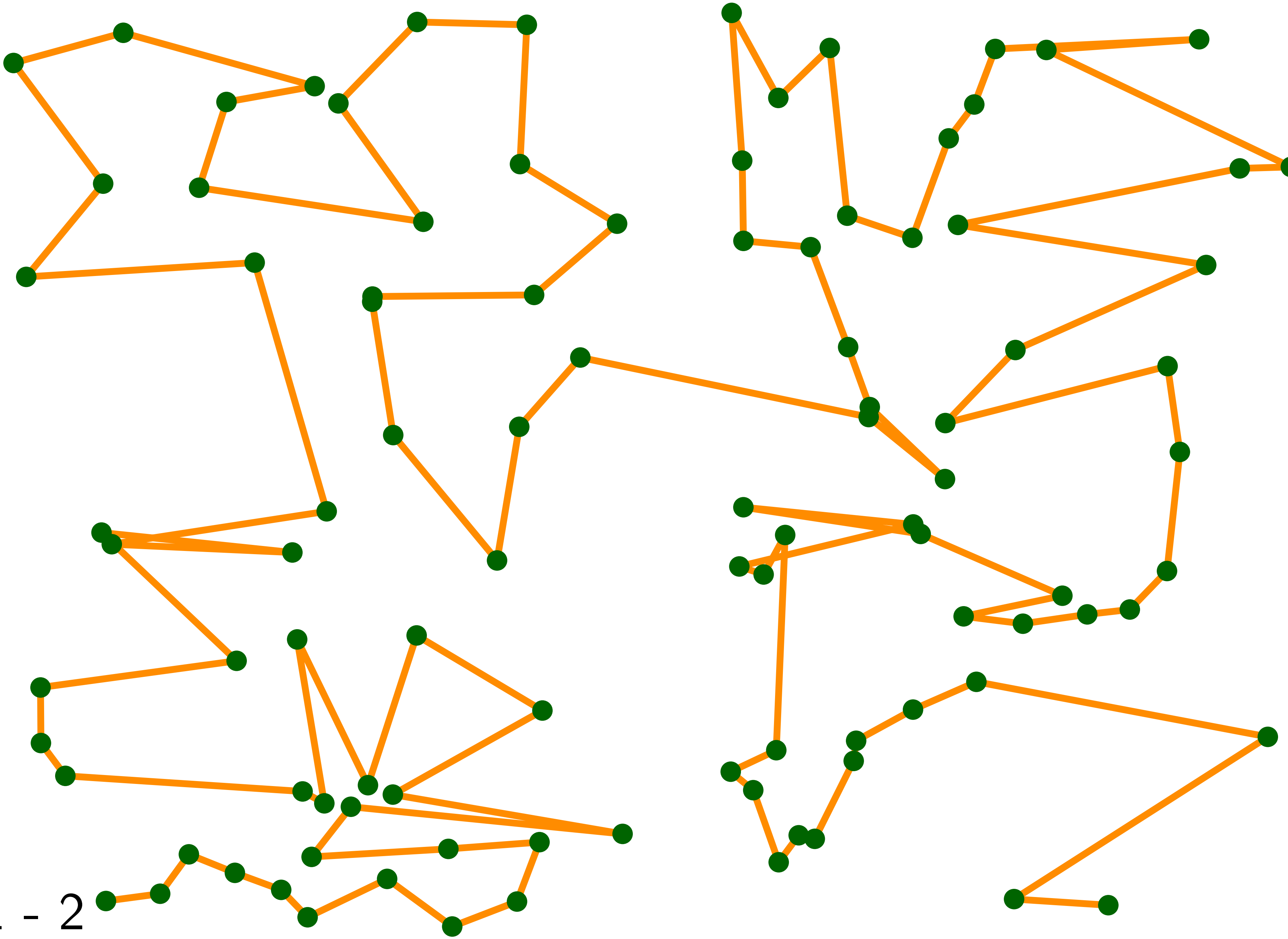
non locality of memory access

data structure for point location

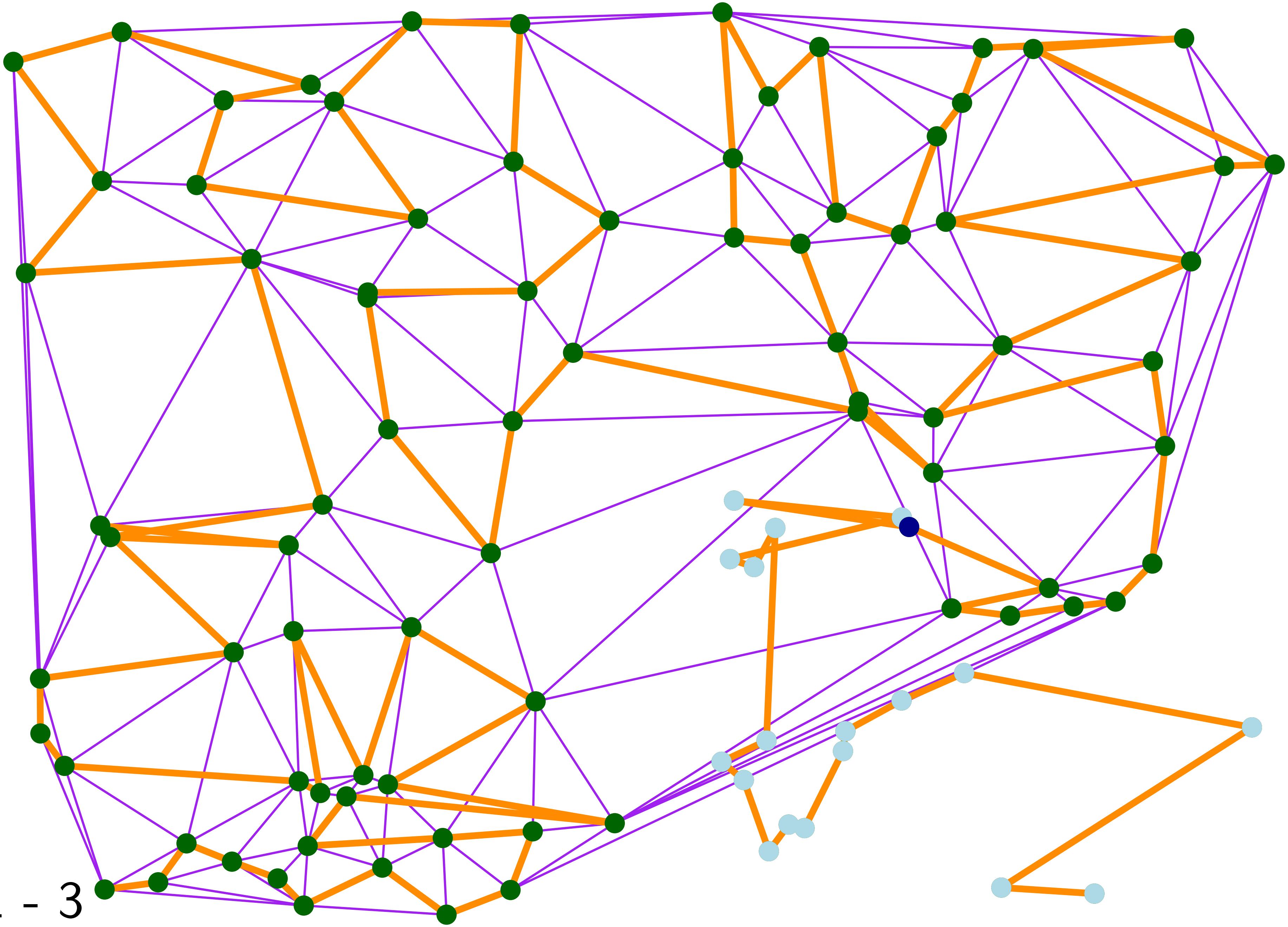
$\longrightarrow$ Hilbert sort



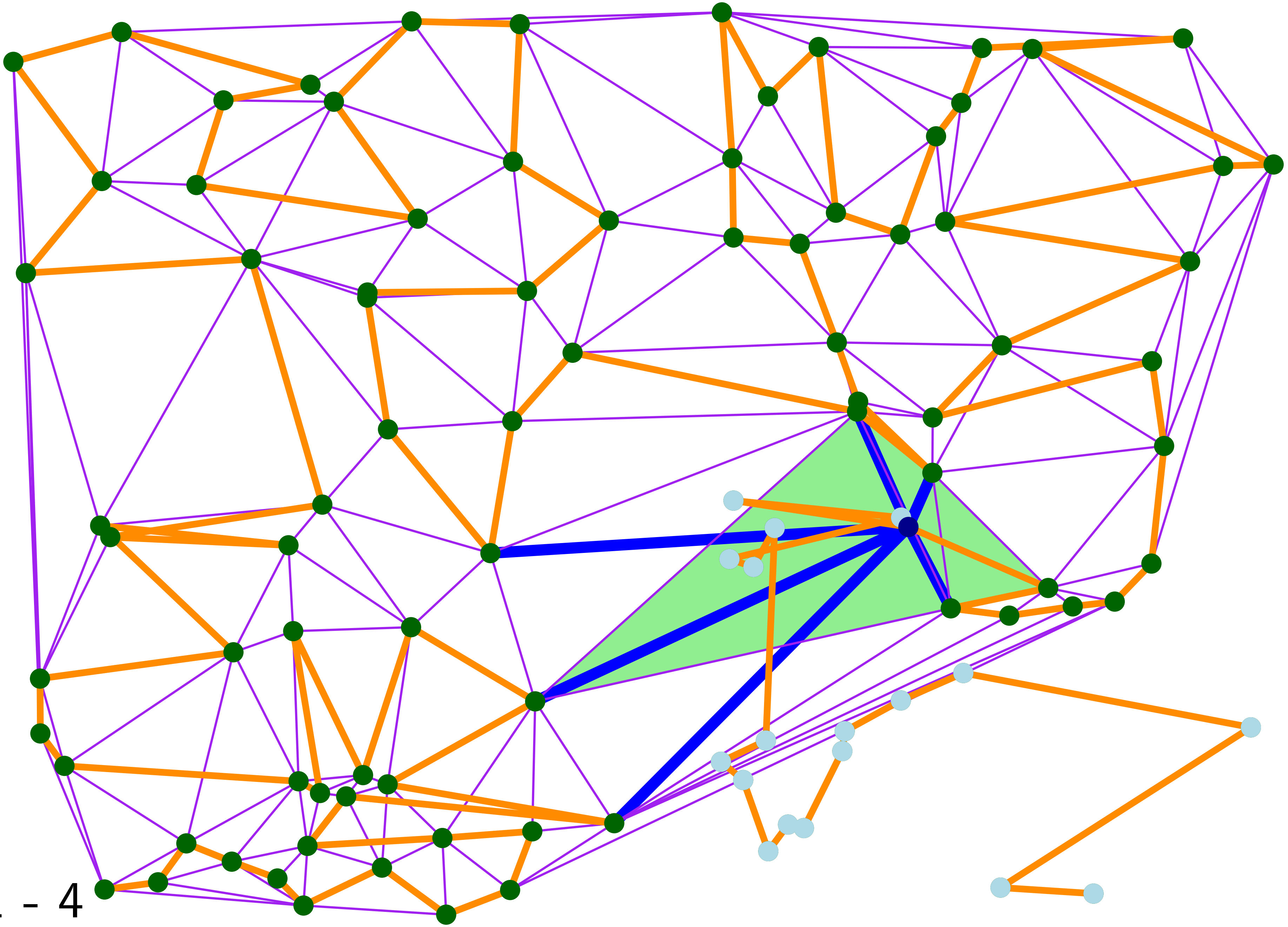
21 - 1



21 - 2



21 - 3



21 - 4

Drawbacks of random order

non locality of memory access

