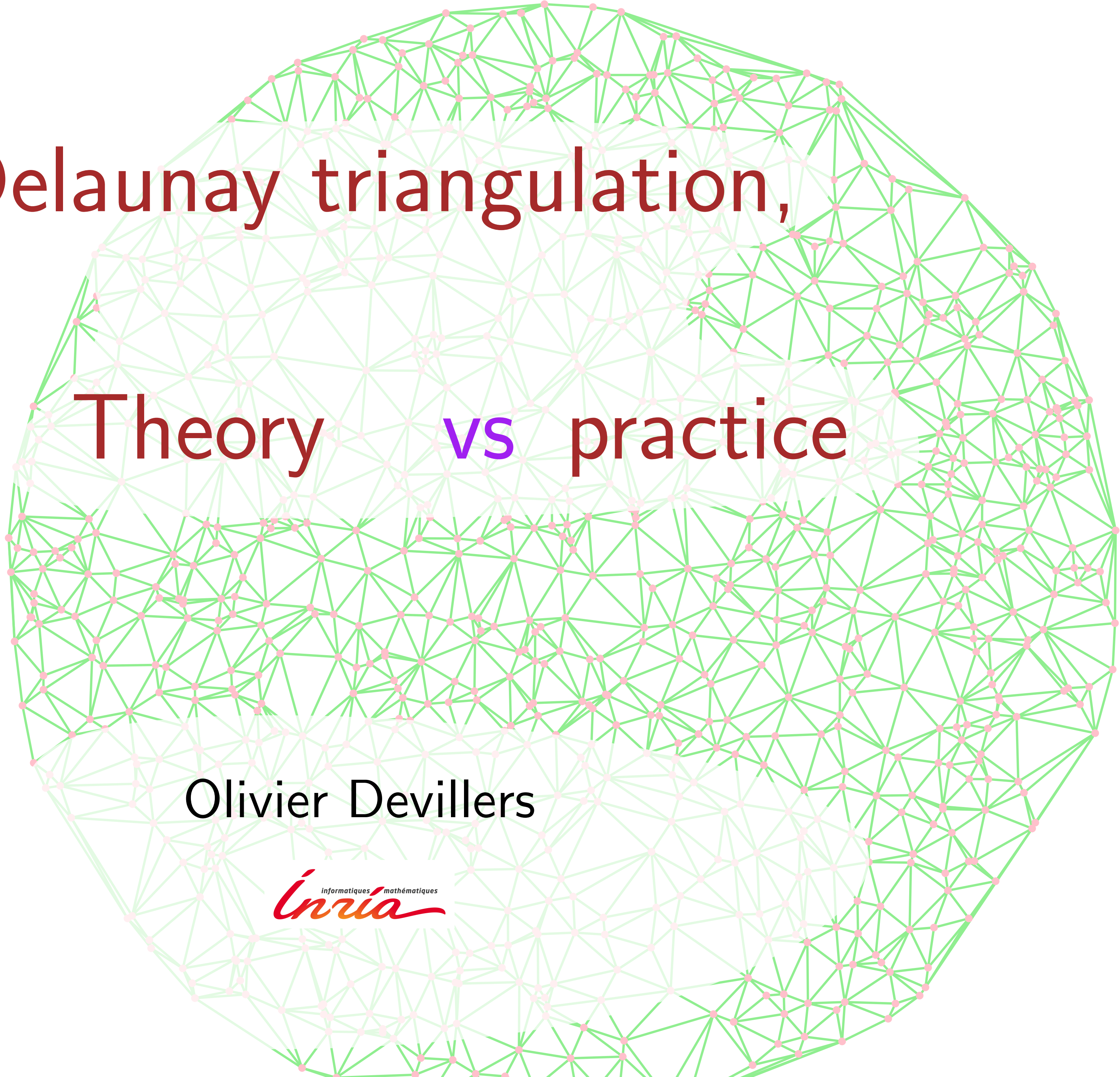


Delaunay triangulation,

Theory vs practice

Olivier Devillers

Delaunay triangulation,

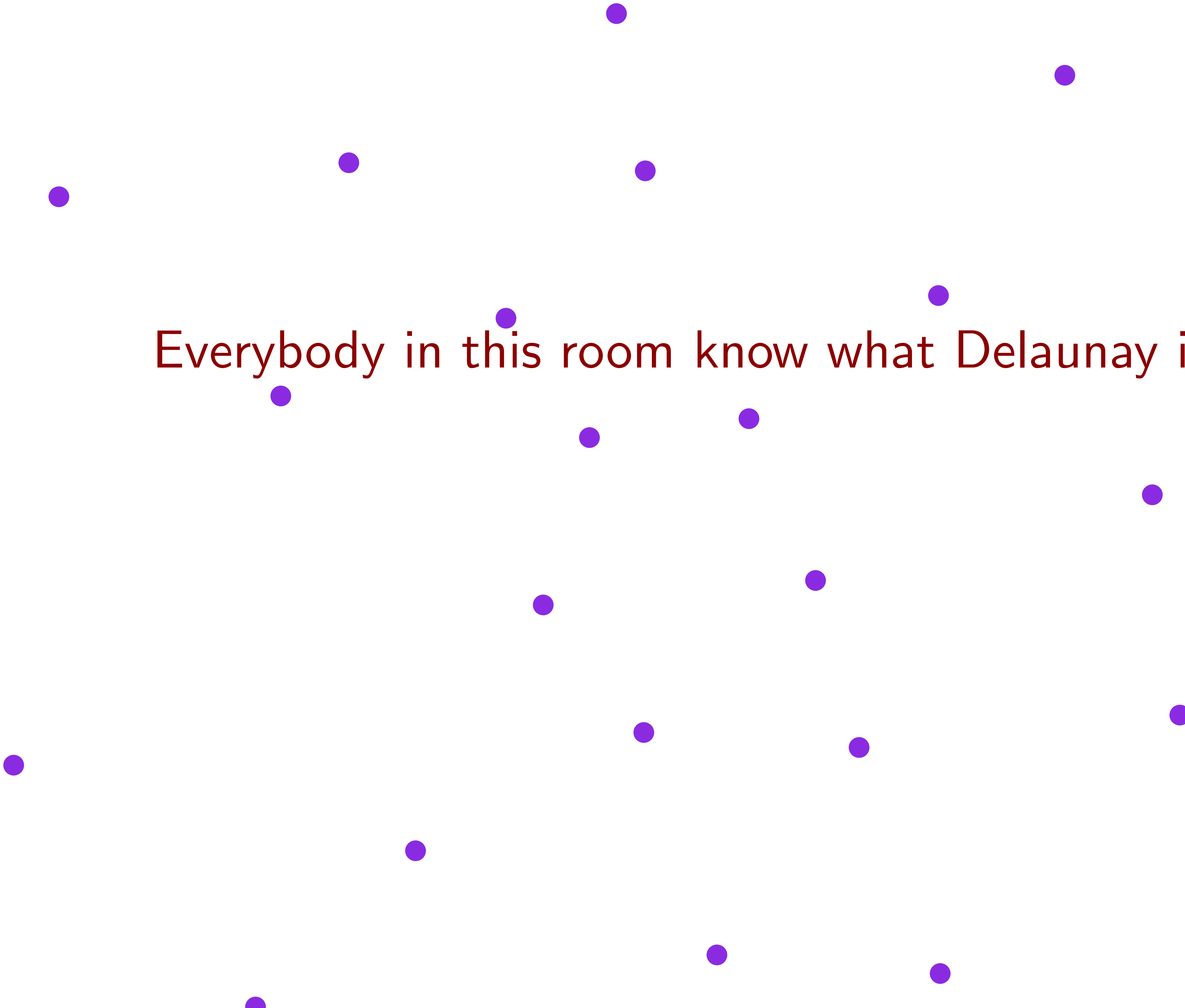
Theory vs practice

http://www.inria.fr/sophia/members/Olivier.Devillers/EuroCG2012/

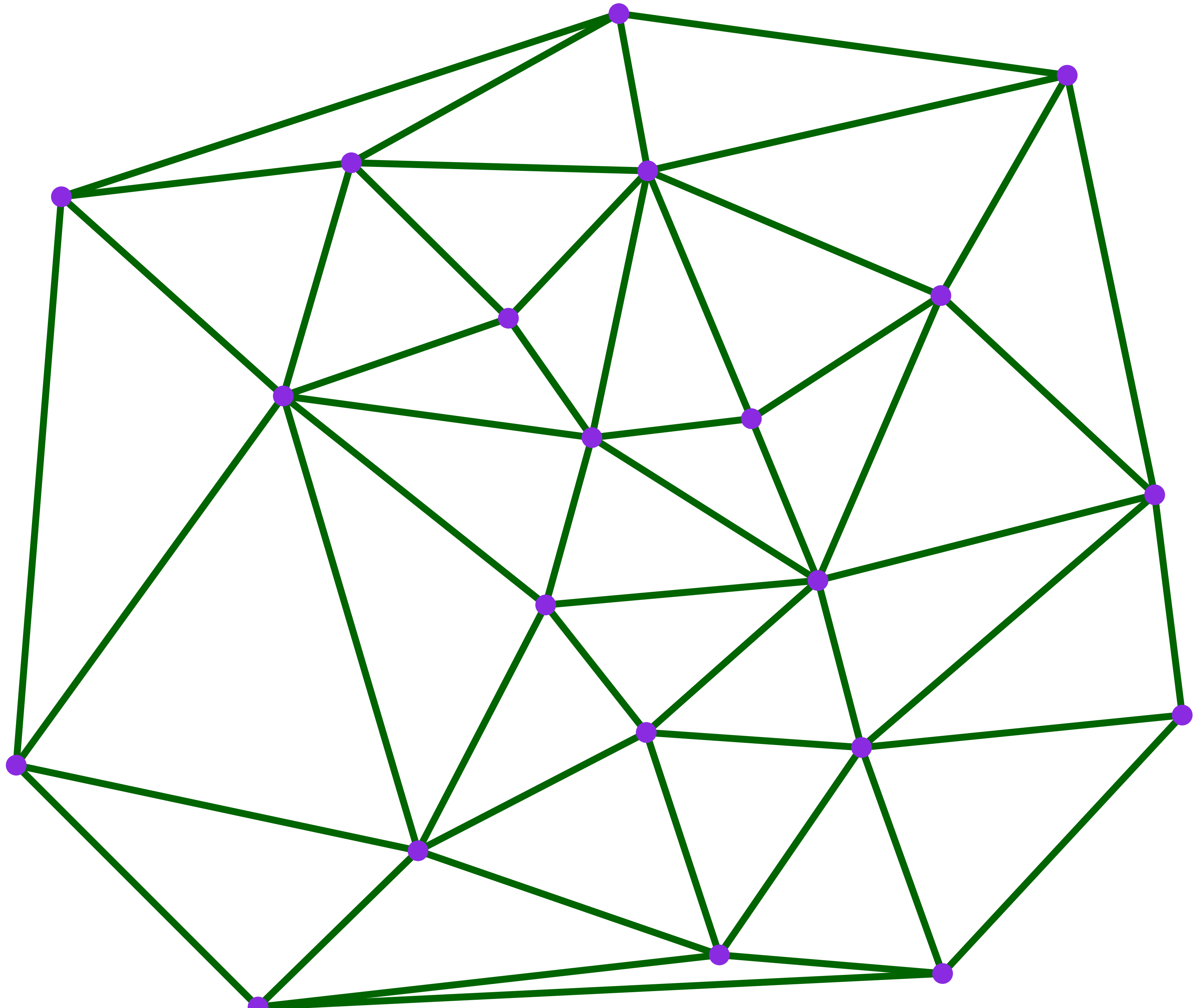
Bibliographical notes

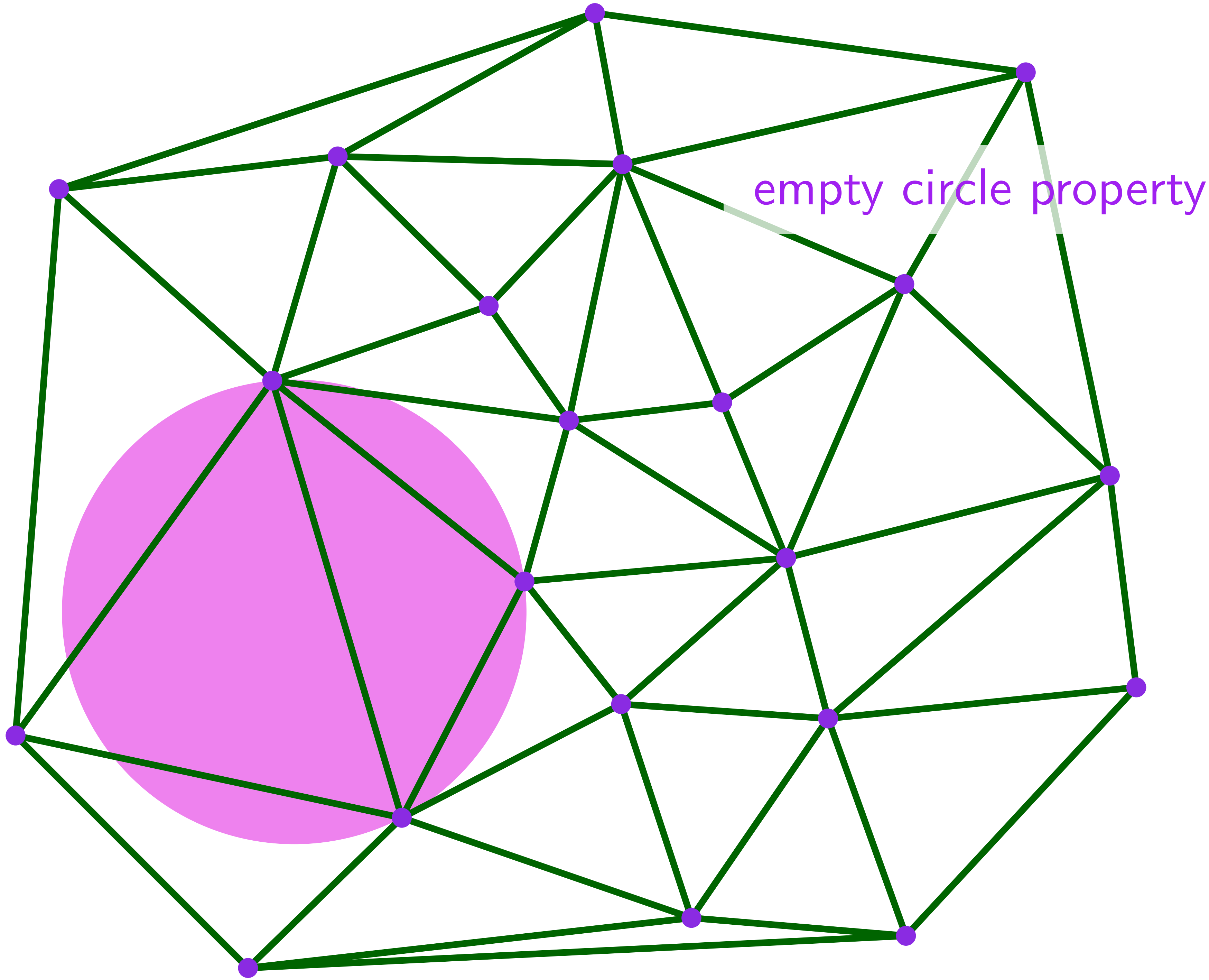
[References, given in the abstract]

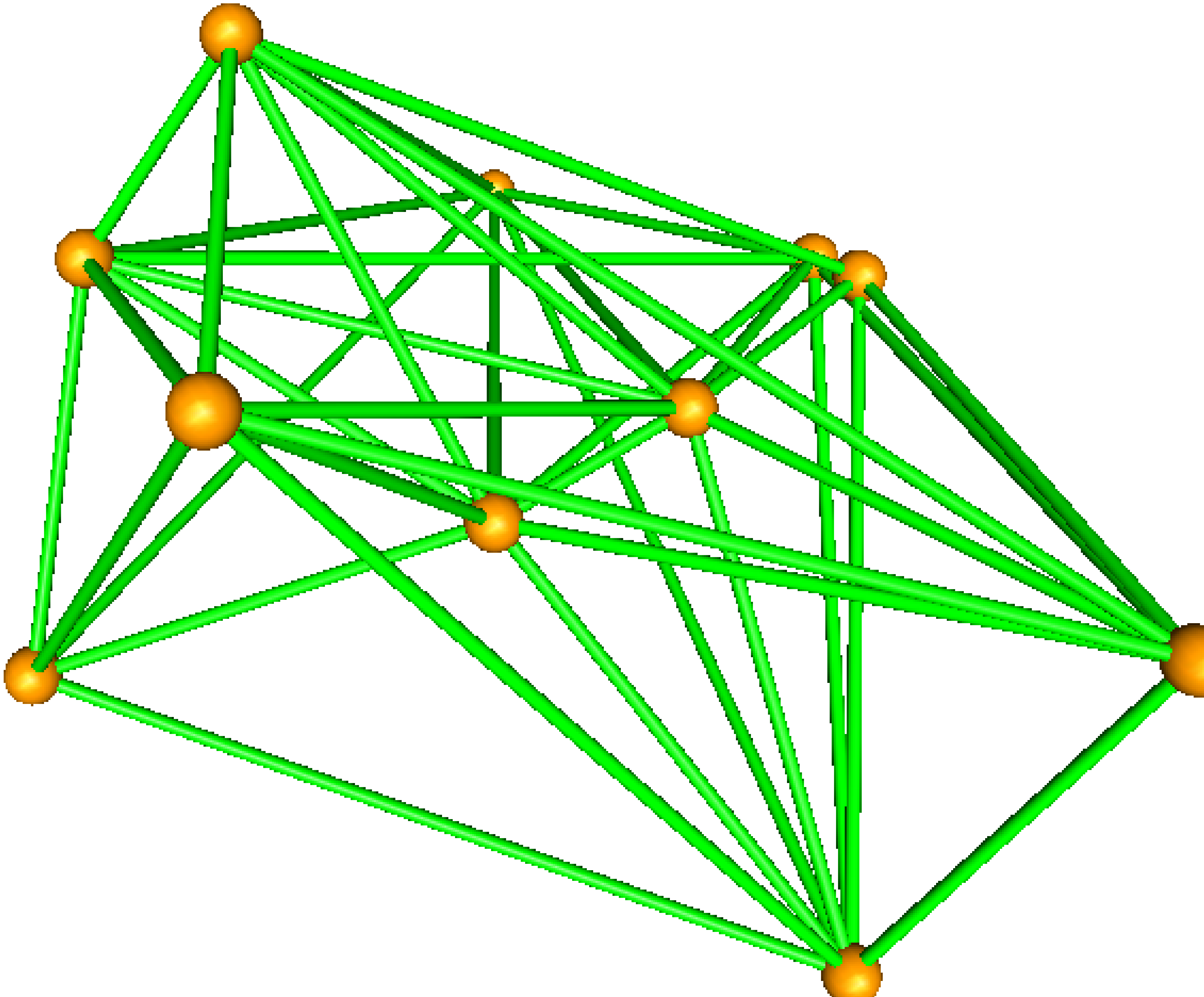
Benchmarks (`src code`)

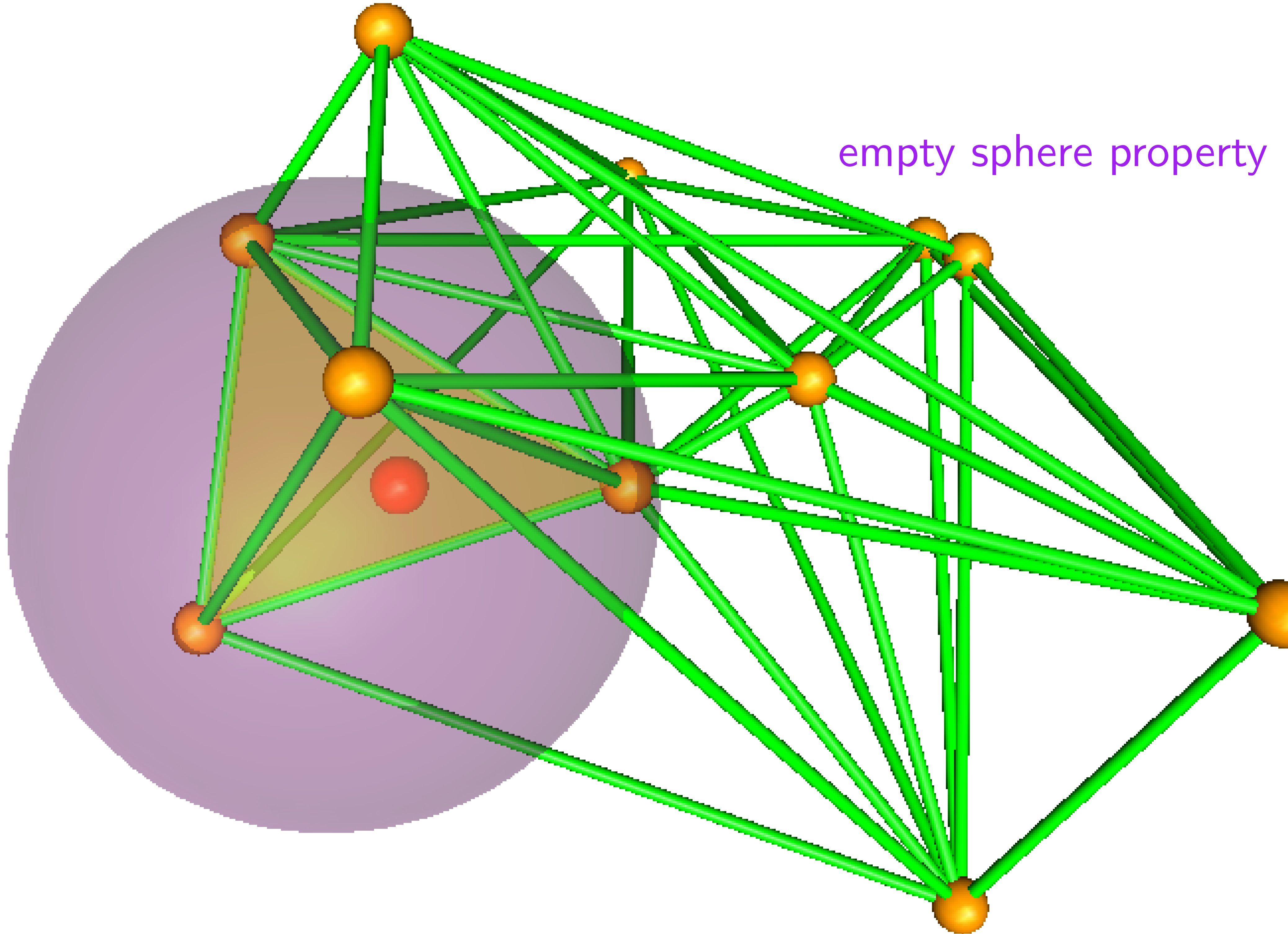


Everybody in this room know what Delaunay is!




empty circle property





empty sphere property

A (partial) history of Delaunay algorithms

(and of computational geometry)

A (partial) history of Delaunay algorithms

(and of computational geometry)

Earlier algorithms [1970...]

A (partial) history of Delaunay algorithms
(and of computational geometry)

Earlier algorithms [1970. . .]

Incremental	2D/3D
Gift wrapping	2D

A (partial) history of Delaunay algorithms

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Earlier algorithms [1970. . .]

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Gift wrapping	2D

Non optimal

"simple"

actually coded

linked to applications: meshes, reconstruction

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A (partial) history of Delaunay algorithms
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Divide & conquer 2D

Plane sweep 2D

A (partial) history of Delaunay algorithms
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Worst case algorithms [1980...]

Randomized algorithms [1990...]

A (partial) history of Delaunay algorithms

(and of computational geometry)

Earlier algorithms [1970...]

Worst case algorithms [1980...]

Randomized algorithms [1990...]

- Delaunay tree
- Clarkson & Shor
- Dynamic updates (history graph)
- Delaunay hierarchy
- Spatial sorting (BRIO)

A (partial) history of Delaunay algorithms

(and of computational geometry)

Earlier algorithms [1970...]

Worst case algorithms [1980...]

Randomized algorithms [1990...]

Robustness issues [1995...]

A (partial) history of Delaunay algorithms
(and of computational geometry)

Earlier algorithms [1970...]

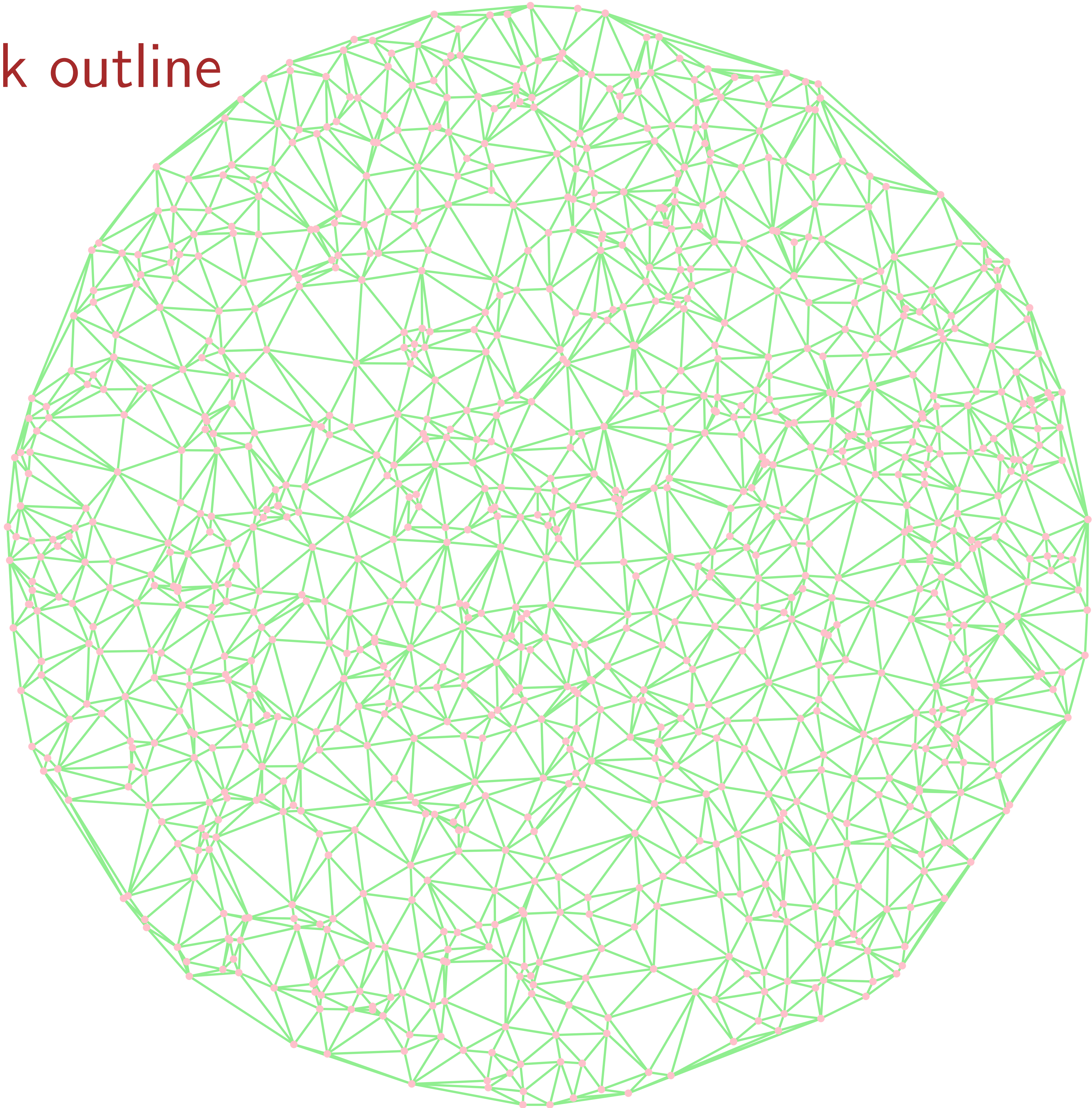
Worst case algorithms [1980...]

Randomized algorithms [1990...]

Robustness issues [1995...]

- Properties checking → VRONI
- Exact computation paradigm → TRIANGLE, CGAL

Talk outline



Talk outline

One word on robustness issues

Basic incremental algorithm

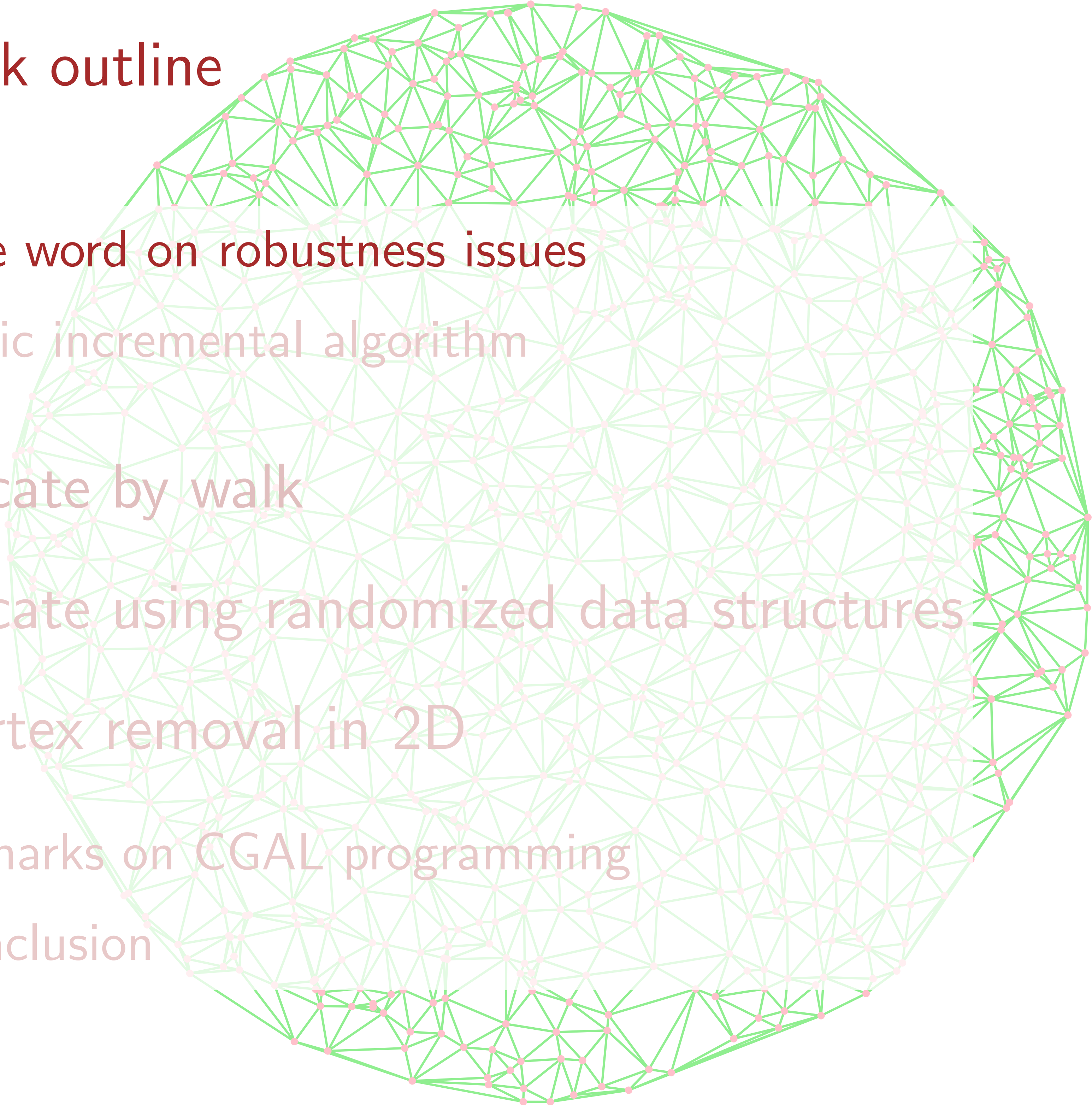
Locate by walk

Locate using randomized data structures

Vertex removal in 2D

Remarks on CGAL programming

Conclusion



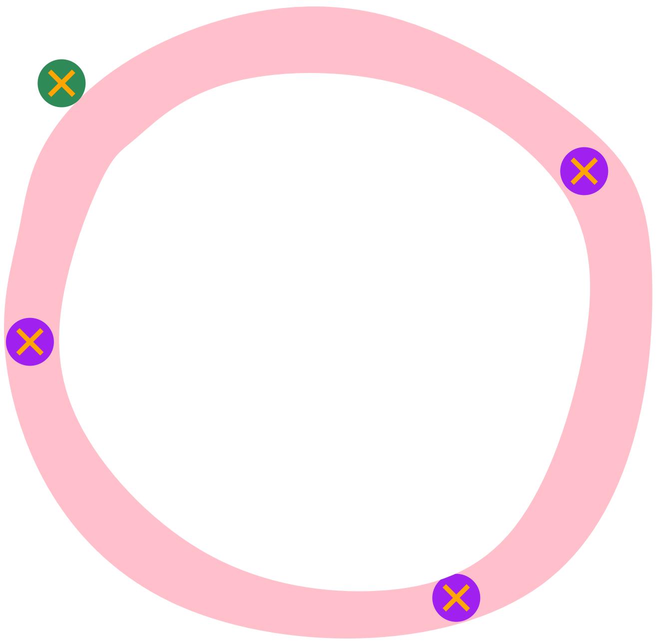
One word on robustness issues

One word on robustness issues

The exact computation paradigm

One word on robustness issues

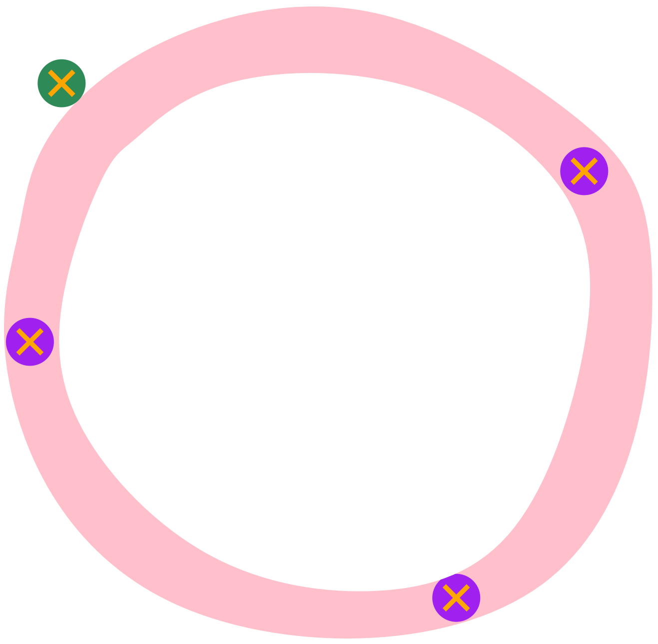
Compute predicates approximately



One word on robustness issues

Compute predicates approximately

Certify result



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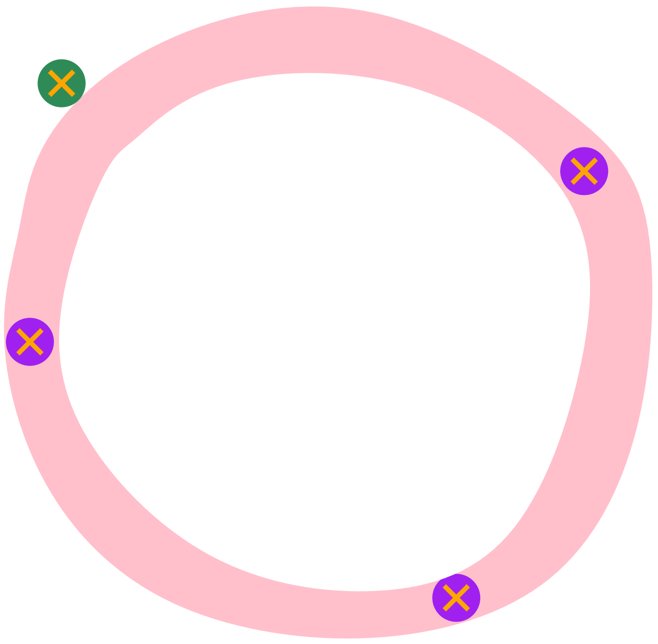

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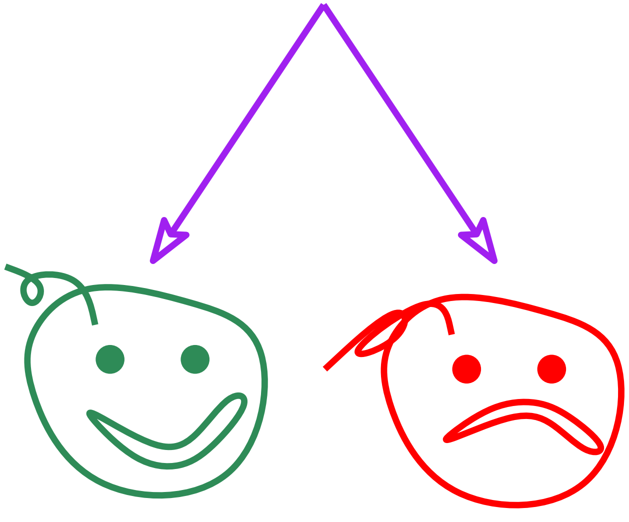
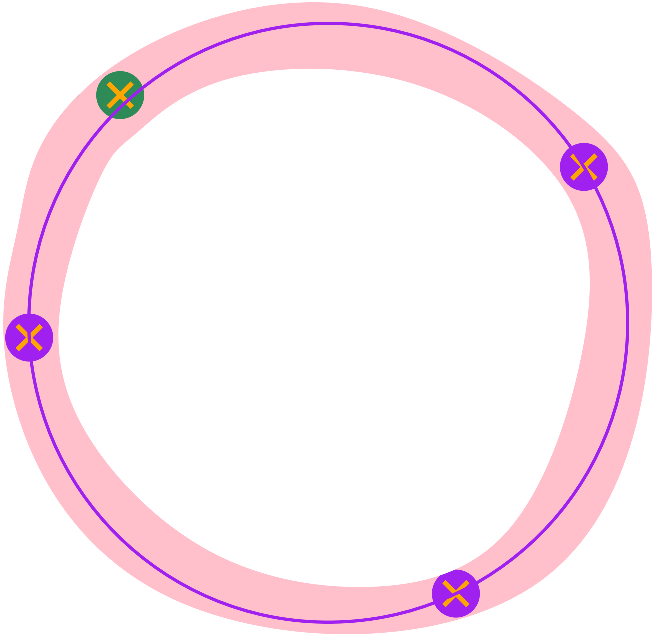
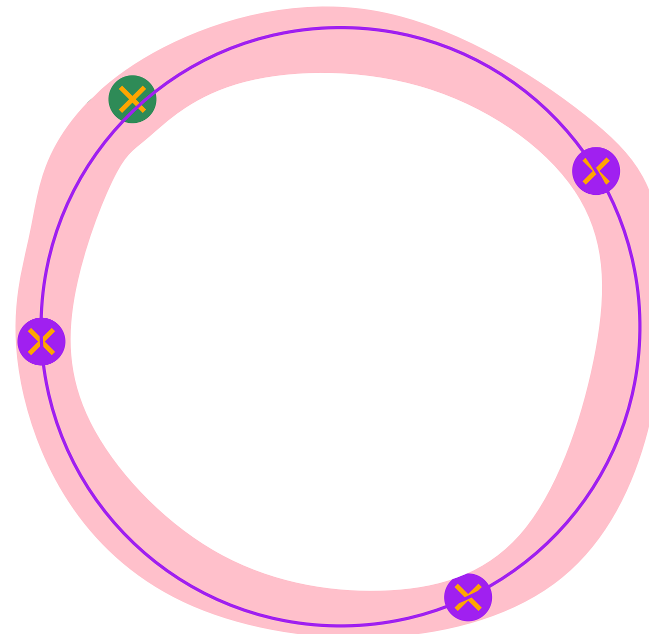


Run exact computation



One word on robustness issues

Compute predicates approximately



Certify result

Extra cost

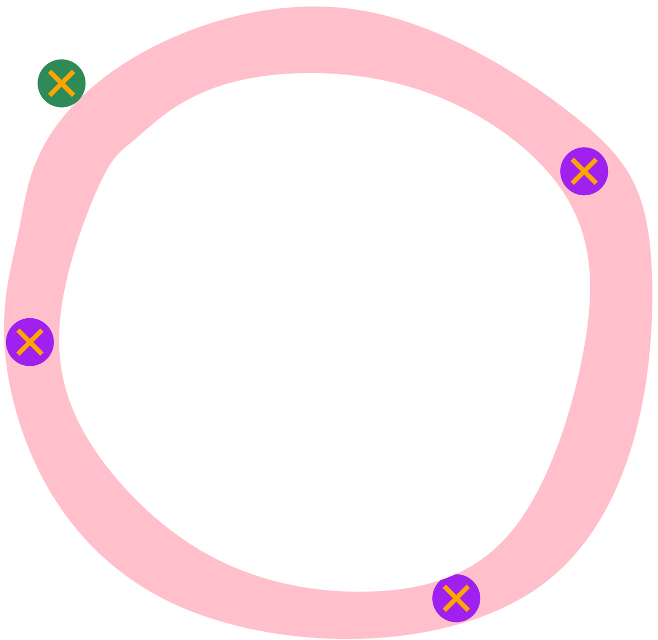
Run exact computation

One word on robustness issues

Compute predicates approximately

Certify result

Extra cost



Delaunay 2D 10M points

`Exact_predicates_inexact_constructions_kernel`

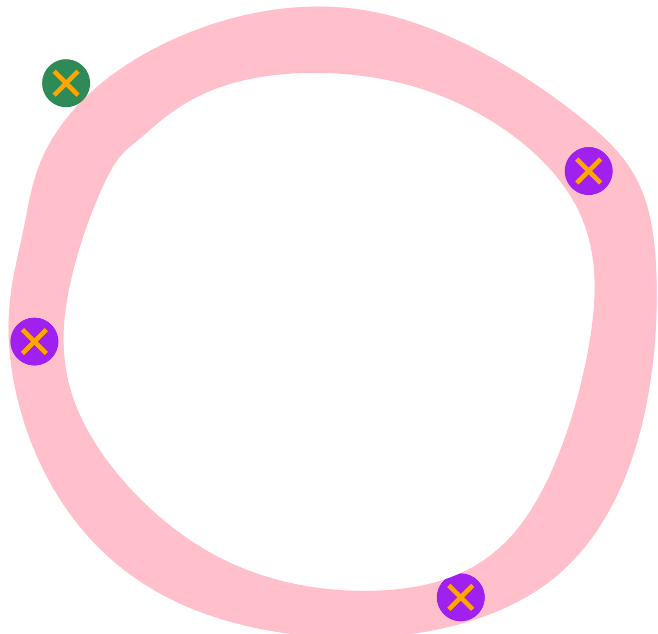
10.6 seconds

`Cartesian<double>`

9.7 seconds

One word on robustness issues

Compute predicates approximately



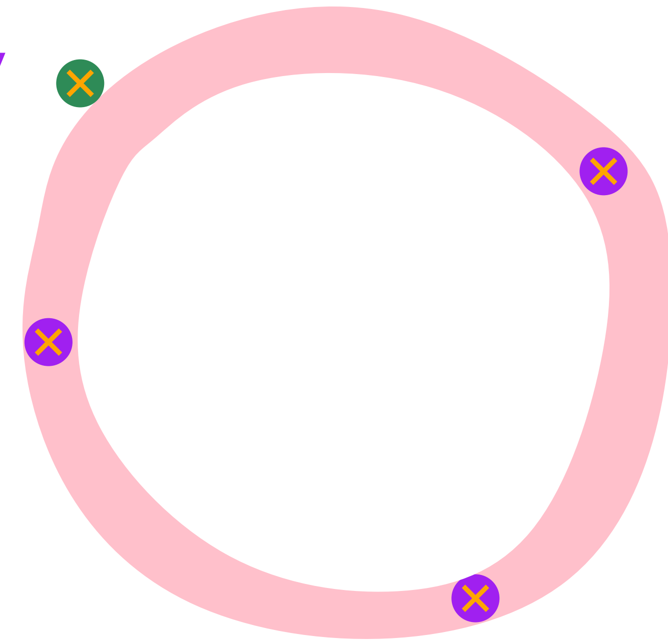
Certify result

Extra cost

	Delaunay 2D 10M points	3D
`Exact_predicates_inexact_constructions_kernel`	10.6 seconds	82 seconds
`Cartesian<double>`	9.7 seconds	75 seconds