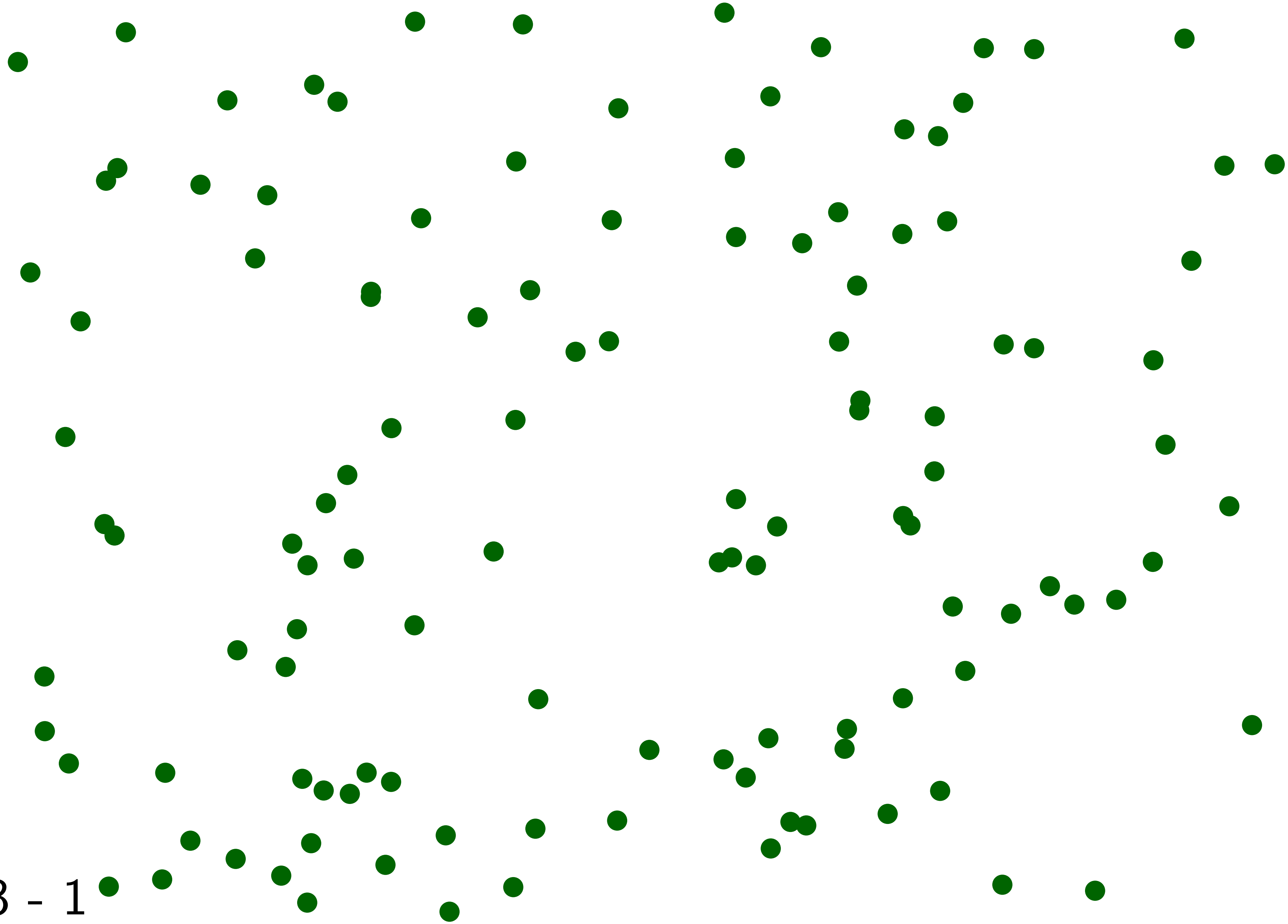
data structure for point location

$\longrightarrow$ Hilbert sort

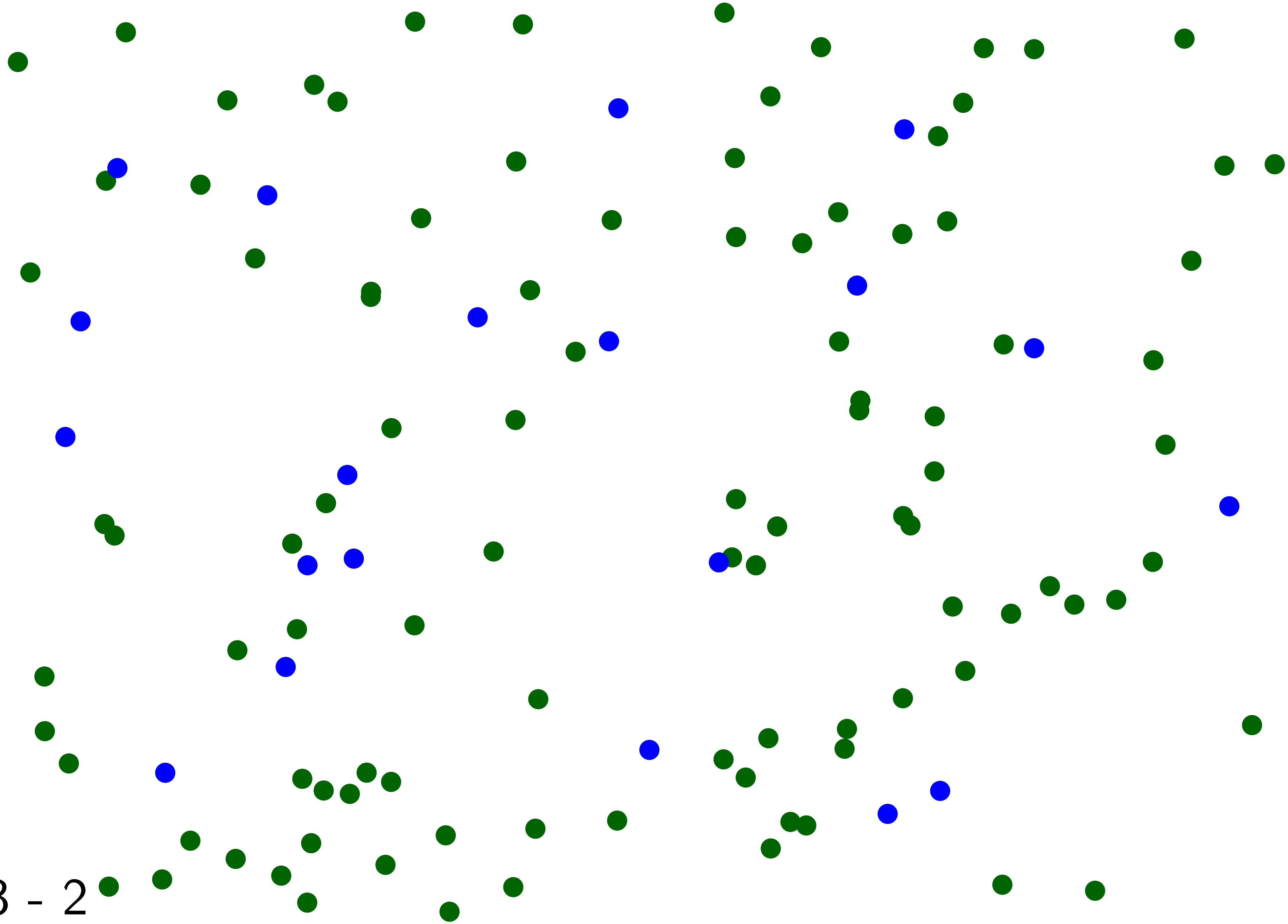
Walk should be fast

Last point is not at all a random point

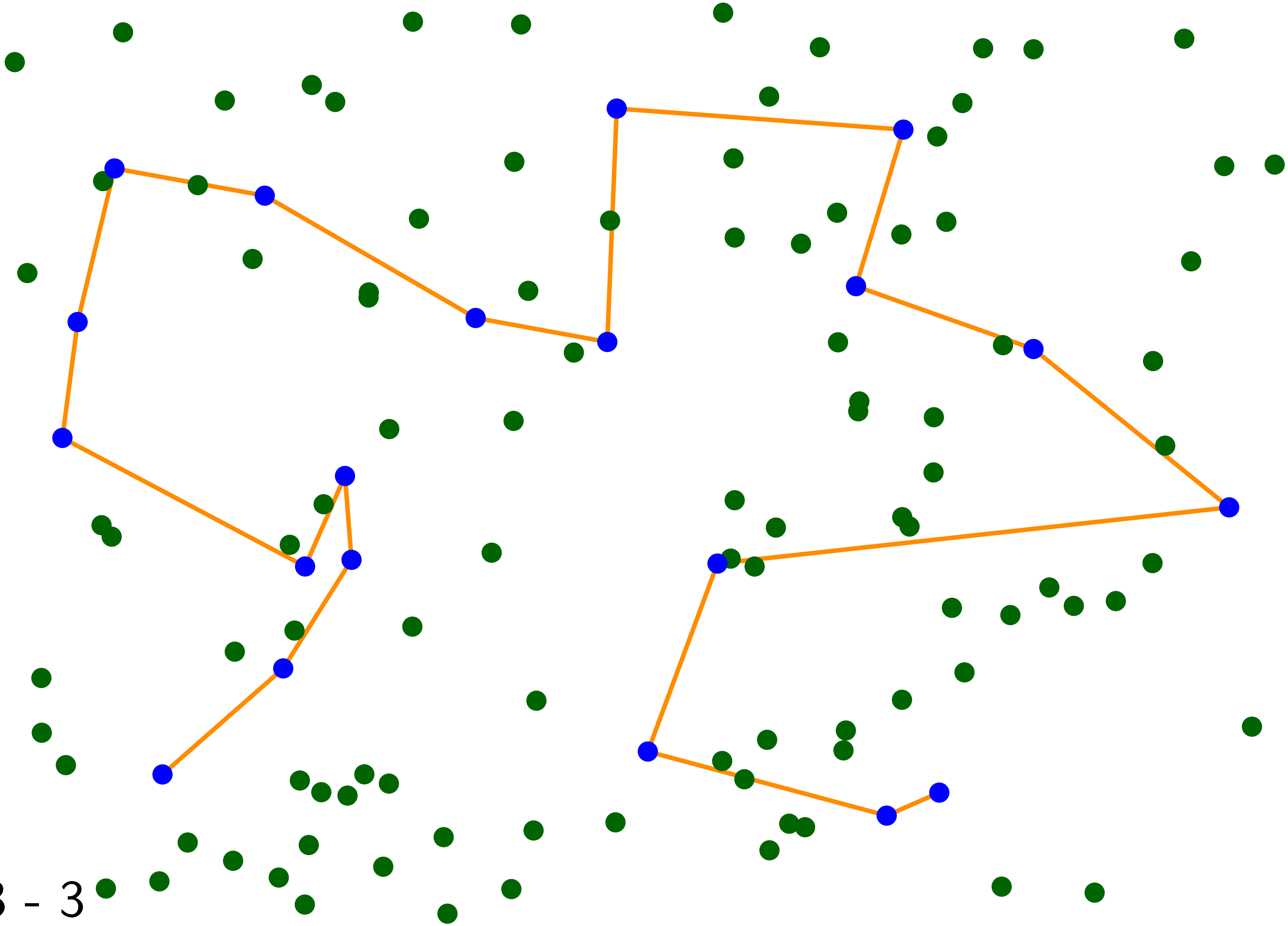
$\longrightarrow$ no control of degree of last point



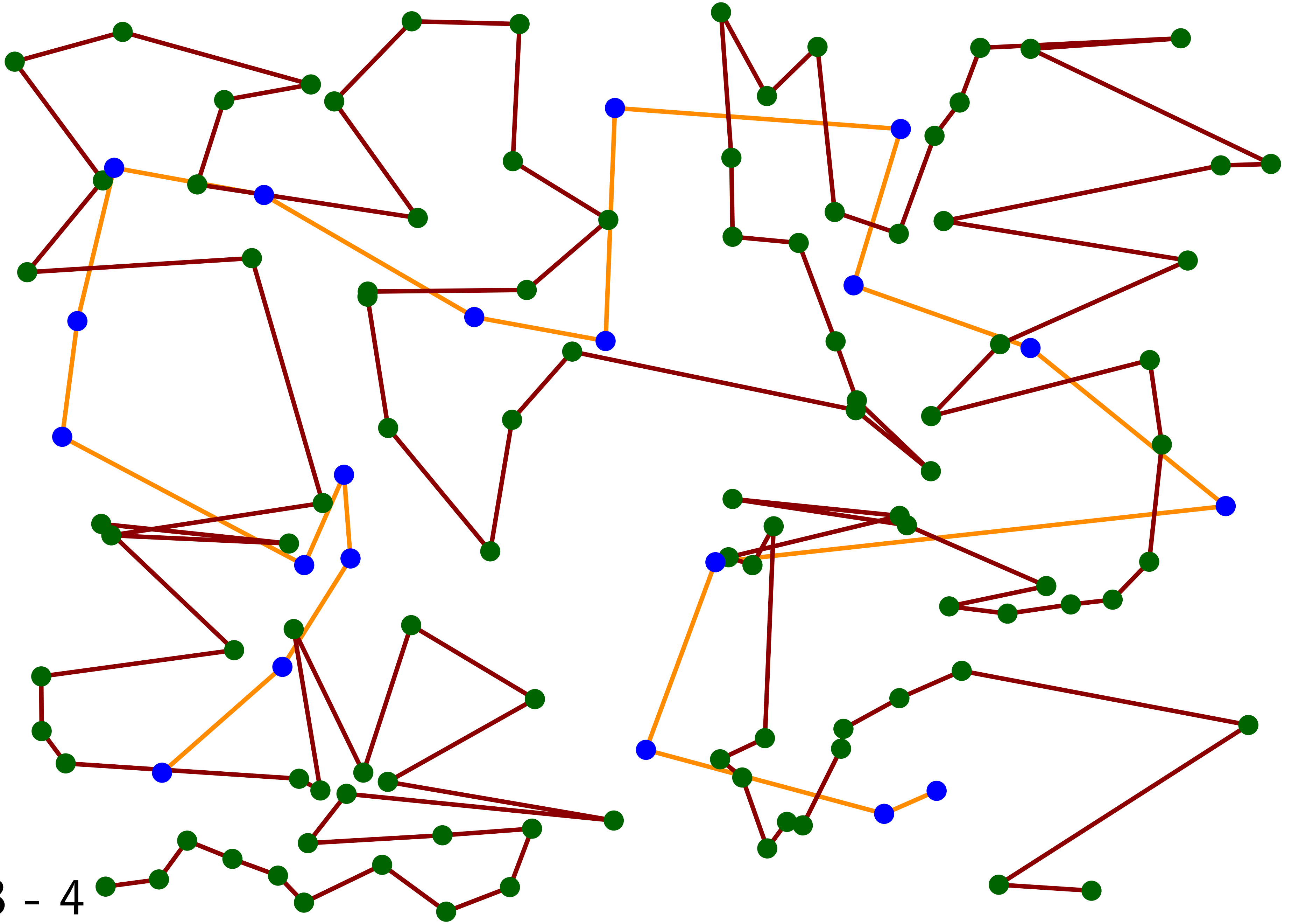
23 - 1



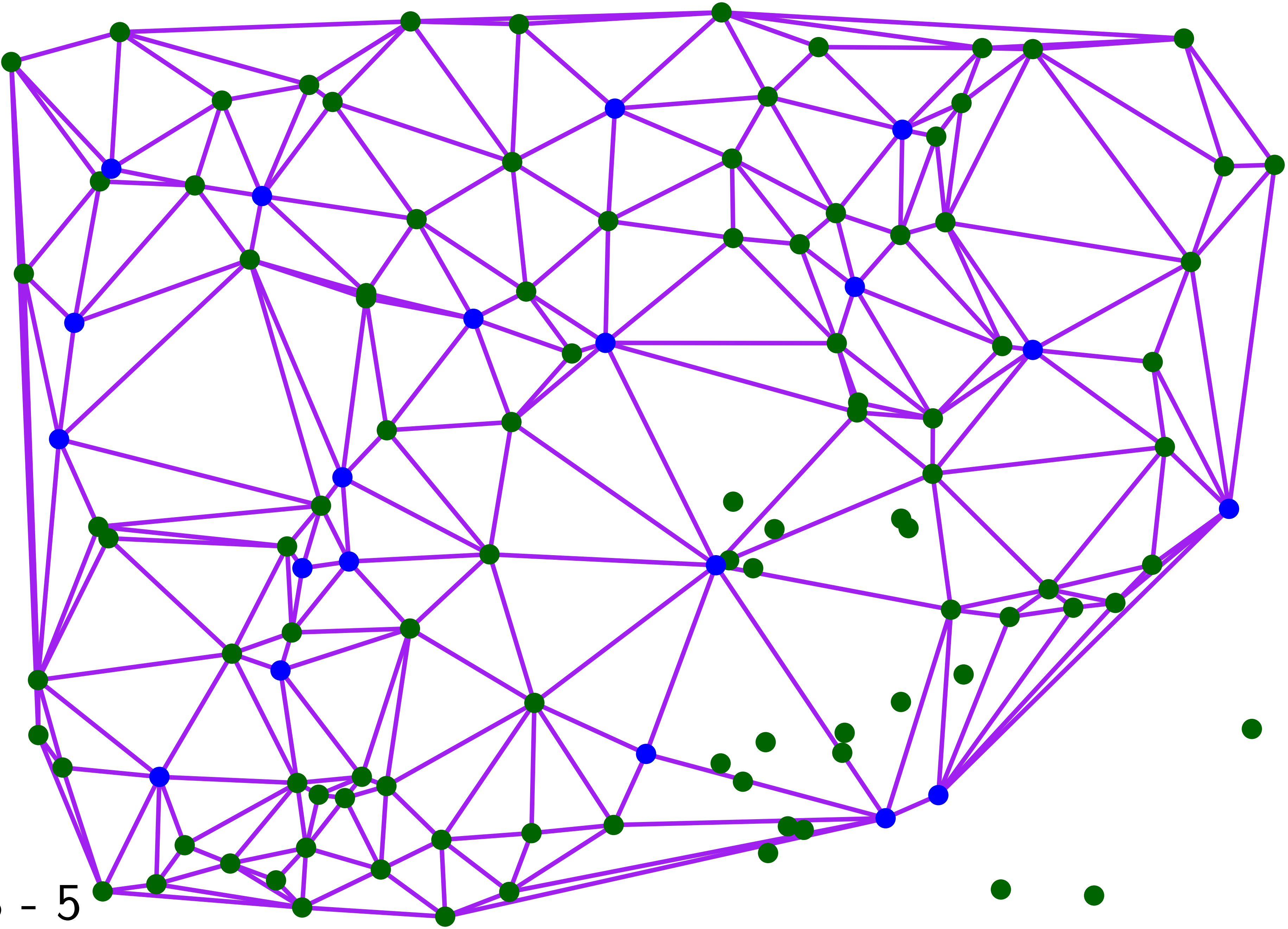
23 - 2



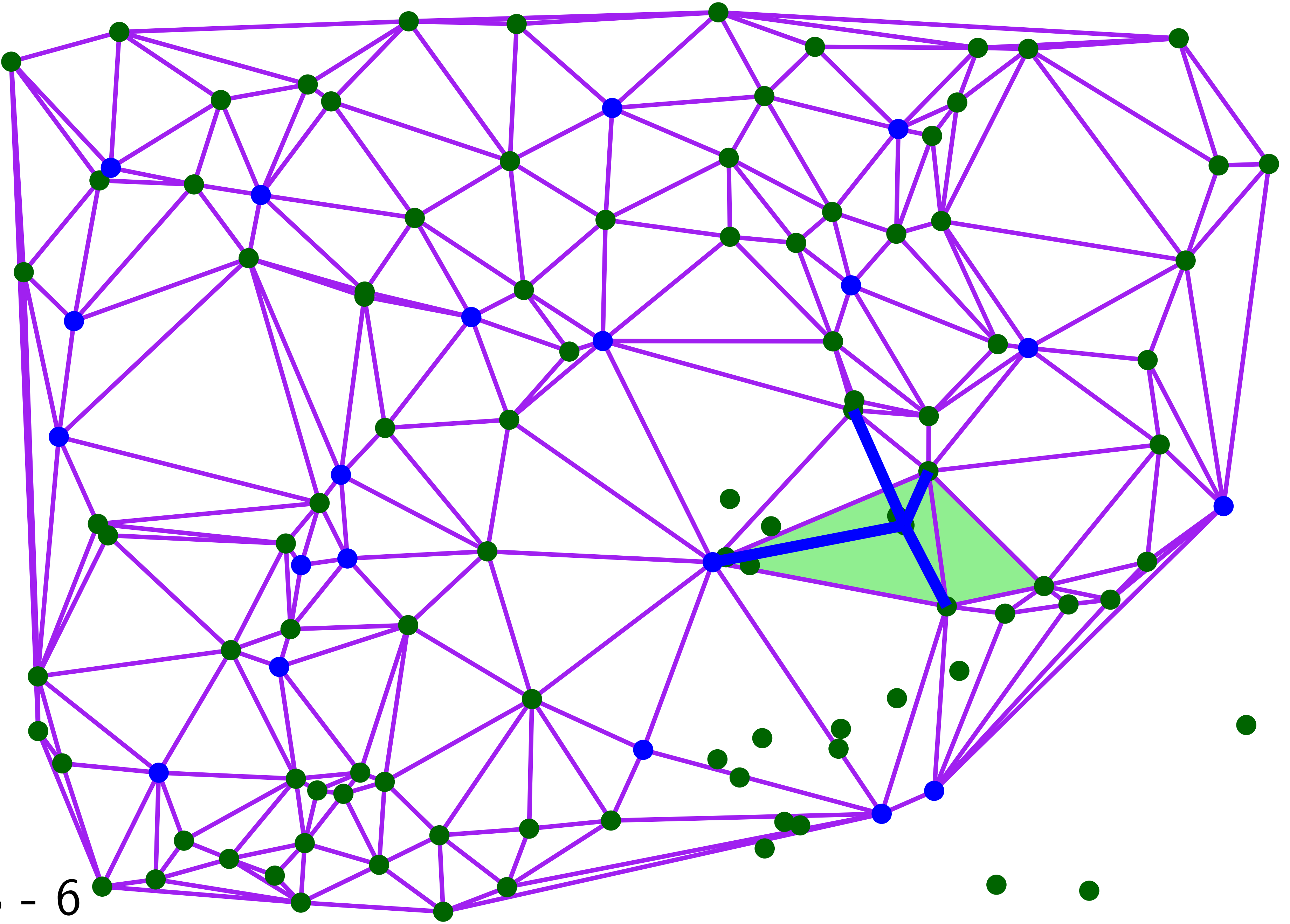
23 - 3



23 - 4

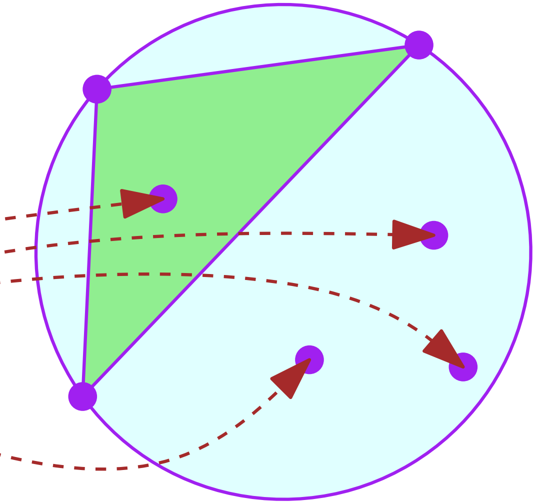


23 - 5

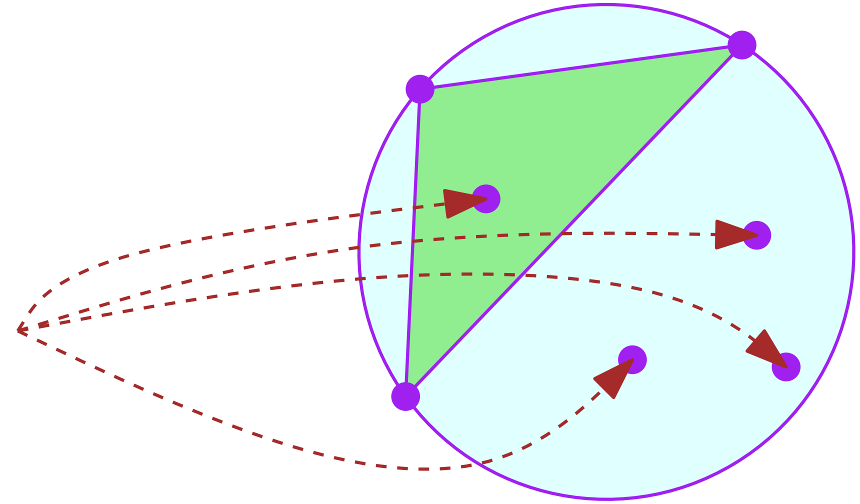


23 - 6

Triangle Δ with j stoppers

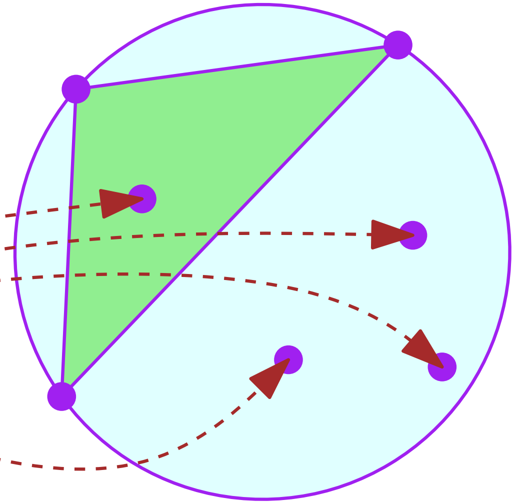


Triangle Δ with j stoppers



Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$

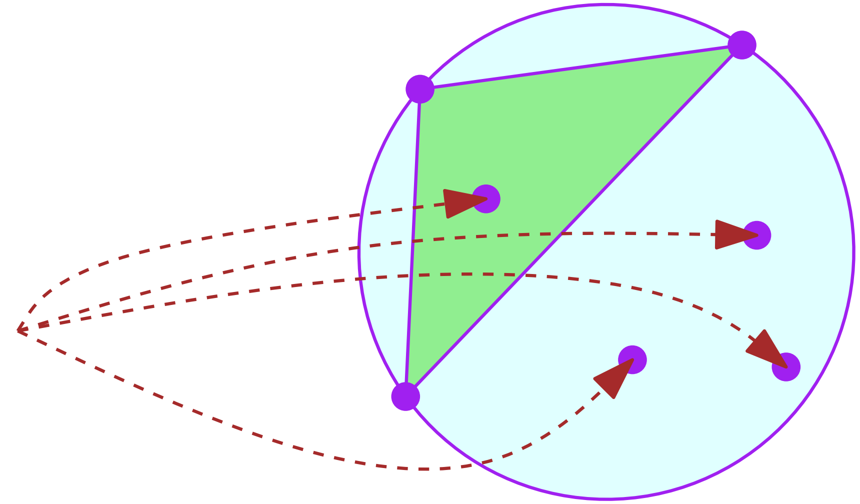
Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