One word on robustness issues

Compute predicates approximately



Certify result

Extra cost

Delaunay 2D 10M points

3D

`Exact_predicates_inexact_constructions_kernel`

10.6 seconds 82 seconds

`Cartesian<double>`

may loop (or crash)

All benchmarks

2.3 GHz, 16 GByte workstation

CGAL 3.9 (Release mode)

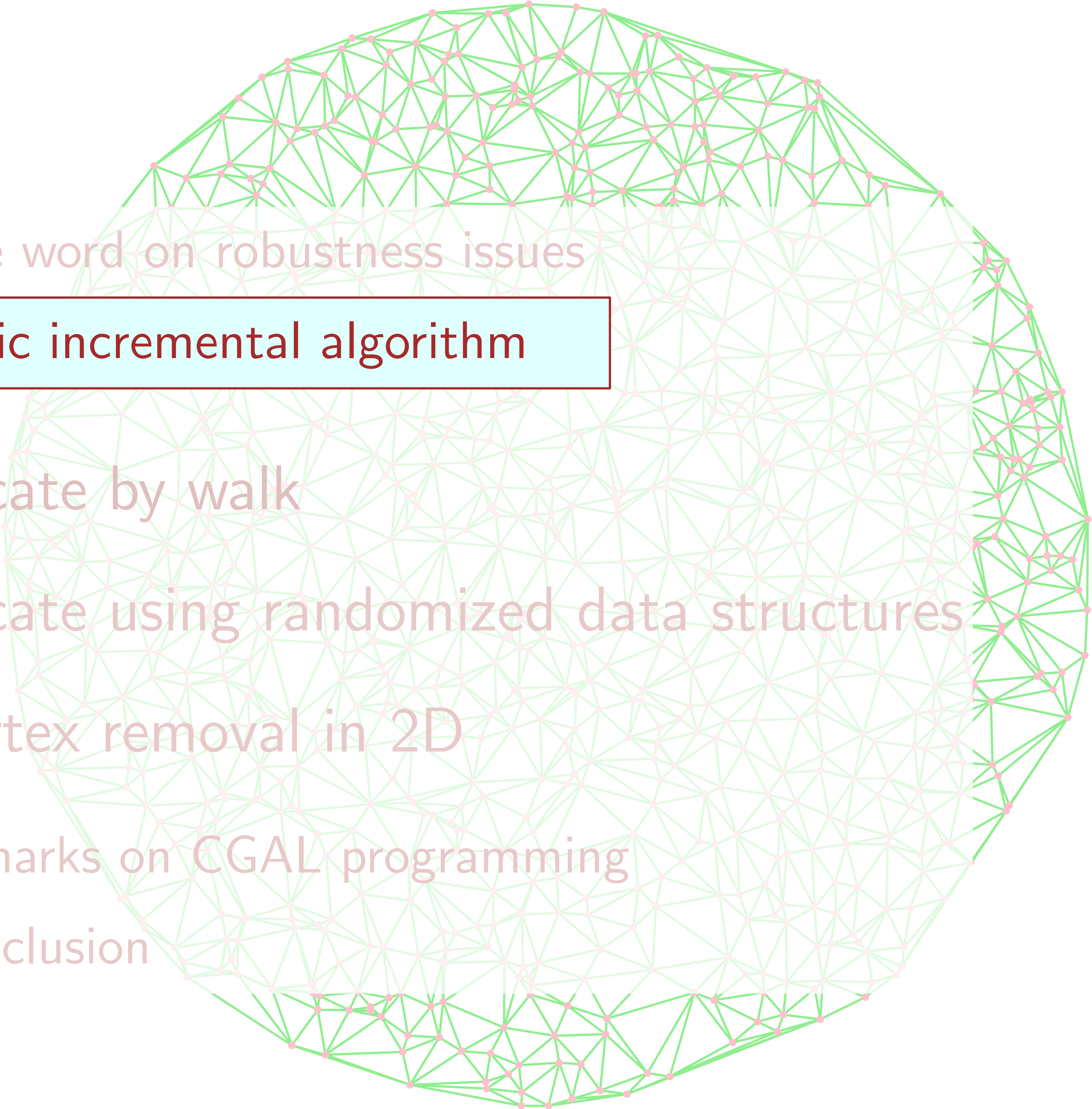
`Exact_predicates_inexact_constructions_kernel`

`Exact_predicates_inexact_constructions_kernel`

10.6 seconds 82 seconds

`Cartesian<double>`

9.7 seconds 75 seconds



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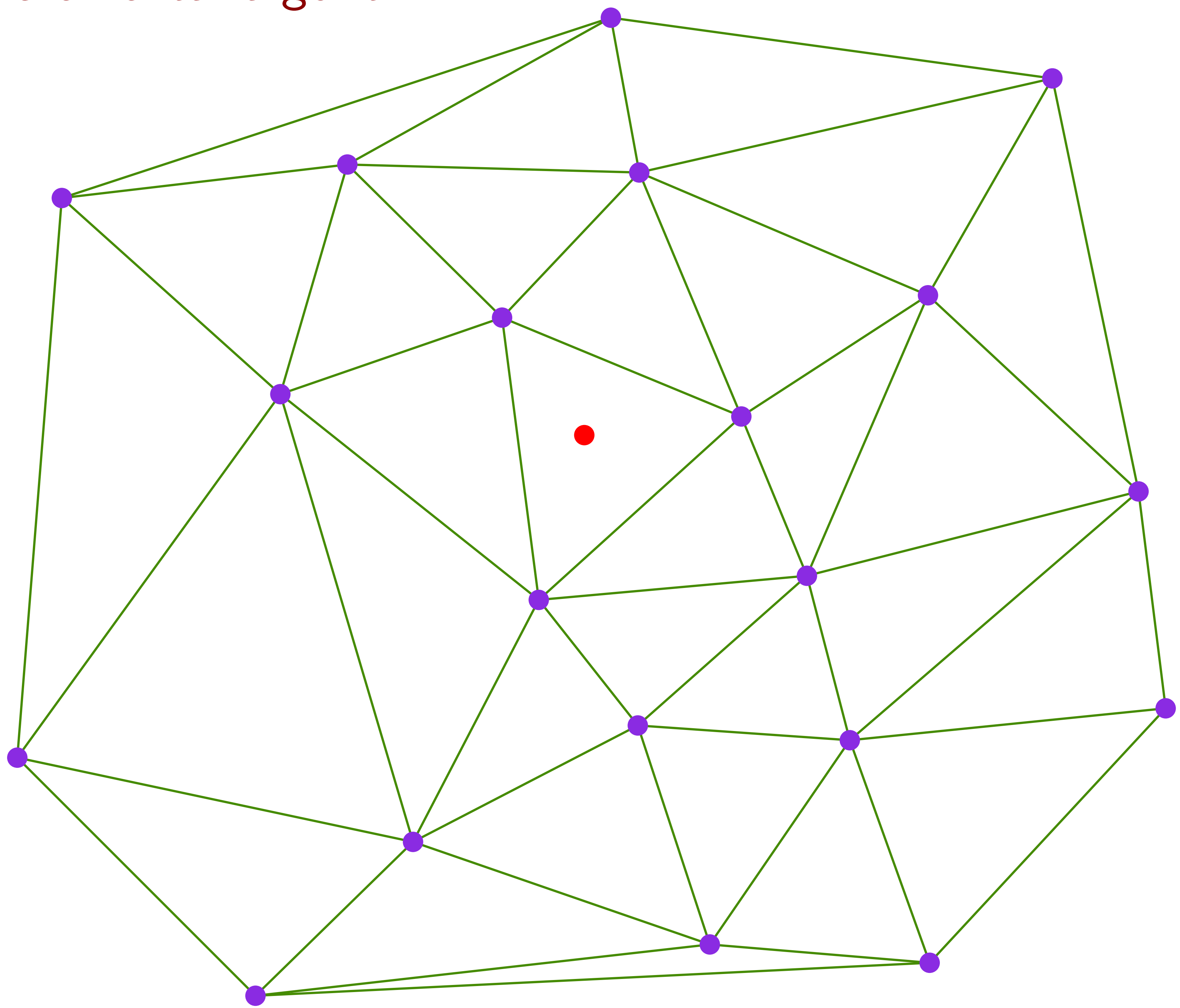
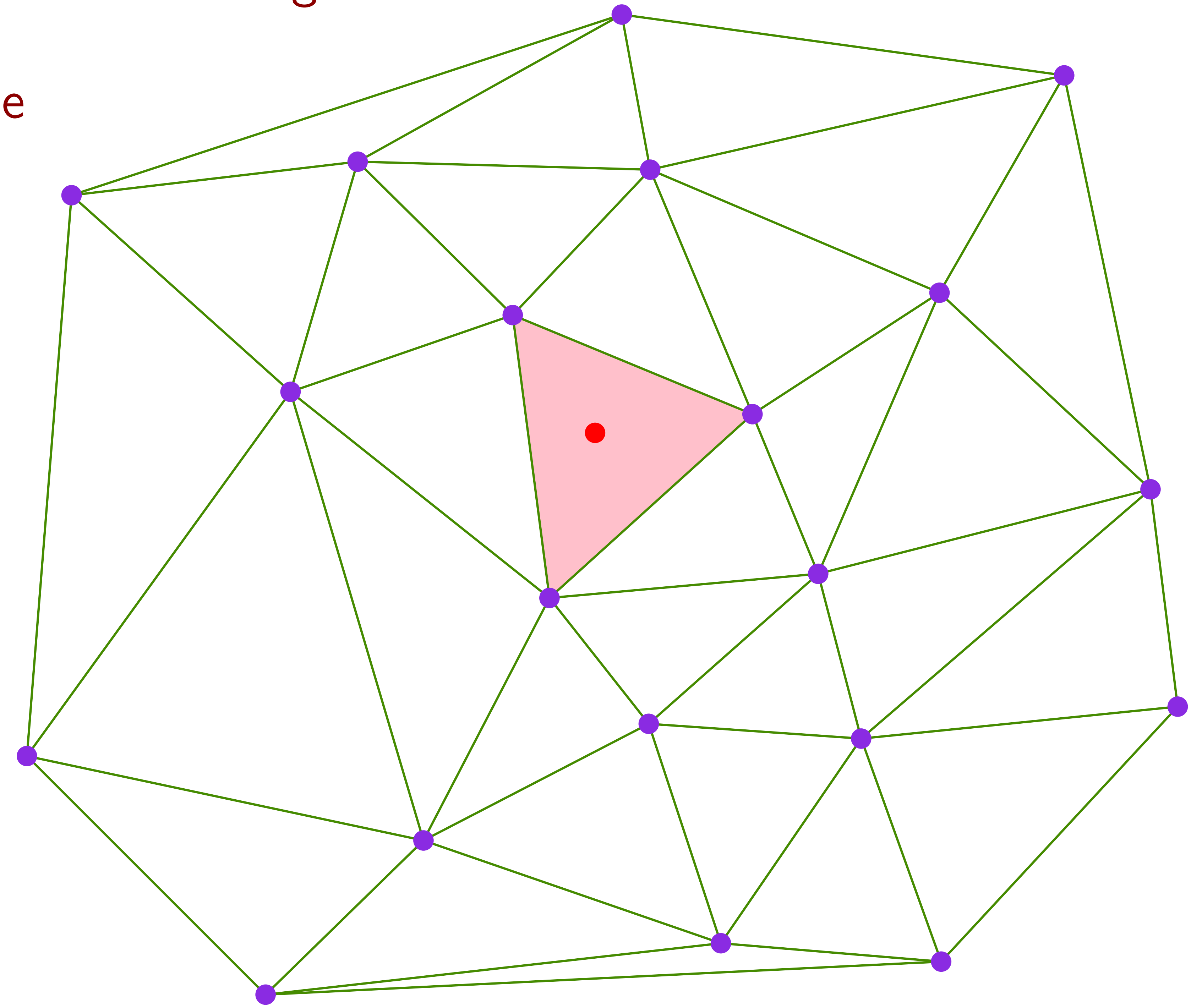


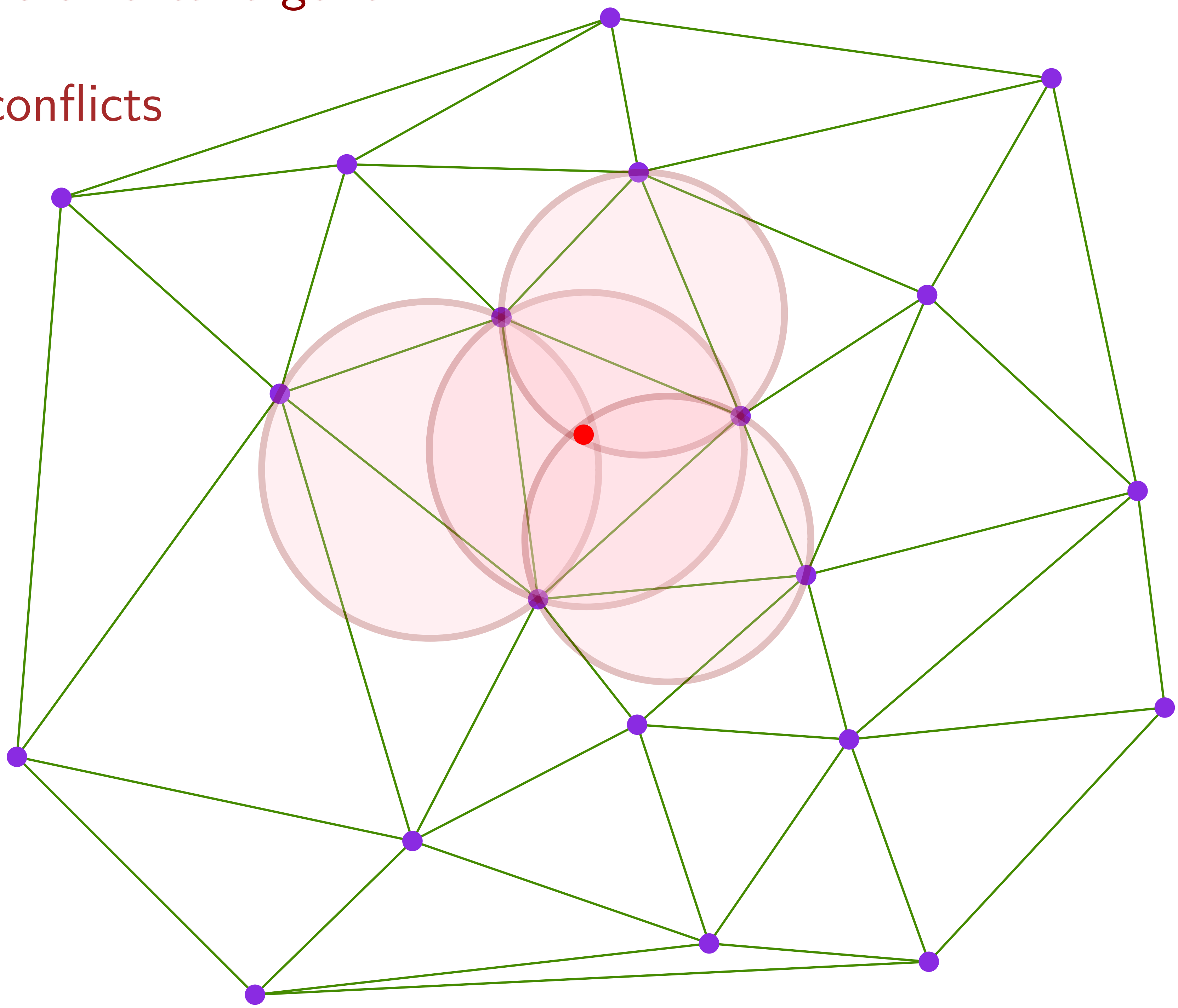
Basic incremental algorithm

Locate



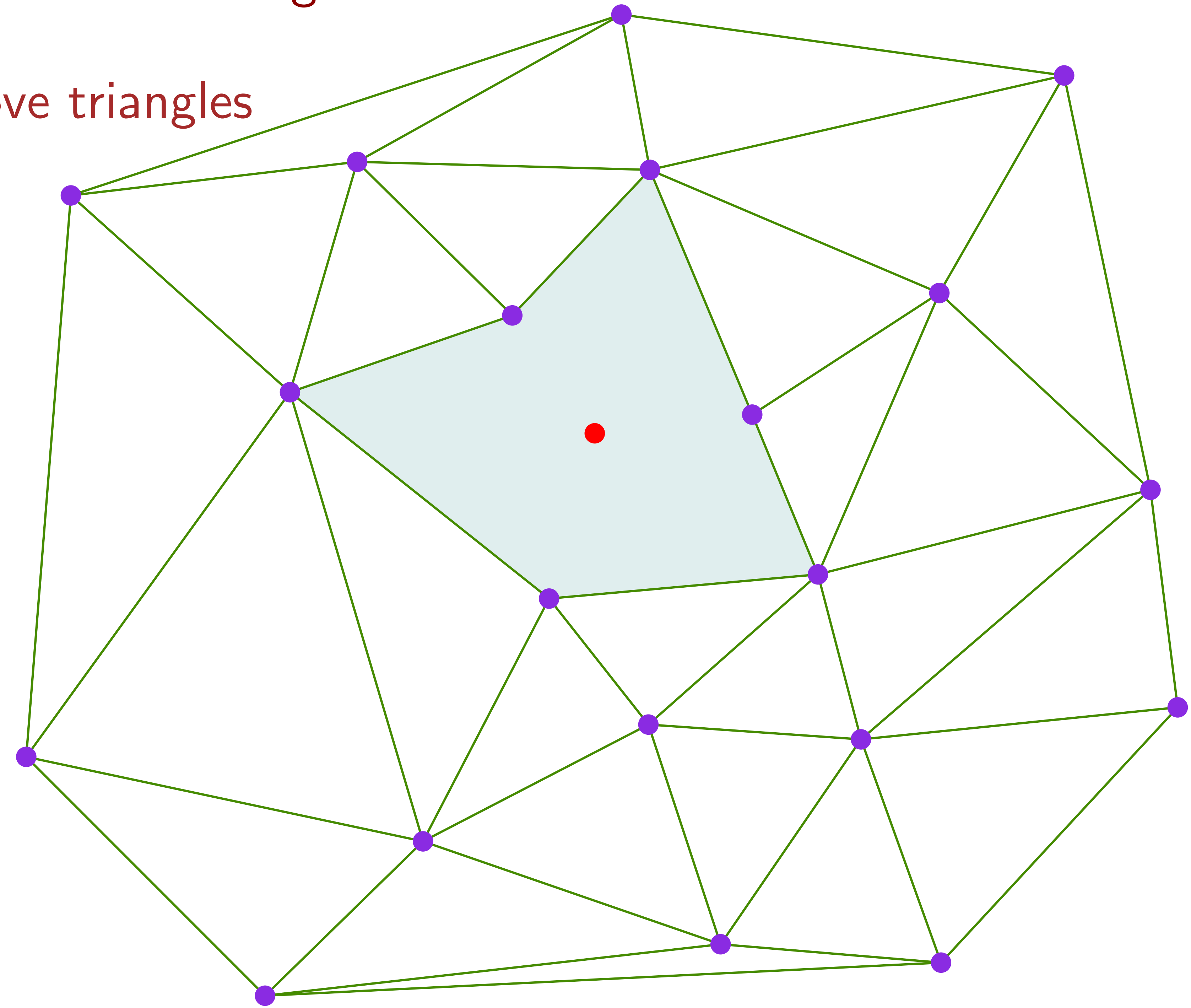
Basic incremental algorithm

Find conflicts



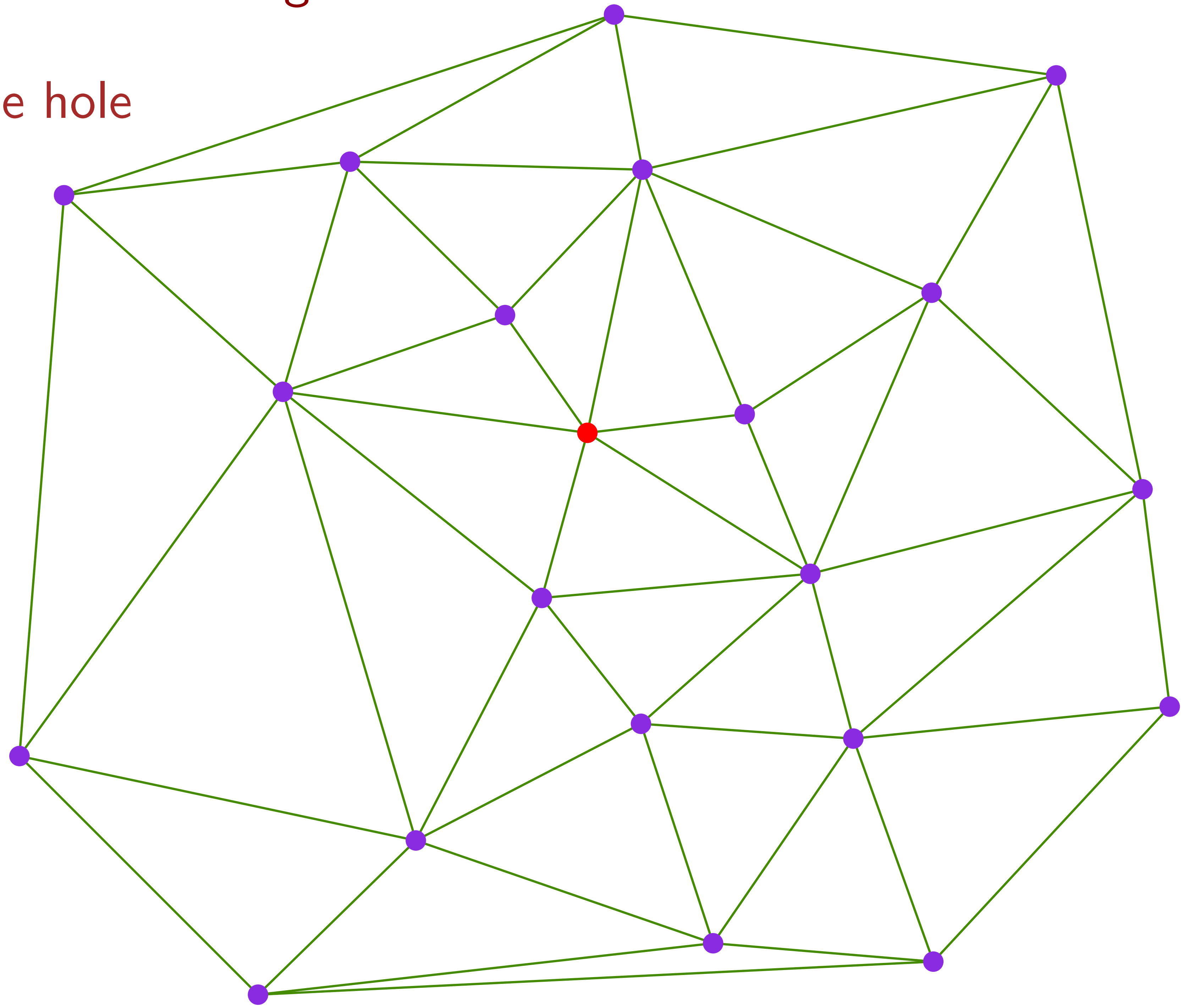
Basic incremental algorithm

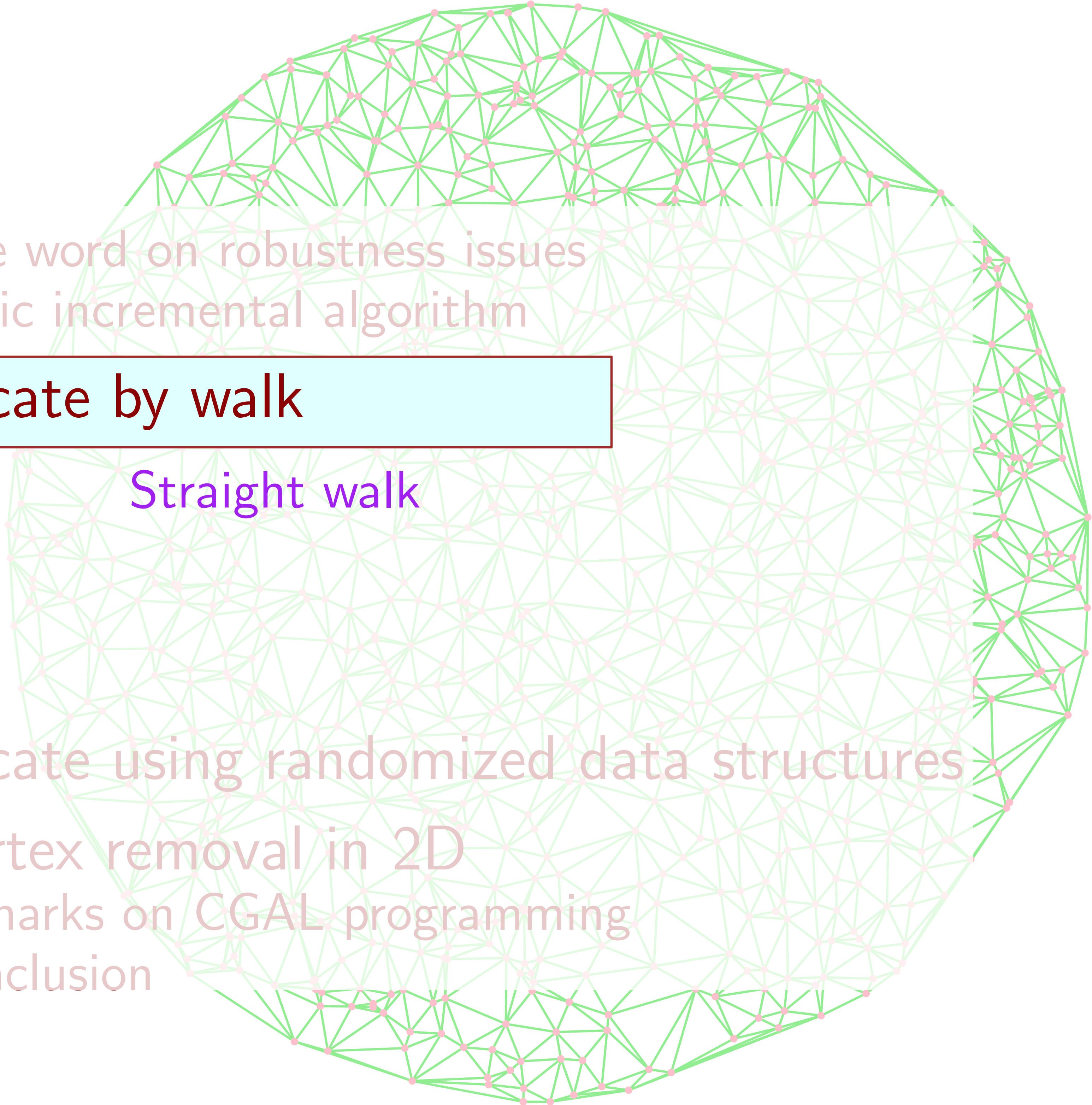
Remove triangles



Basic incremental algorithm

Fill the hole





One word on robustness issues

Basic incremental algorithm

Locate by walk

Straight walk

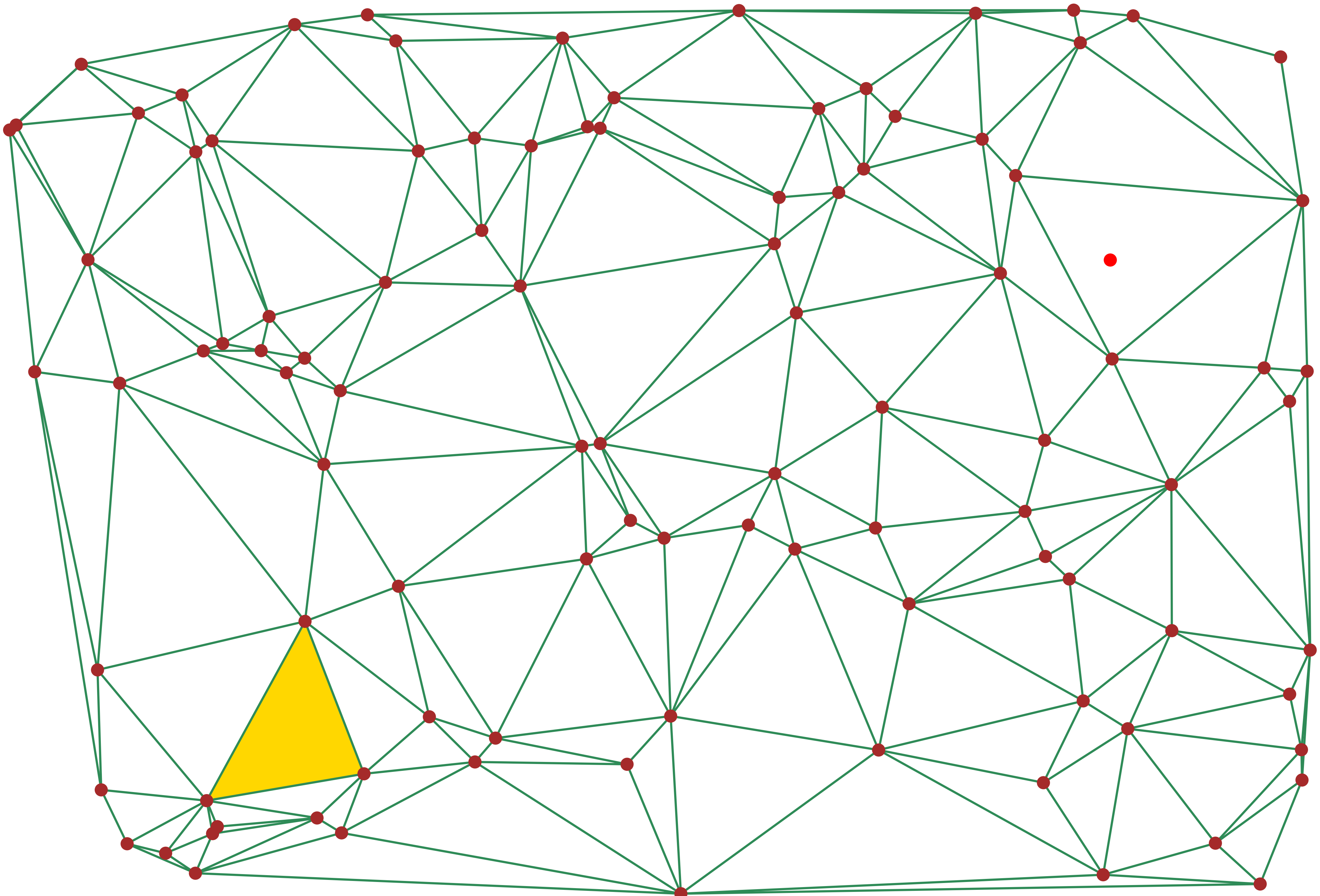
Locate using randomized data structures

Vertex removal in 2D

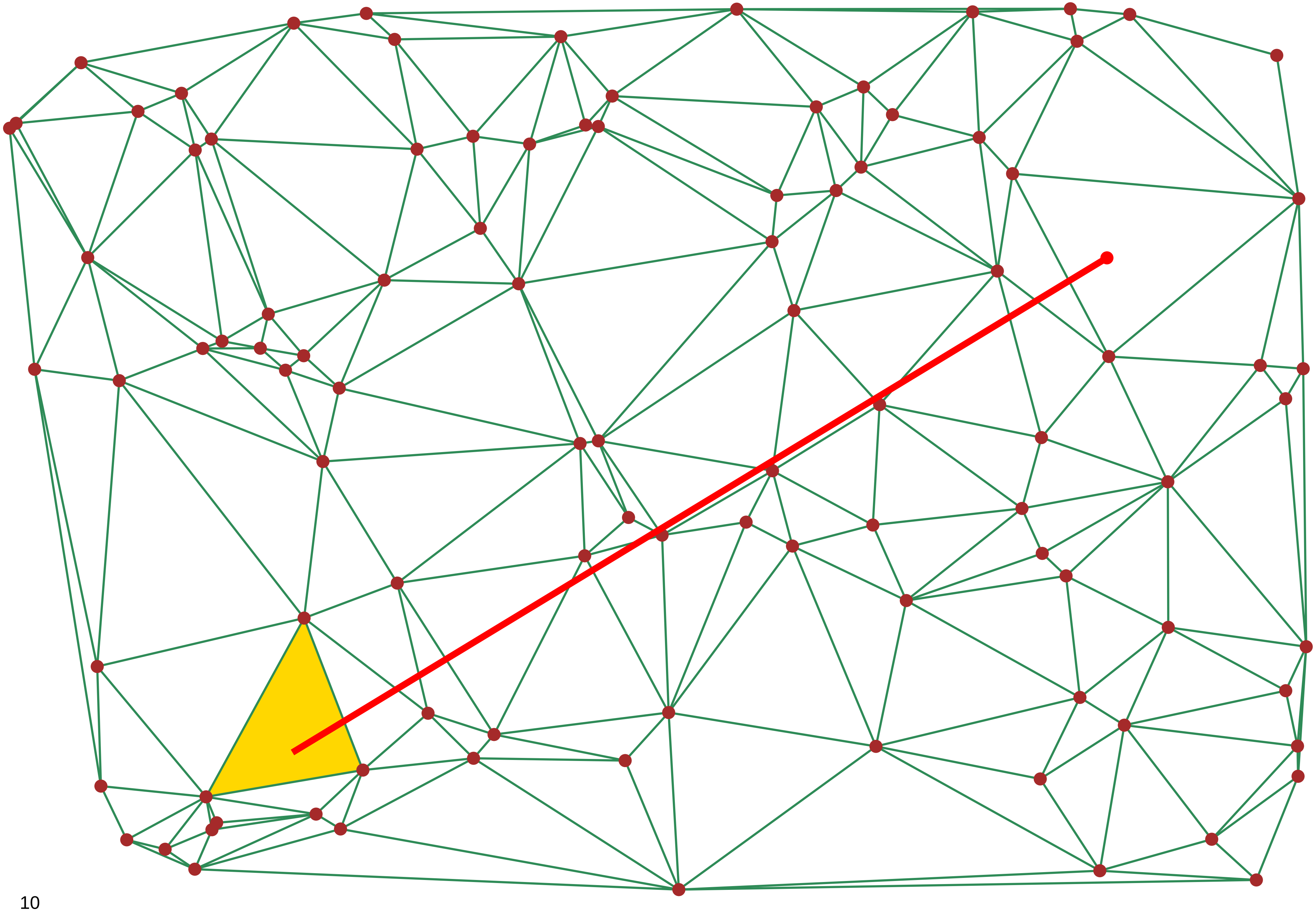
Remarks on CGAL programming

Conclusion

Locate by walk - straight walk



Locate by walk - straight walk



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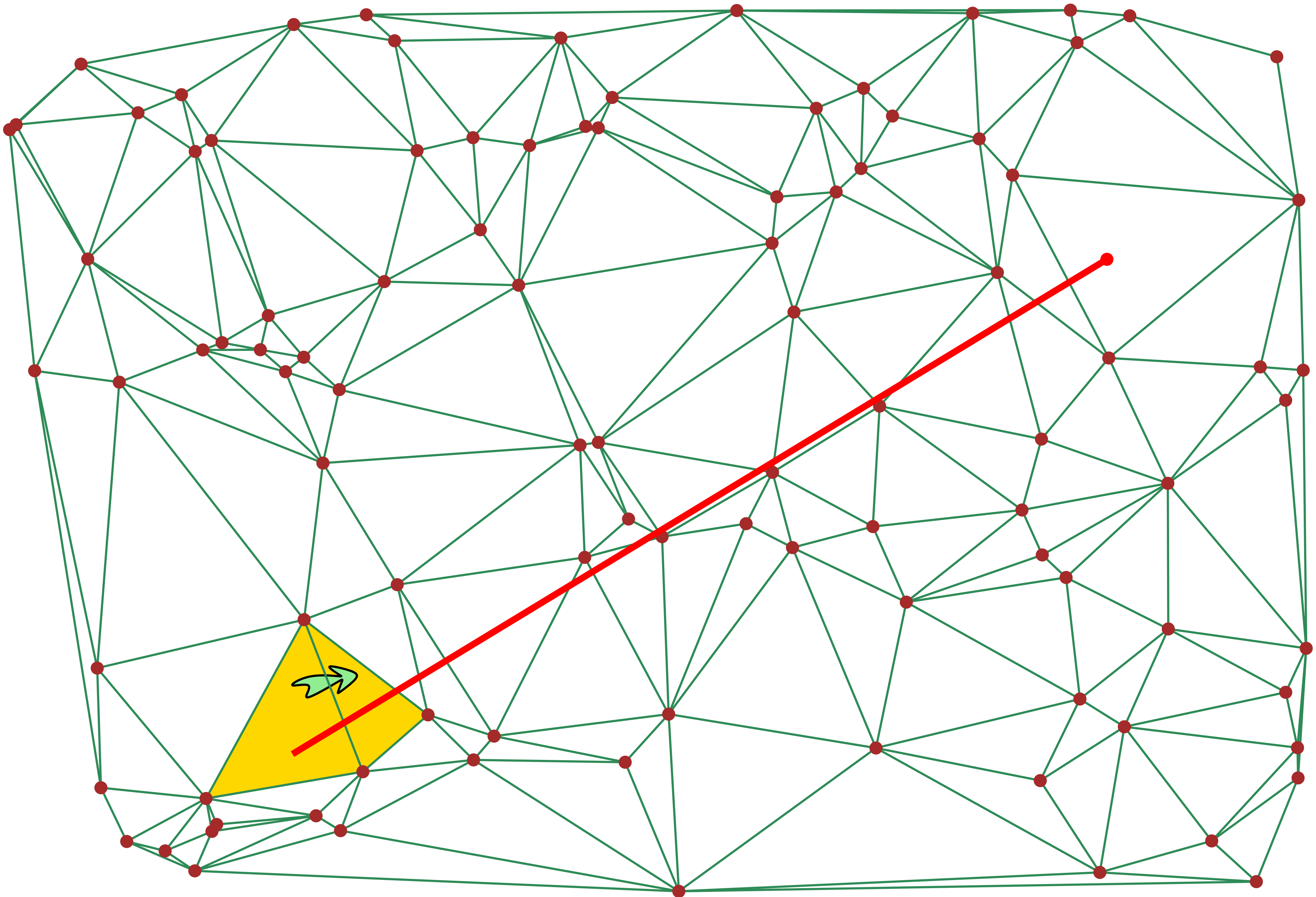

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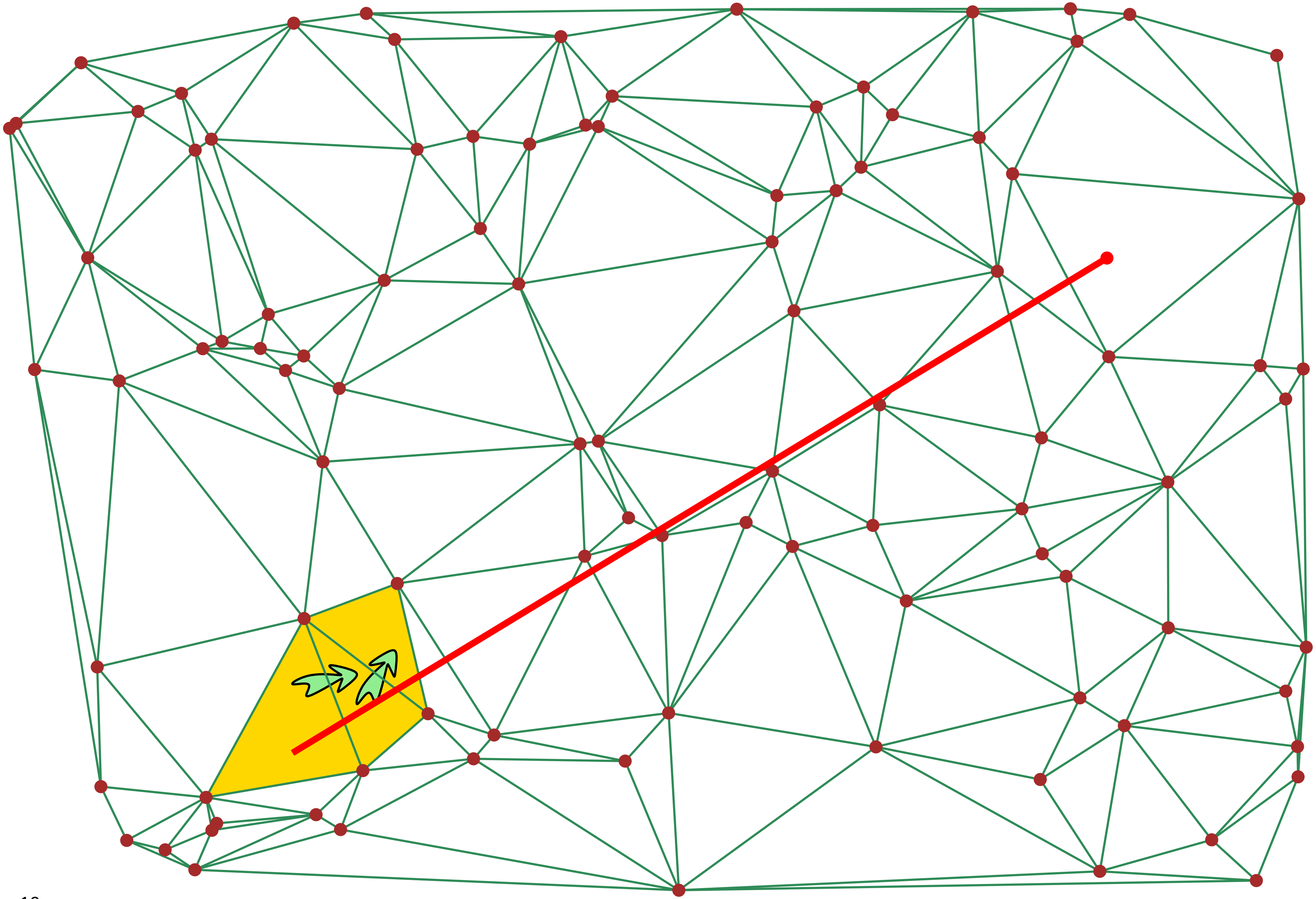

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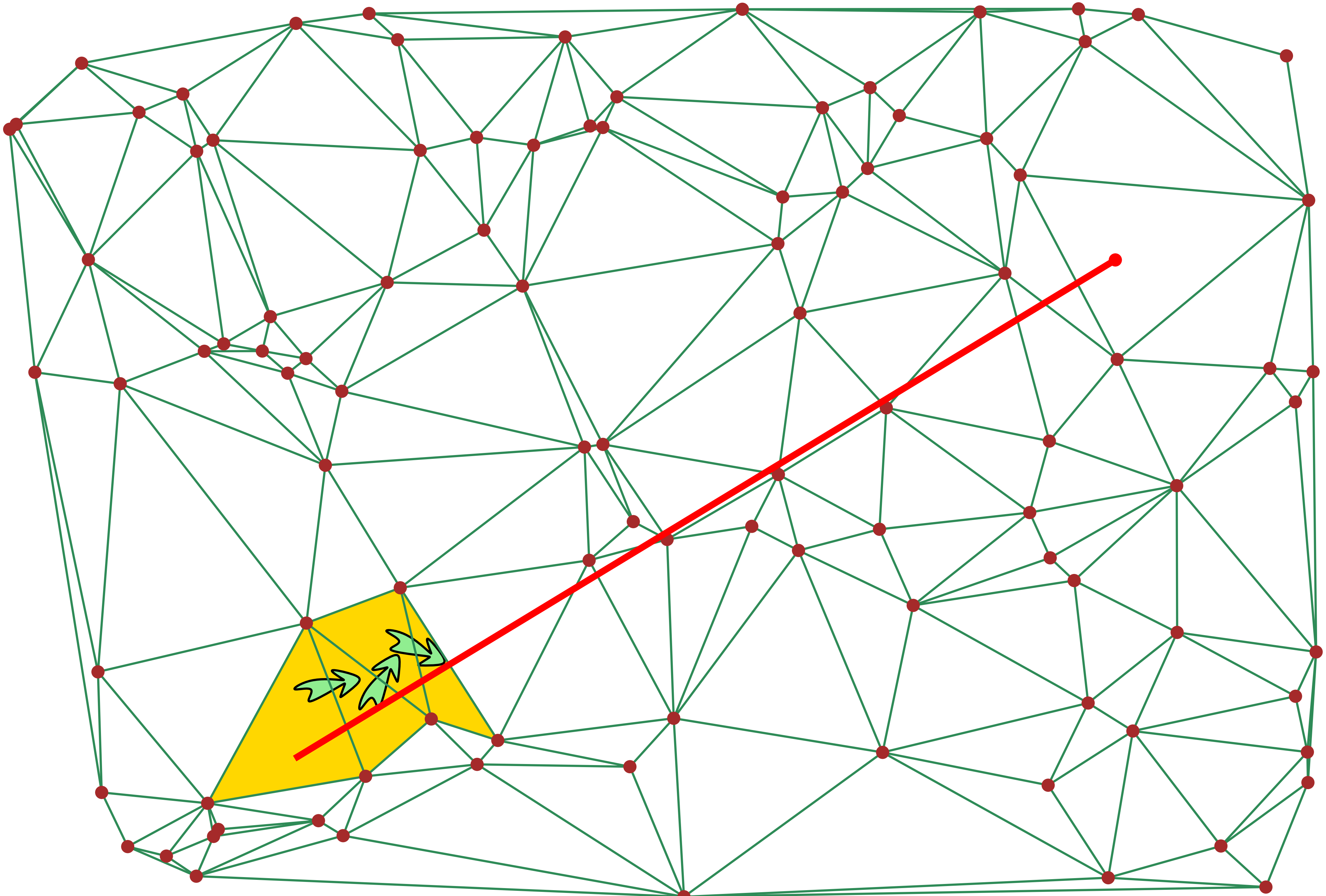
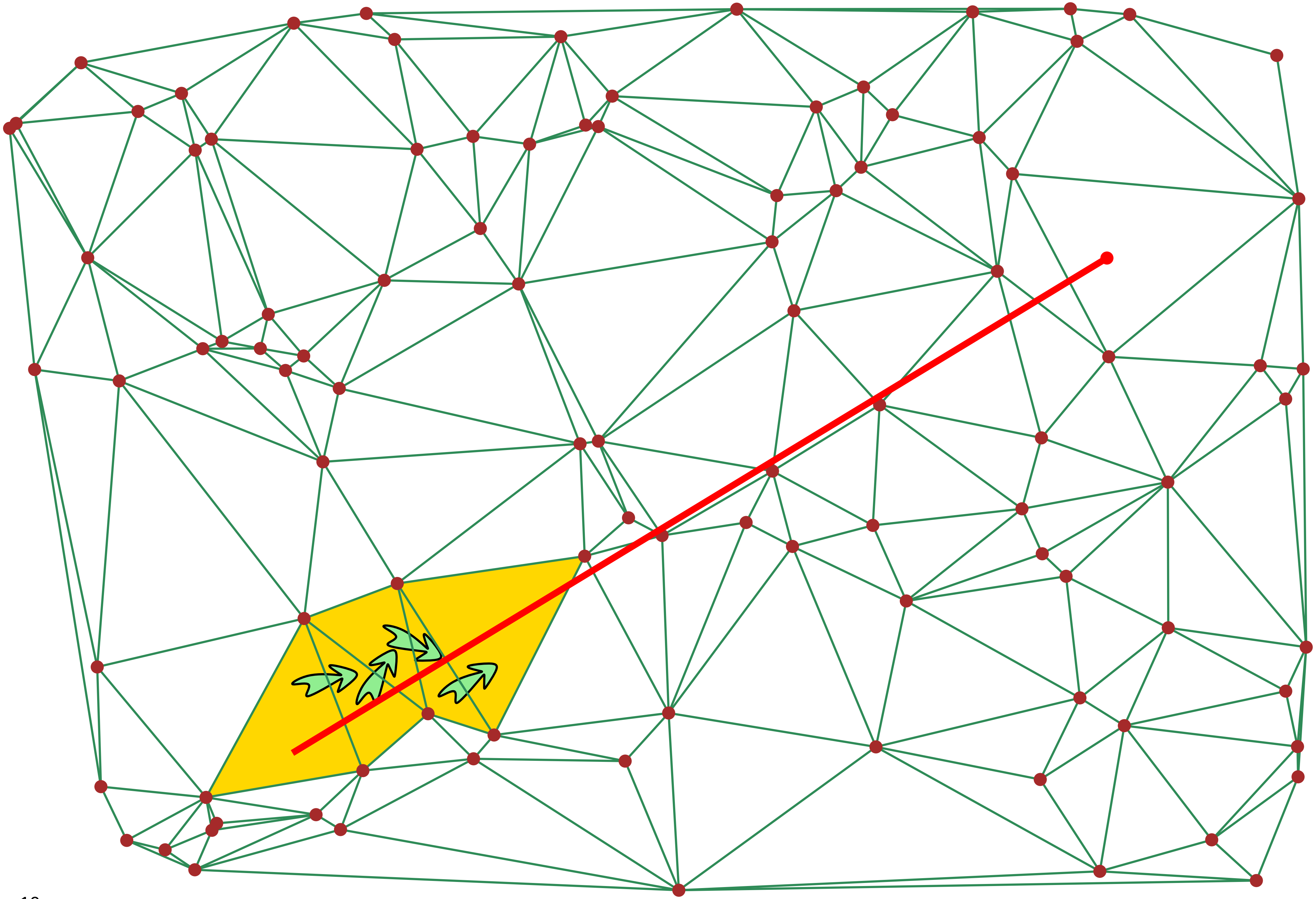
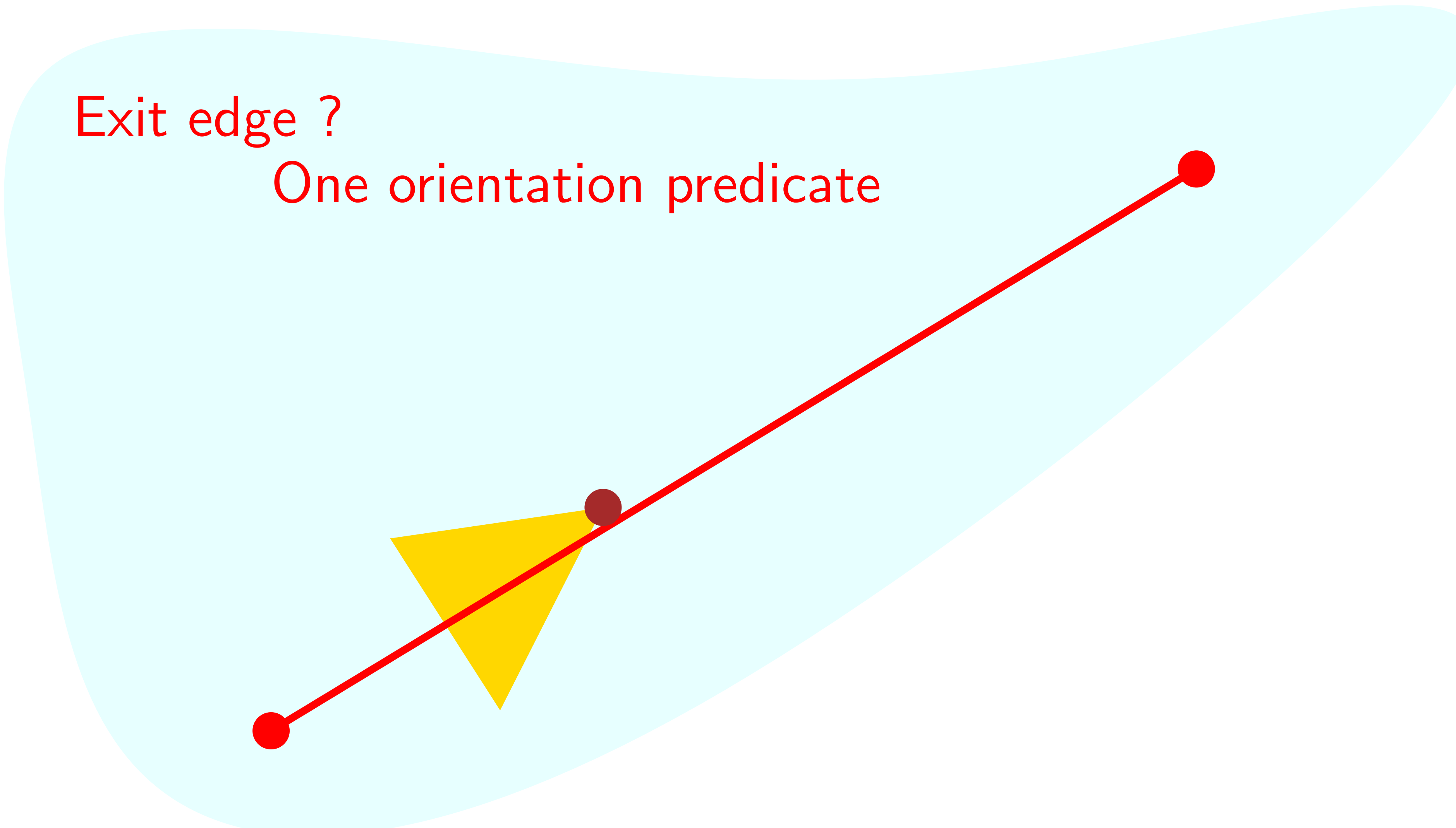
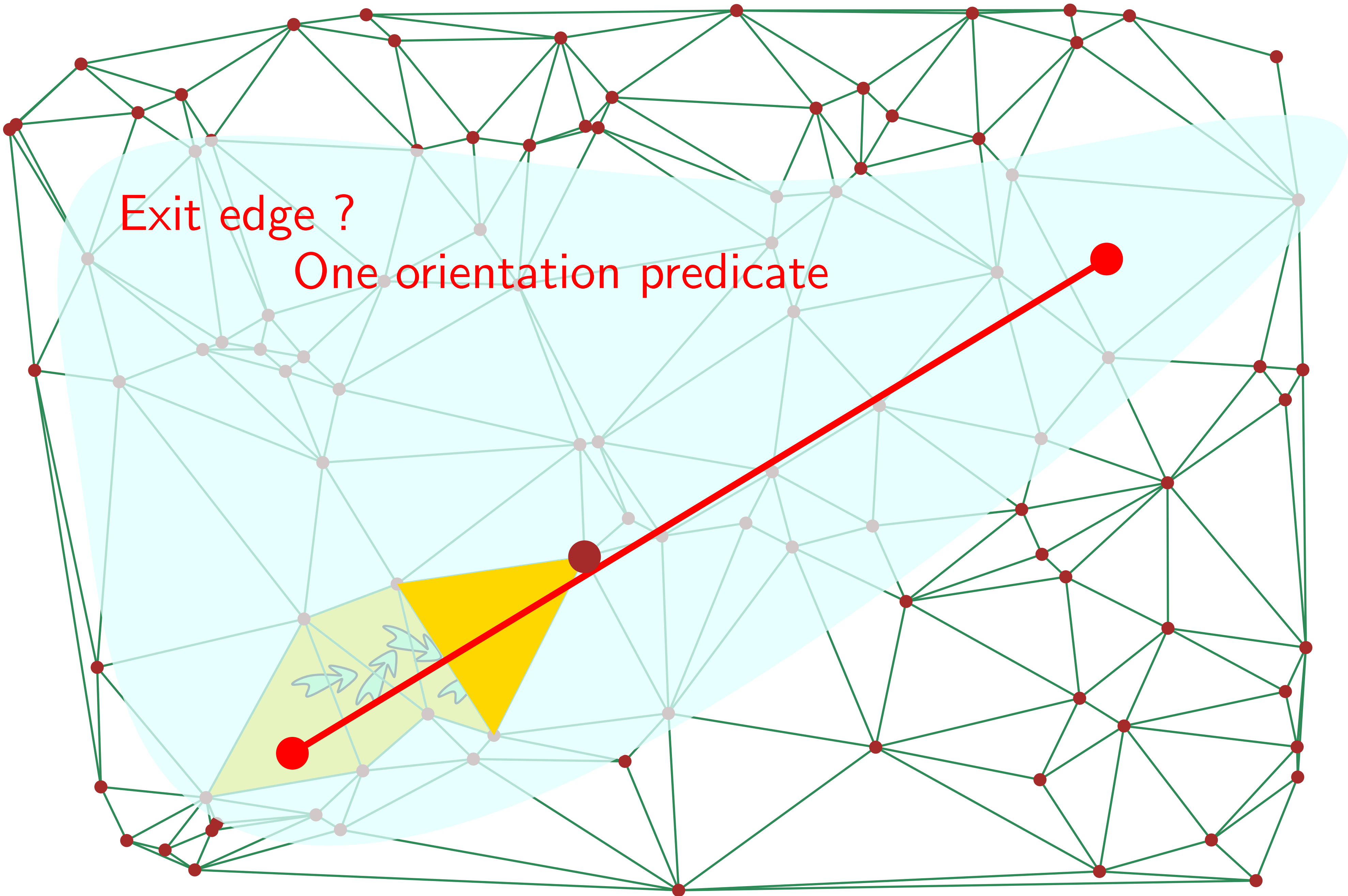


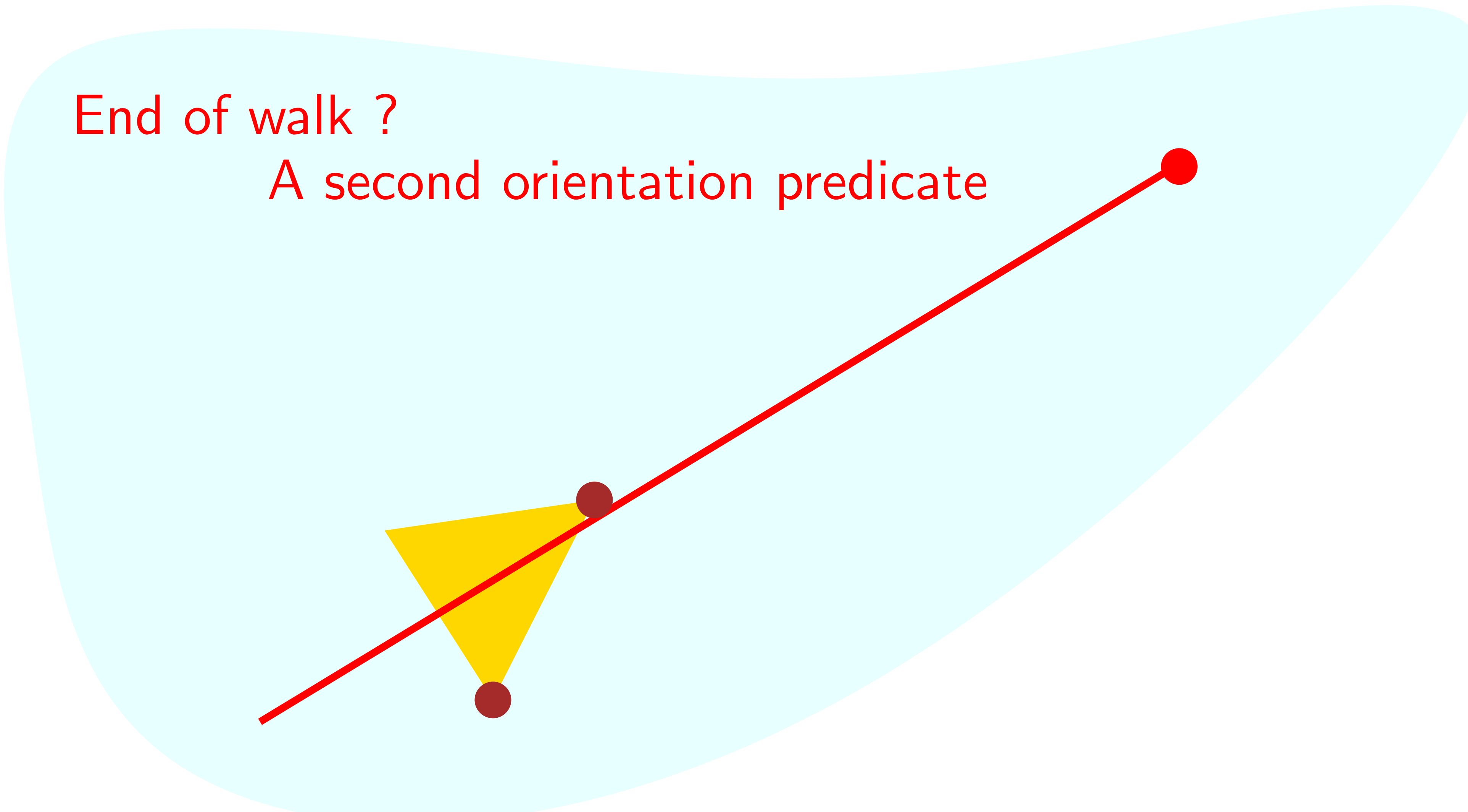
Locate by walk - straight walk



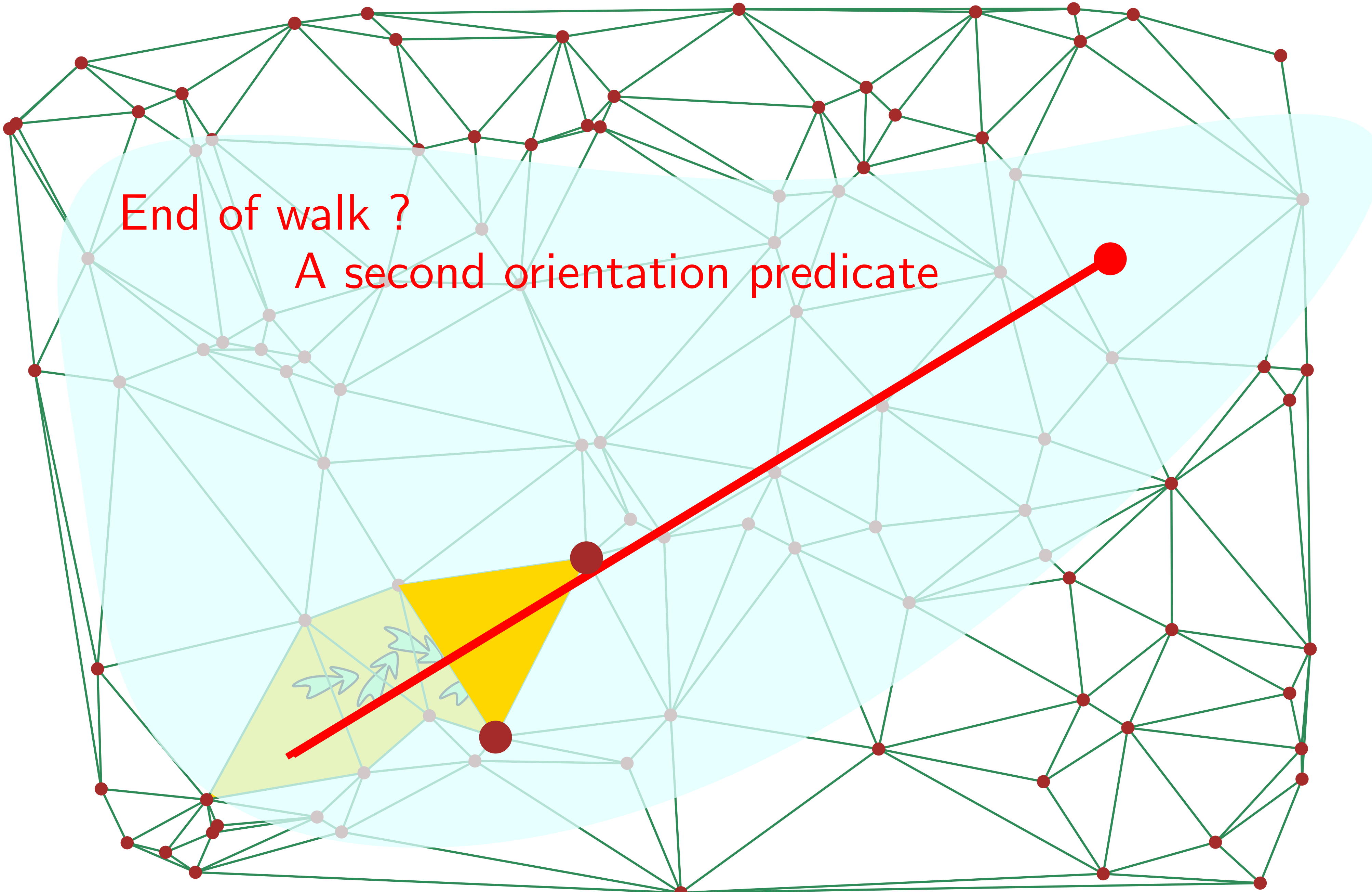

Exit edge ?
One orientation predicate

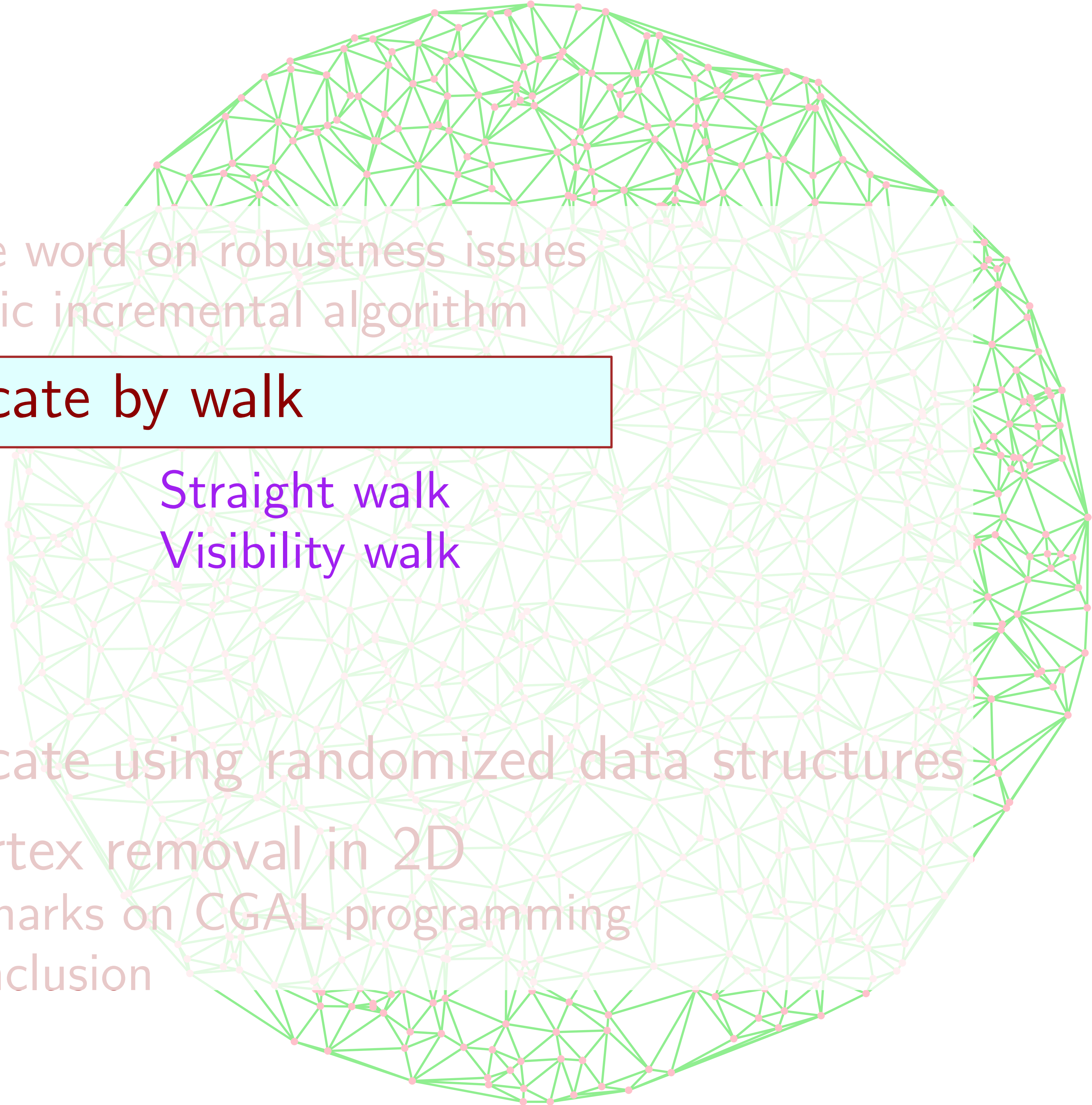
Locate by walk - straight walk




End of walk ?
A second orientation predicate

Locate by walk - straight walk





One word on robustness issues

Basic incremental algorithm

Locate by walk

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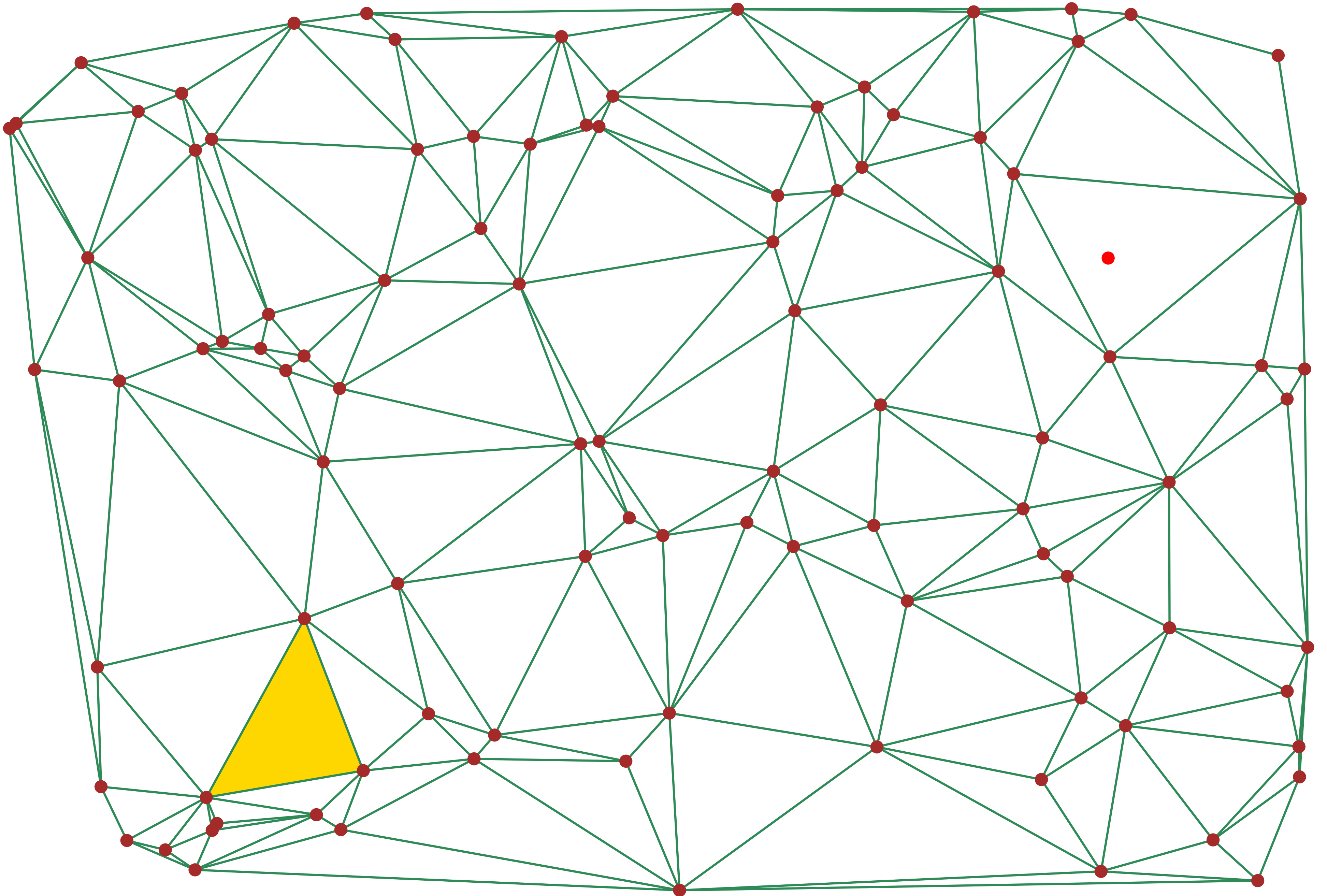
Locate by walk - visibility walk



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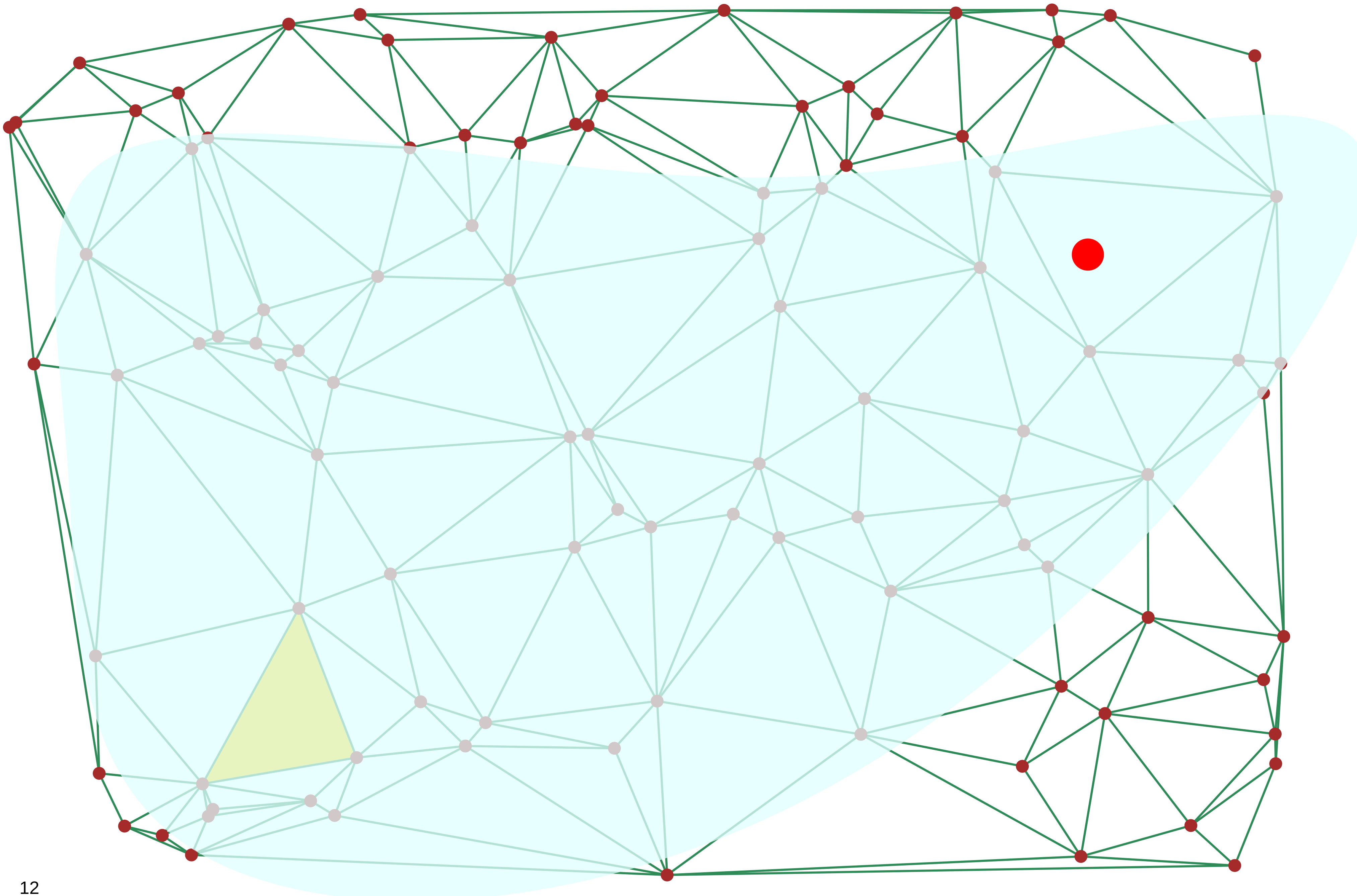

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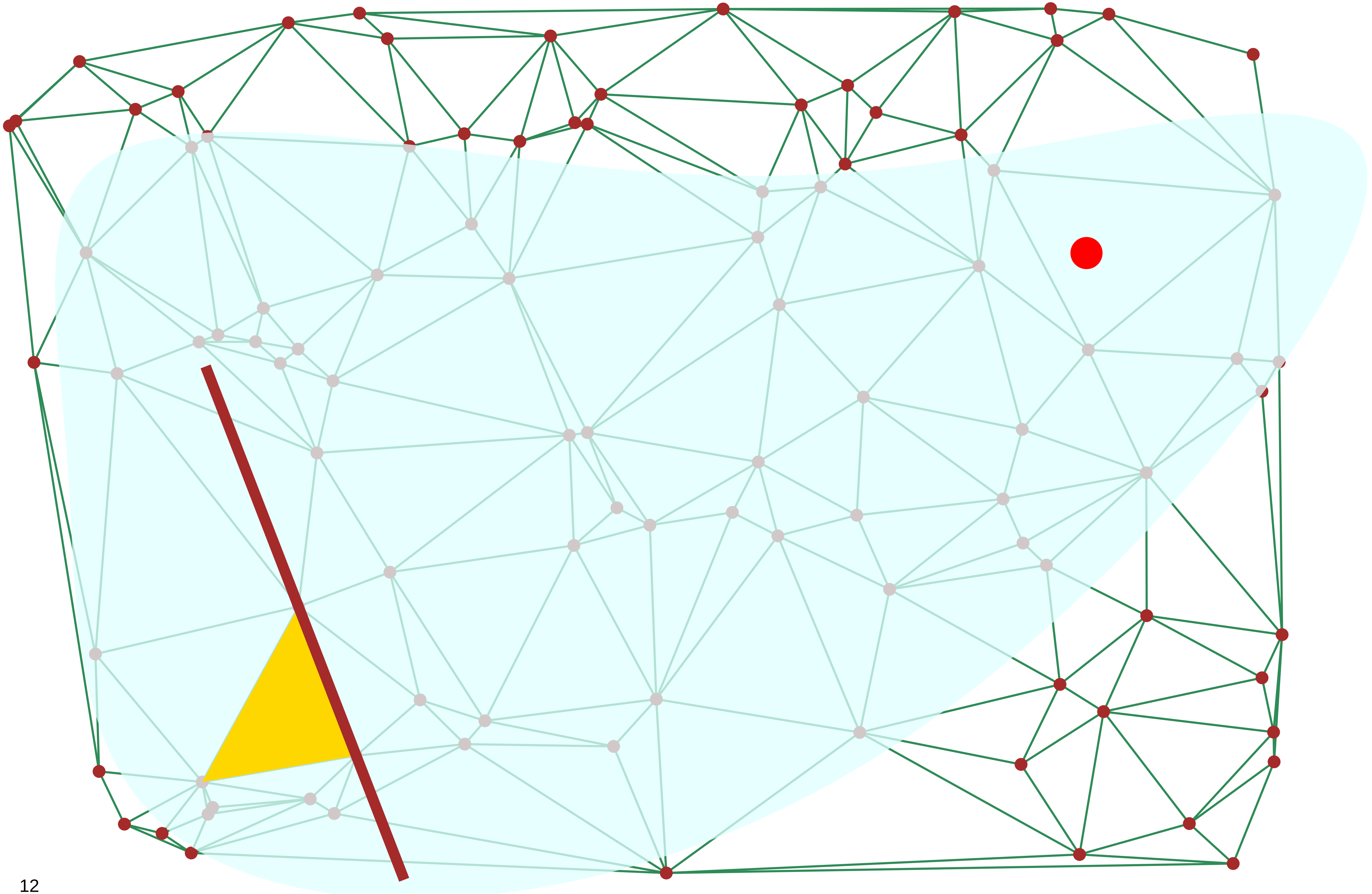

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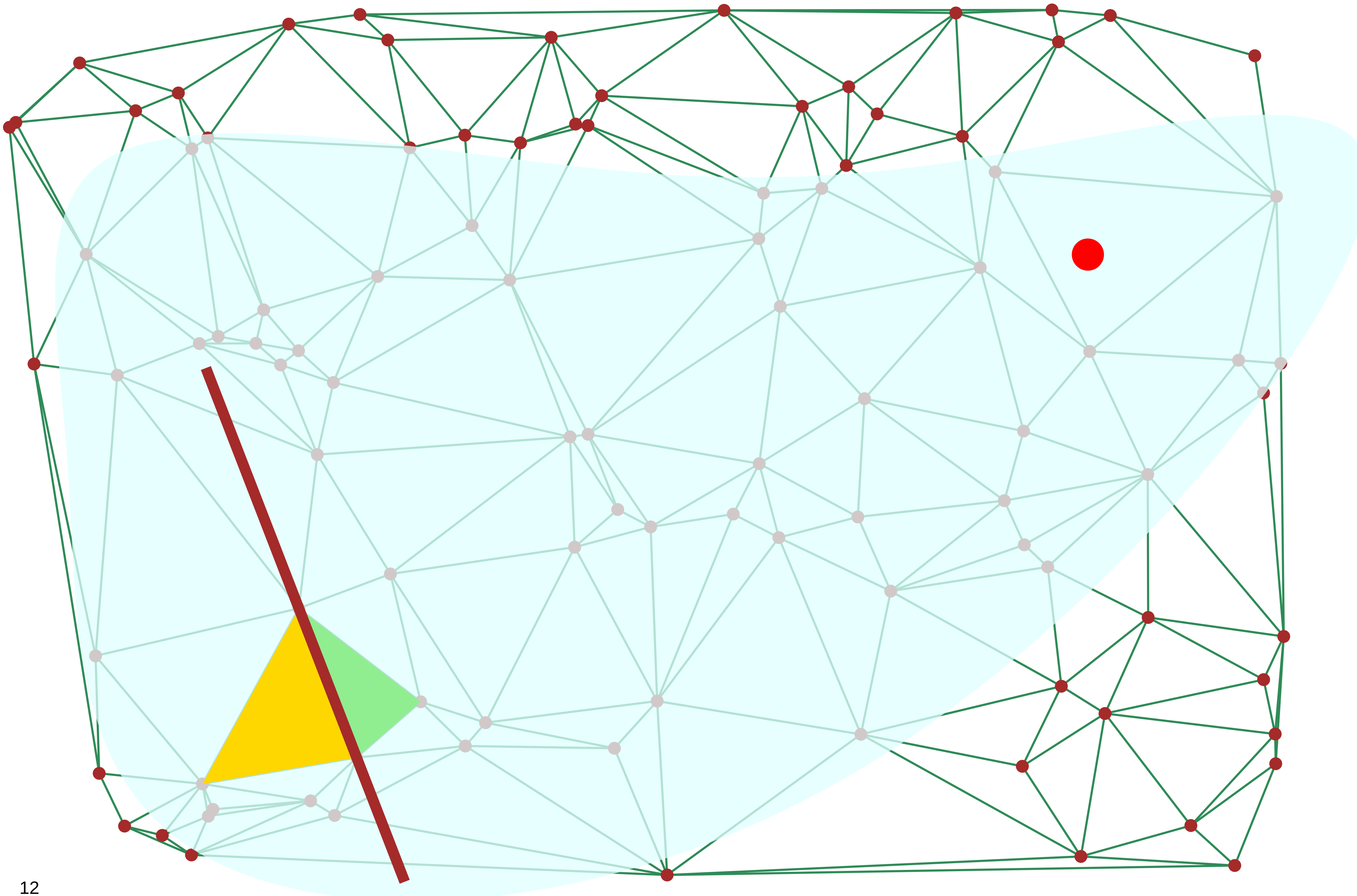

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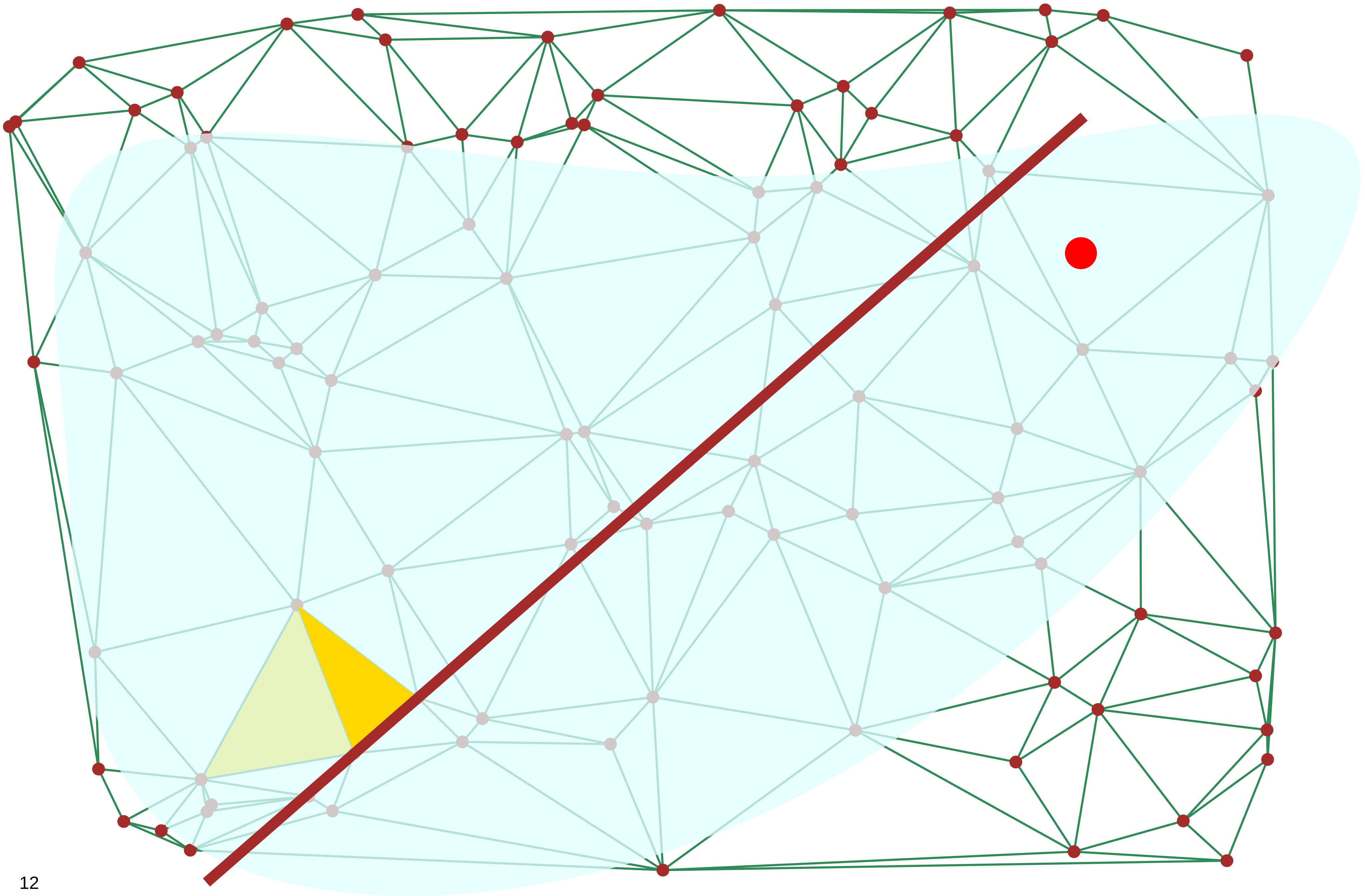