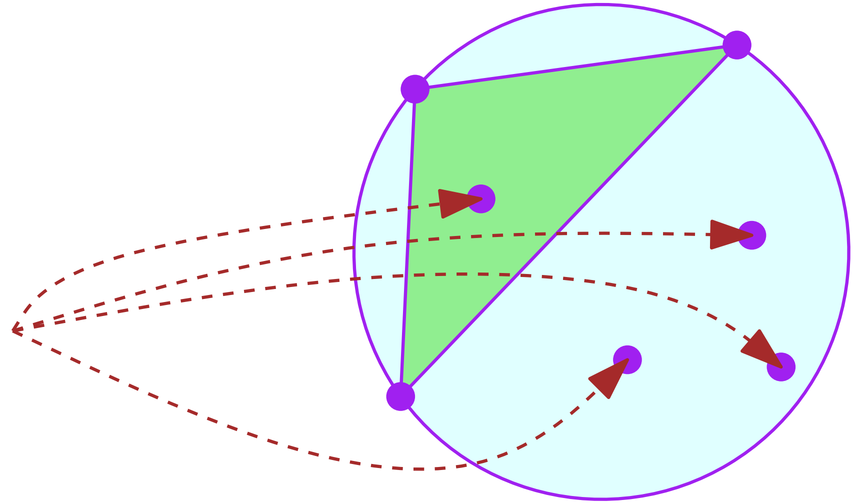
Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

Triangle Δ with j stoppers



Probability that it exists during the construction

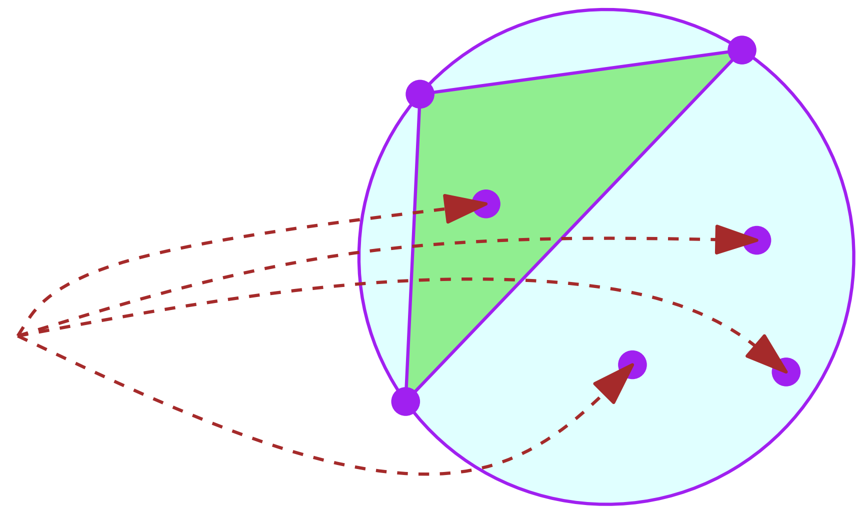
$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\simeq O\left(\sum \frac{nj^2}{j^4}\right) = O(n)$$

Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\simeq O\left(\sum j \frac{nj^2}{j^4}\right) = O(n \log n)$$

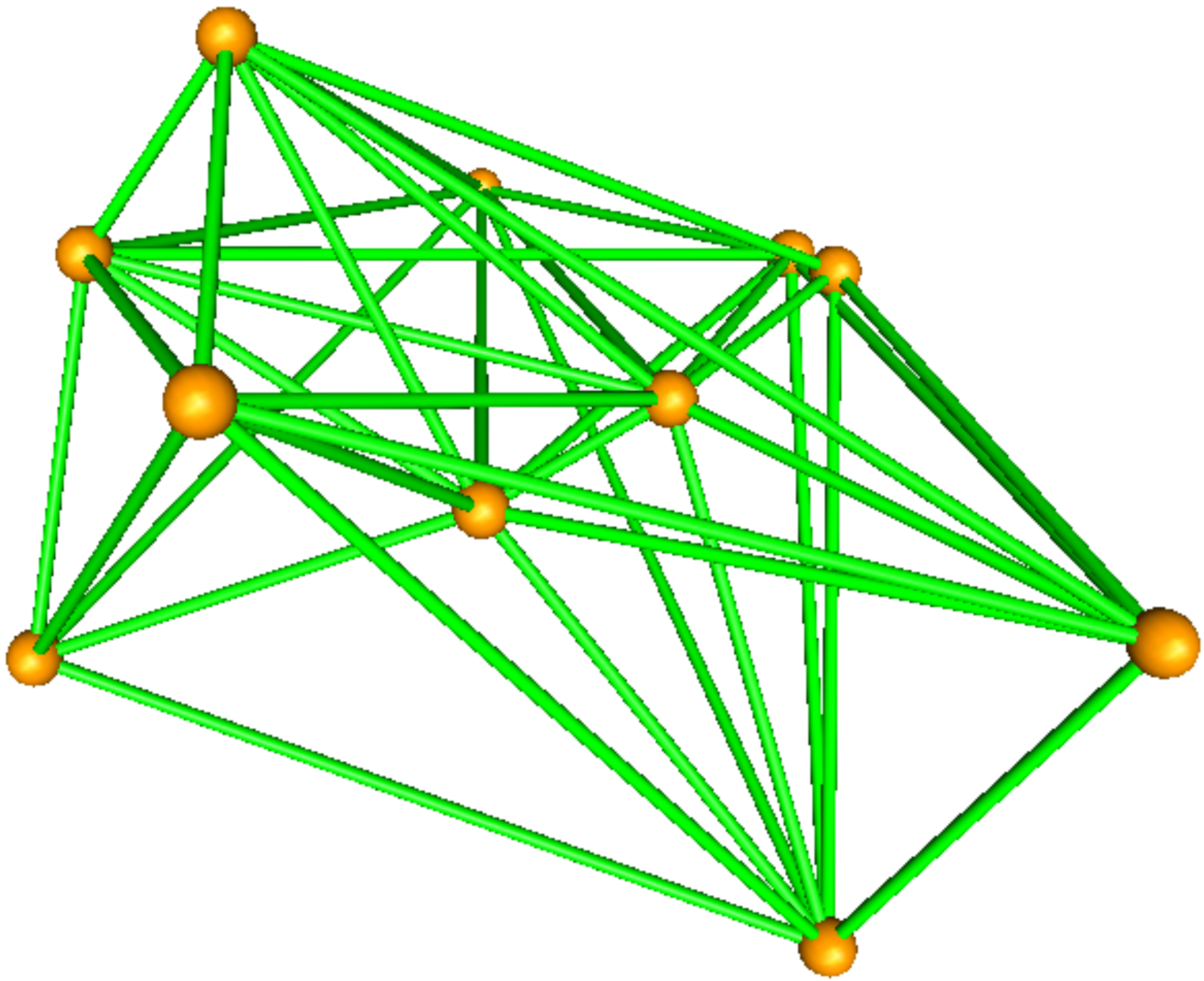
Delaunay 2D 1M random points

locate using Delaunay hierarchy	6 seconds
random order (visibility walk)	157 seconds
x-order	3 seconds
Hilbert order	0.8 seconds
Biased order (Spatial sorting)	0.7 seconds

Delaunay 2D 100K parabola points

locate using Delaunay hierarchy	0.3 seconds
random order (visibility walk)	128 seconds
x-order	632 seconds
Hilbert order	46 seconds
Biased order (Spatial sorting)	0.3 seconds

3D



3D

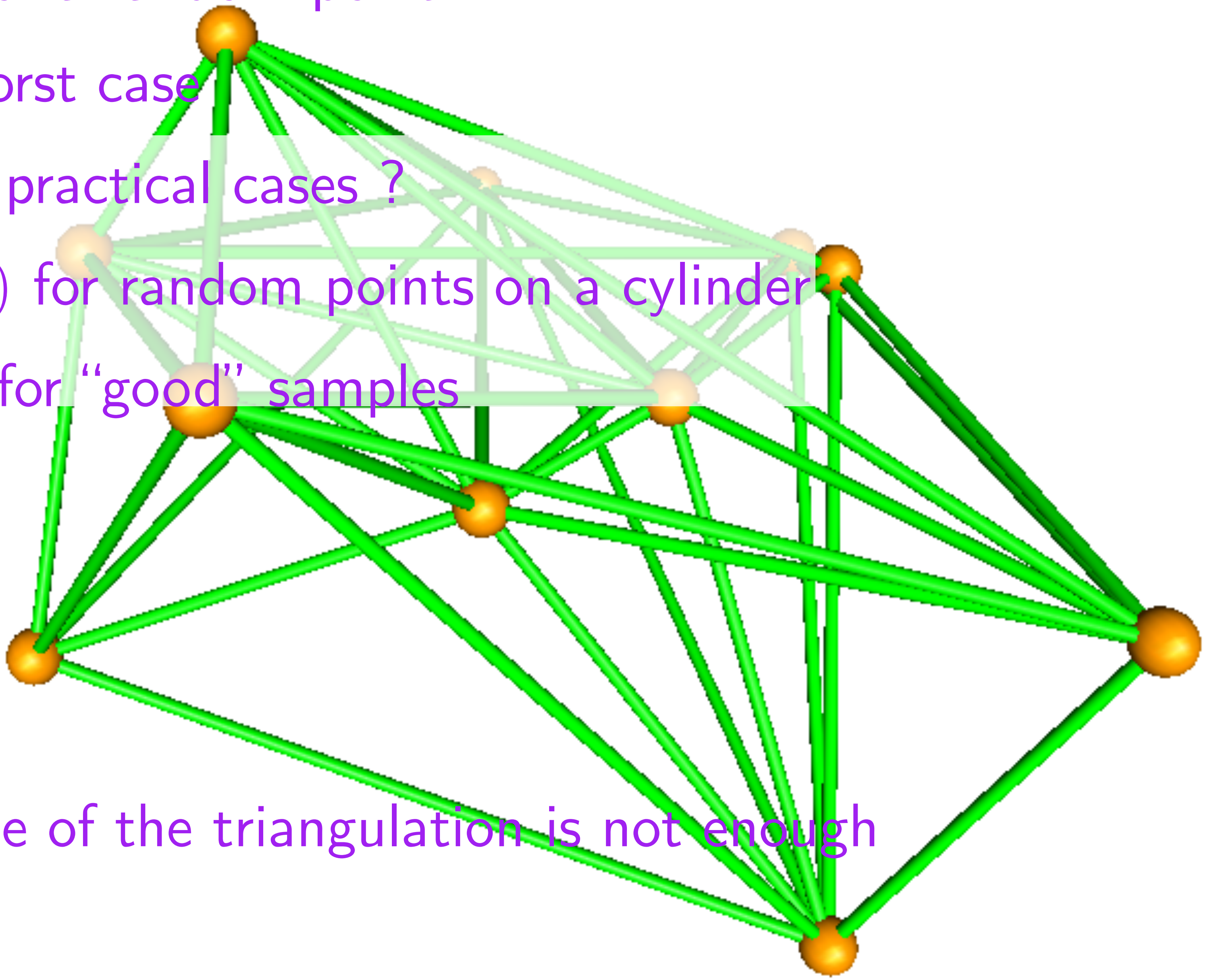
Degree of a random point?