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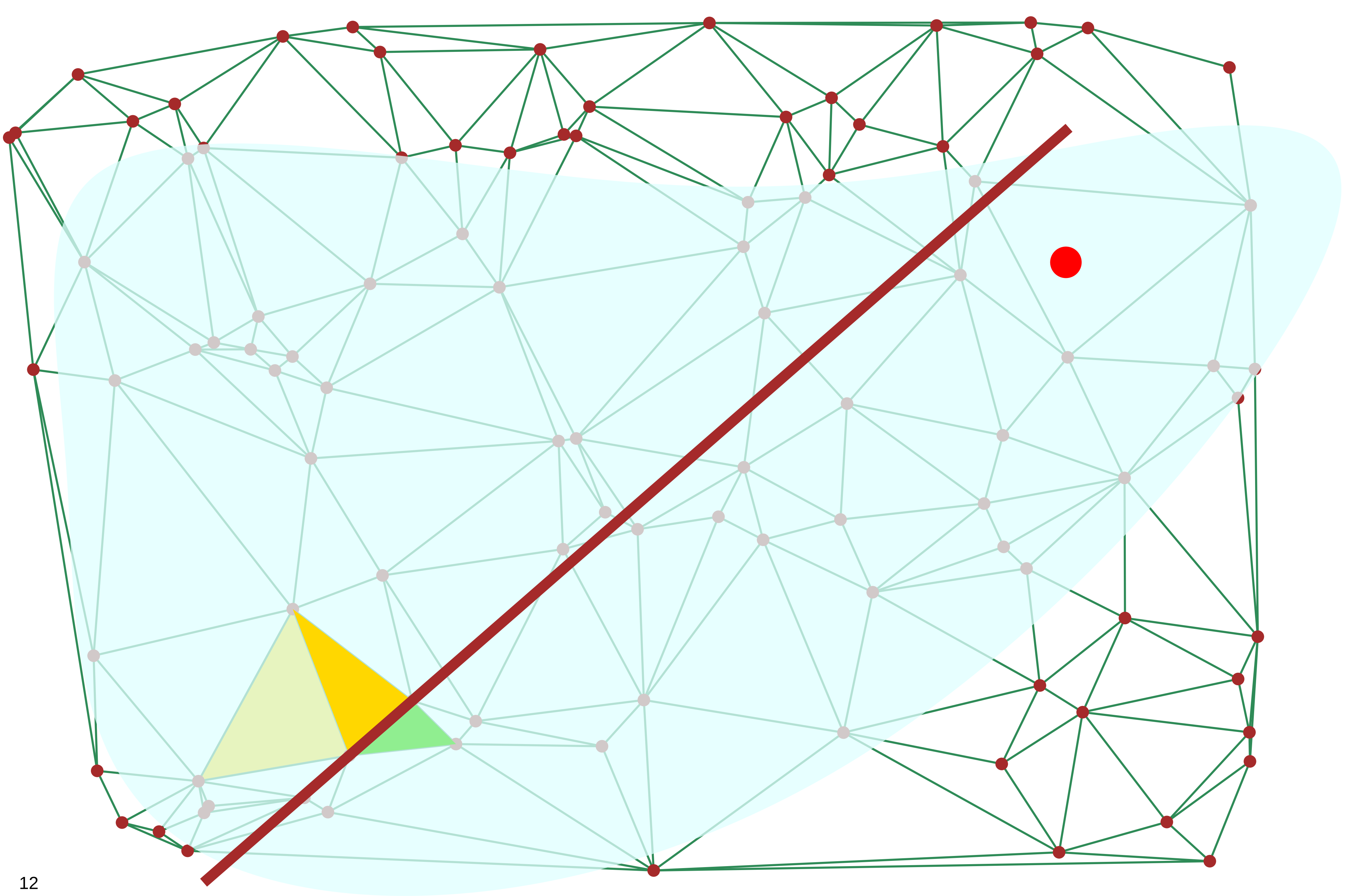

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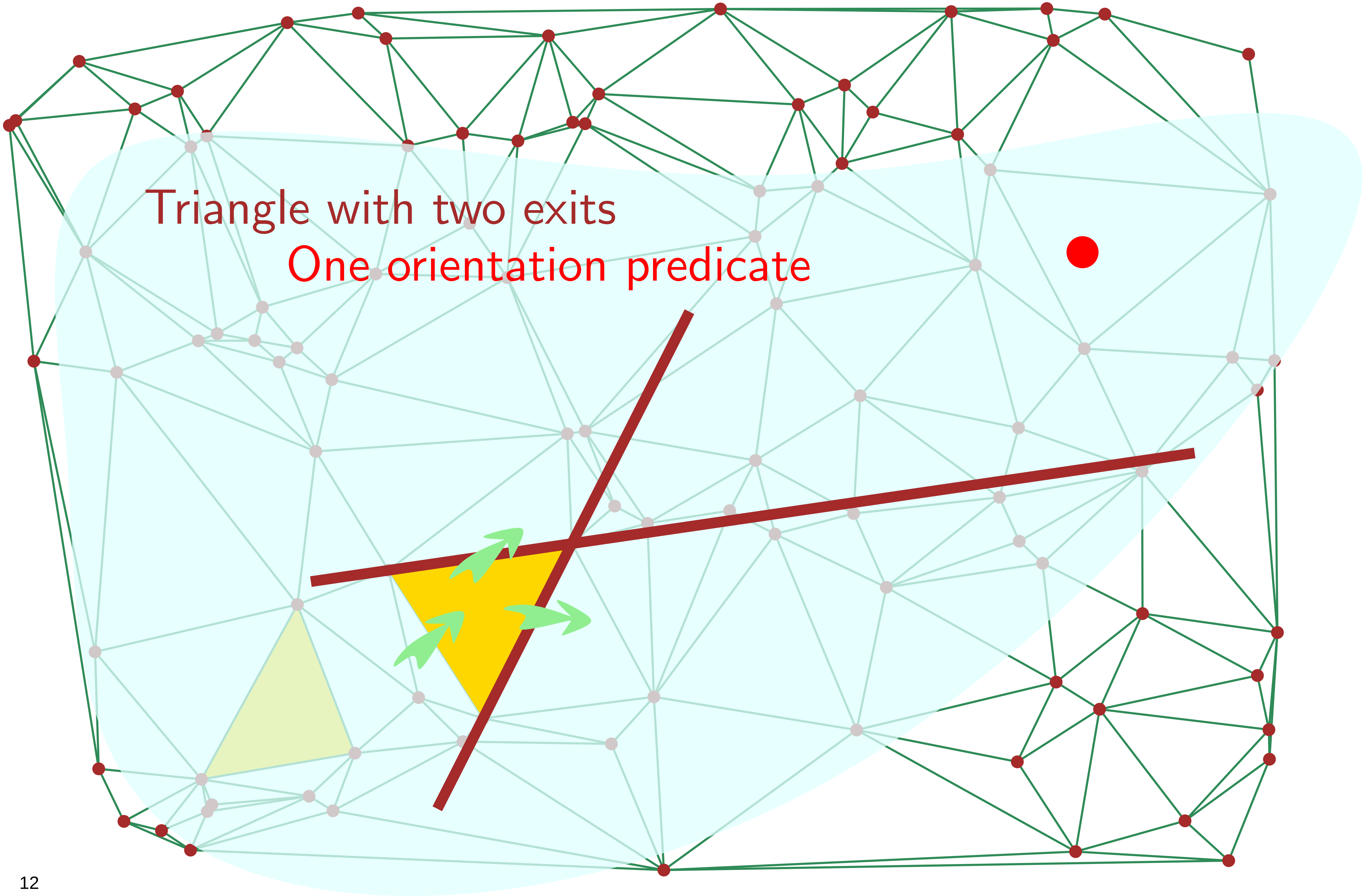

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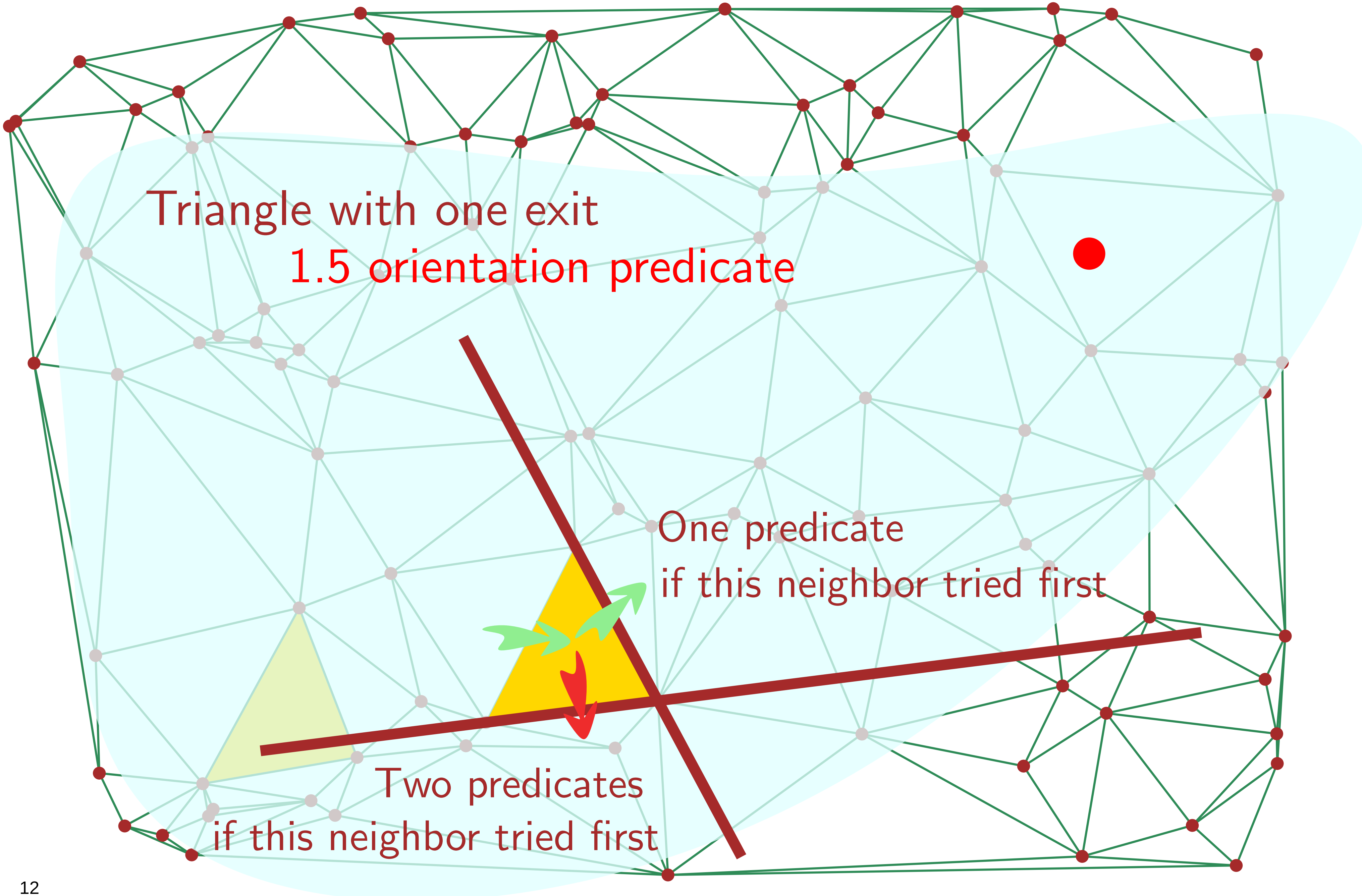

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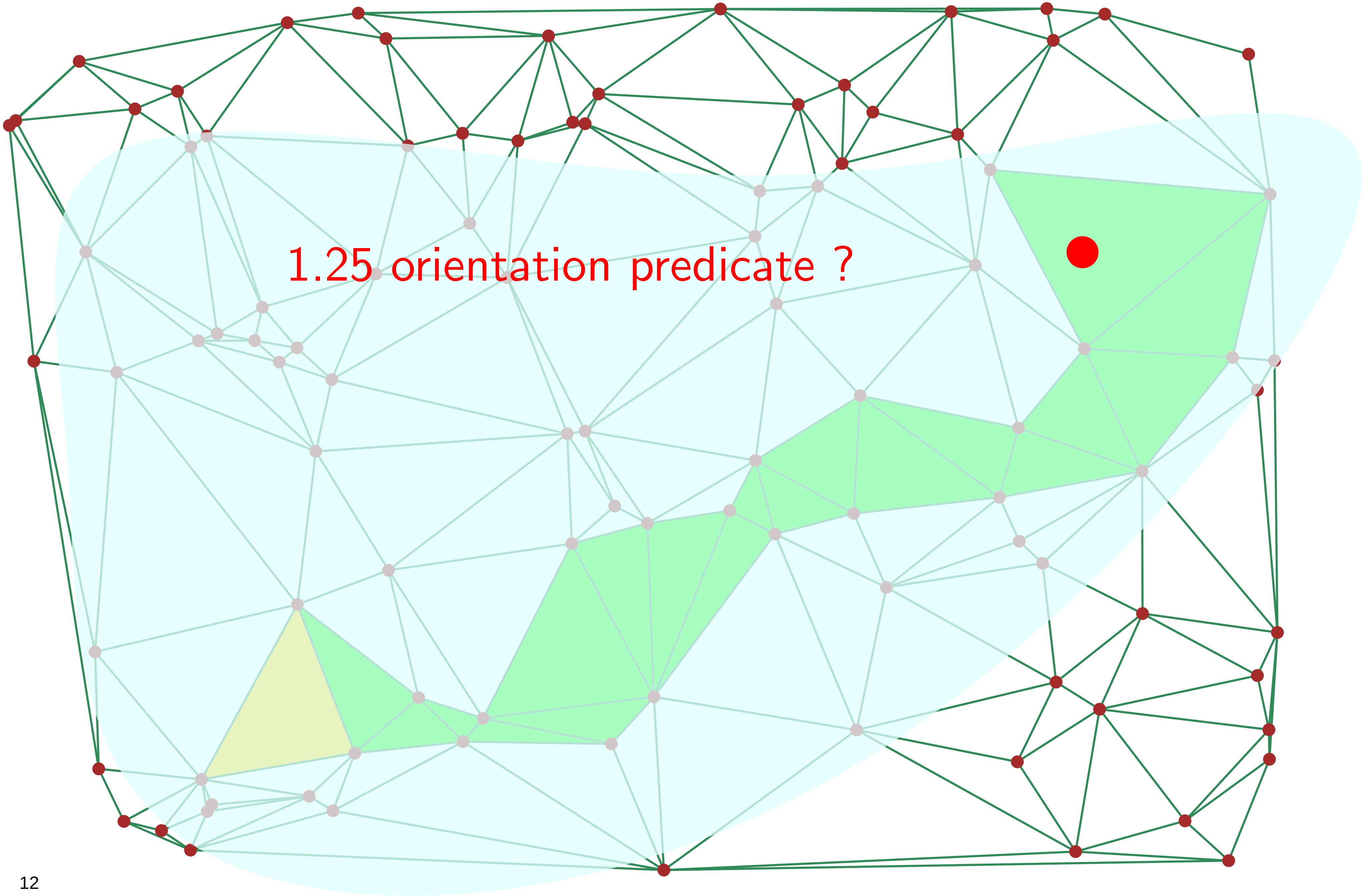


Locate by walk

Visibility vs straight walk

Locate by walk

Visibility vs straight walk 2D and 3D

less predicates per crossed edge

similar number of crossed edges

experimental / theoretical

Locate by walk

Visibility vs straight walk

Speed improvement ?

Locate by walk

Visibility vs straight walk

Speed improvement ?

Walk in Delaunay 1 Mpoints

Straight: 324 μs

Visibility: 285 μs

3D: 97 μs

Locate by walk

Visibility vs straight walk

Speed improvement ?

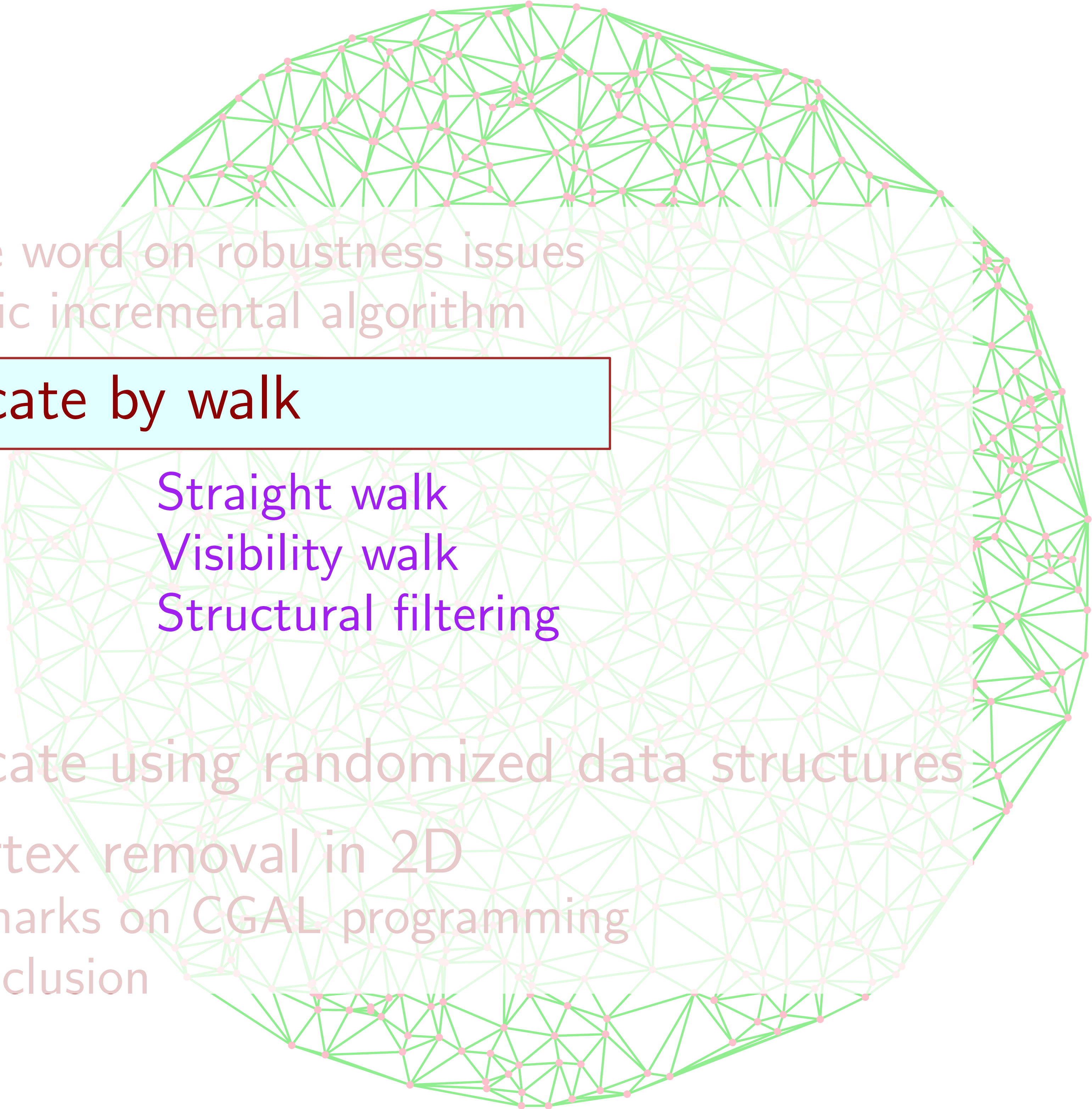
Walk in Delaunay 1 Mpoints

Straight: 324 μs

Visibility: 285 μs

3D: 97 μs

Much easier to code



One word on robustness issues

Basic incremental algorithm

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- Visibility walk
- Structural filtering

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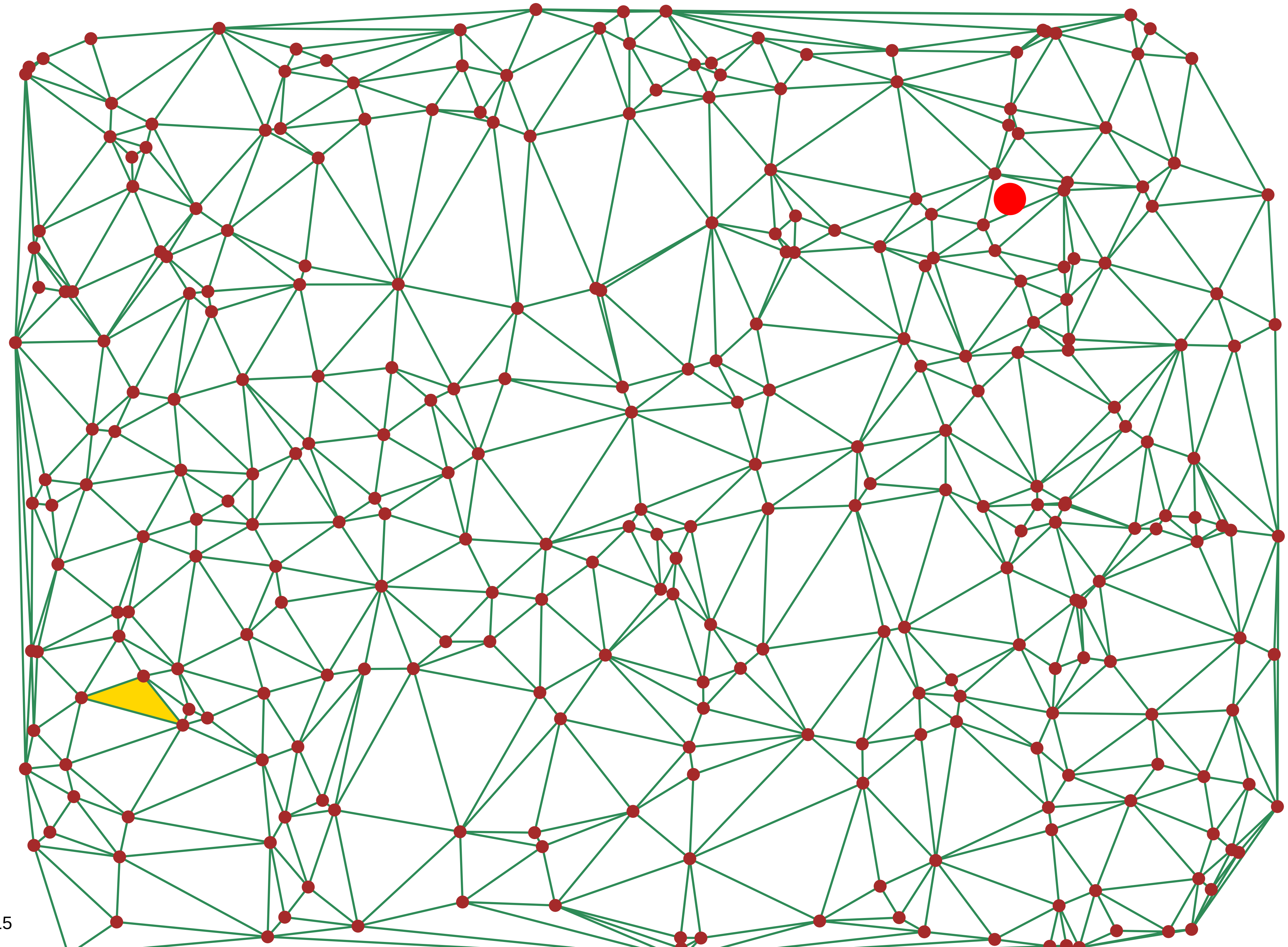
Vertex removal in 2D

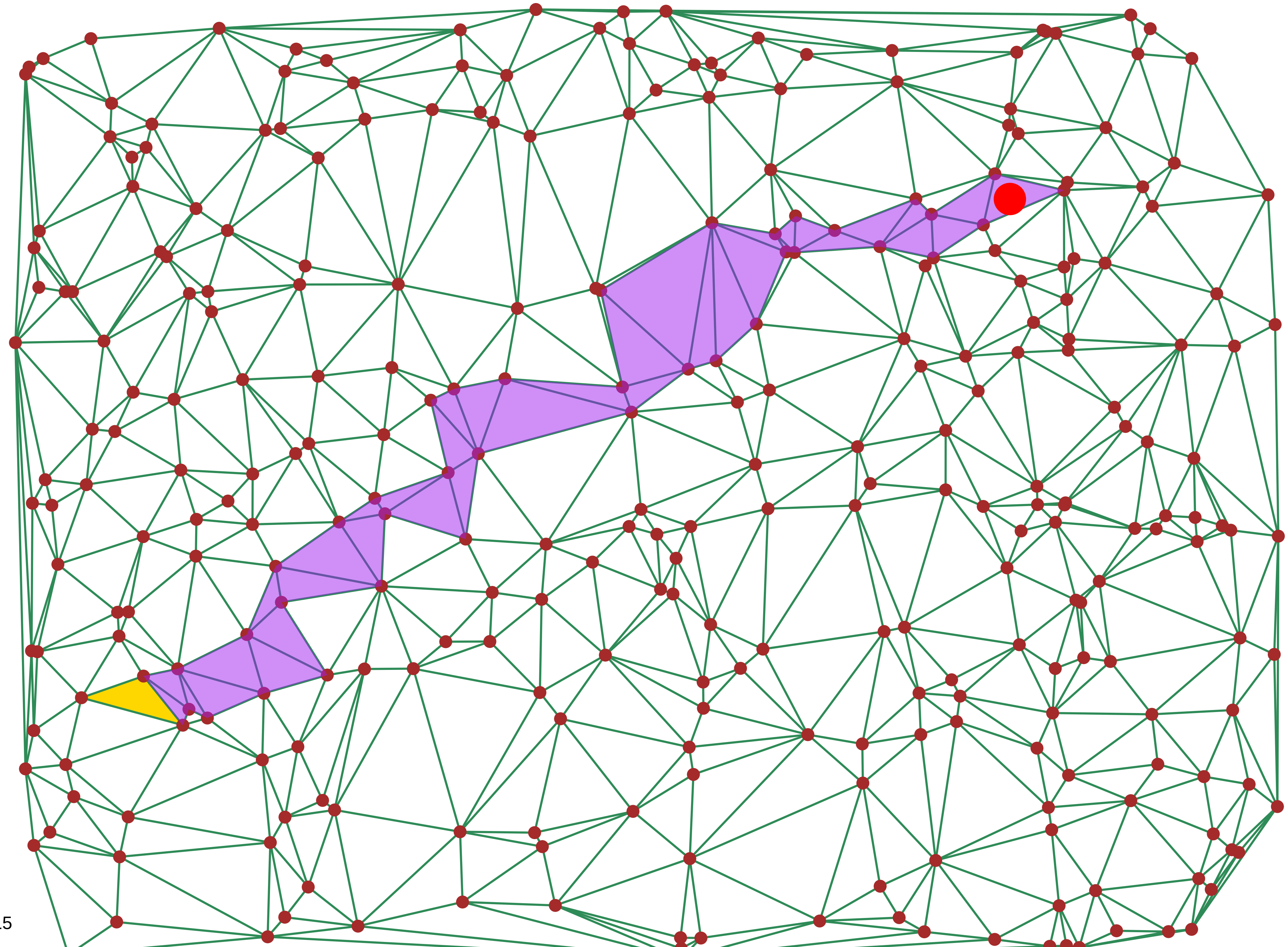
Remarks on CGAL programming

Conclusion

Locate by visibility walk - structural filtering



Locate by visibility walk - structural filtering



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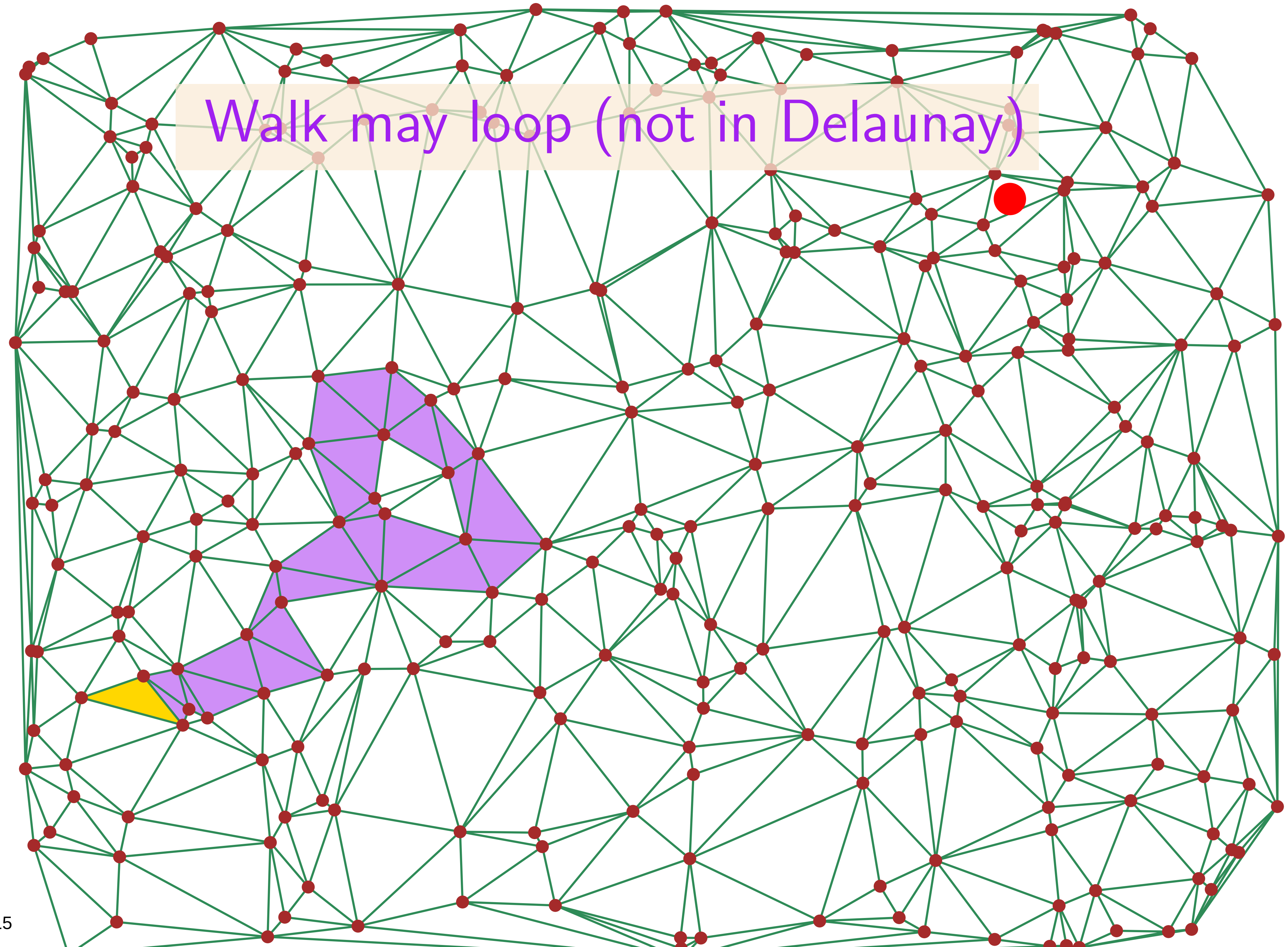
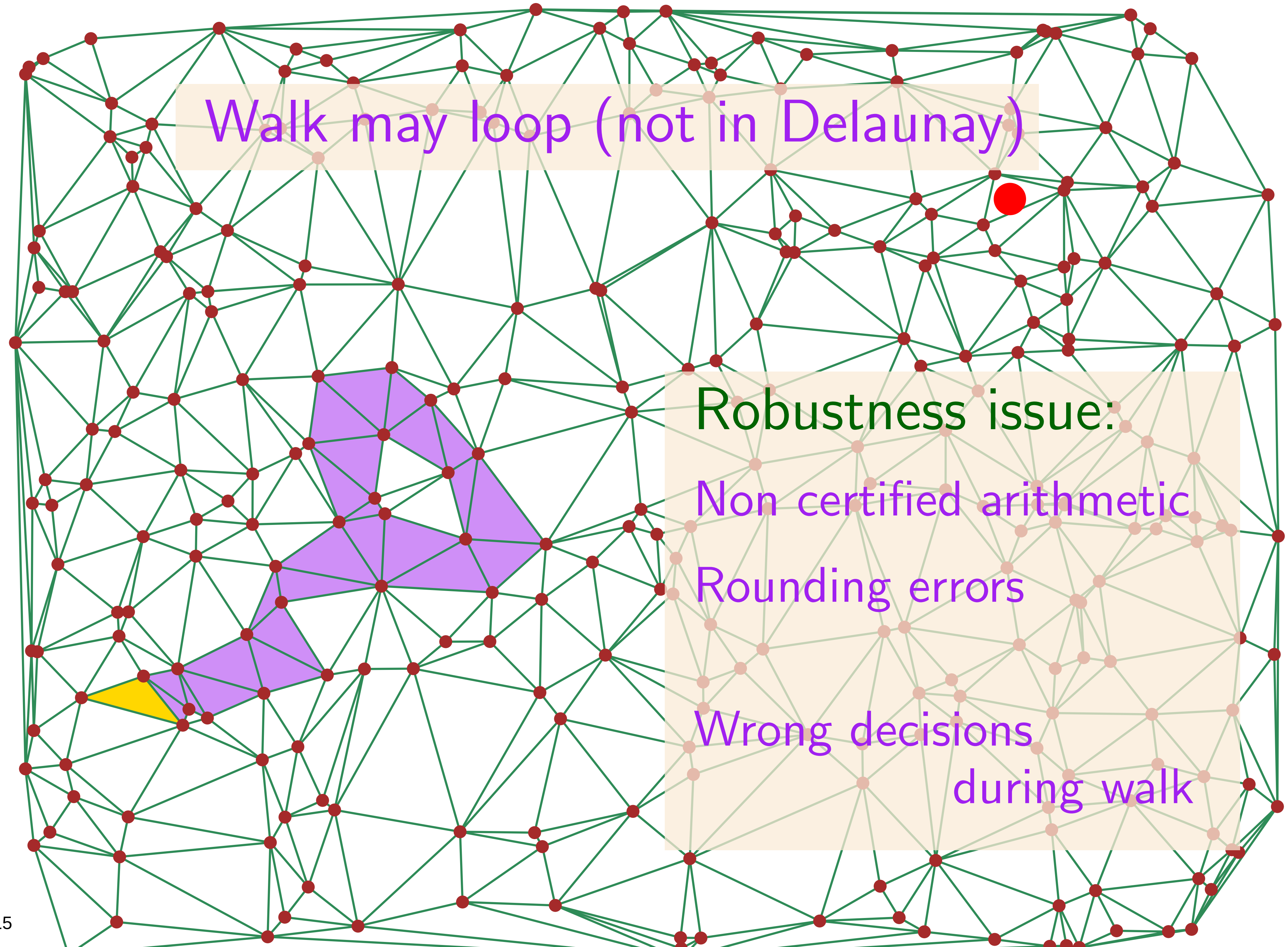
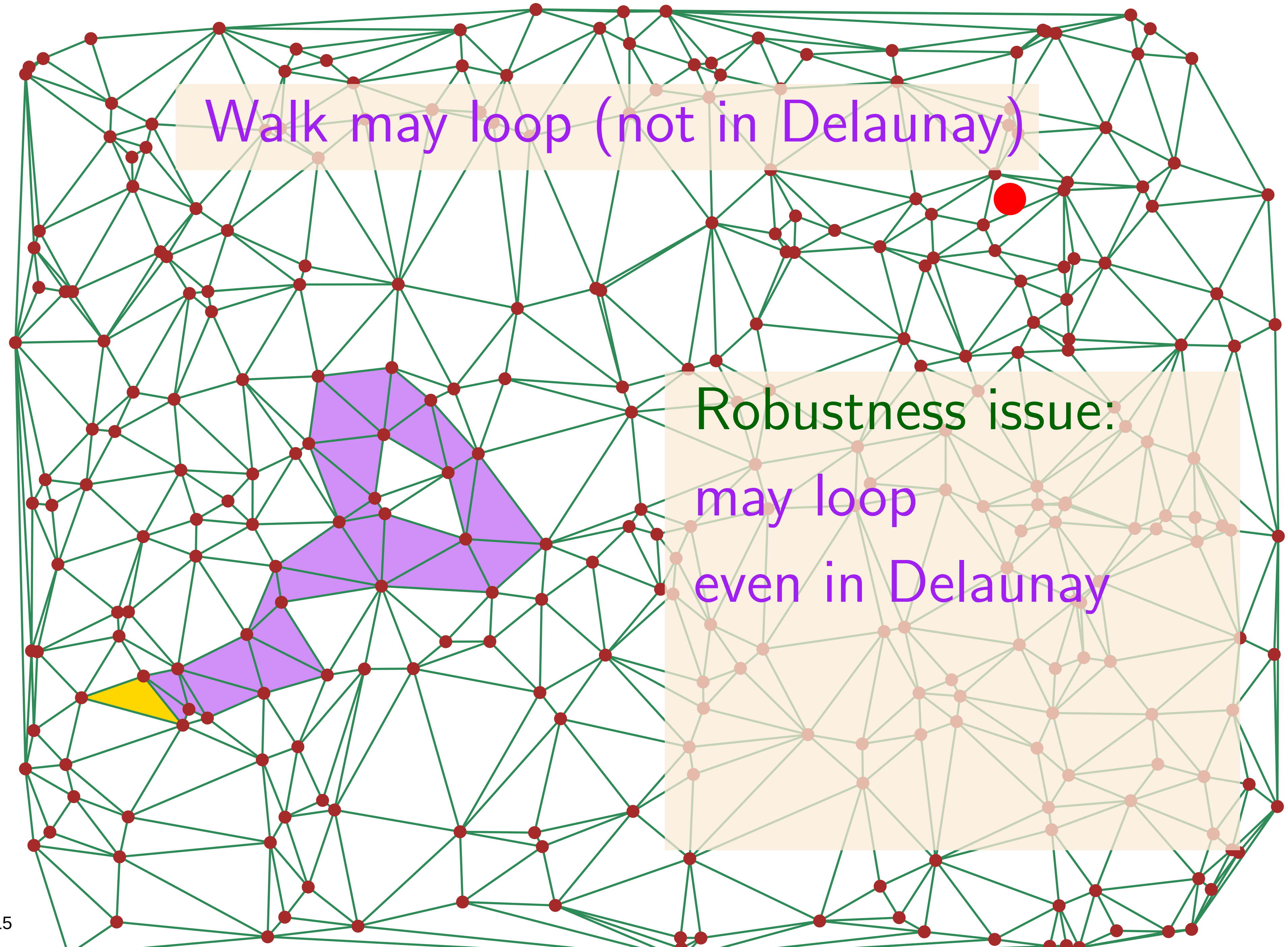


Locate by visibility walk - structural filtering



Locate by visibility walk - structural filtering



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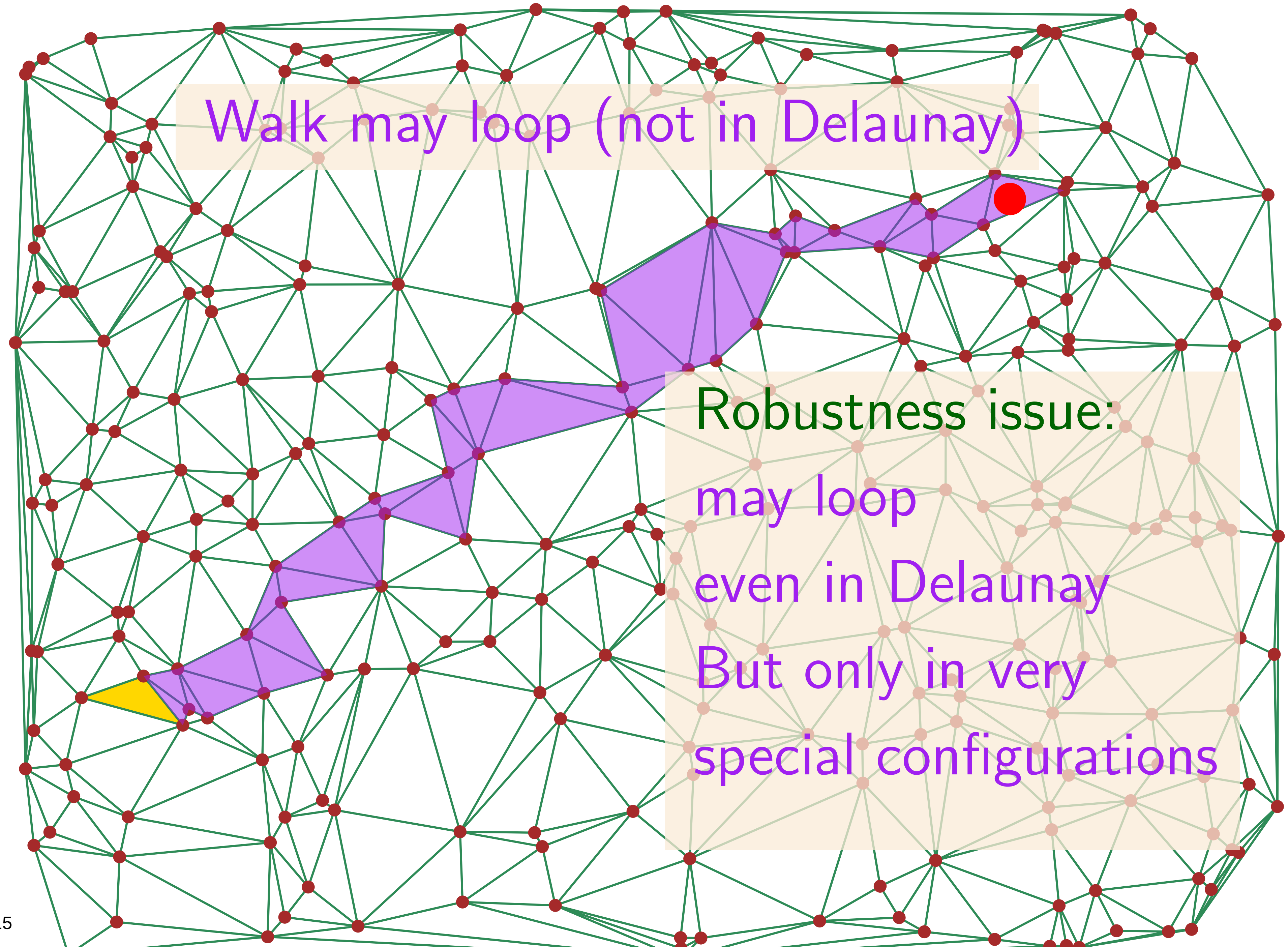
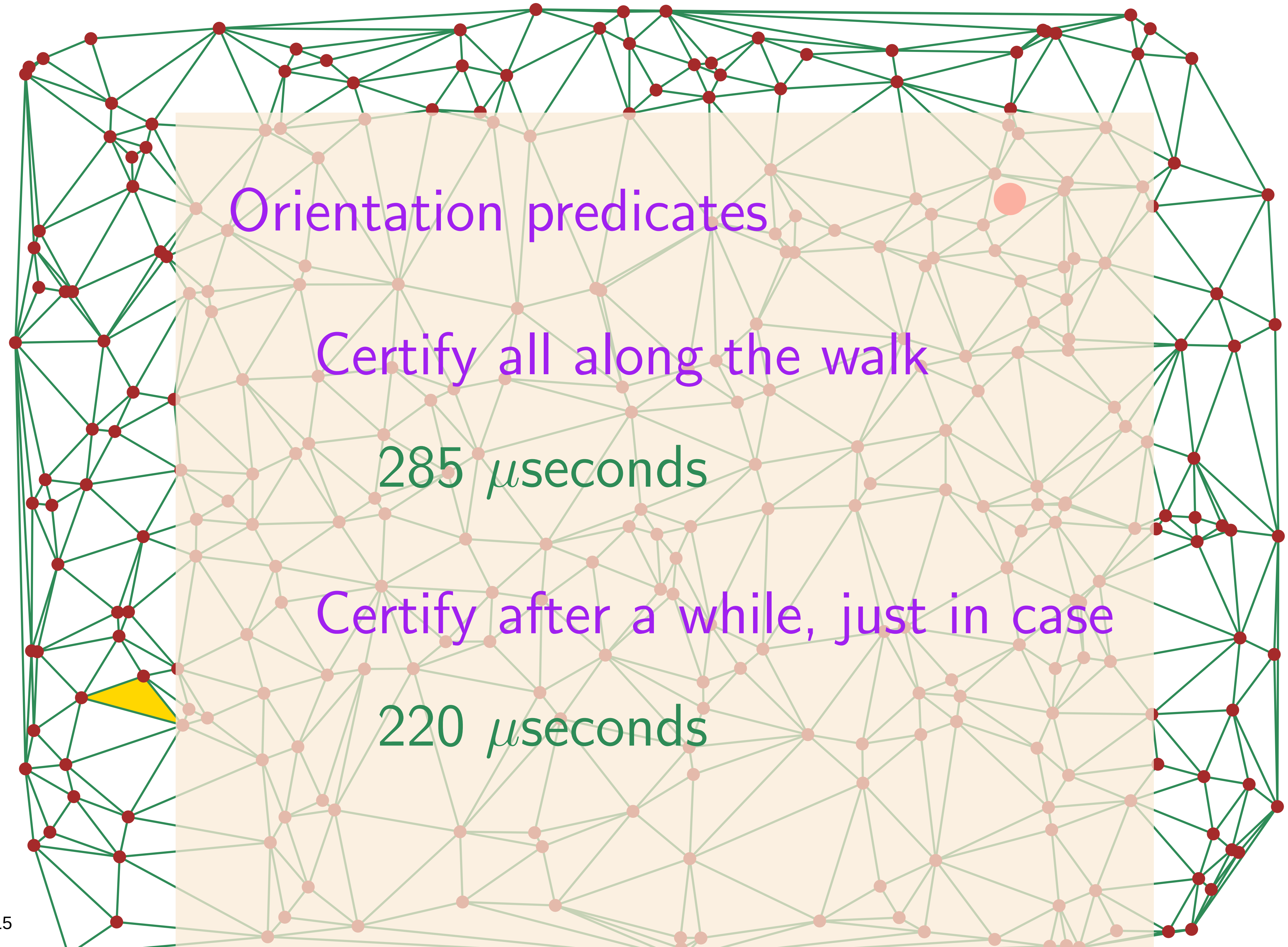