$O(n)$ worst case

$O(1)$ in practical cases ?

$O(\log n)$ for random points on a cylinder

$O(\sqrt{n})$ for "good" samples



Final size of the triangulation is not enough

Randomization

Avoiding point location

Delaunay randomized construction

$O(n)$

Delaunay randomized construction

$O(n)$ + point location

Delaunay randomized construction

$O(n)$ + point location

Use additional information to save on point location

Delaunay randomized construction

$O(n)$ + point location

Use additional information to save on point location

e.g. points are sorted by spatial sort

Delaunay randomized construction

$O(n)$ + point location

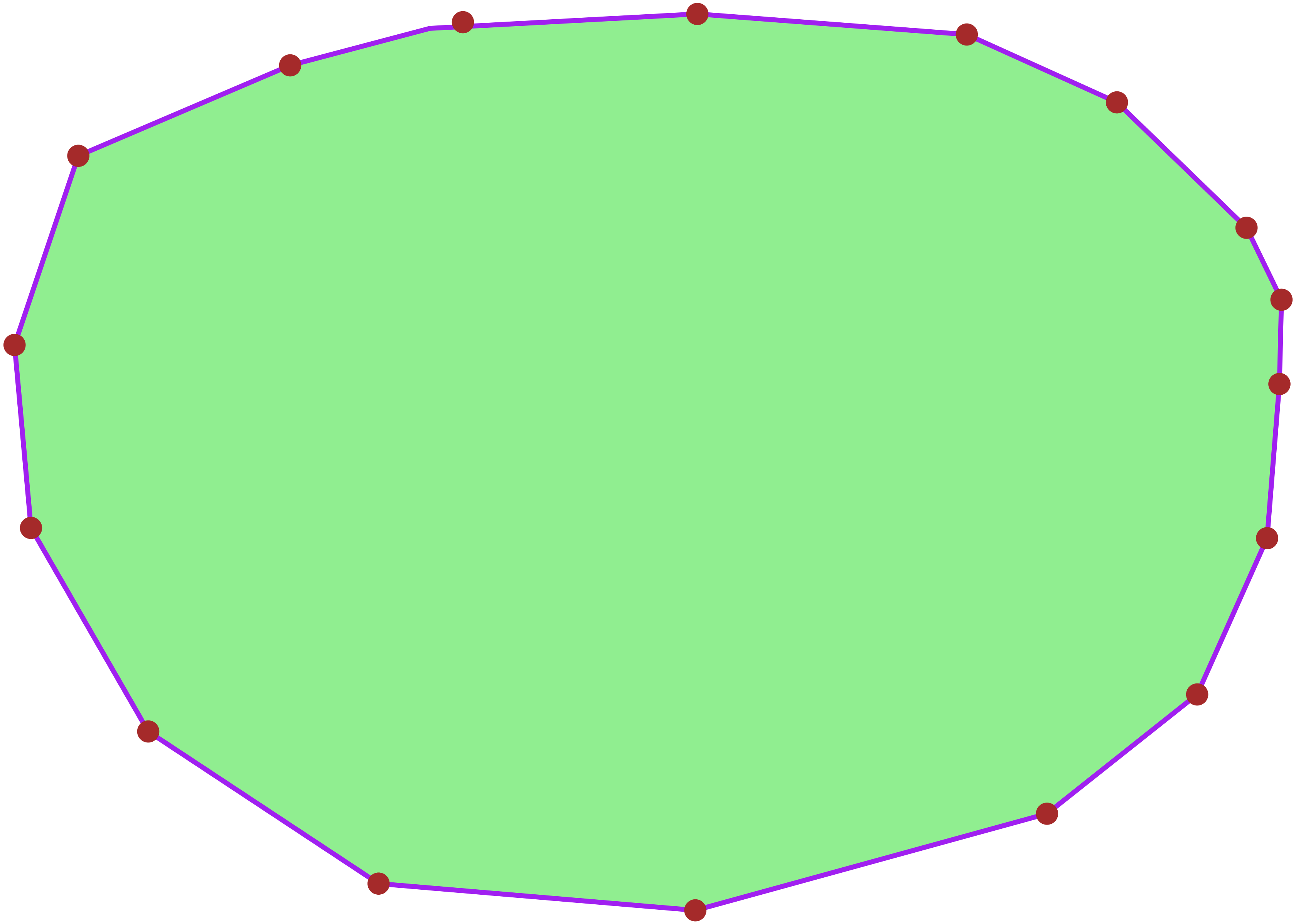
Use additional information to save on point location