Locate by visibility walk - structural filtering



Orientation predicates

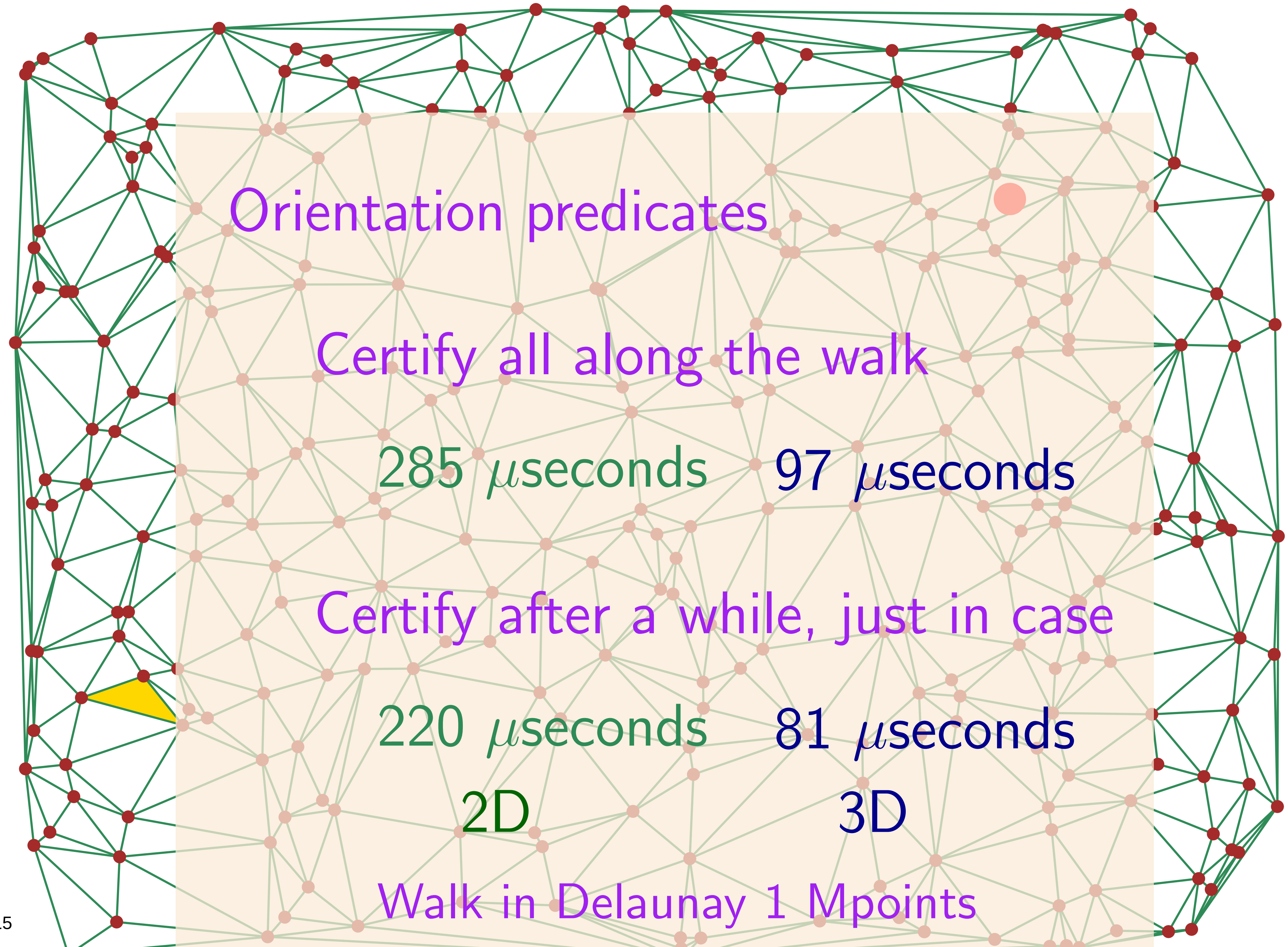
Certify all along the walk

285 μseconds

Certify after a while, just in case

220 μseconds

Locate by visibility walk - structural filtering



Orientation predicates

Certify all along the walk

285 μseconds 97 μseconds

Certify after a while, just in case

220 μseconds 81 μseconds

2D 3D

Walk in Delaunay 1 Mpoints

One word on robustness issues

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- Visibility walk
- Structural filtering
- Walk shape

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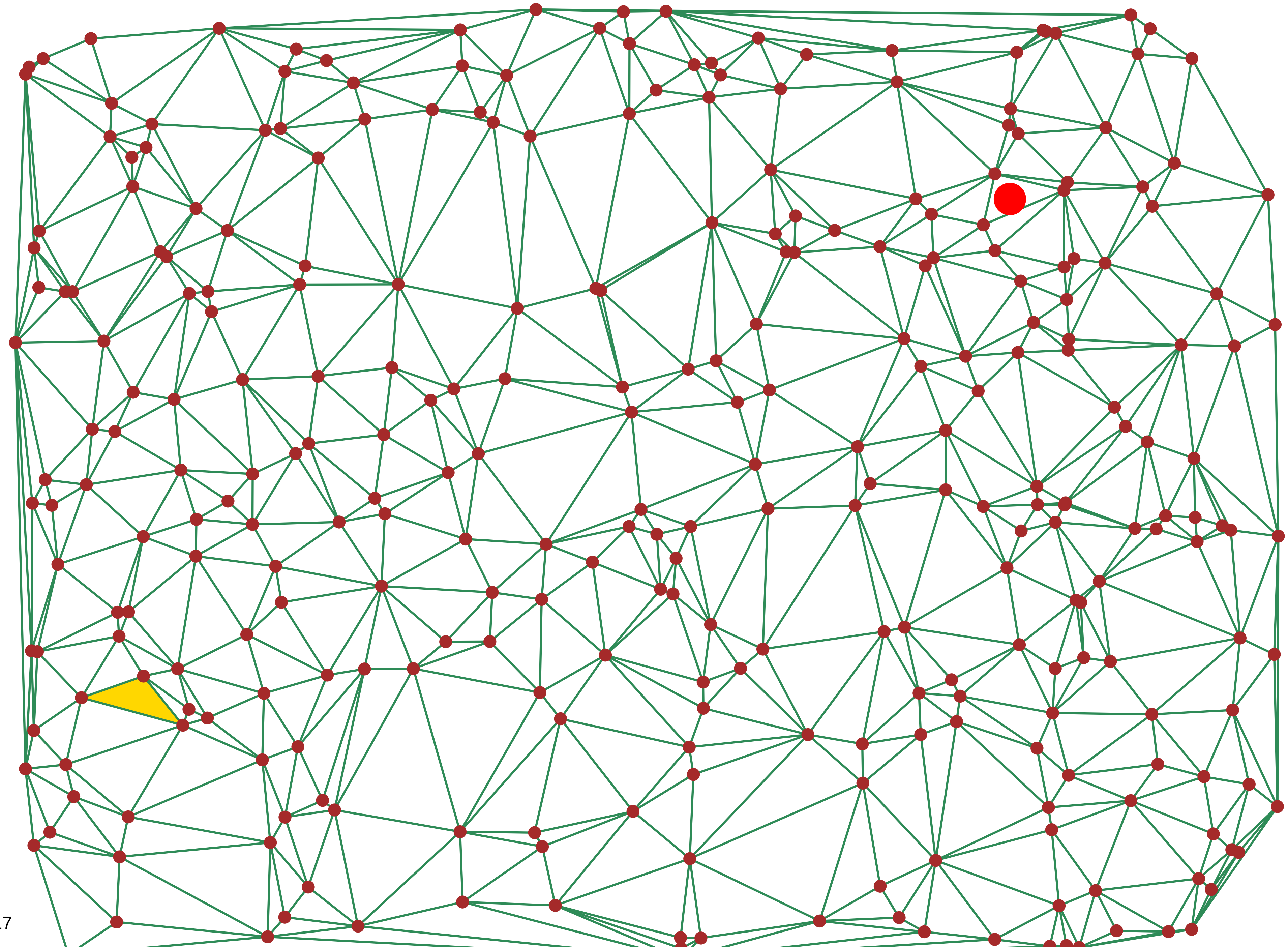
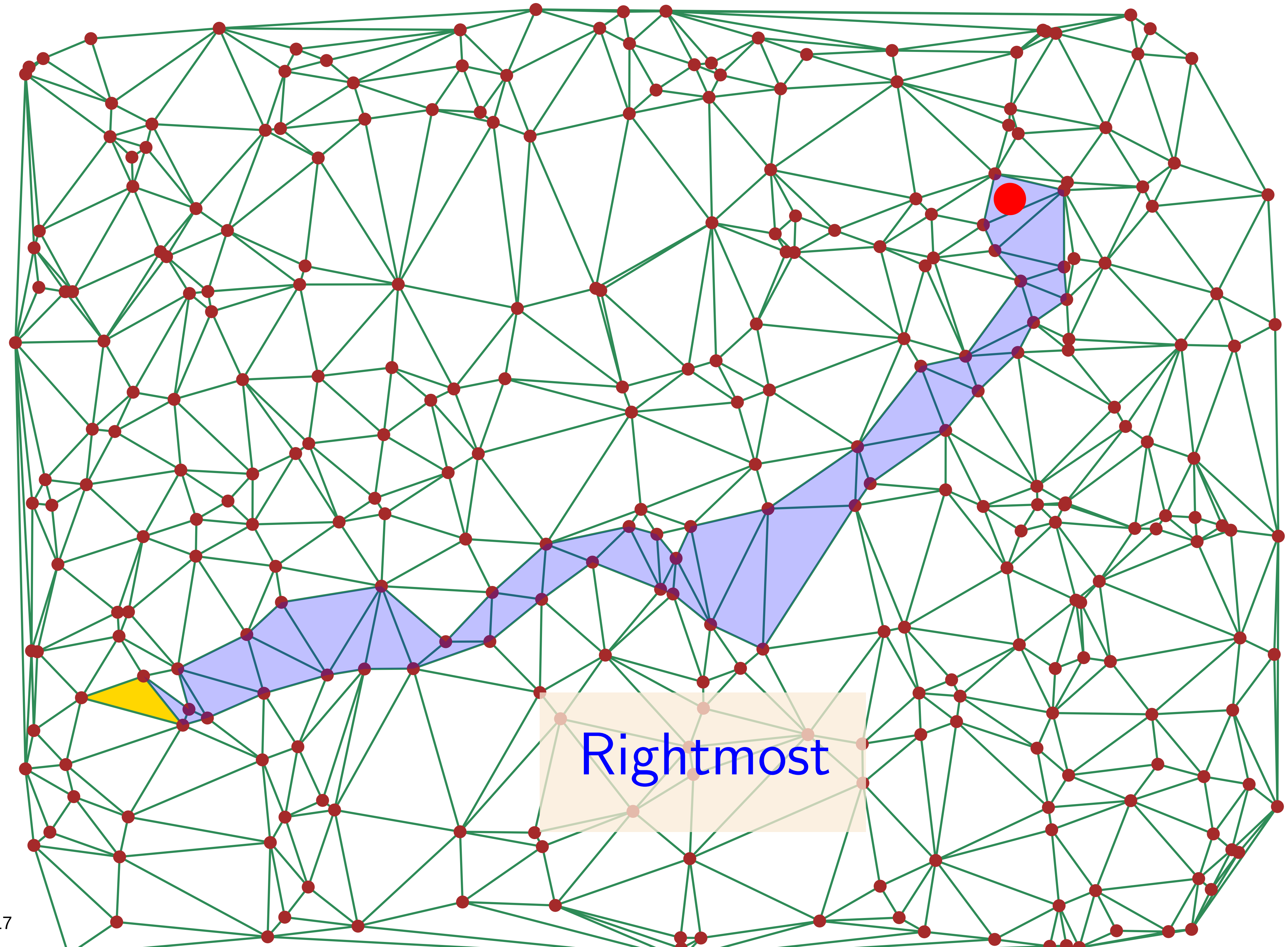
Conclusion

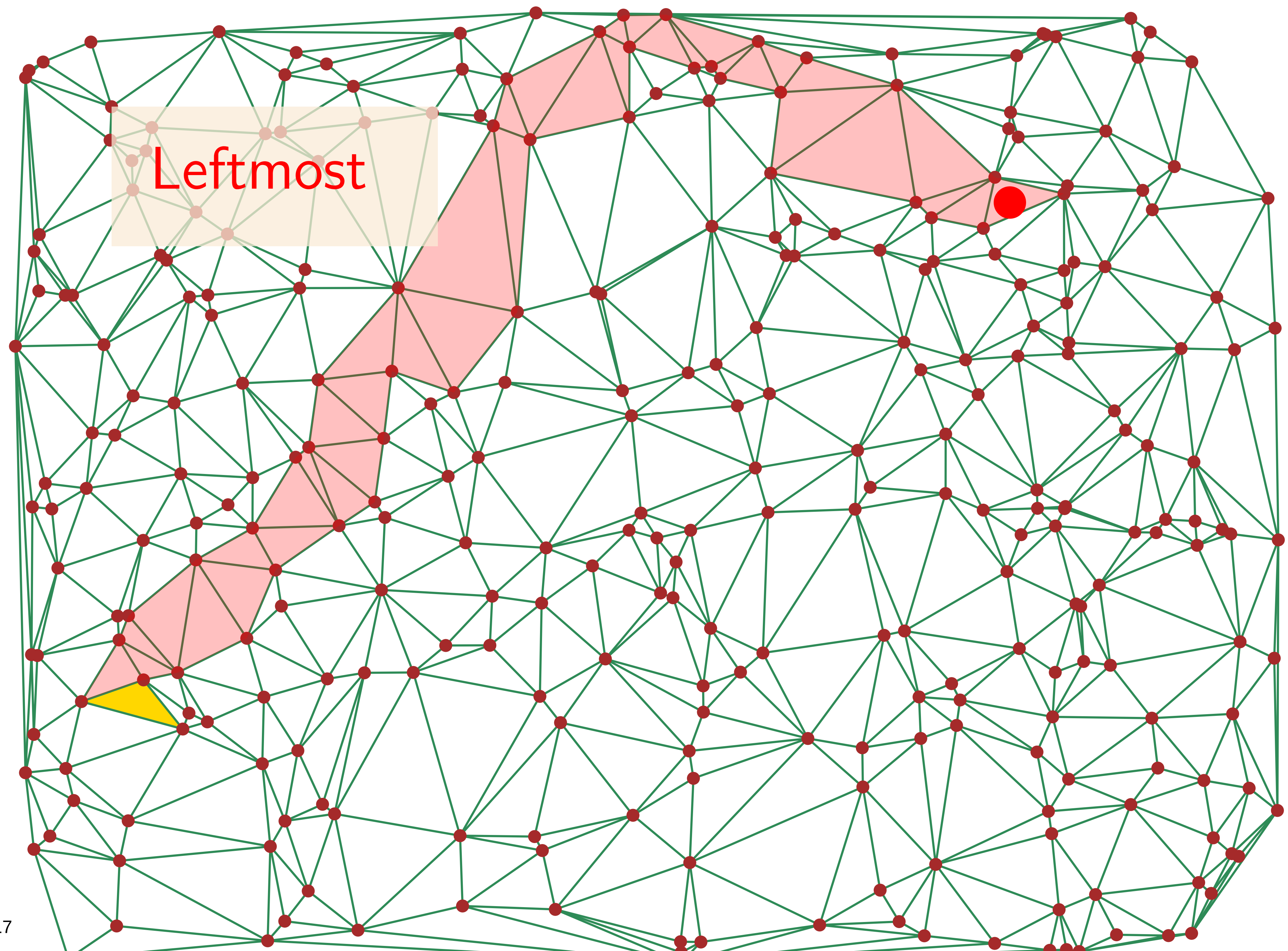
Locate by visibility walk - walk shape



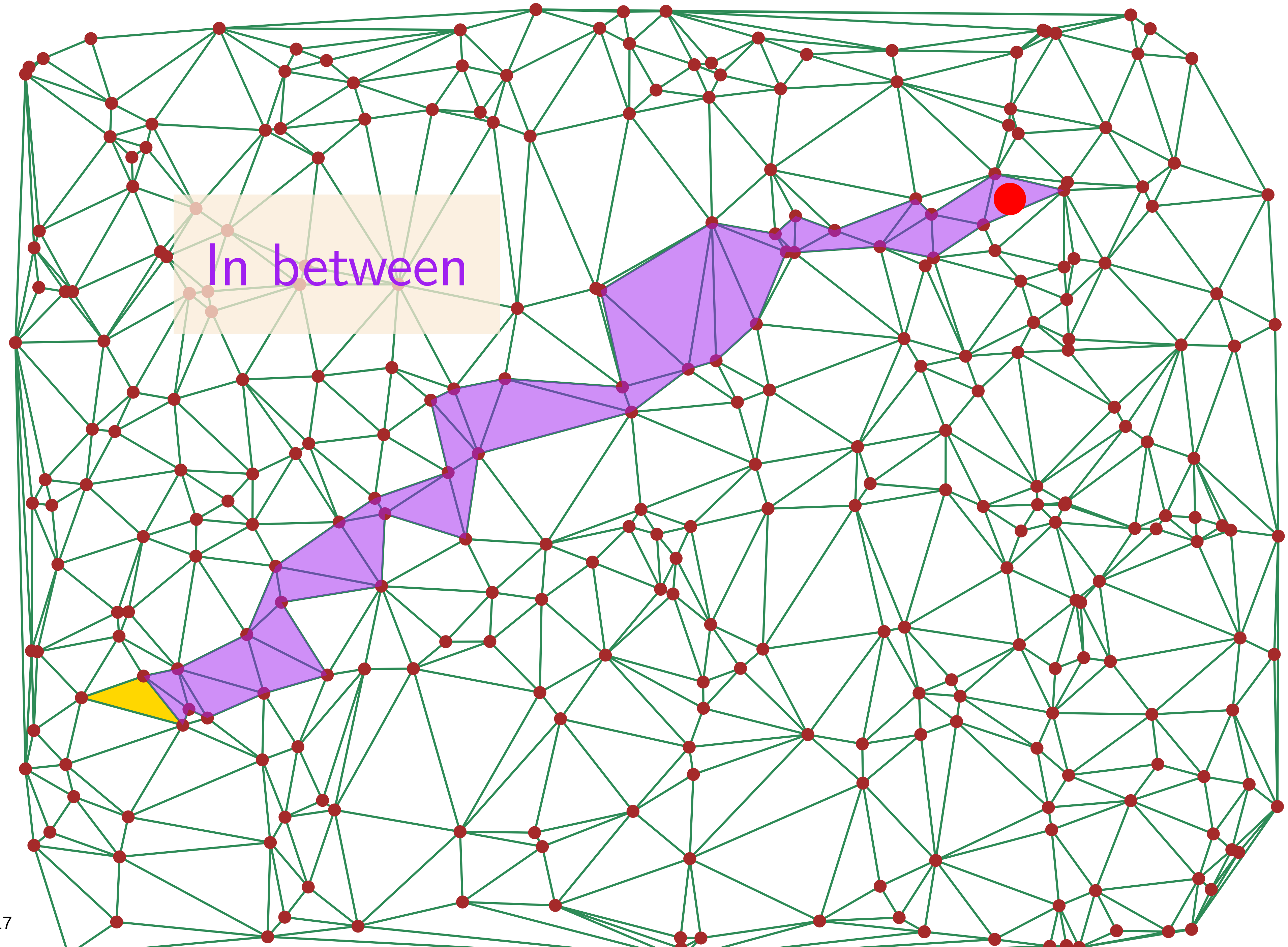
Locate by visibility walk - walk shape



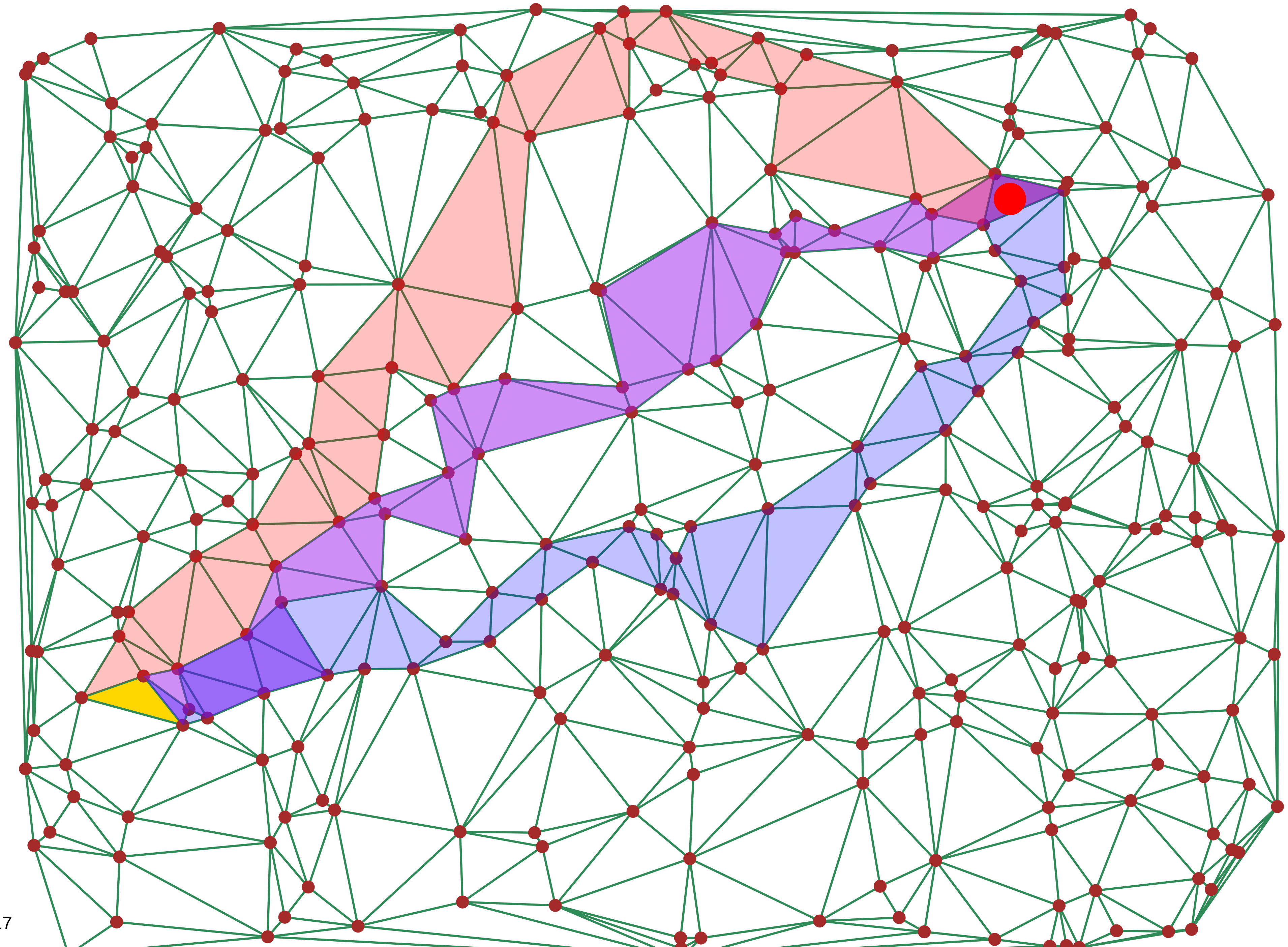
Locate by visibility walk - walk shape



Locate by visibility walk - walk shape

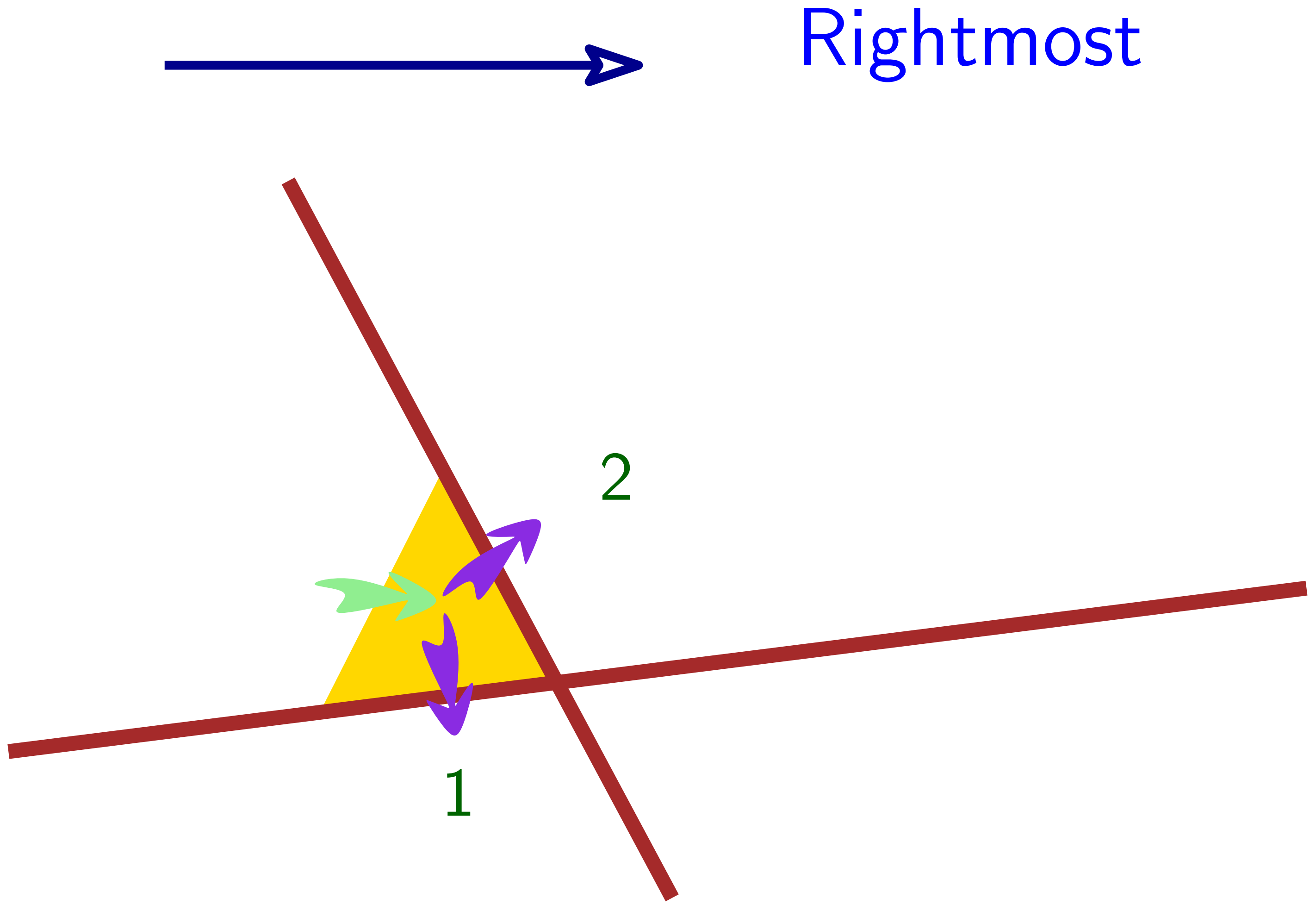


Locate by visibility walk - walk shape



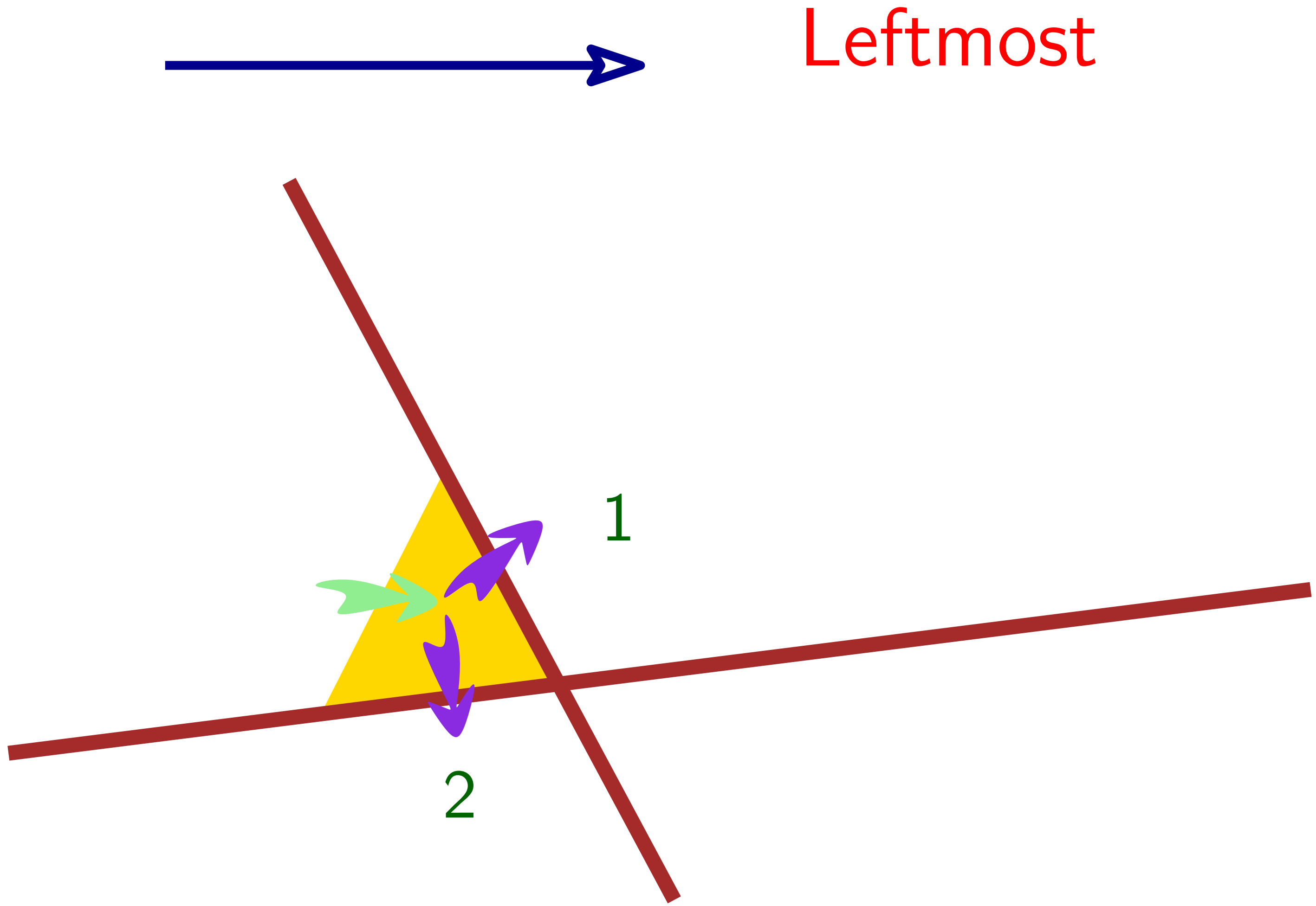
Locate by visibility walk - walk shape

Turn counterclockwise from previous



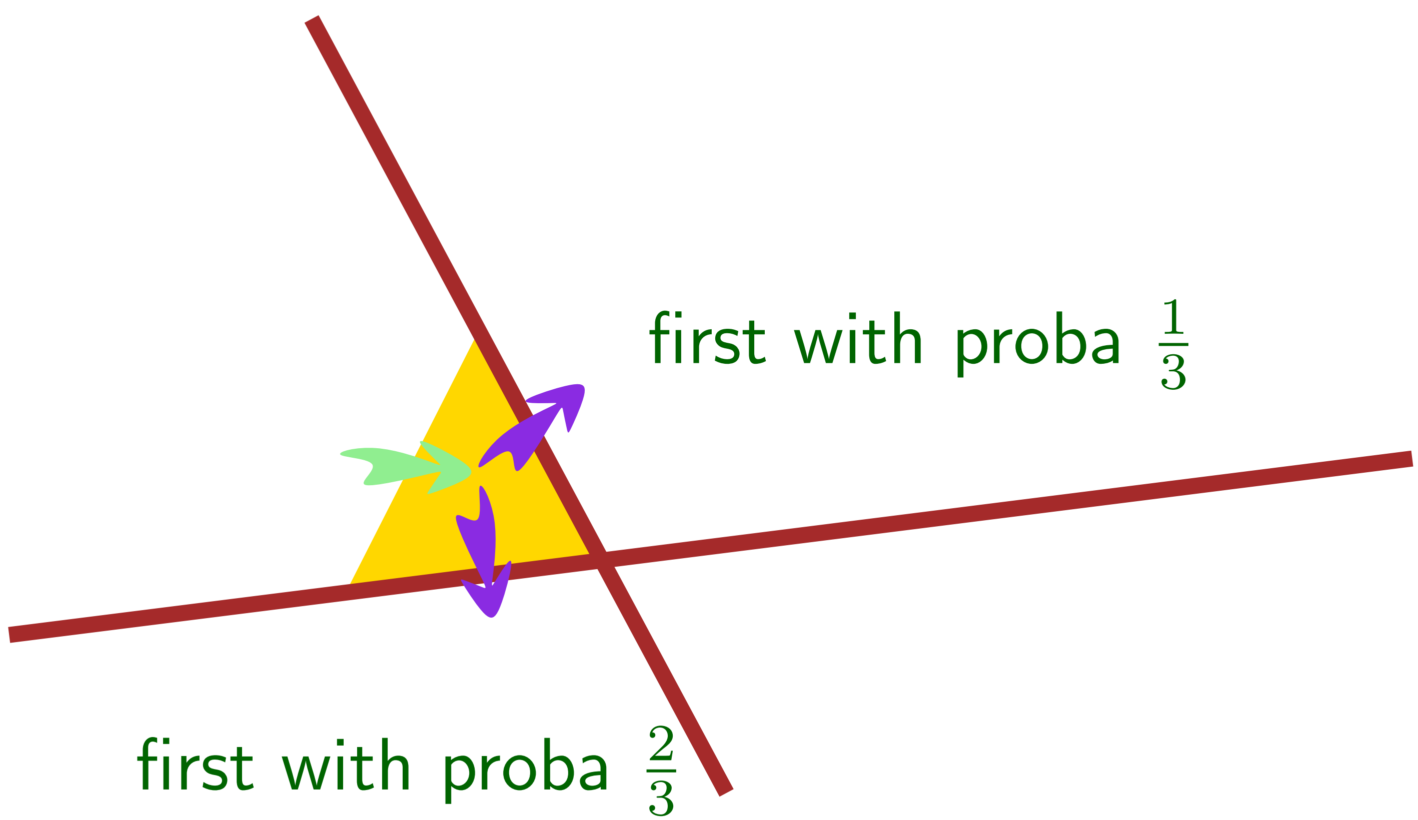
Locate by visibility walk - walk shape

Turn clockwise from previous



Locate by visibility walk - walk shape

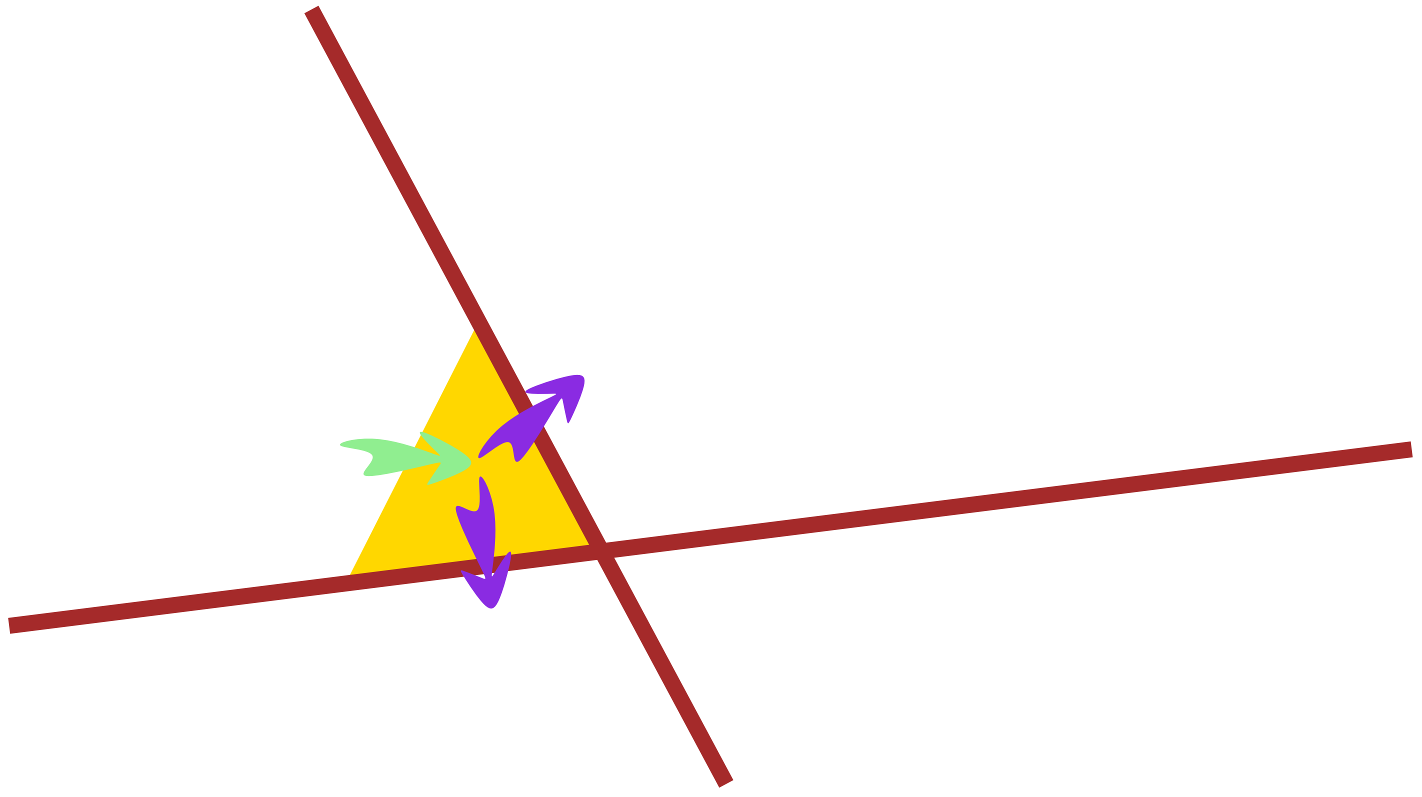
Balance left and right turns



Locate by visibility walk - walk shape

Walk in Delaunay 1 Mpoints

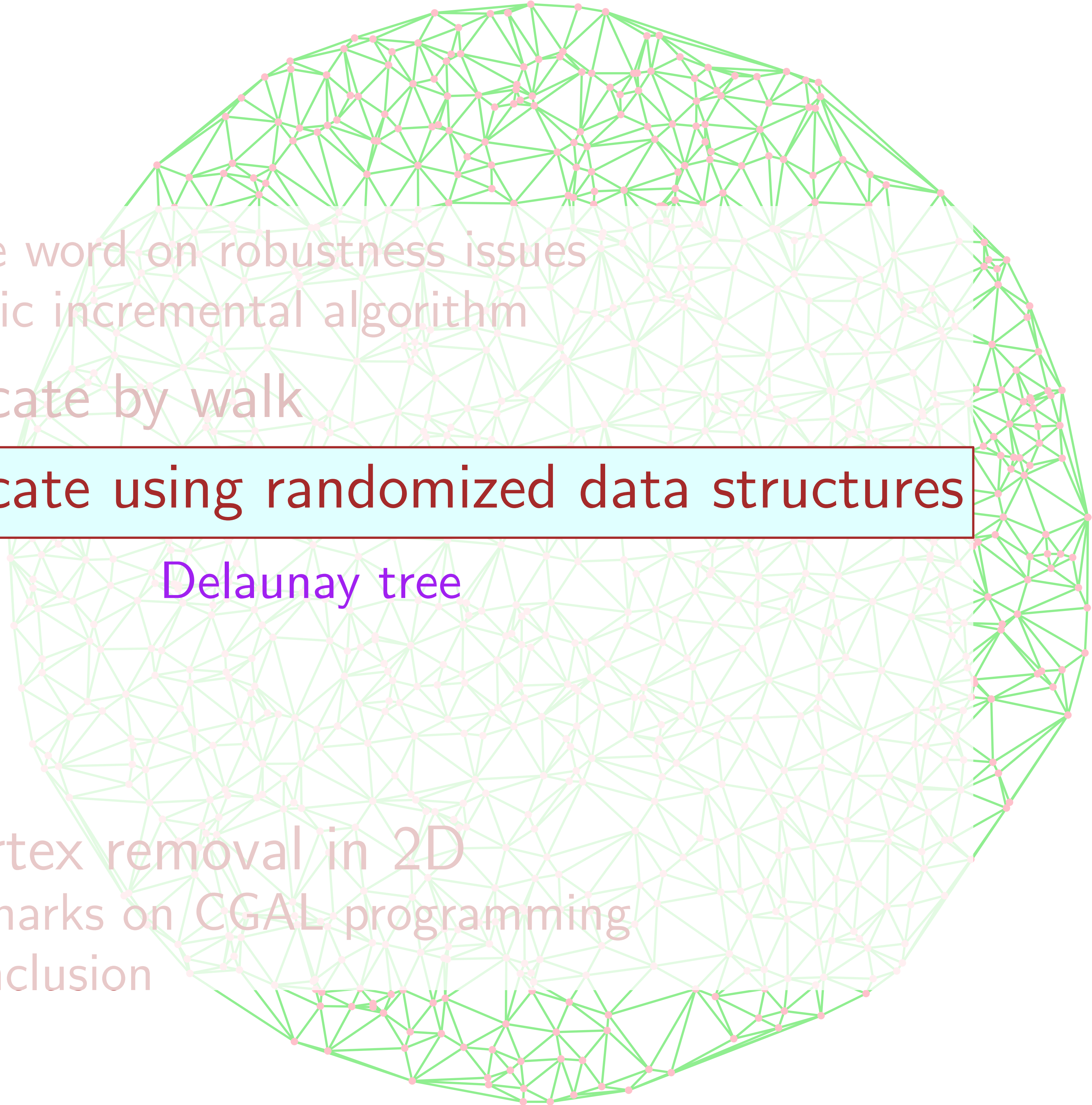
Leftmost	220 μseconds
Balanced	188 μseconds



Locate by visibility walk

Walk in Delaunay 1 Mpoints

Straight walk	324 μseconds
Visibility walk	285 μseconds
Structural filtering	220 μseconds
Balanced walk	188 μseconds



One word on robustness issues

Basic incremental algorithm

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Locate using randomized data structures

Delaunay tree

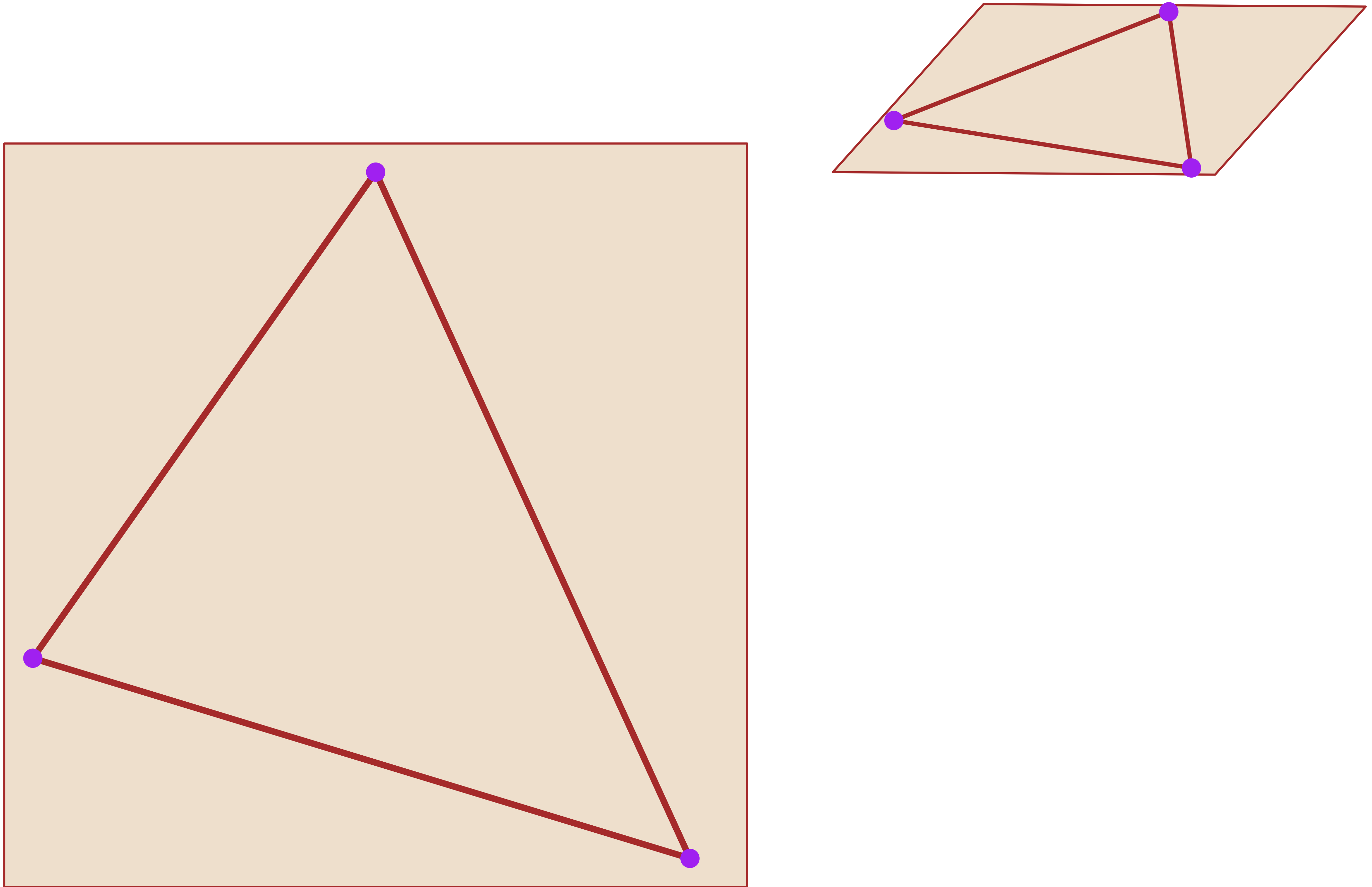
Vertex removal in 2D

Remarks on CGAL programming

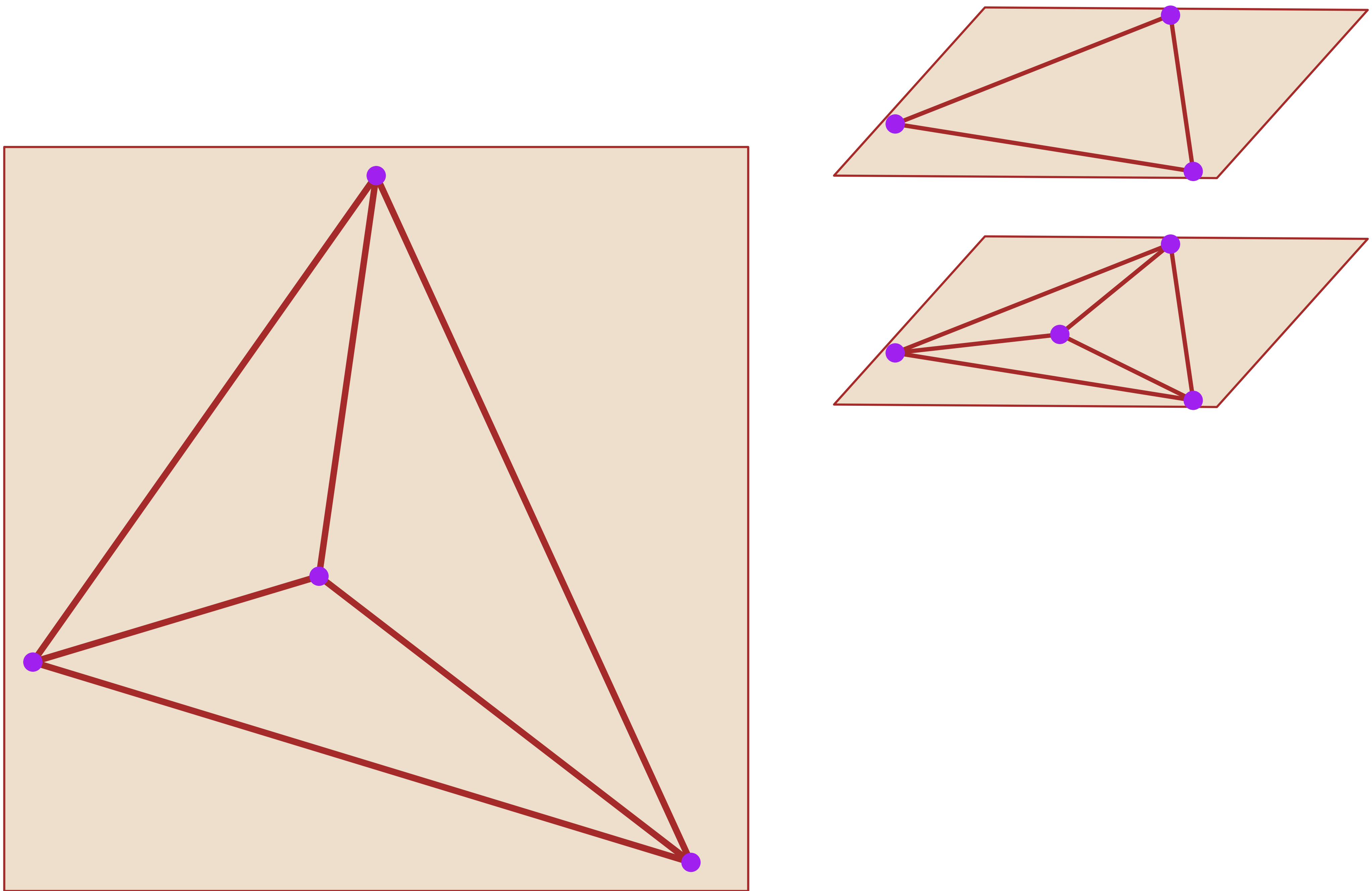
Conclusion

Data structures to locate - the Delaunay tree

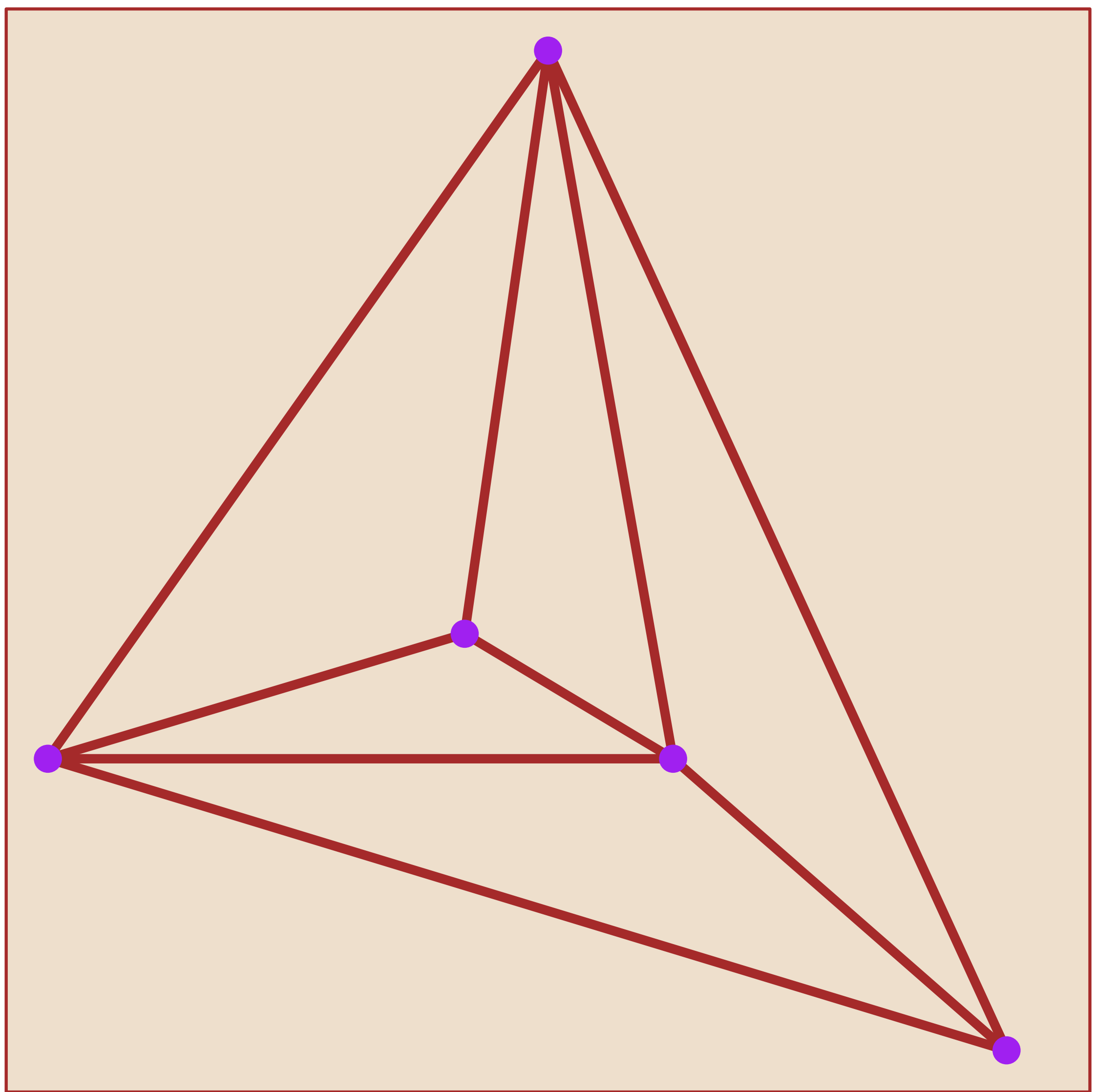
Data structures to locate - the Delaunay tree

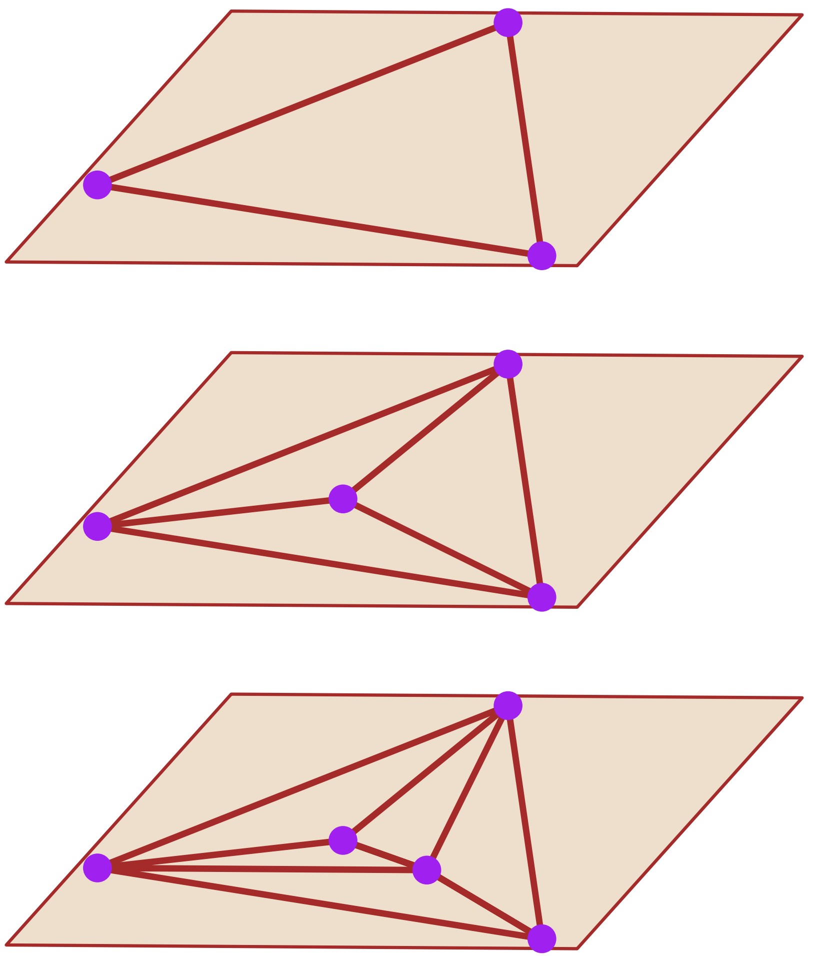


Data structures to locate - the Delaunay tree

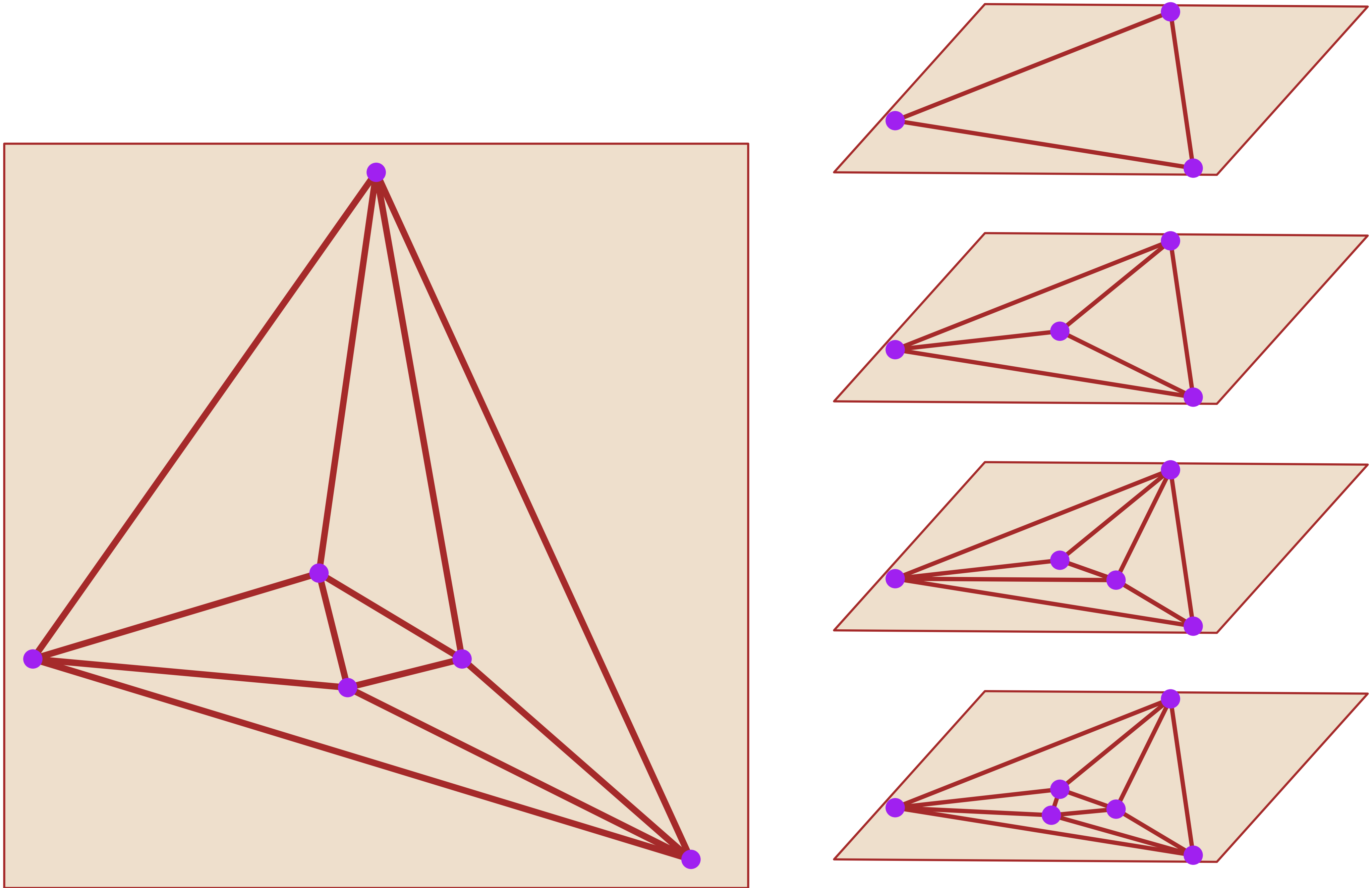


Data structures to locate - the Delaunay tree

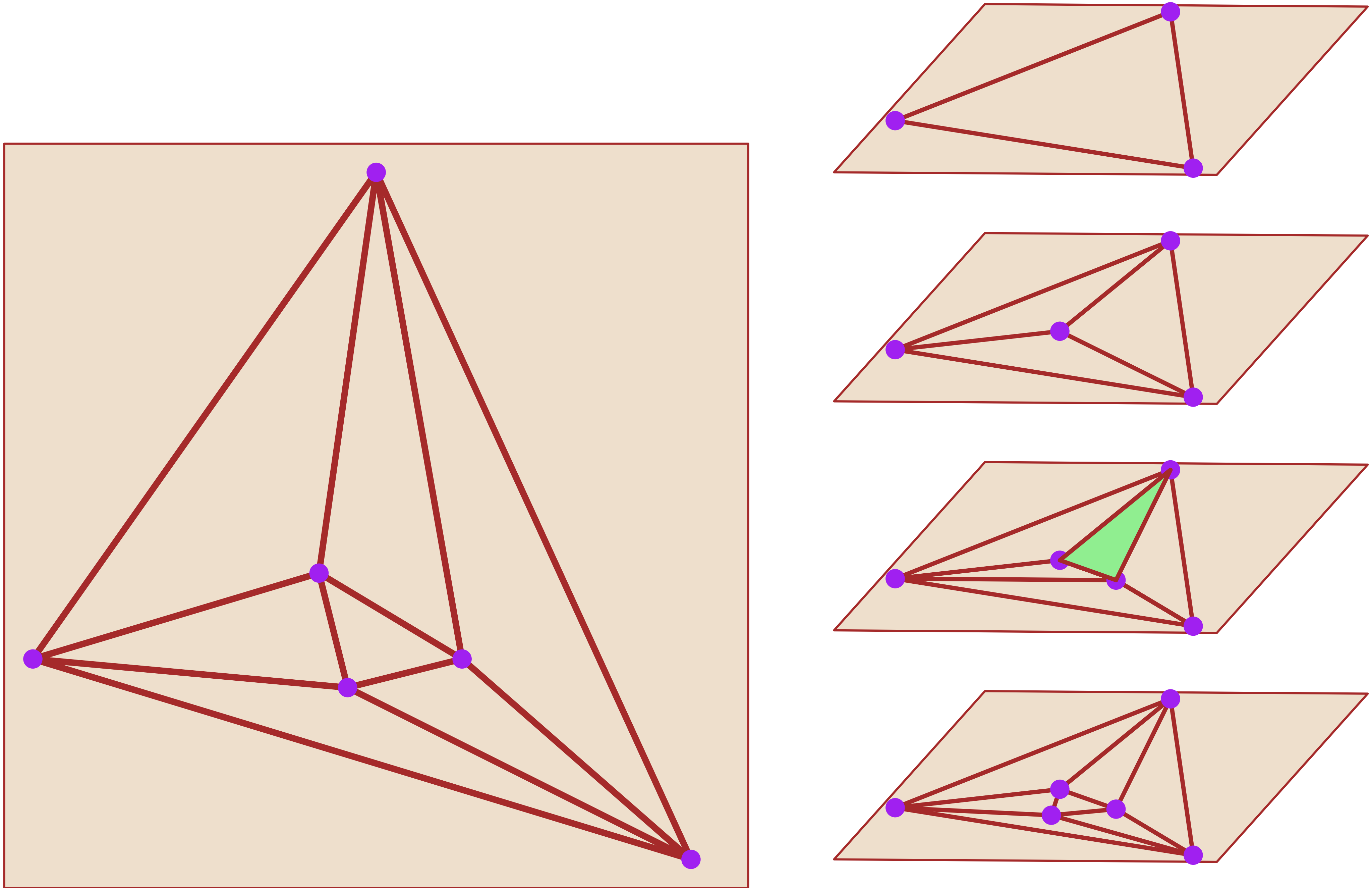




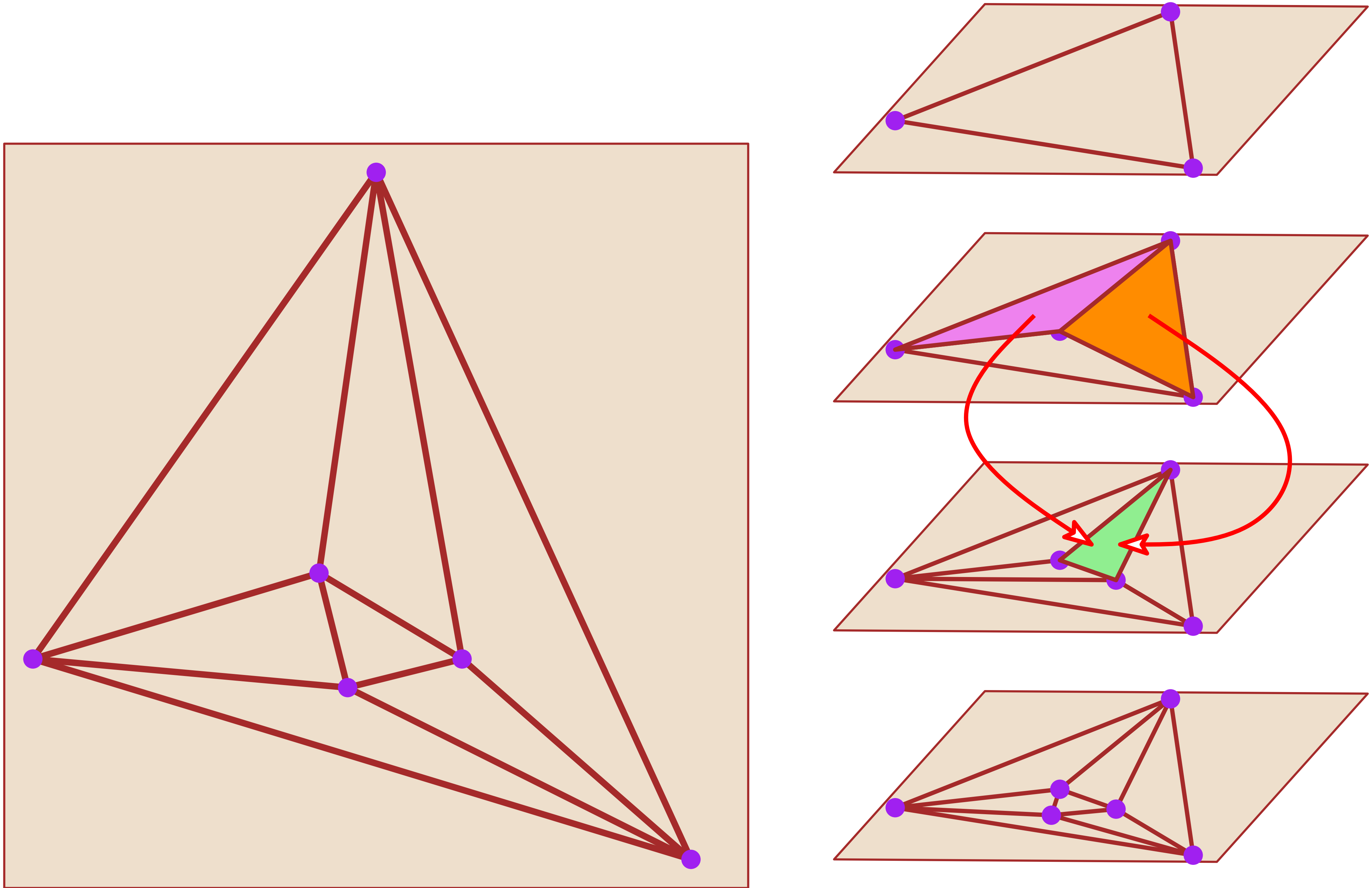
Data structures to locate - the Delaunay tree



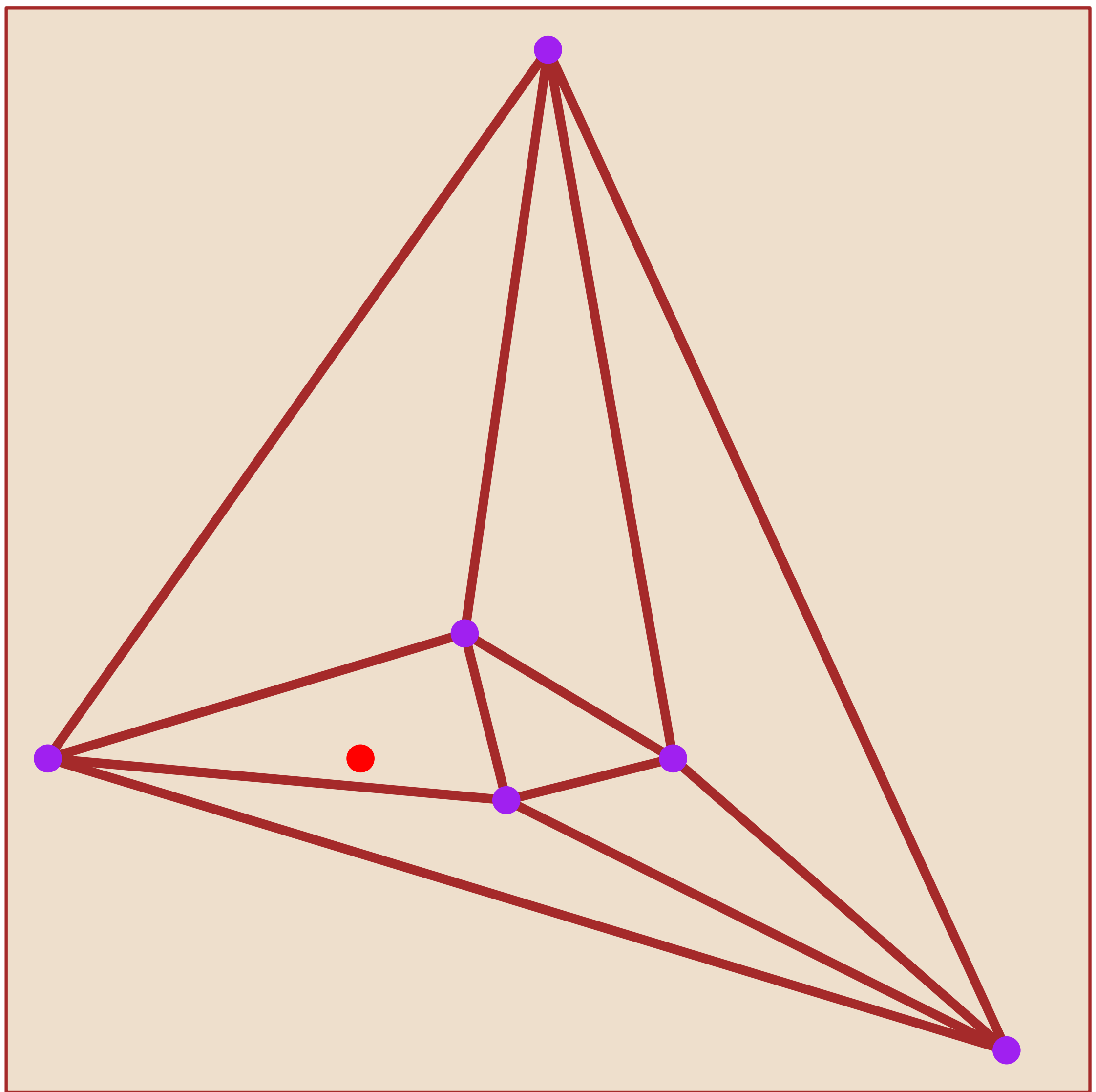
Data structures to locate - the Delaunay tree

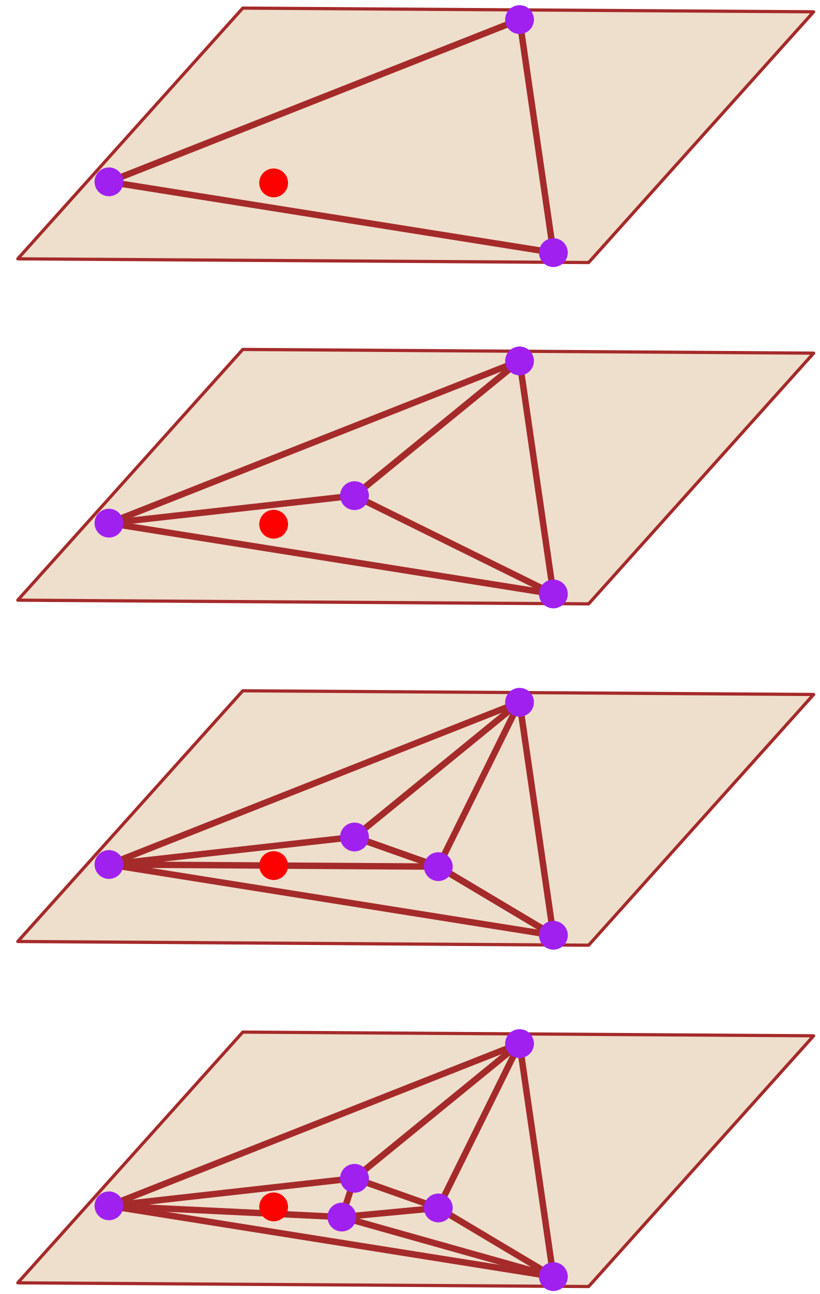


Data structures to locate - the Delaunay tree

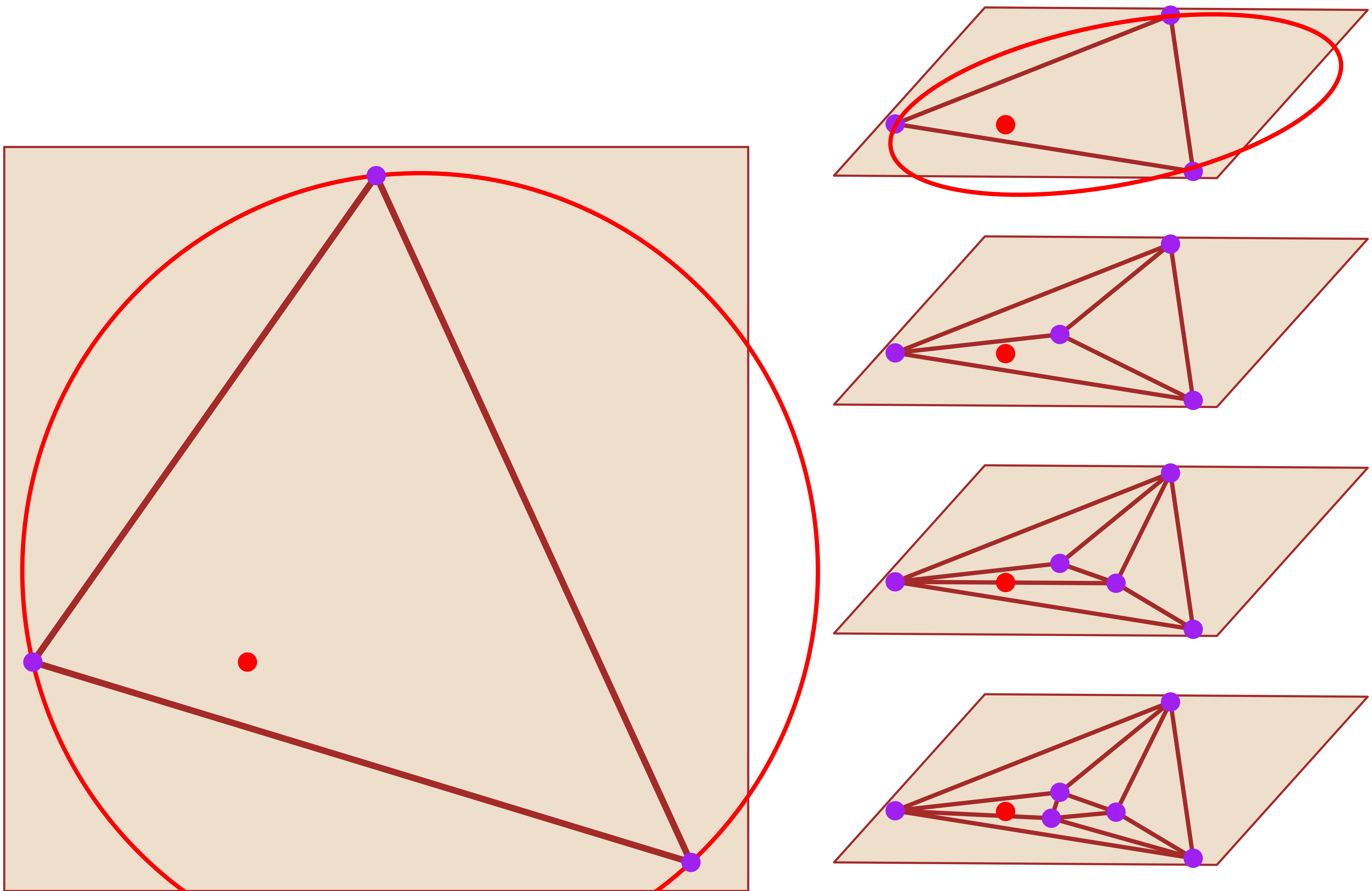


Data structures to locate - the Delaunay tree

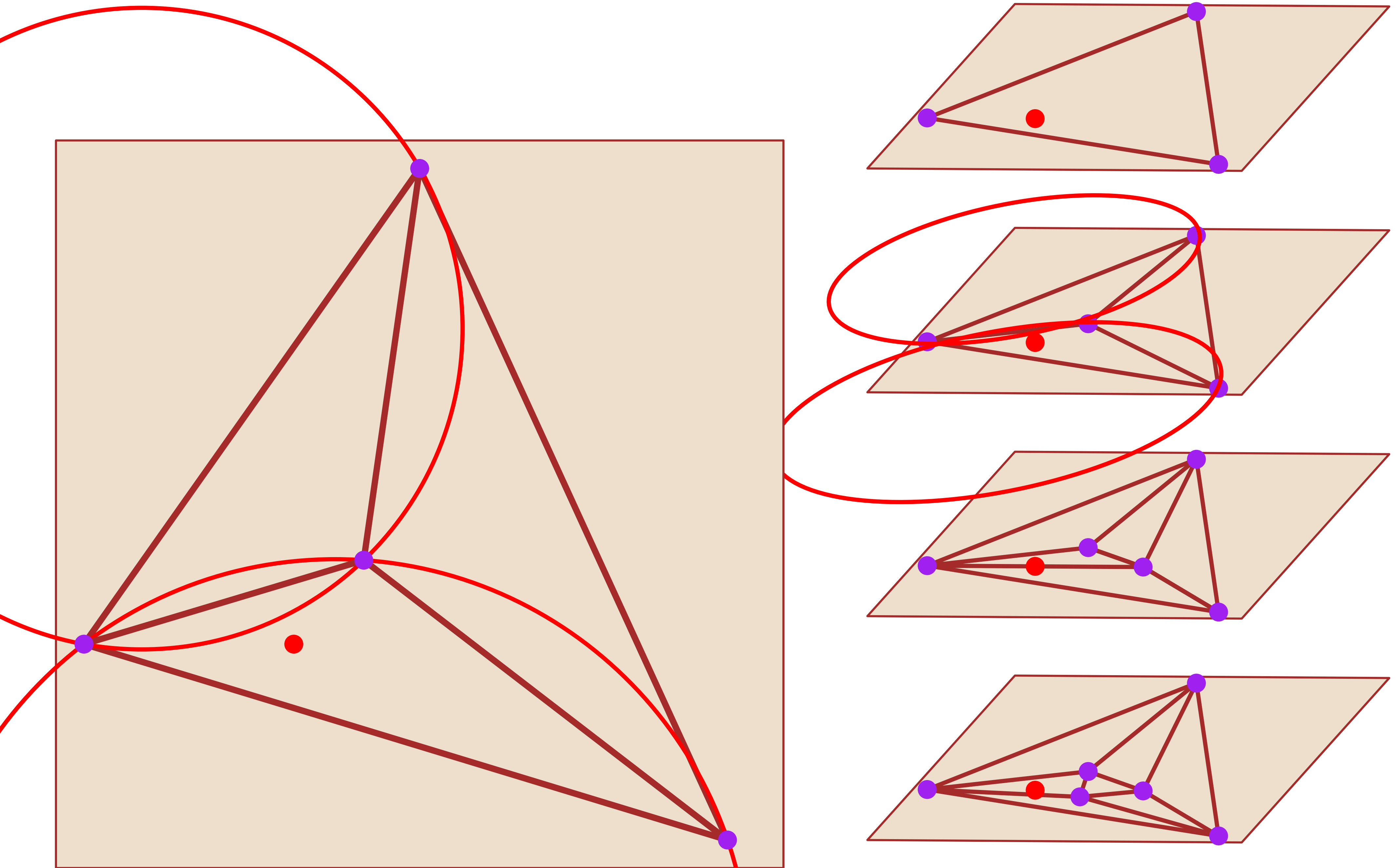




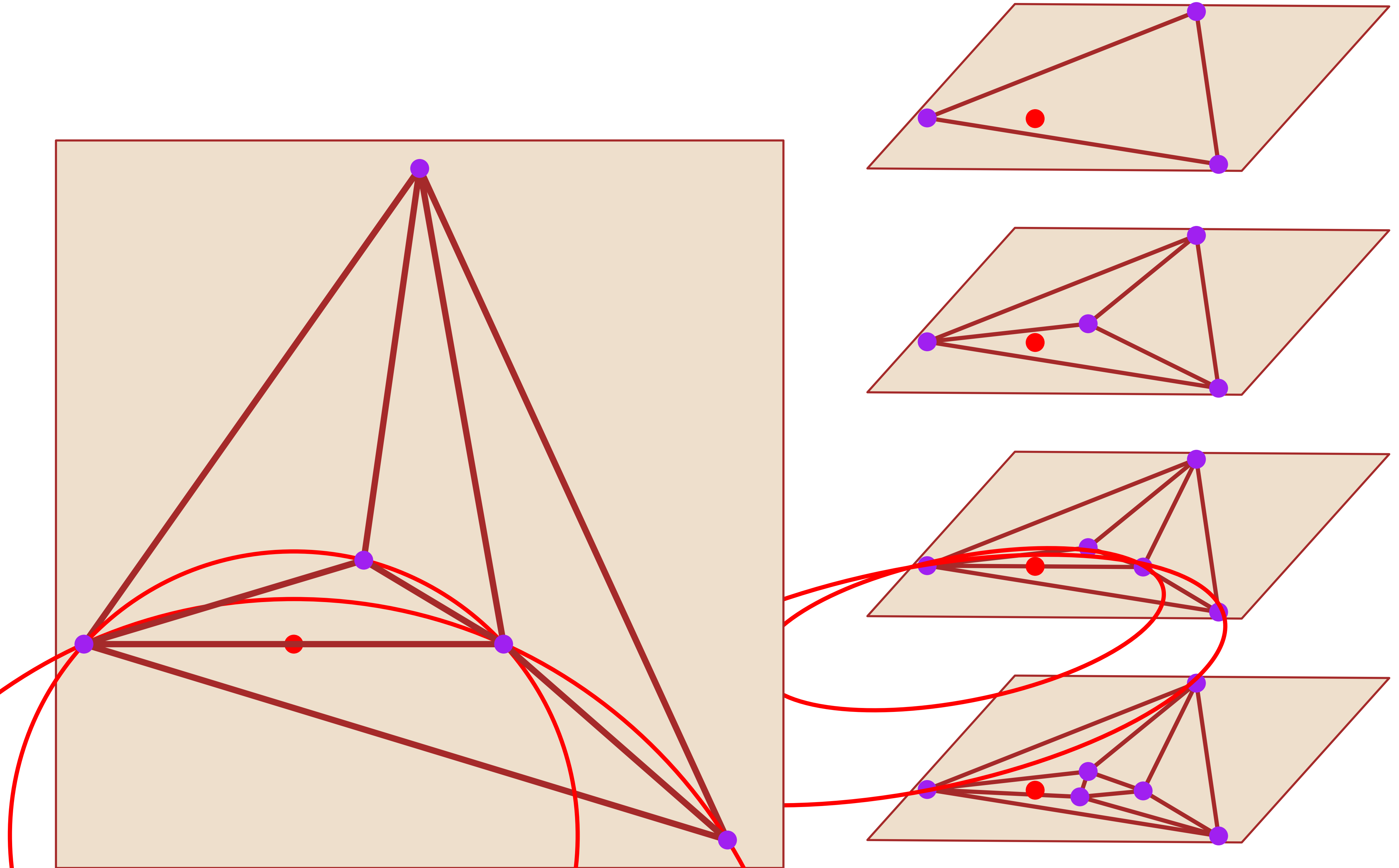
Data structures to locate - the Delaunay tree