e.g. points are sorted by spatial sort

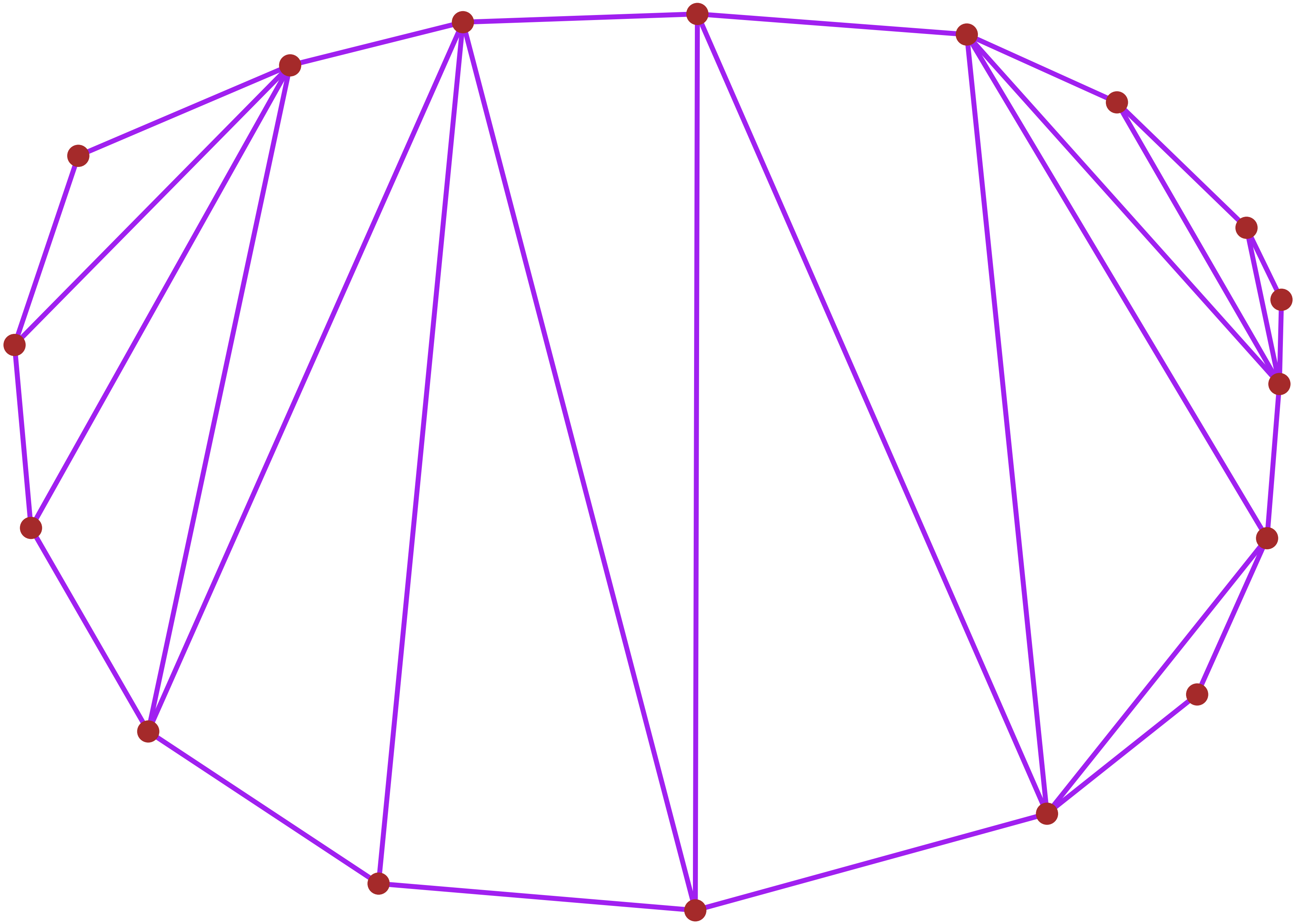
Delaunay of points in convex position

Splitting Delaunay

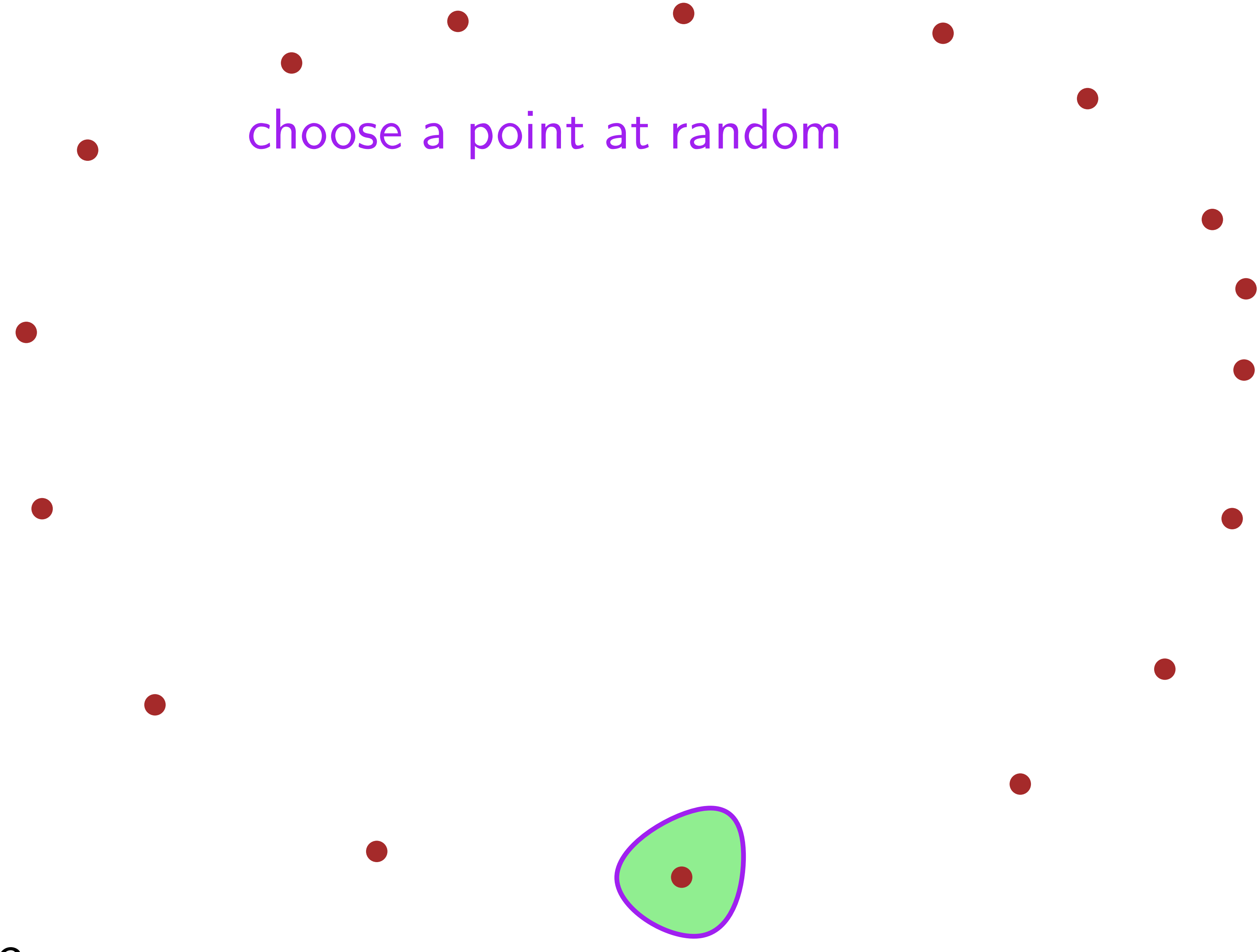
Delaunay of points in convex position



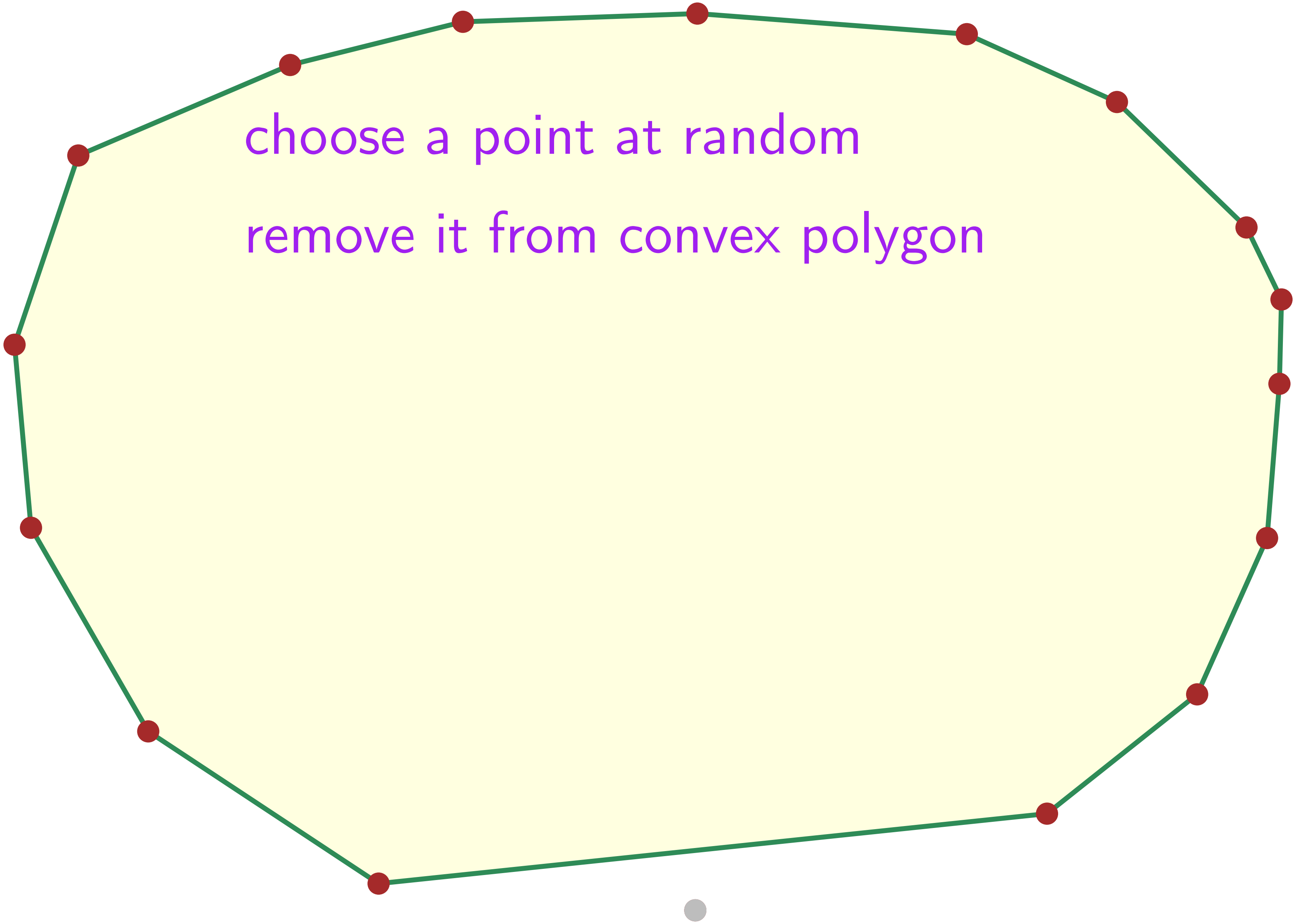
Delaunay of points in convex position



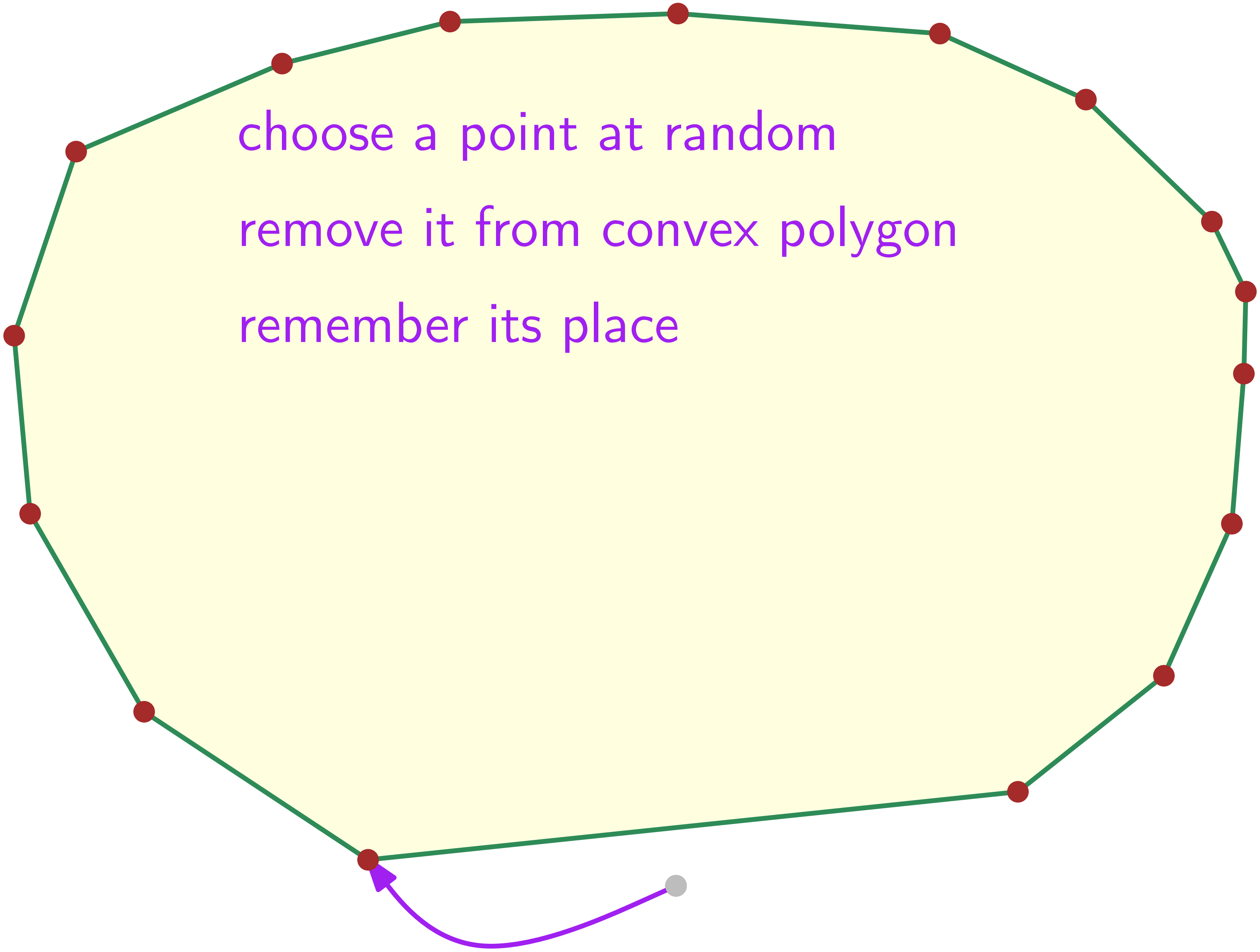
Delaunay of points in convex position



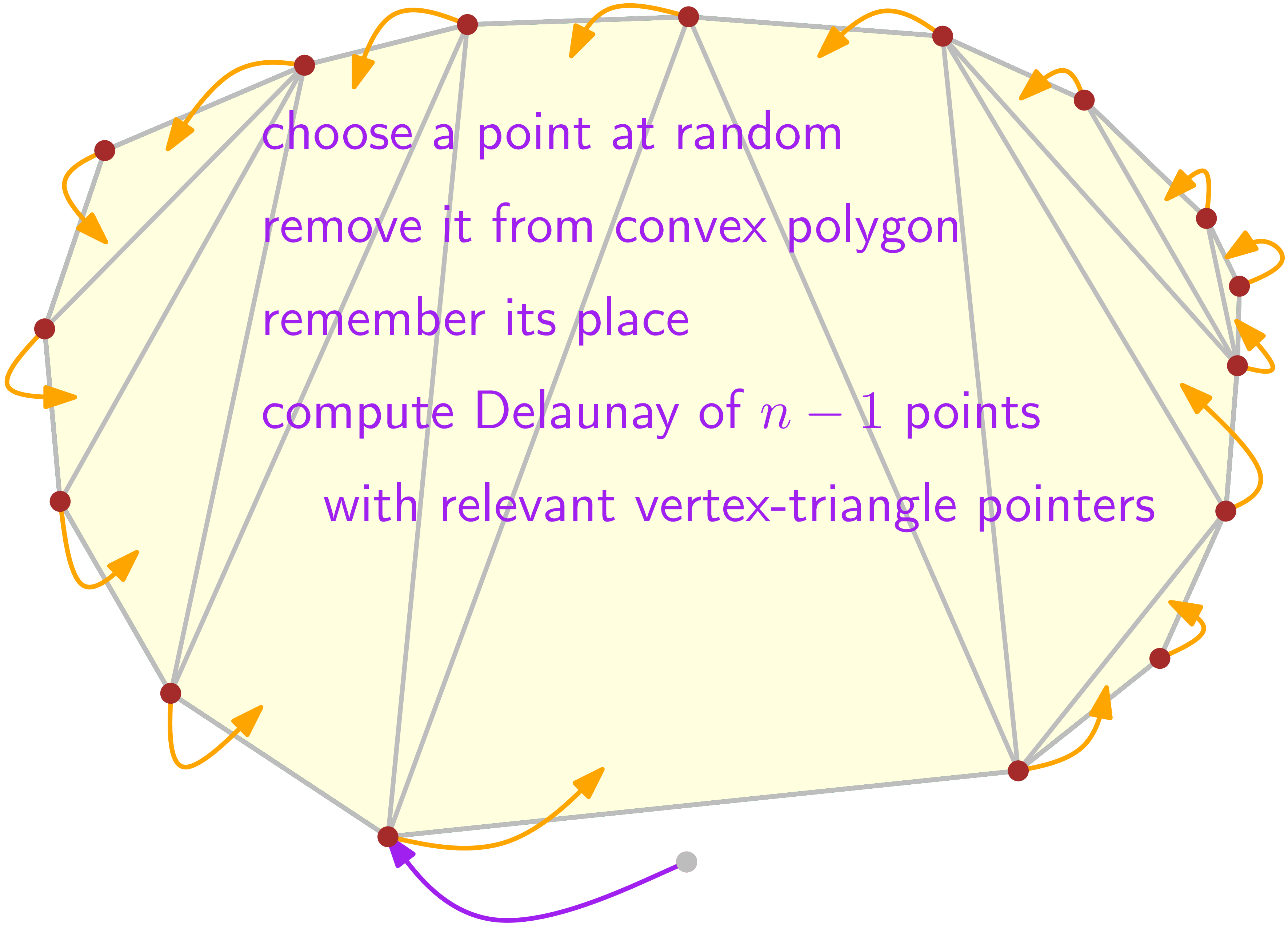
Delaunay of points in convex position