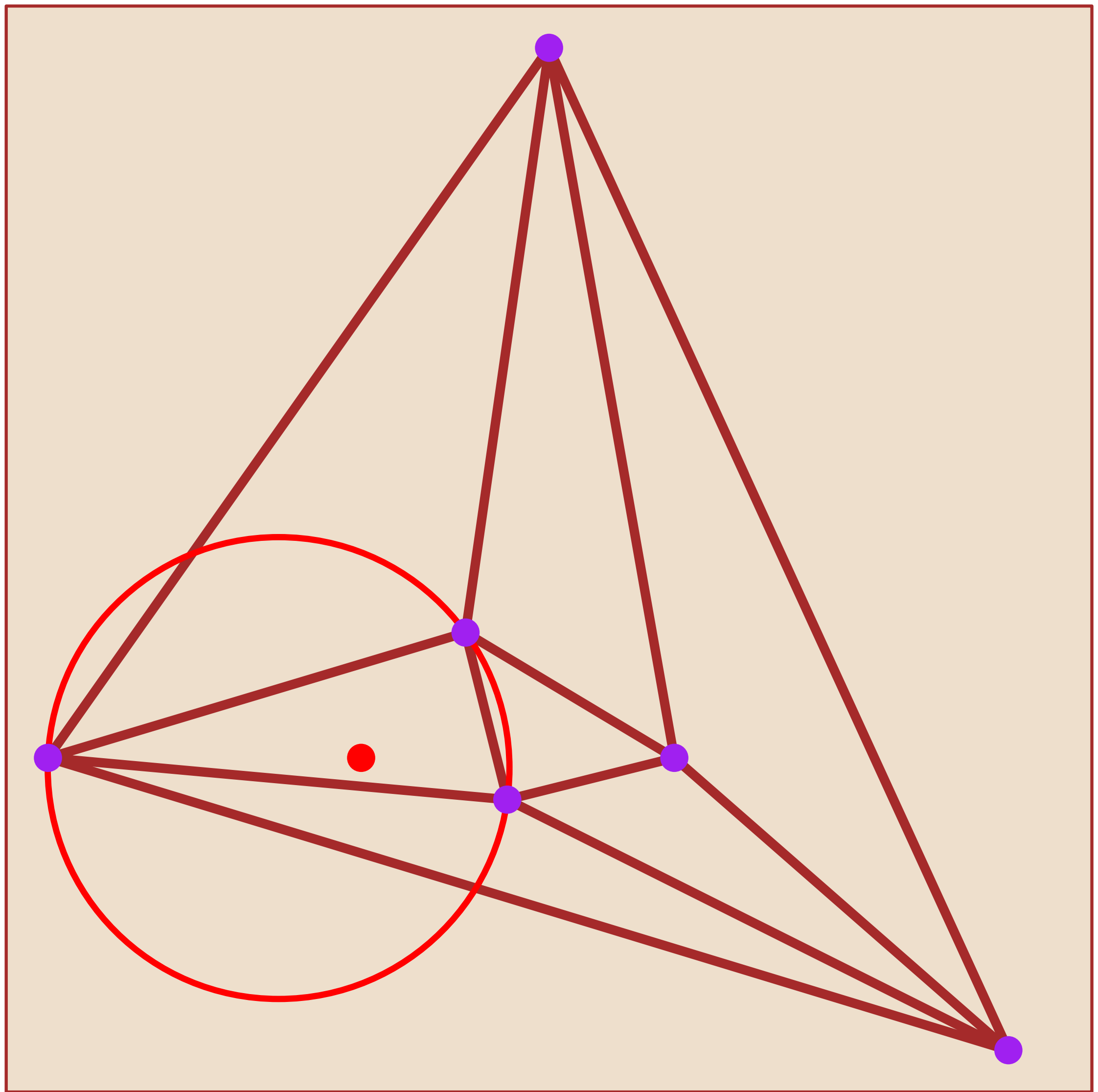
Data structures to locate - the Delaunay tree

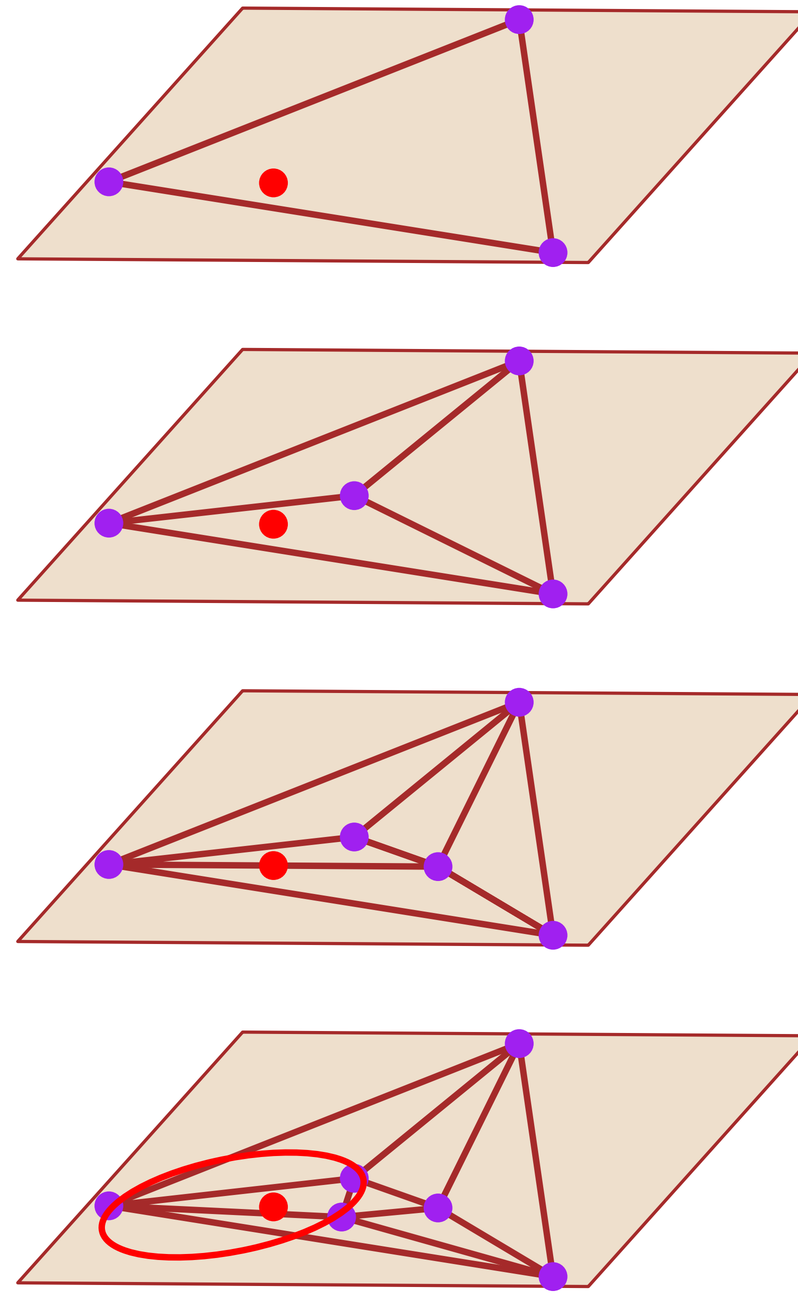


Data structures to locate - the Delaunay tree



Data structures to locate - the Delaunay tree

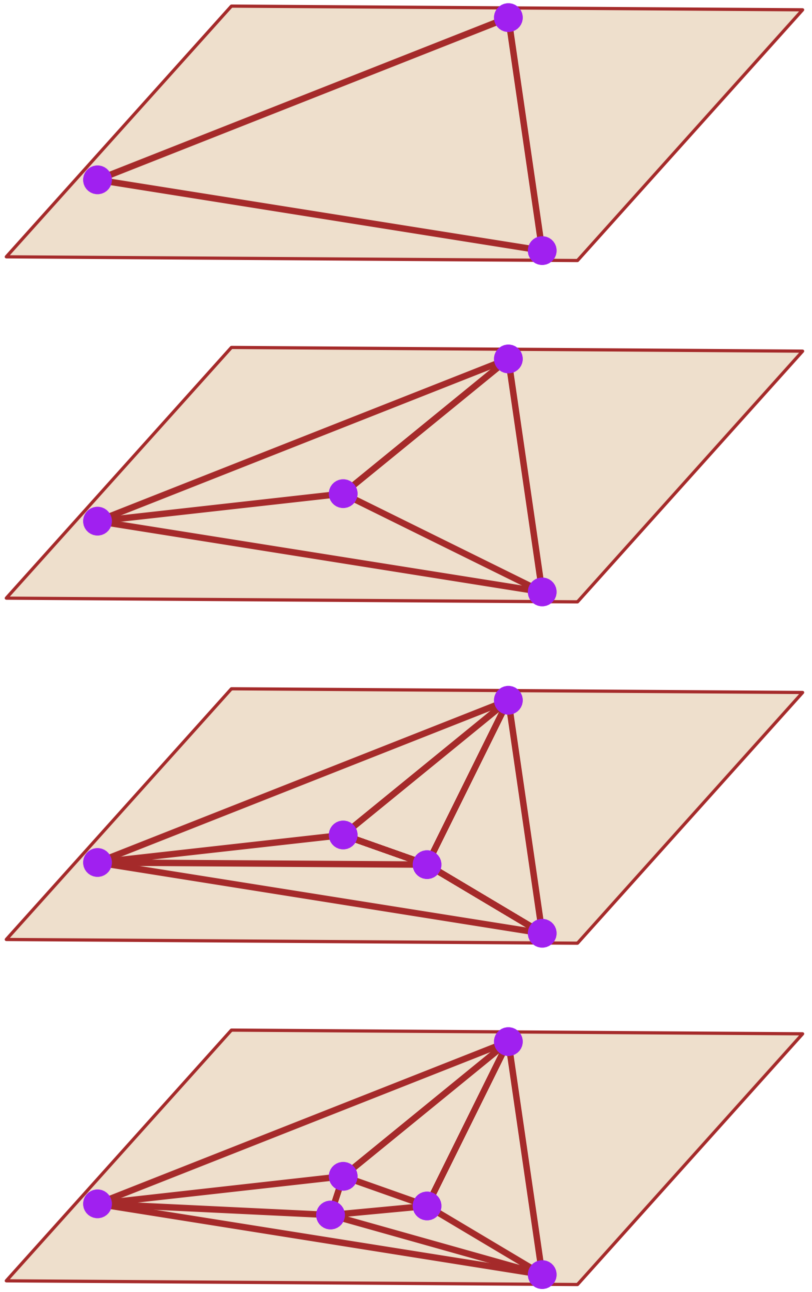




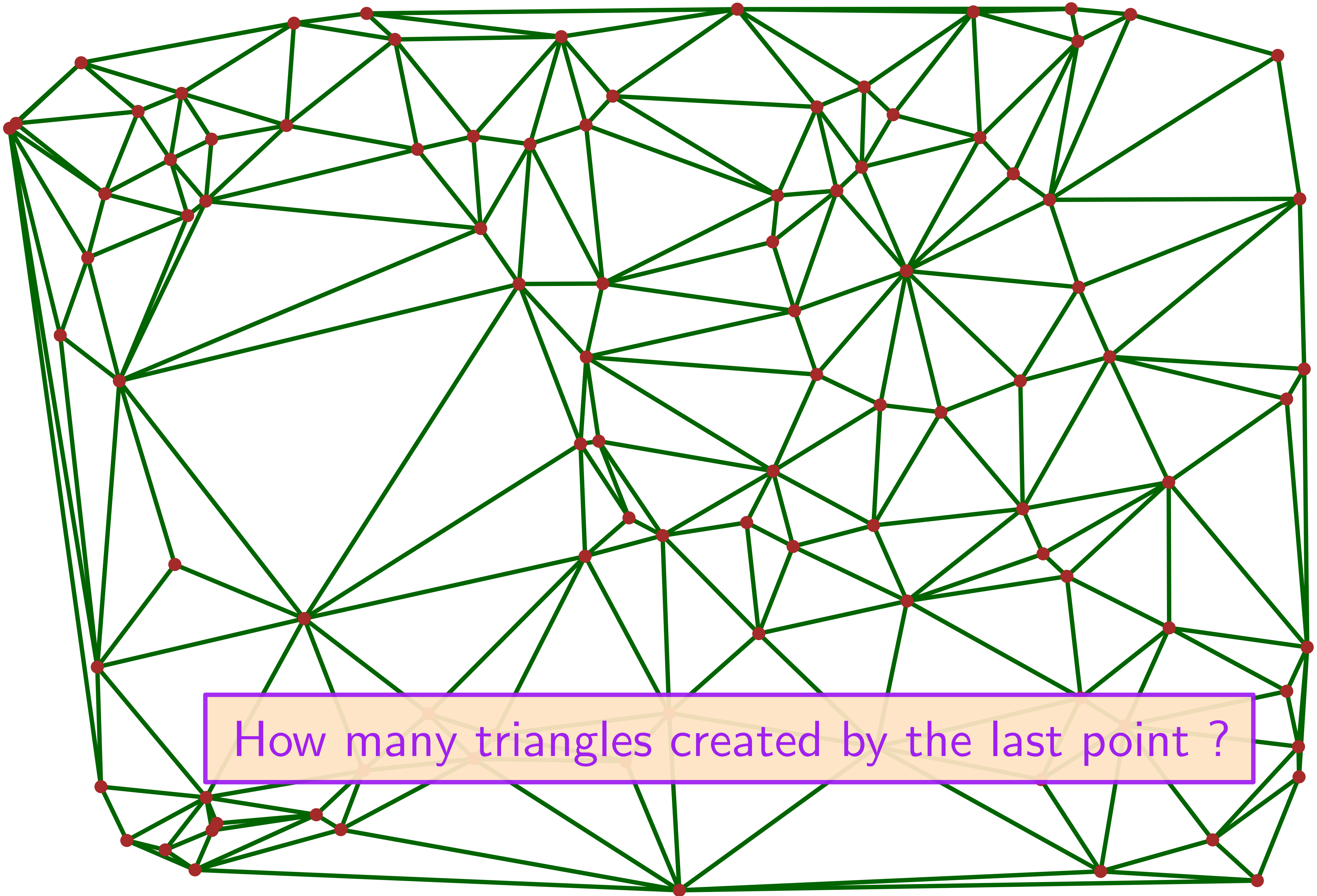
Data structures to locate - the Delaunay tree

locate based on incircle predicate

$\sharp$ triangles in the Delaunay tree

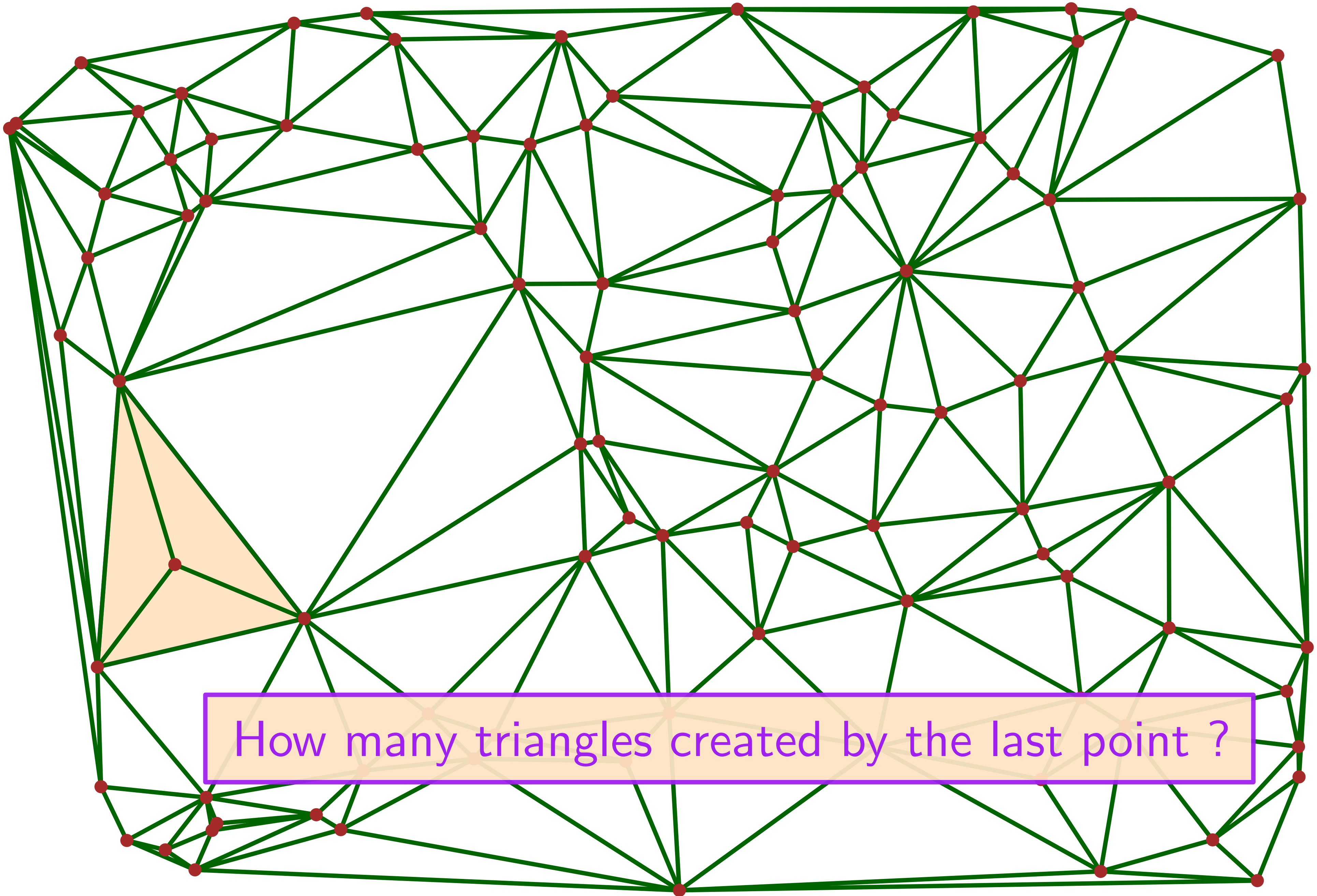


Data structures to locate - the Delaunay tree



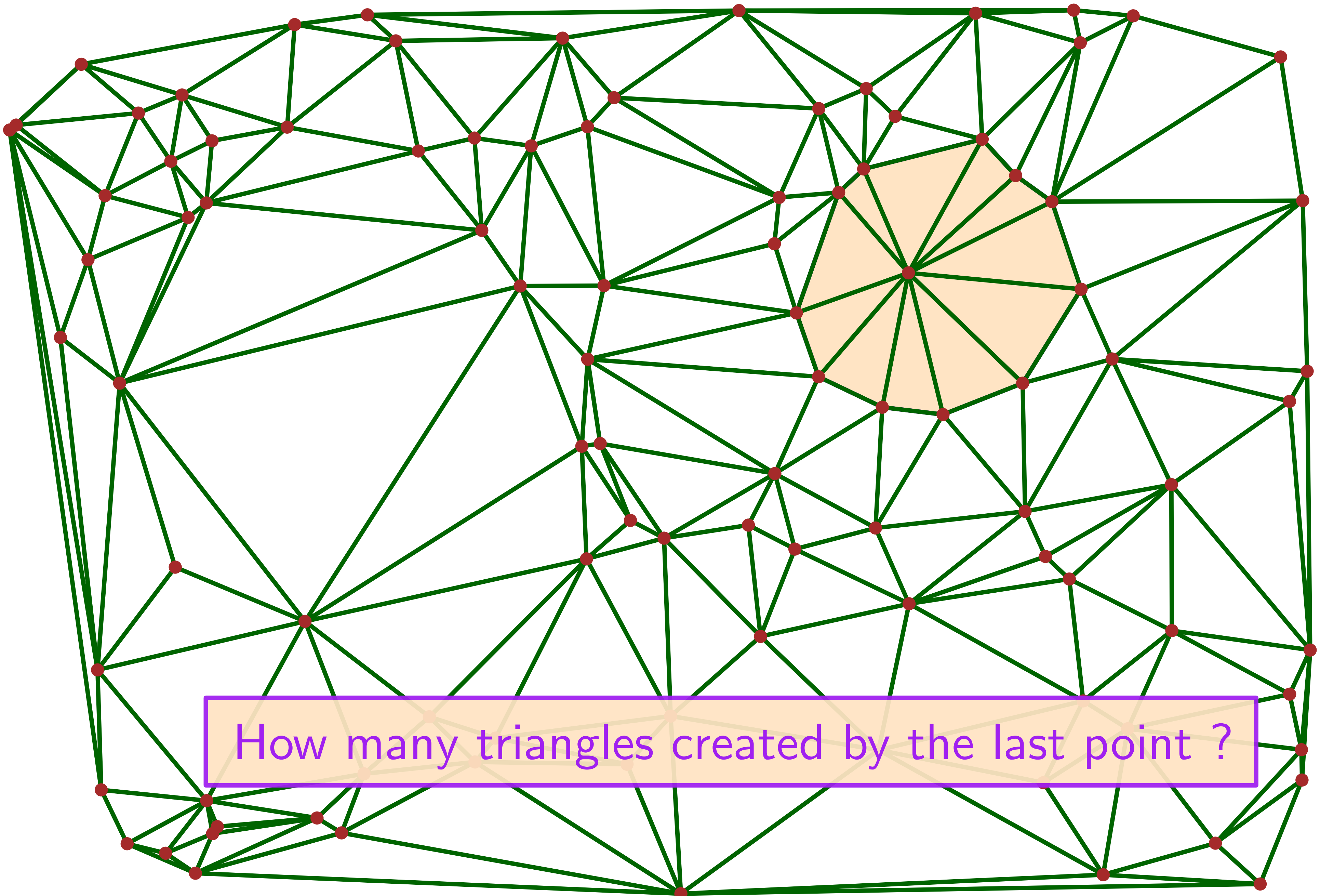
How many triangles created by the last point ?

Data structures to locate - the Delaunay tree

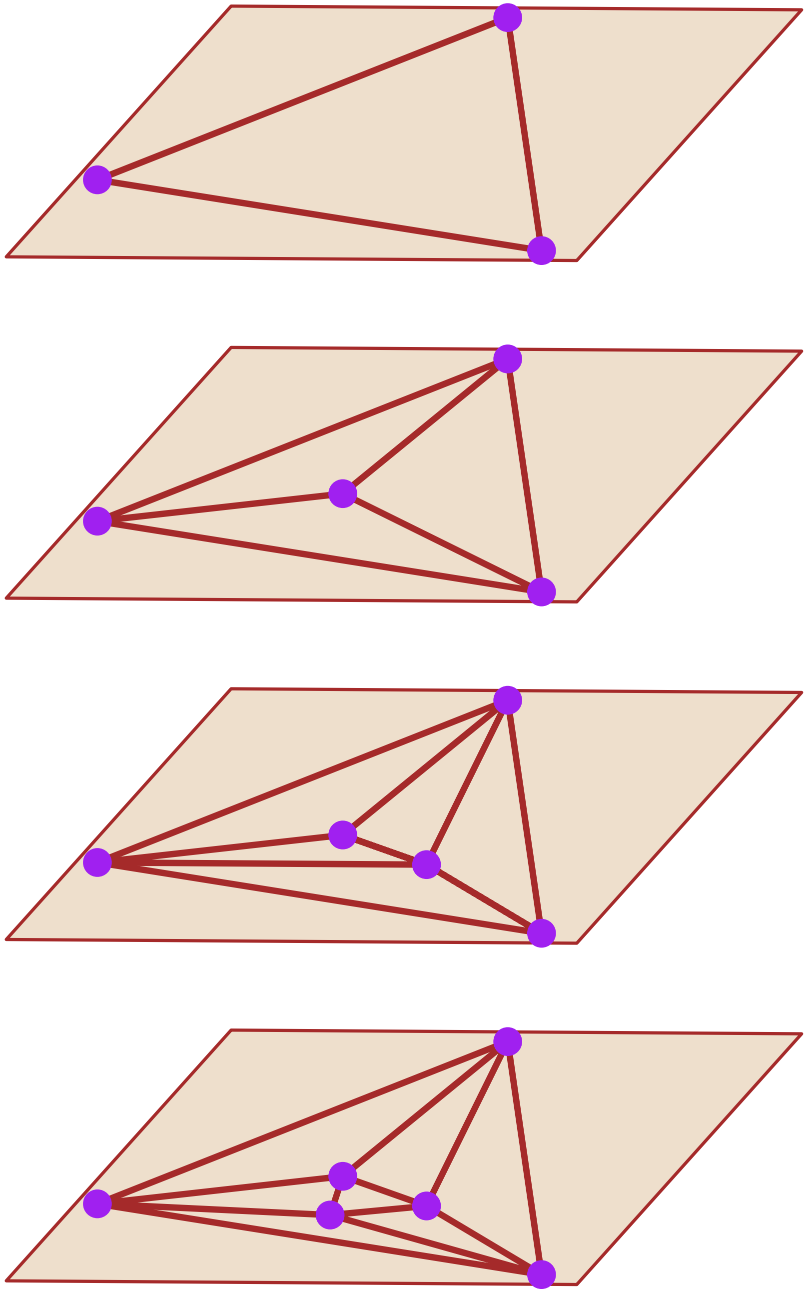


How many triangles created by the last point ?

Data structures to locate - the Delaunay tree



Data structures to locate - the Delaunay tree



locate based on incircle predicate

$\sharp$ triangles in the Delaunay tree

$= 6n$ (randomized)

One word on robustness issues

Basic incremental algorithm

Locate by walk

Locate using randomized data structures

Delaunay tree

Delaunay hierarchy

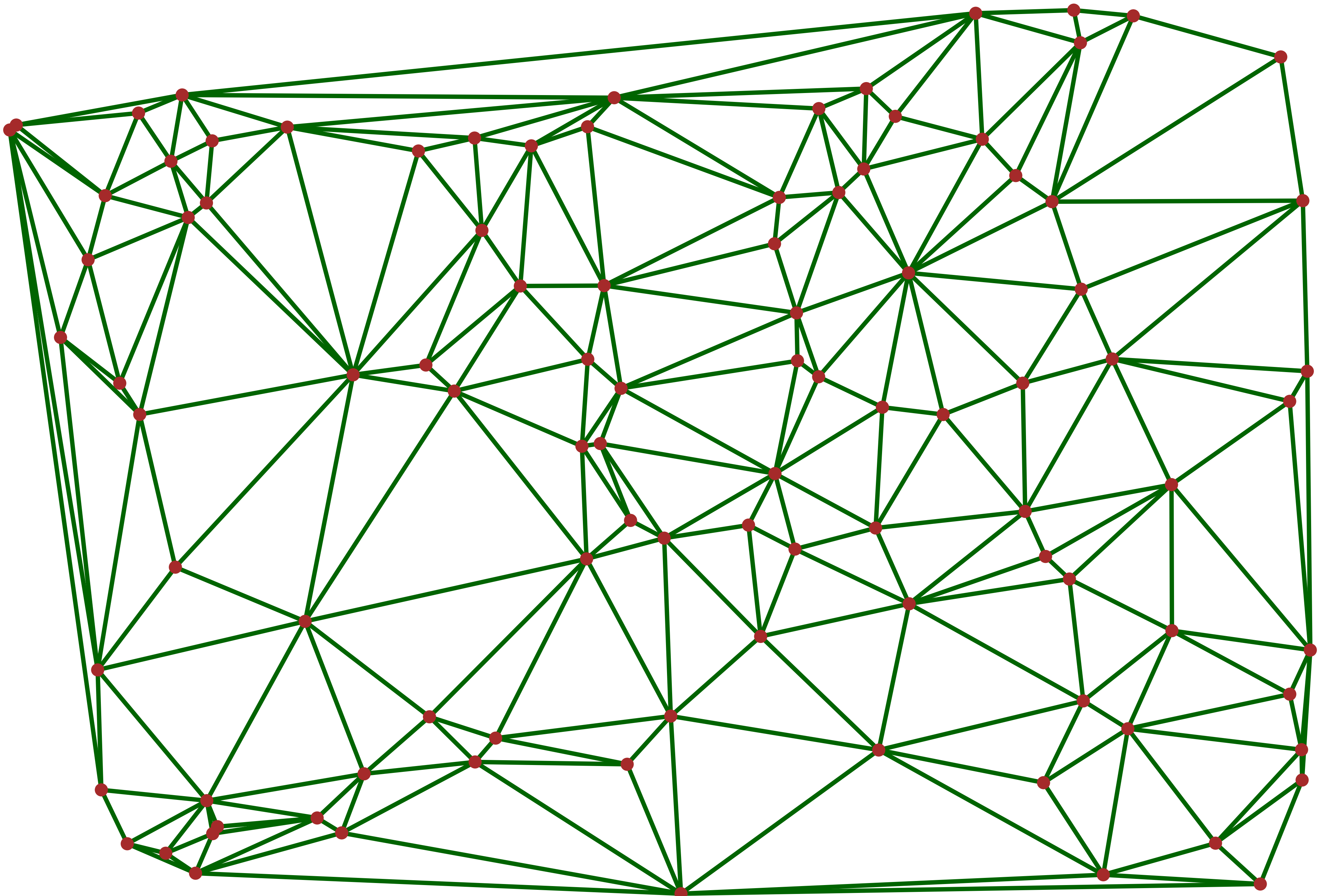
Vertex removal in 2D

Remarks on CGAL programming

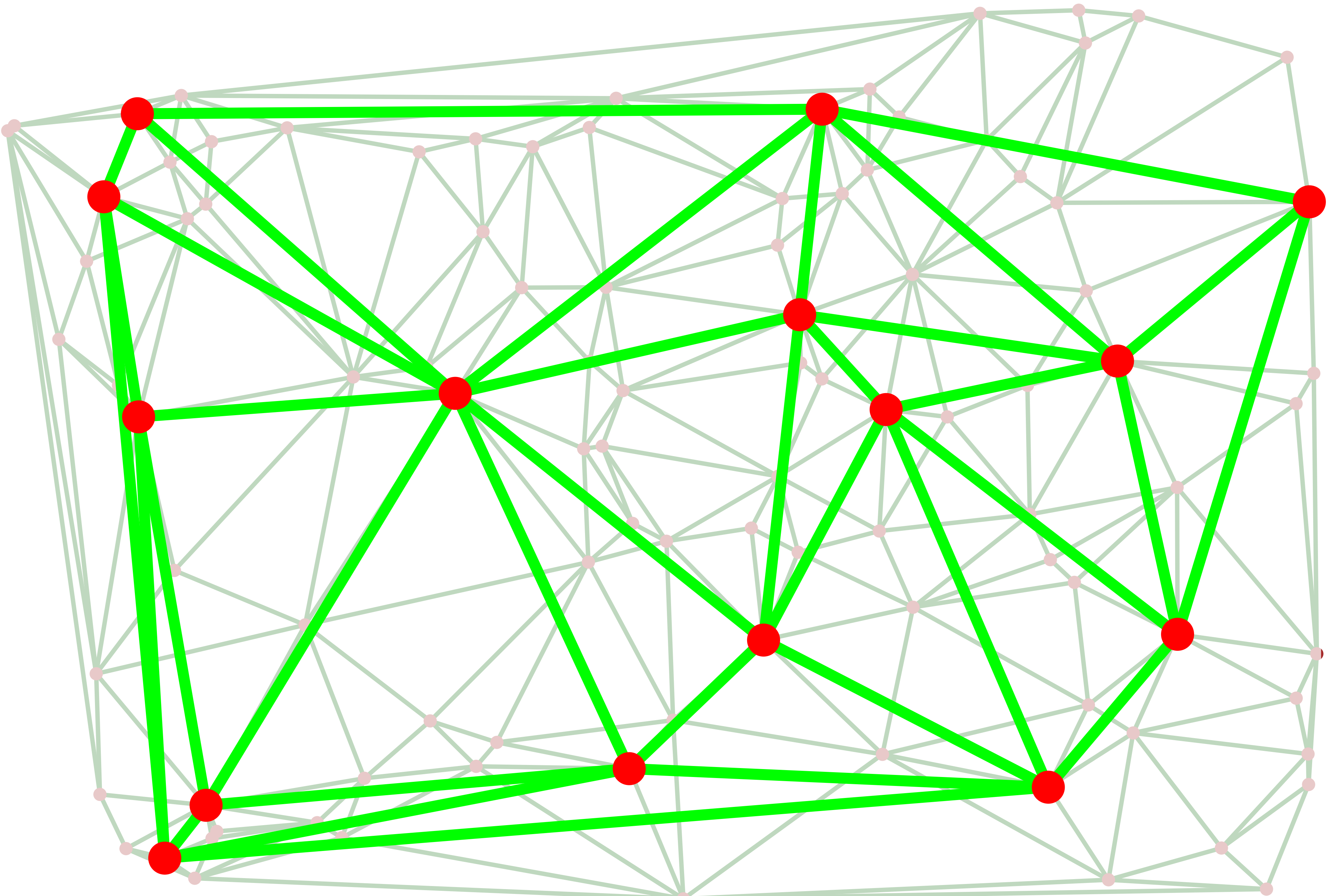
Conclusion

Data structures to locate - the Delaunay hierarchy

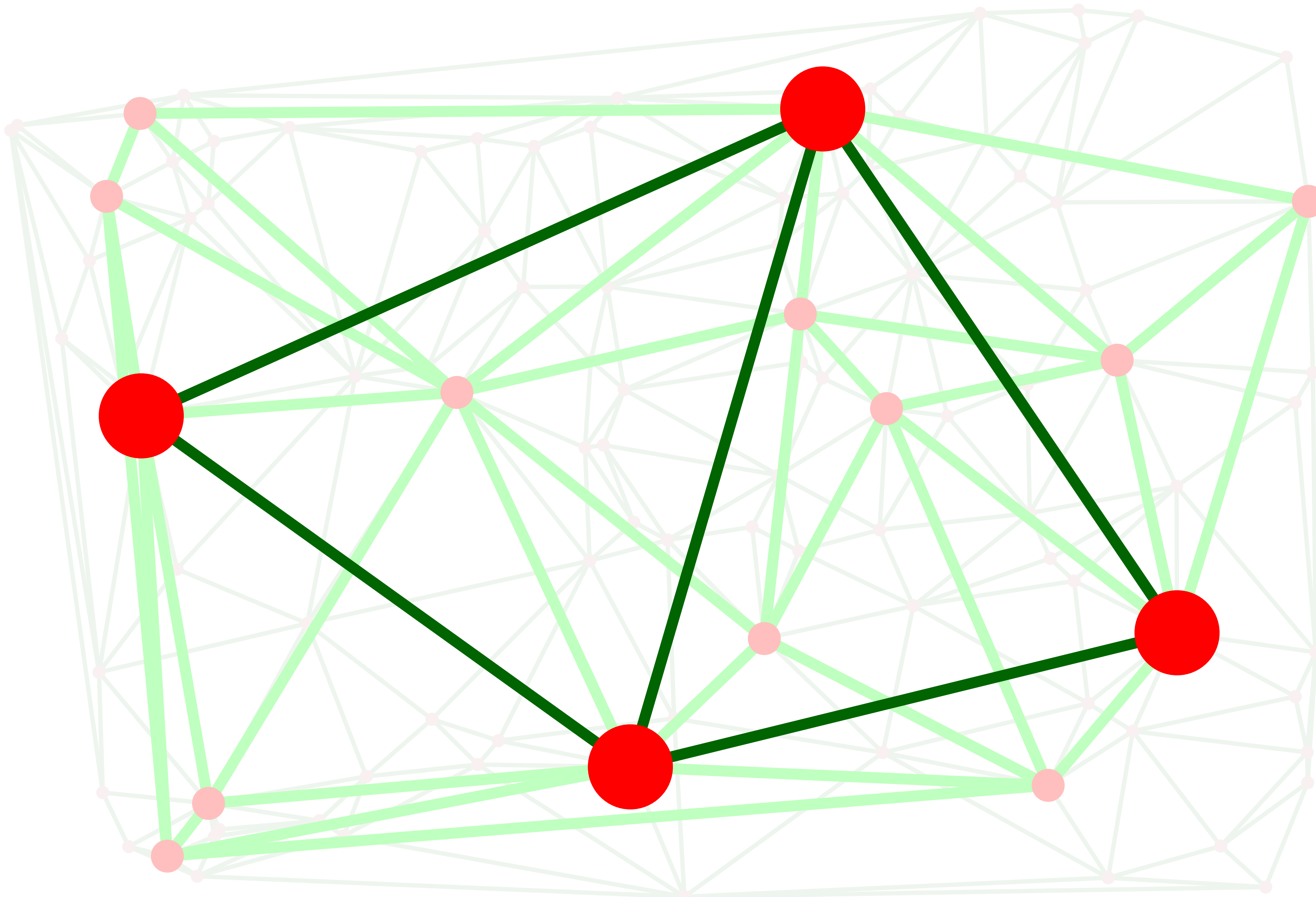
Data structures to locate - the Delaunay hierarchy



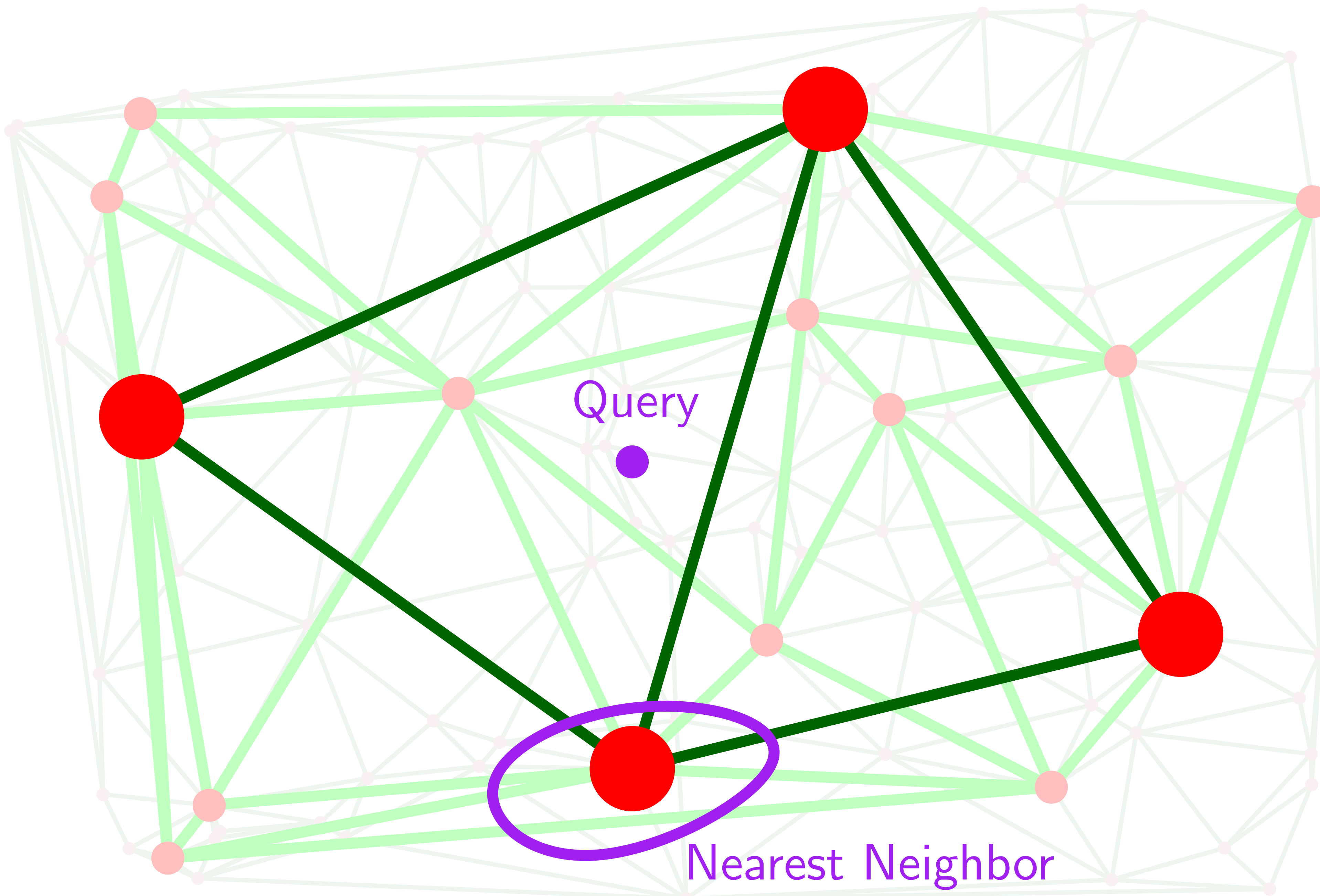
Data structures to locate - the Delaunay hierarchy



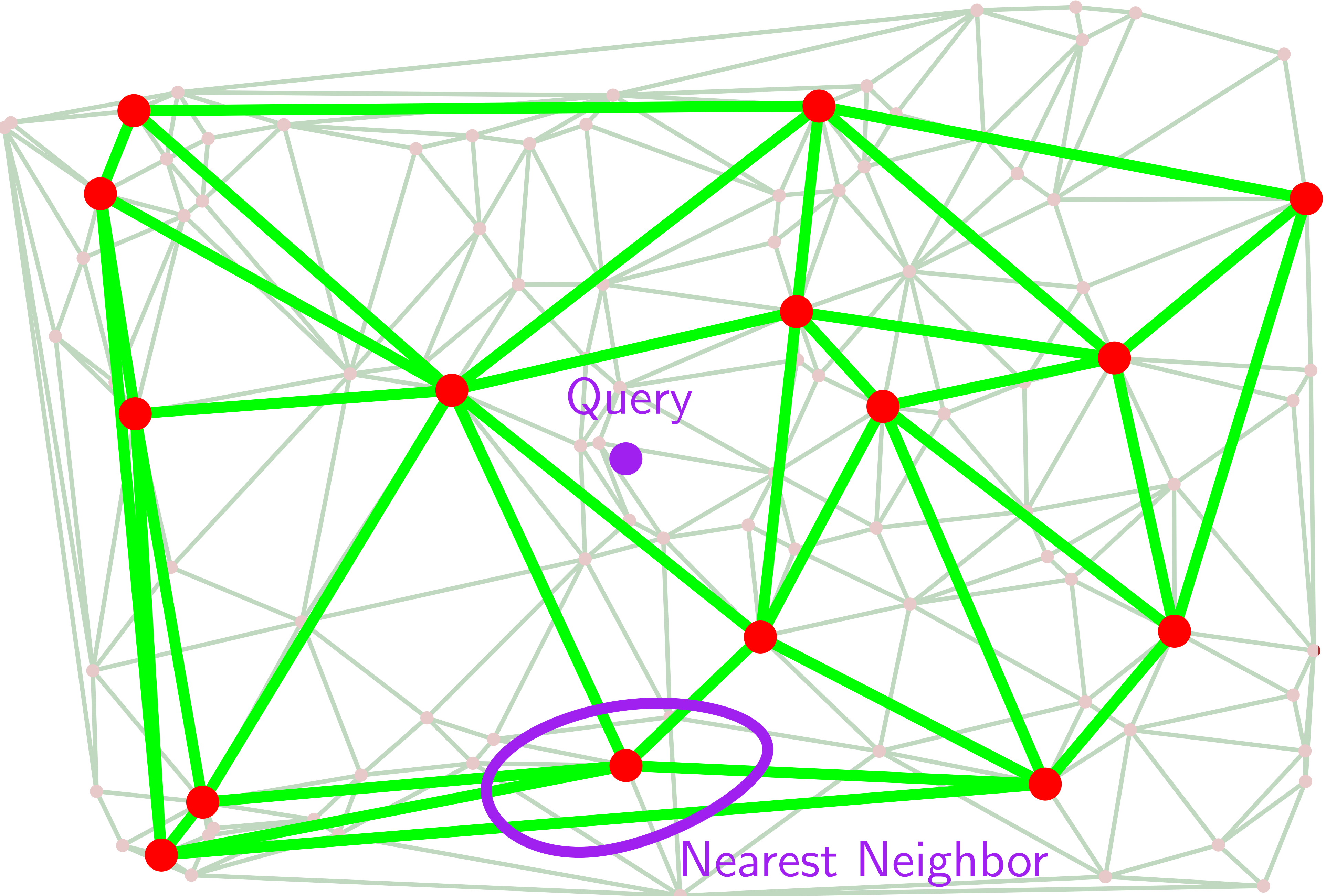
Data structures to locate - the Delaunay hierarchy