Delaunay of points in convex position



Delaunay of points in convex position



choose a point at random

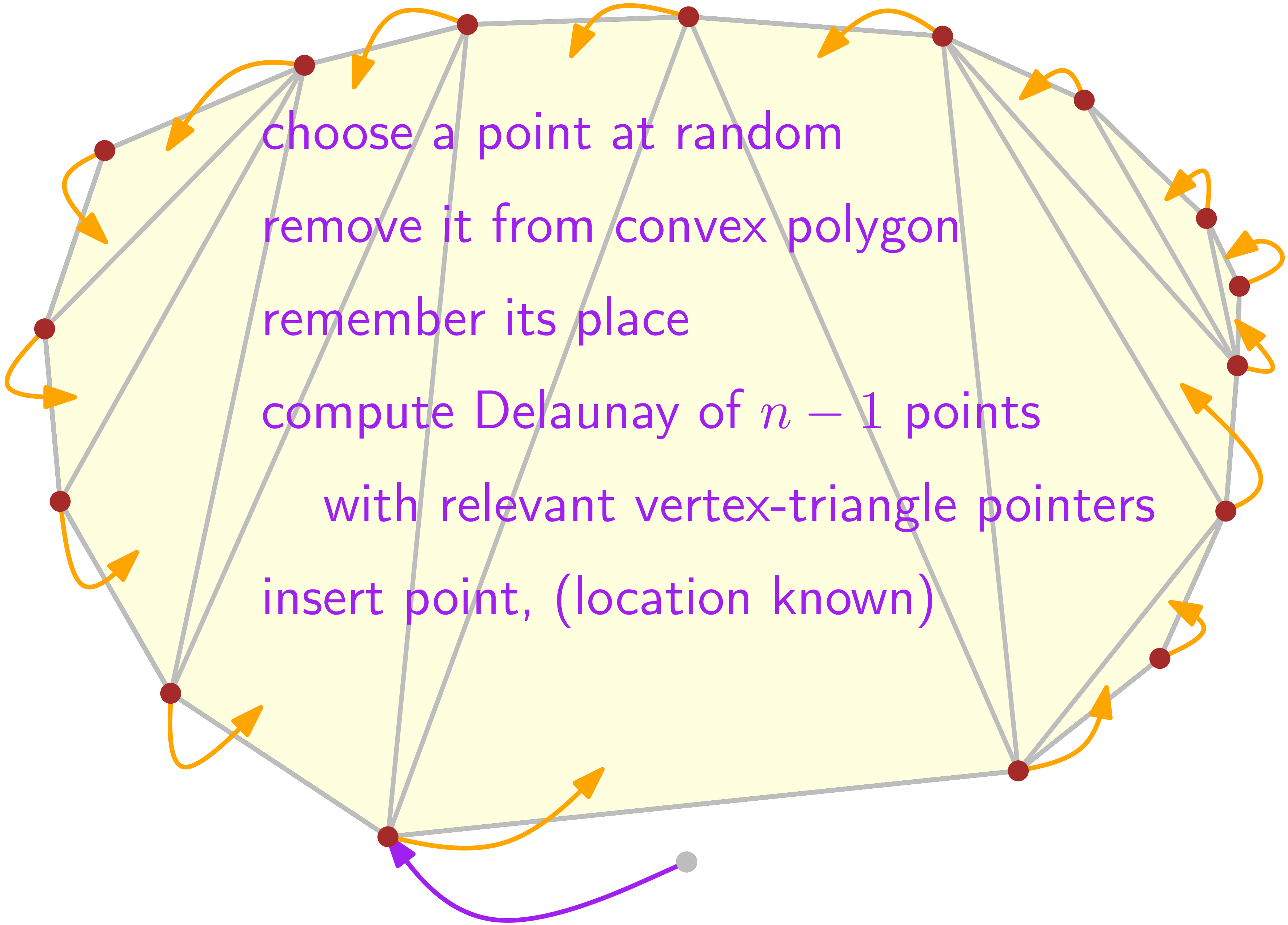
remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

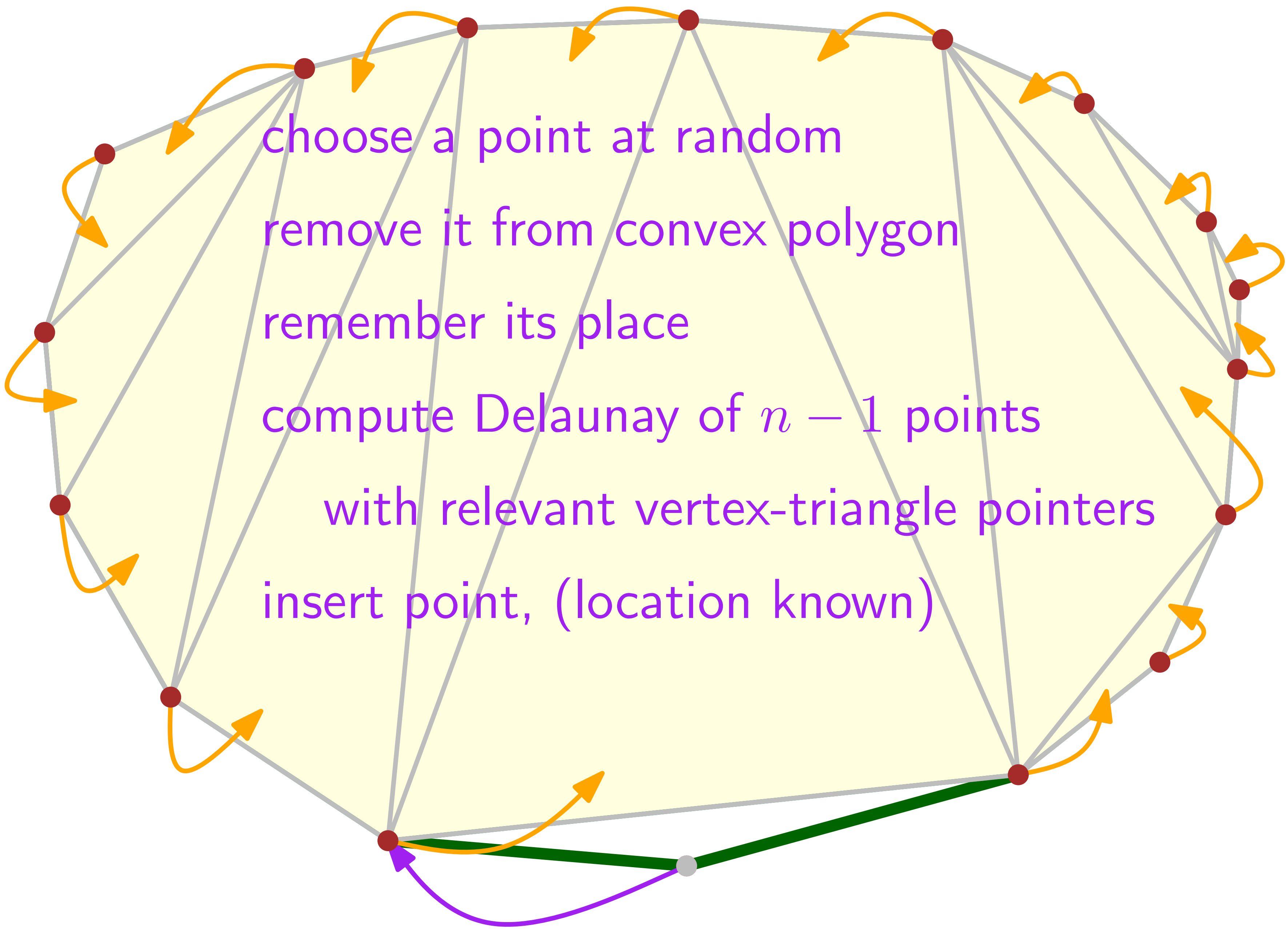
remember its place

compute Delaunay of $n-1$ points

  with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



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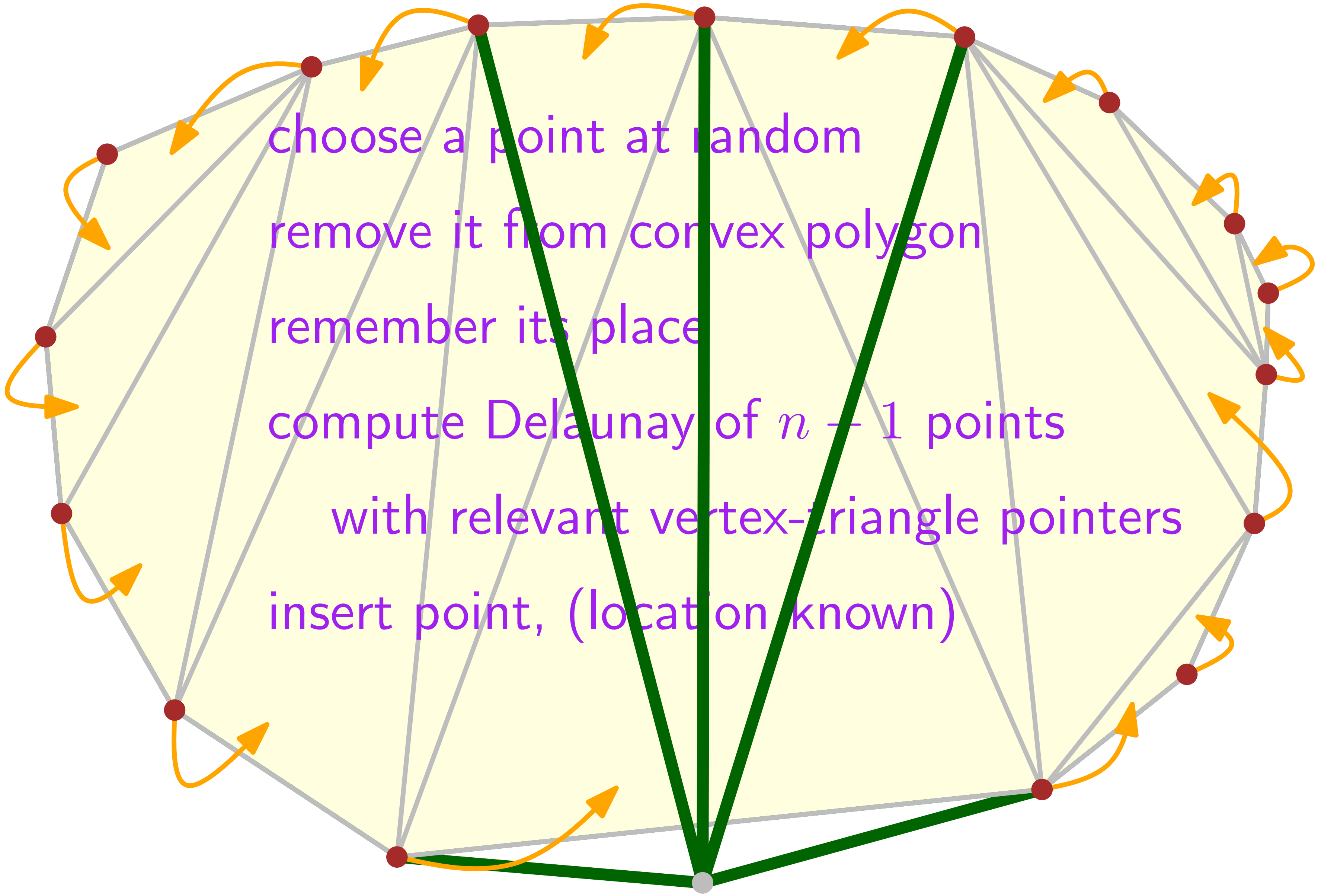
Delaunay of points in convex position



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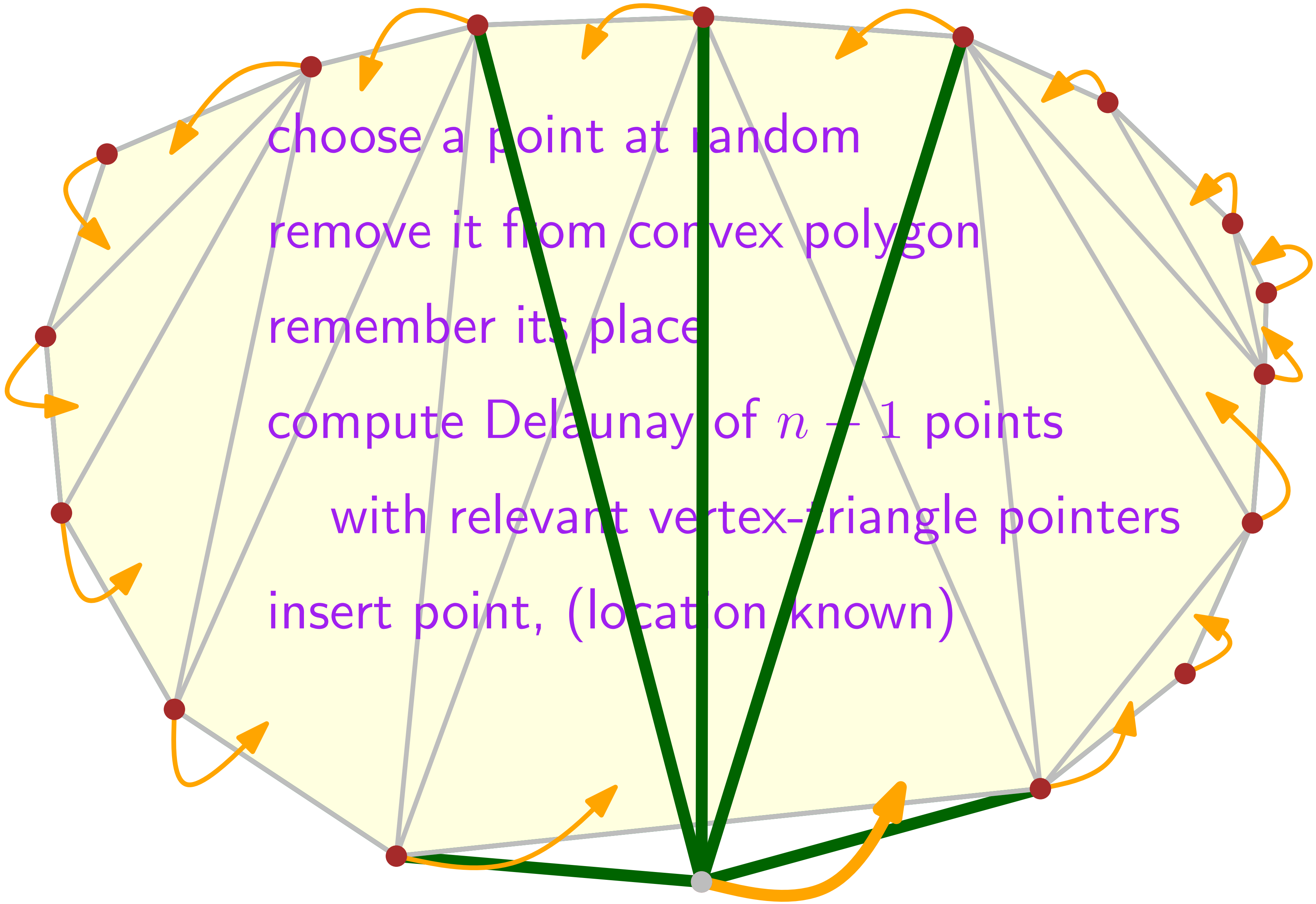
Delaunay of points in convex position



choose a point at random

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

choose a point at random $O(1)$ [model]

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known) $\quad O(d^\circ p)$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known) $O(d^\circ p) = O(1)$

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon $O(1)$

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers $f(n-1)$

insert point, (location known) $O(d^\circ p) = O(1)$

Delaunay of points in convex position

Analysis

- choose a point at random
- remove it from convex polygon
- remember its place

$\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} O(1)$

- compute Delaunay of $n-1$ points
  - with relevant vertex-triangle pointers $\quad f(n-1)$
- insert point, (location known) $\quad O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1)$$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points
with relevant vertex-triangle pointers $\quad f(n-1)$

insert point, (location known) $\quad O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1) \ = O(n)$$

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

$\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\} O(1)$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers $\quad f(n-1)$

insert point, (location known) $\quad O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1) \; = O(n)$$

[Chew 86]

Randomization

Randomization

Randomized incremental constructions

Simple algorithms

non trivial analysis

good complexities

efficient in practice

Randomization

Randomized incremental constructions

Simple algorithms

non trivial analysis

good complexities

efficient in practice

CGAL

Delaunay hierarchy

Spatial sorting

Randomization

Randomized incremental constructions

Simple algorithms

non trivial analysis

good complexities

efficient in practice

CGAL

Delaunay hierarchy

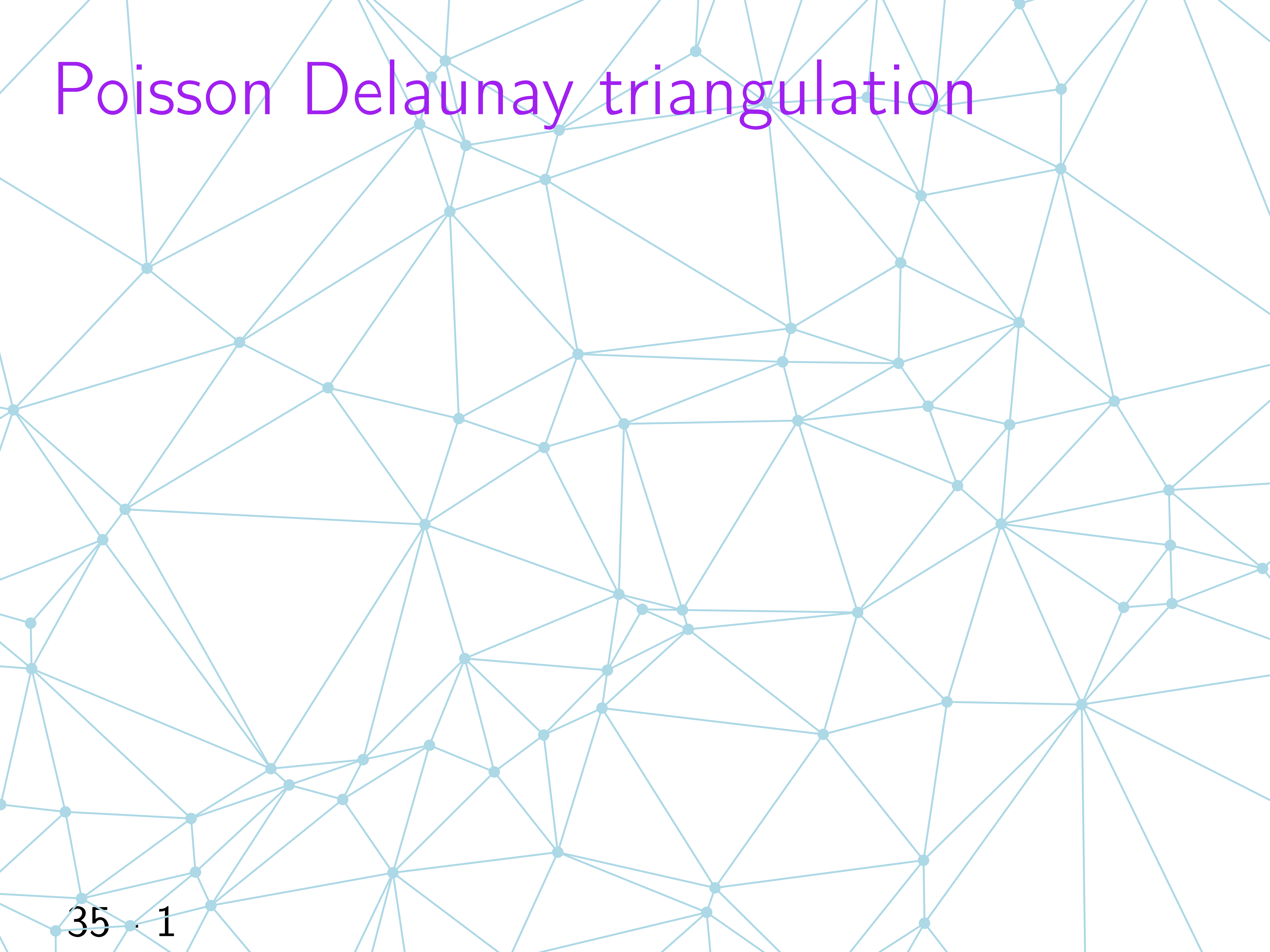
Spatial sorting

Other tools

divide and conquer

ϵ nets   Good sample with high probability

Poisson Delaunay triangulation



Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Very convenient

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

$$\mathbb{P}\left[|X \cap A| = 0\right] = e^{-\mathrm{vol}(A)}$$

$$\mathbb{E}\left[|X \cap A|\right] = \sum_{0}^{\infty} k \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)} = \mathrm{vol}(A)$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

$$\mathbb{E}\left[\sum_{q\in X}\mathbb{1}_{[P(X,q)]}\right]$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum ⟶ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right] = n\int_{\mathbb{R}^2} \mathbb{P}\left[P(X\cup\{q\},q)\right] \mathrm{d}q$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right] = n \int_{\mathbb{R}^2} \mathbb{P}\left[P(X \cup \{q\}, q)\right] \mathrm{d}q$$

e.g.,

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[NN_X(0)=q]}\right]$$

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$$= n\int_{\mathbb{R}^2} e^{-n\pi\|q\|^2} \mathrm{d}q$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

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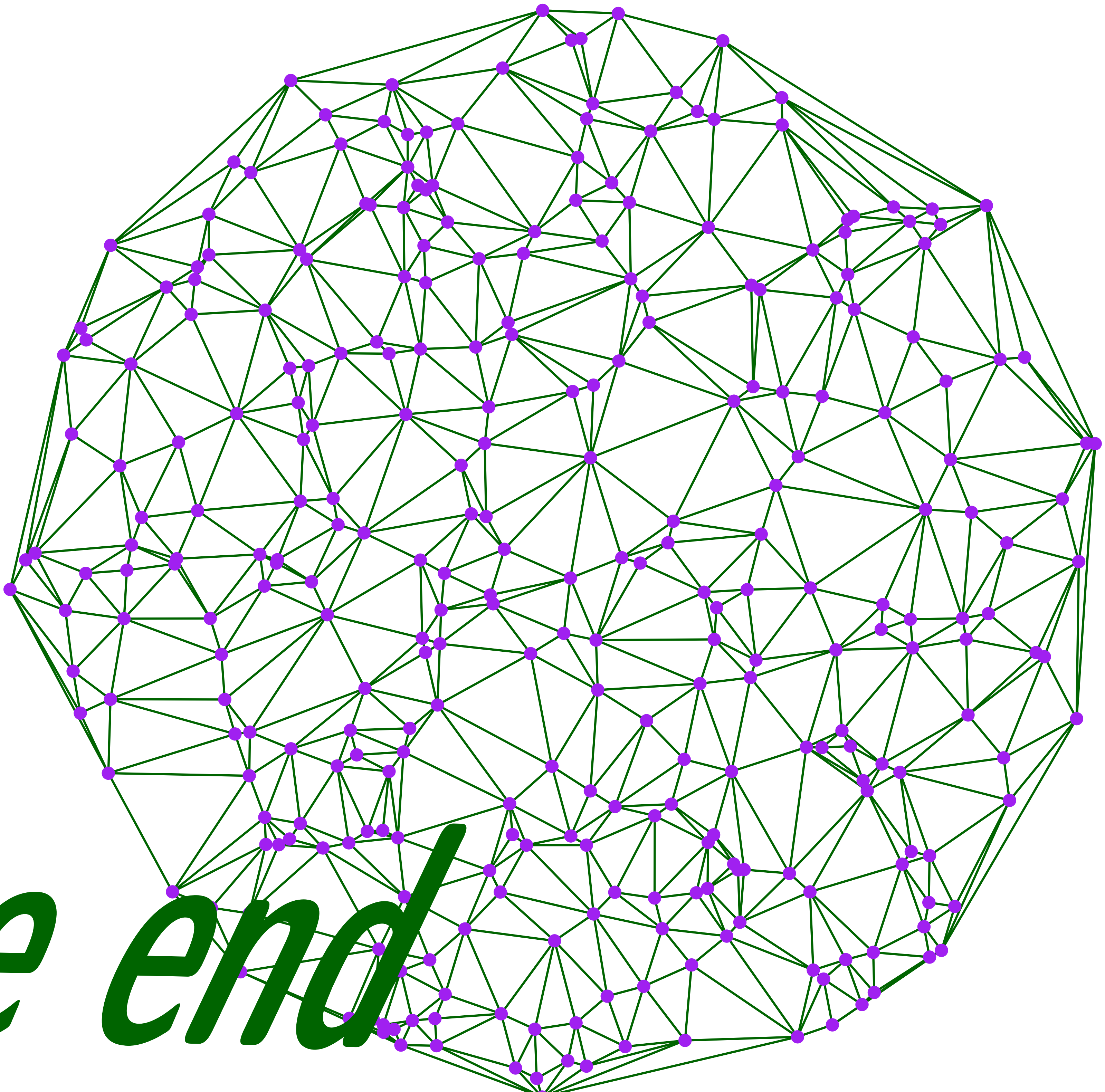
$$= n\int_{\mathbb{R}^2} e^{-n\pi\|q\|^2}\,\mathrm{d}q$$

$$= n\int_0^{2\pi}\int_0^{\infty} e^{-n\pi r^2} r\mathrm{d}\theta\,\mathrm{d}r = n\times 2\pi\times\frac{1}{2n\pi} = 1$$

☺



The end