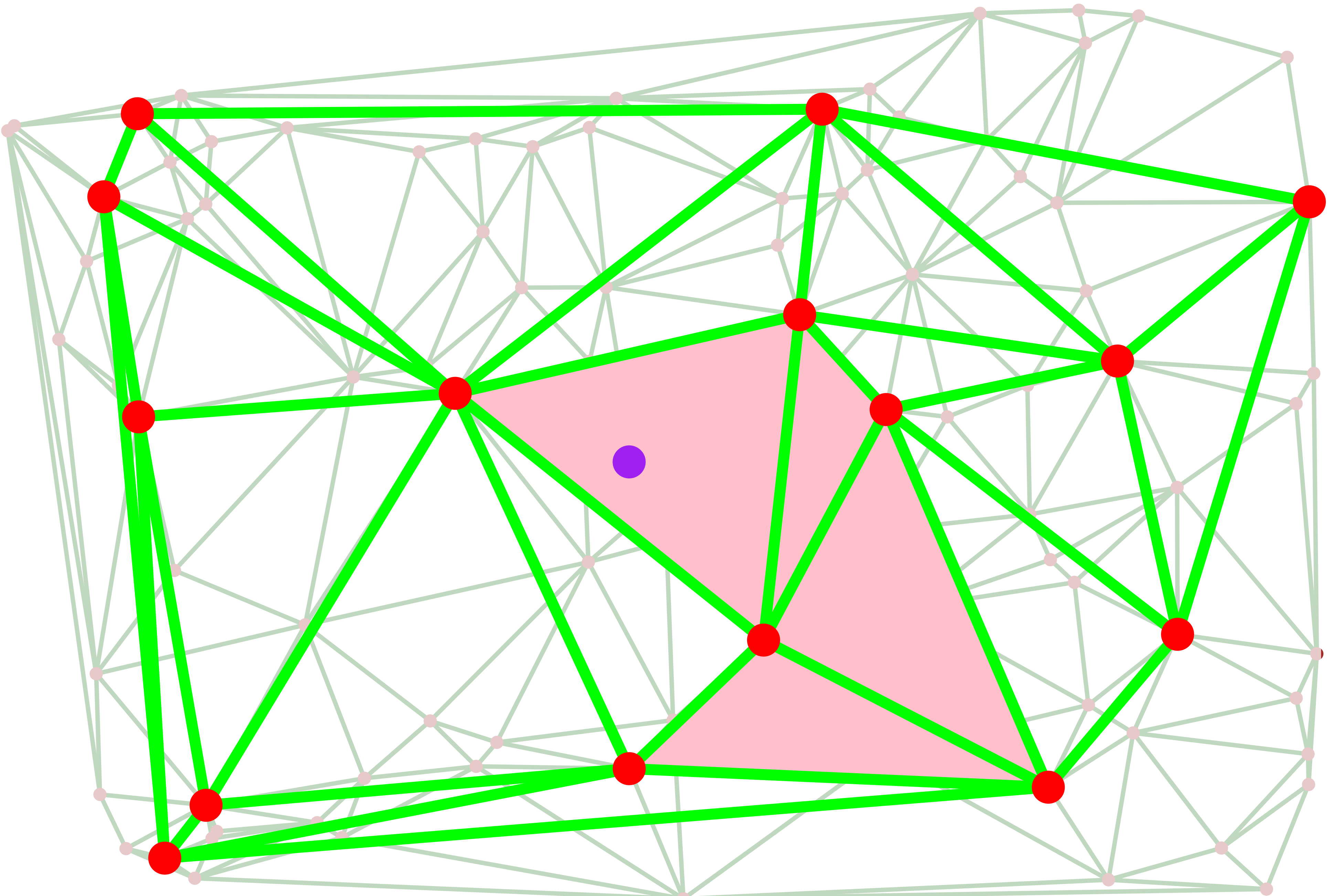
Data structures to locate - the Delaunay hierarchy



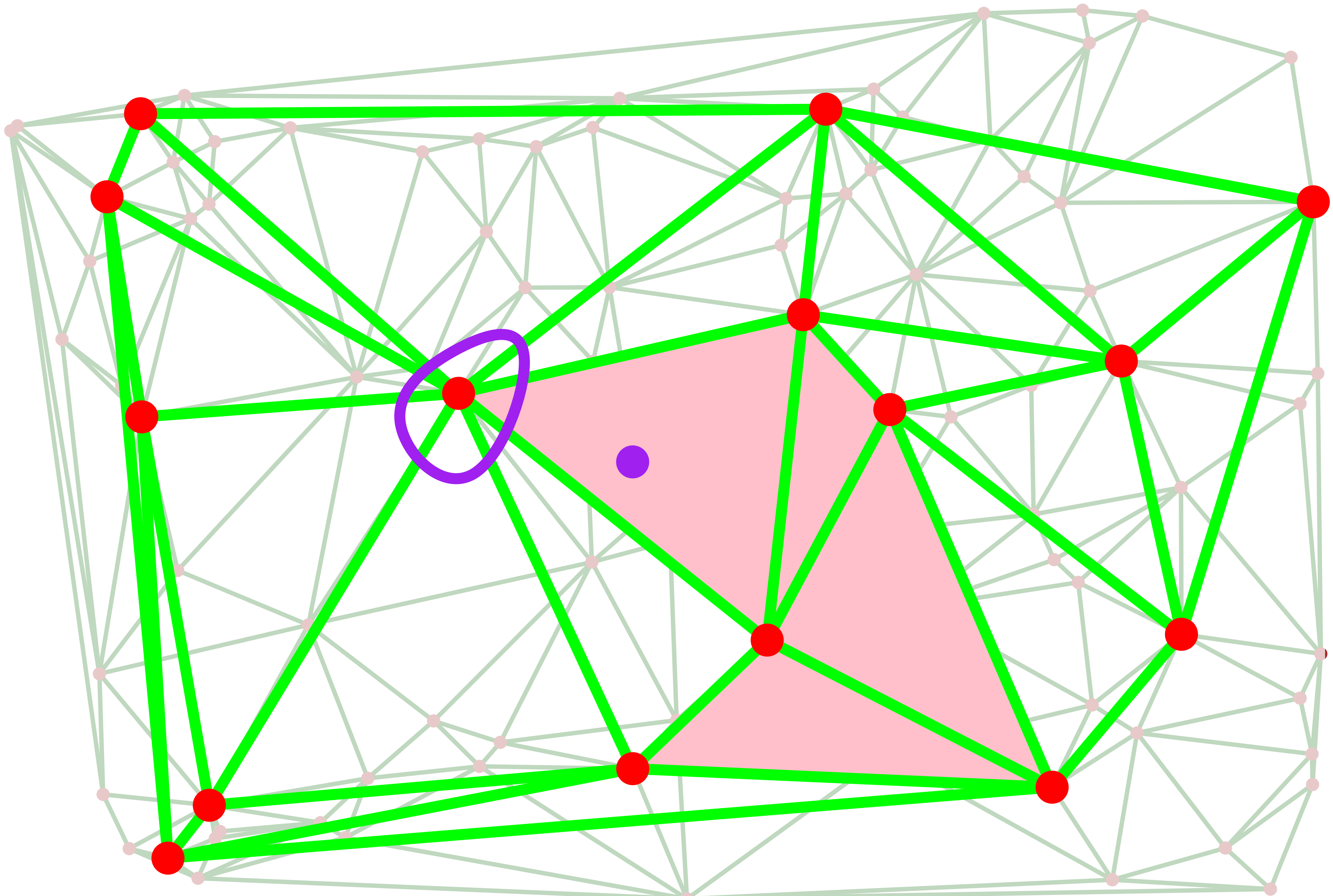
Data structures to locate - the Delaunay hierarchy



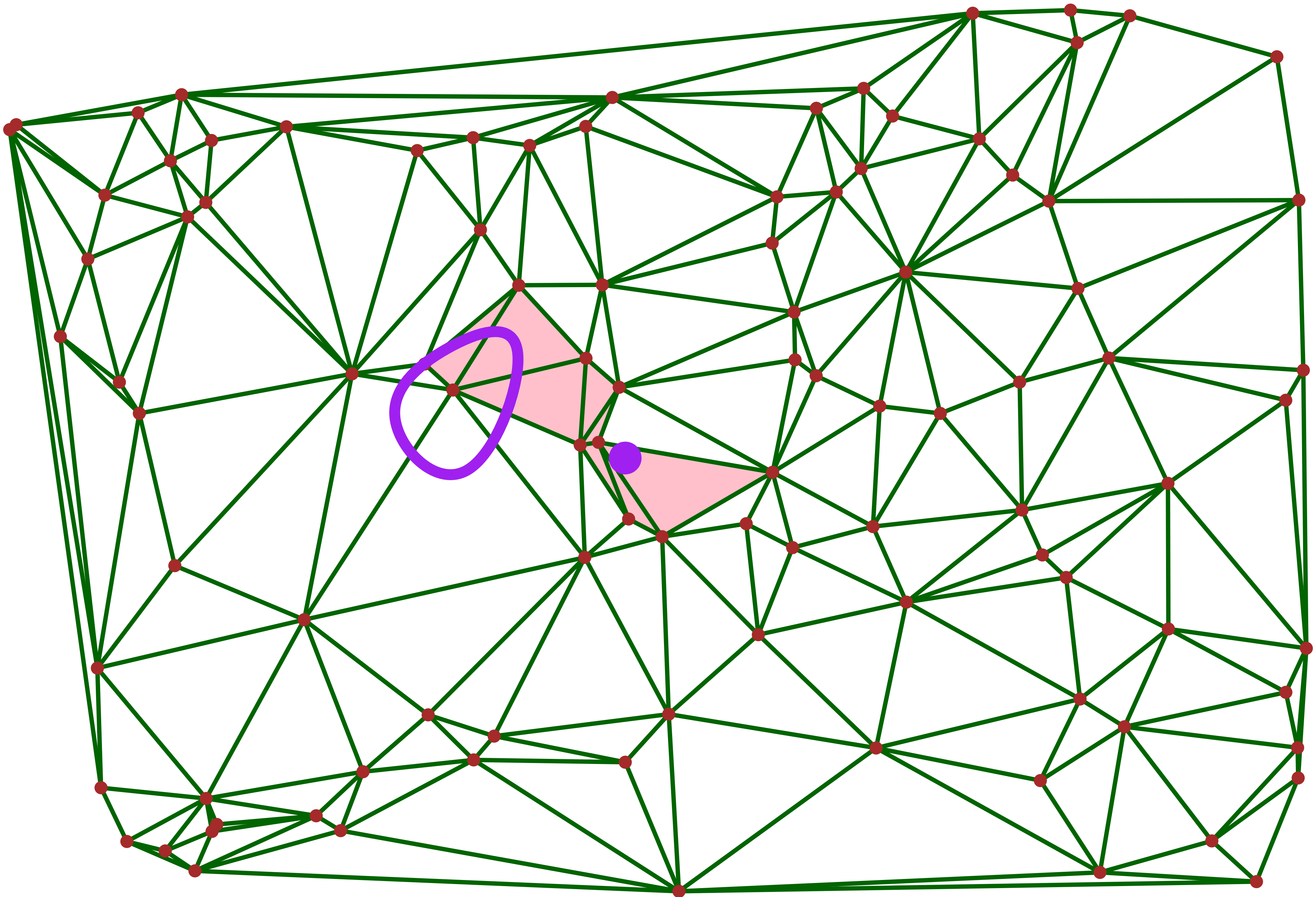
Data structures to locate - the Delaunay hierarchy



Data structures to locate - the Delaunay hierarchy



Data structures to locate - the Delaunay hierarchy



Data structures to locate - the Delaunay hierarchy

The Delaunay tree

locate based on incircle predicate

$\sharp$ triangles in the Delaunay tree

$= 6n$ (randomized)

Data structures to locate - the Delaunay hierarchy

The Delaunay hierarchy

based on orientation predicate

$\sharp$ triangles in the hierarchy

can be chosen

$= 1.03 \times 2n$ (expected)

The Delaunay tree

locate based on incircle predicate

$\sharp$ triangles in the Delaunay tree

$= 6n$ (randomized)

Data structures to locate - the Delaunay hierarchy

The Delaunay hierarchy

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locate based on incircle predicate

$\sharp$ triangles in the Delaunay tree

$= 6n$ (randomized)

$$O(n \log n)$$

Data structures to locate - the Delaunay hierarchy

The Delaunay hierarchy

based on orientation predicate

$\sharp$ triangles in the hierarchy

can be chosen

$= 1.03 \times 2n$ (expected)

The Delaunay tree

locate based on incircle predicate

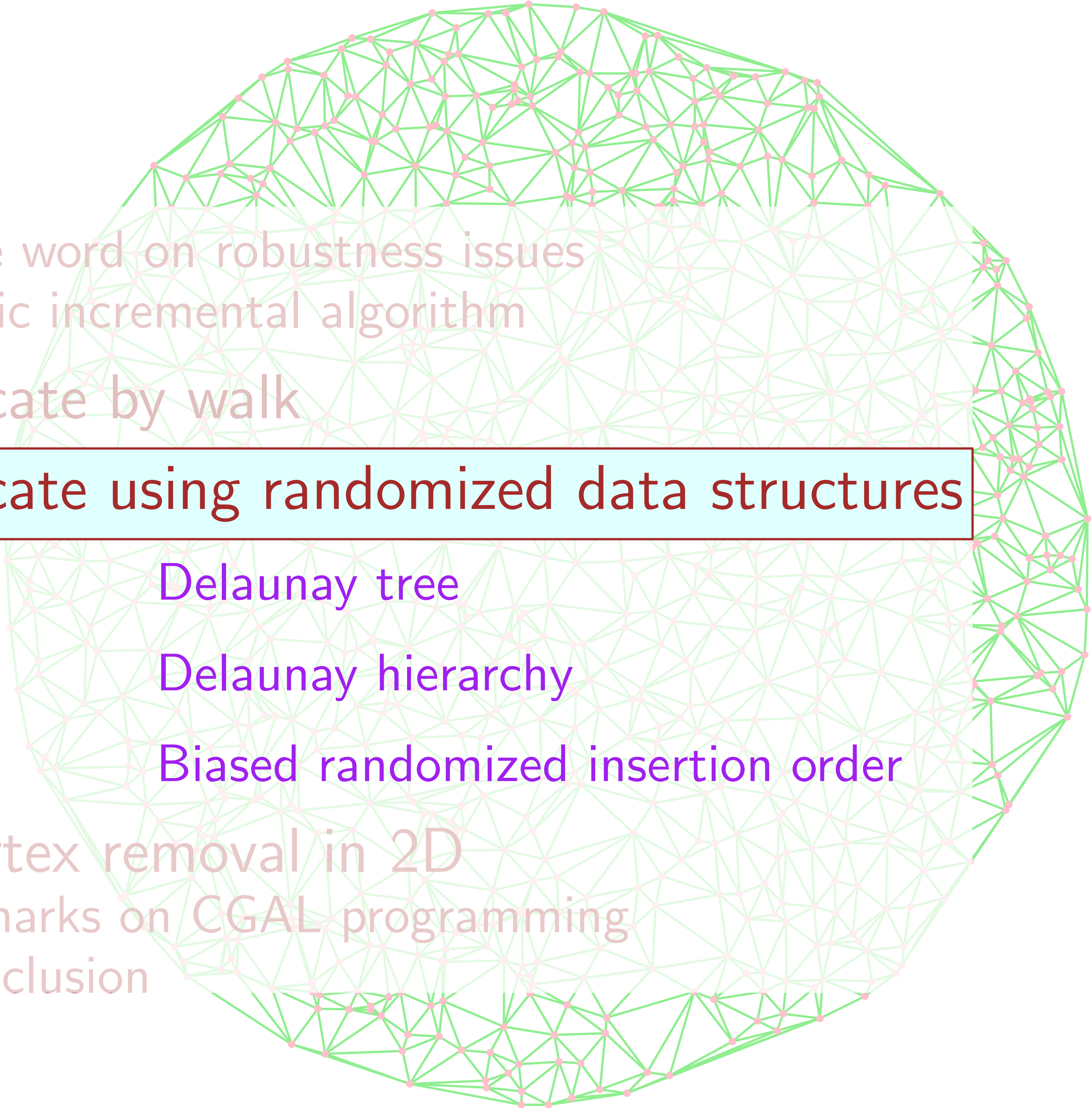
$\sharp$ triangles in the Delaunay tree

$= 6n$ (randomized)

2.3 seconds

17 seconds

50000 random points (original benchmarks in 2000).



One word on robustness issues

Basic incremental algorithm

Locate by walk

Locate using randomized data structures

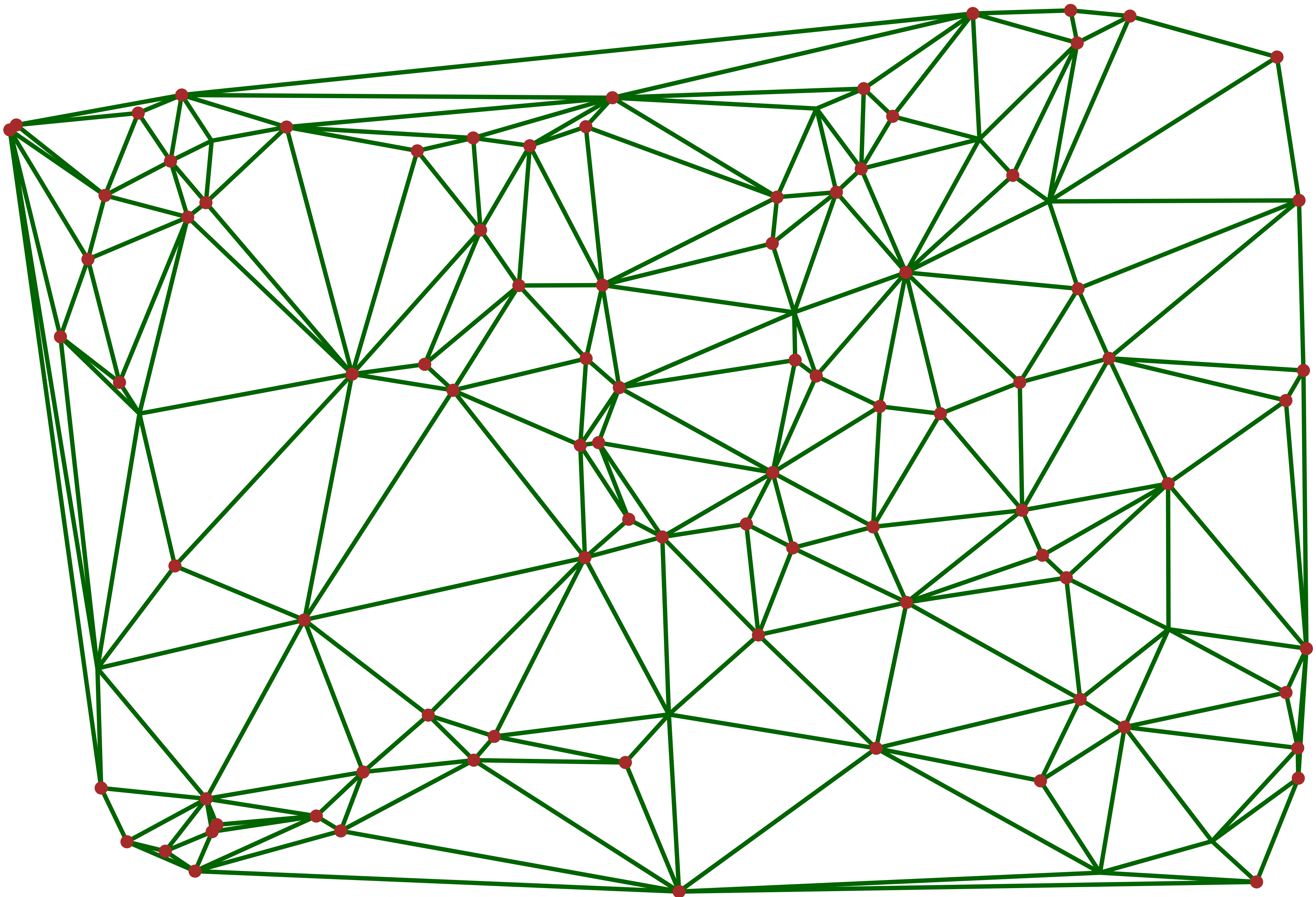
- Delaunay tree
- Delaunay hierarchy
- Biased randomized insertion order

Vertex removal in 2D

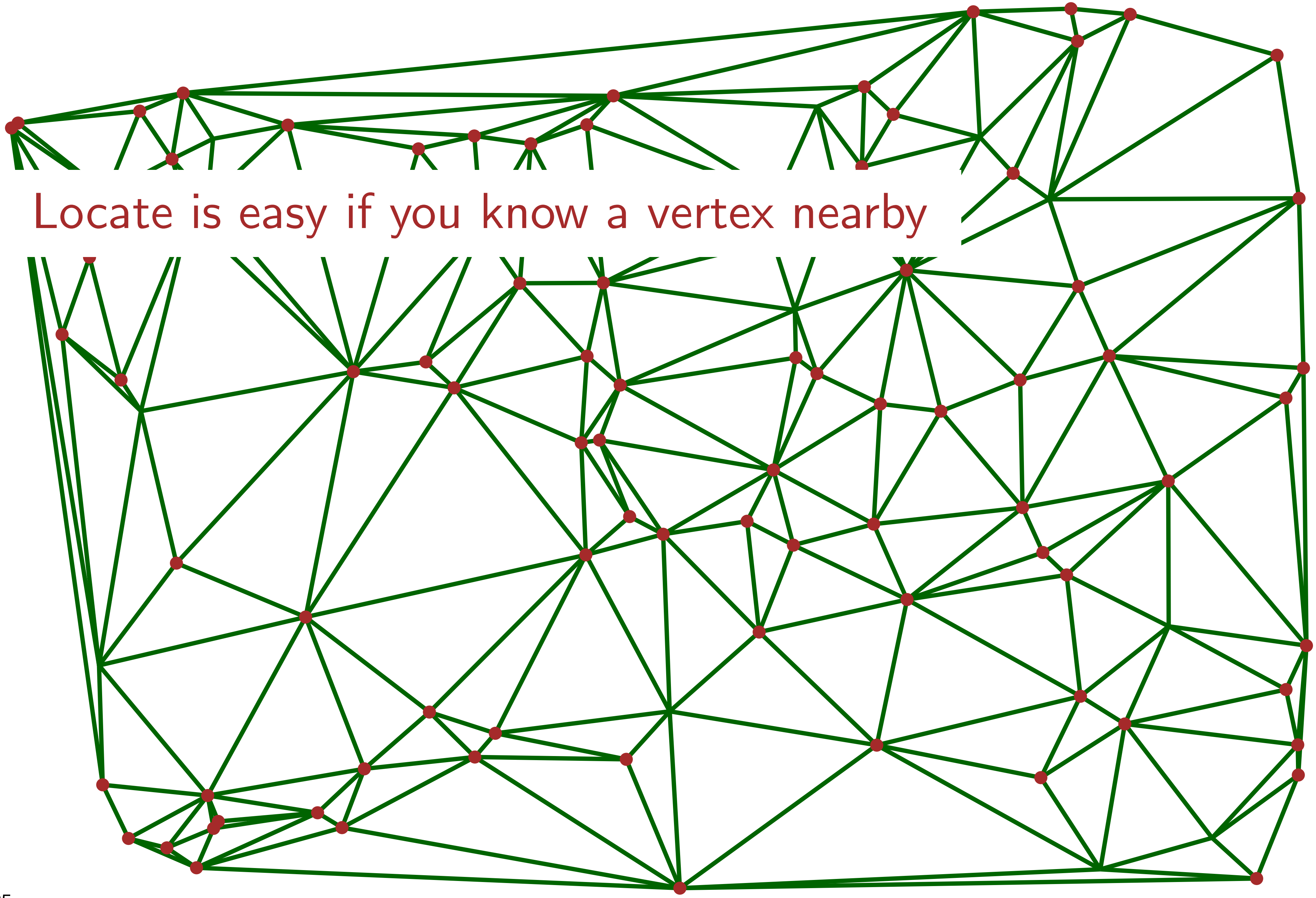
Remarks on CGAL programming

Conclusion

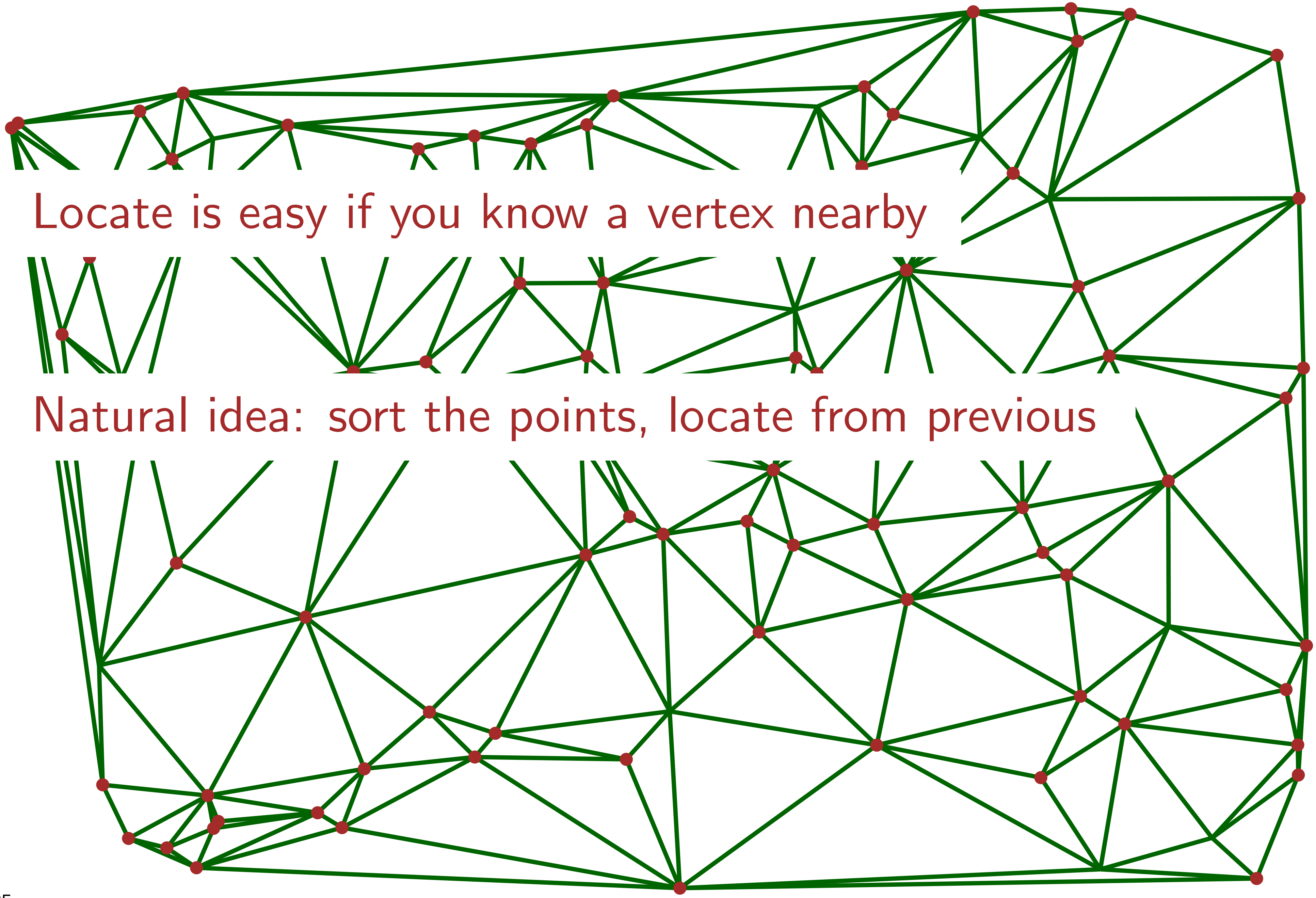
Data structures to locate - biased random insertion order



Data structures to locate - biased random insertion order



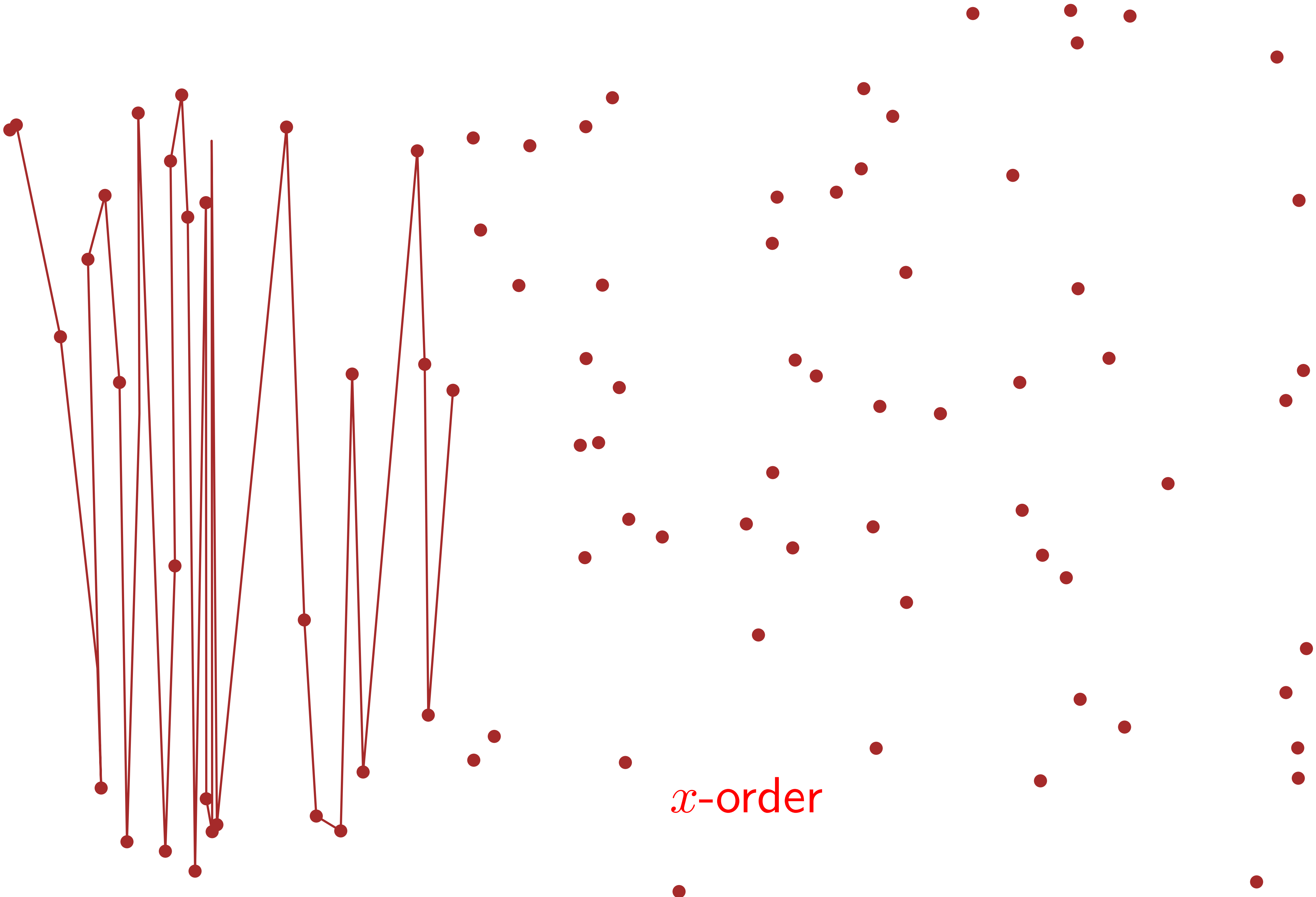
Data structures to locate - biased random insertion order



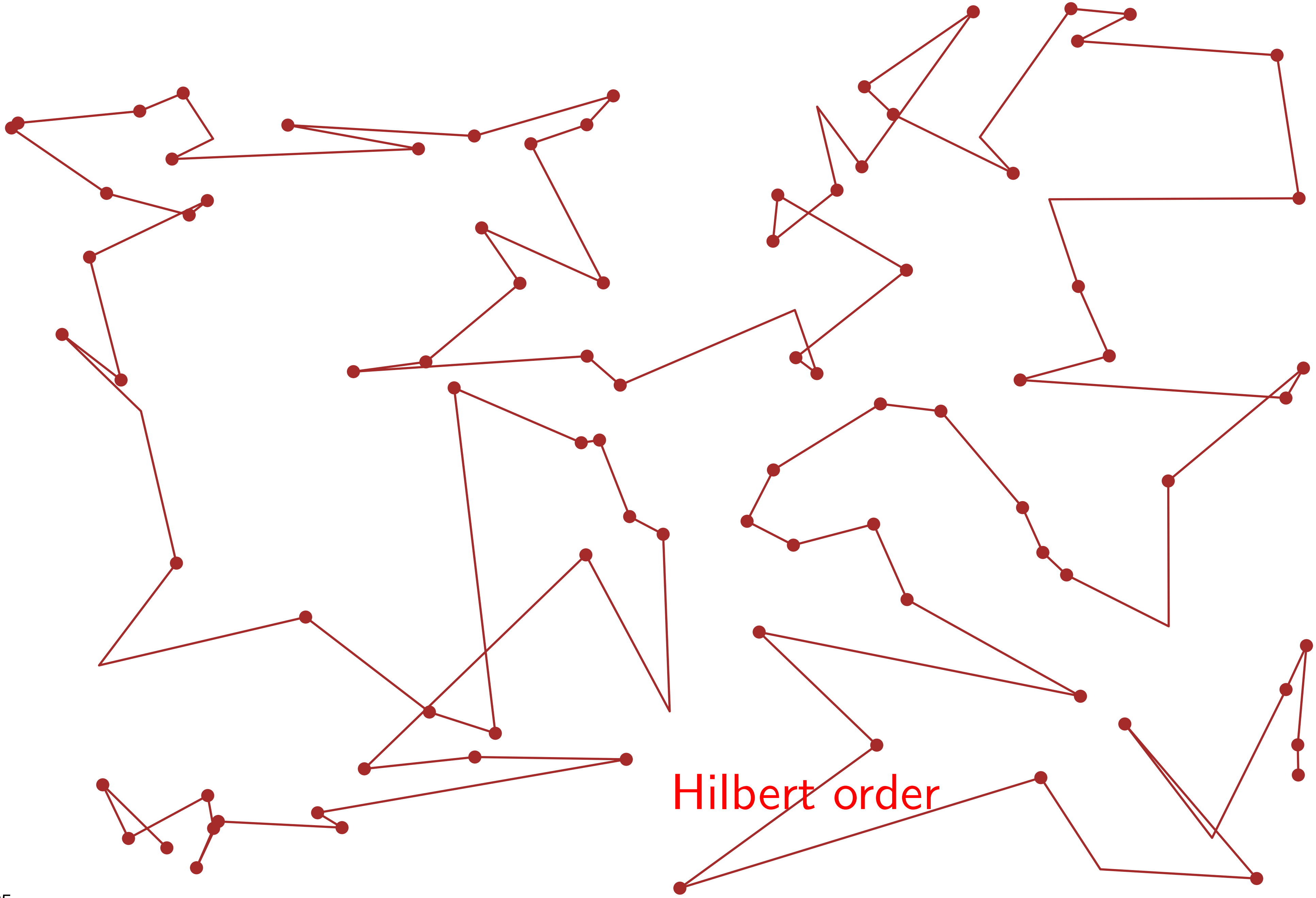
Locate is easy if you know a vertex nearby

Natural idea: sort the points, locate from previous

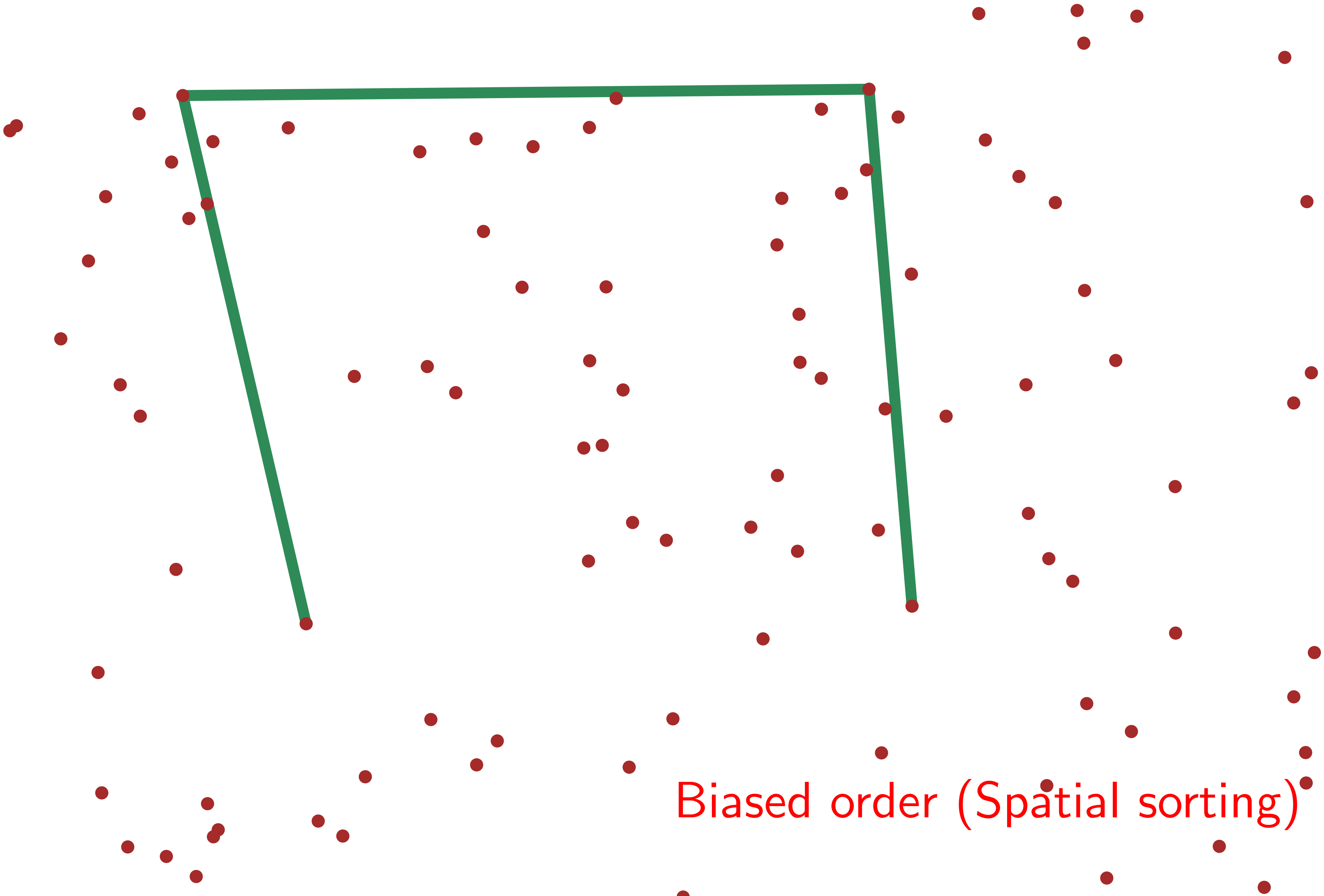
Data structures to locate - biased random insertion order



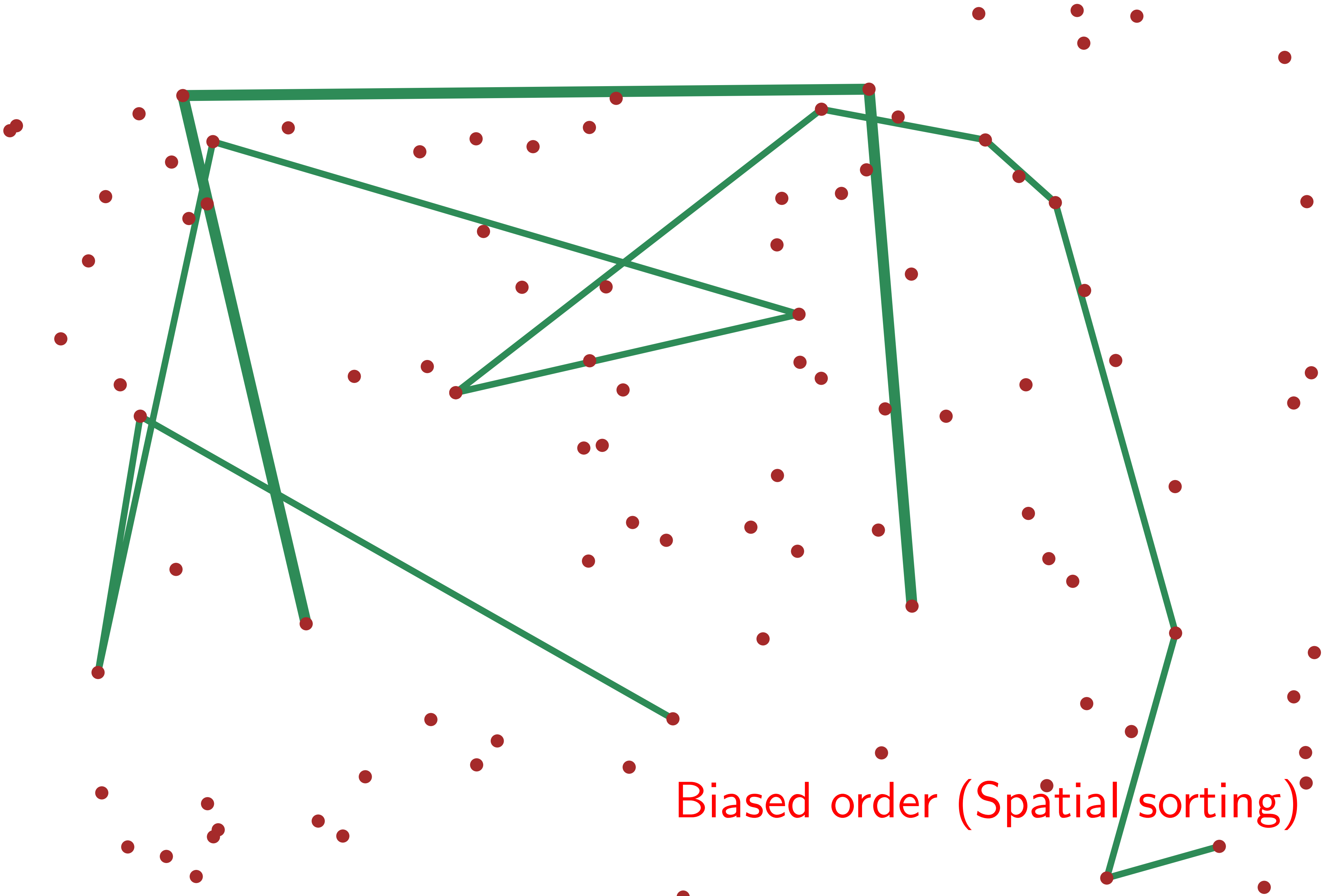
Data structures to locate - biased random insertion order



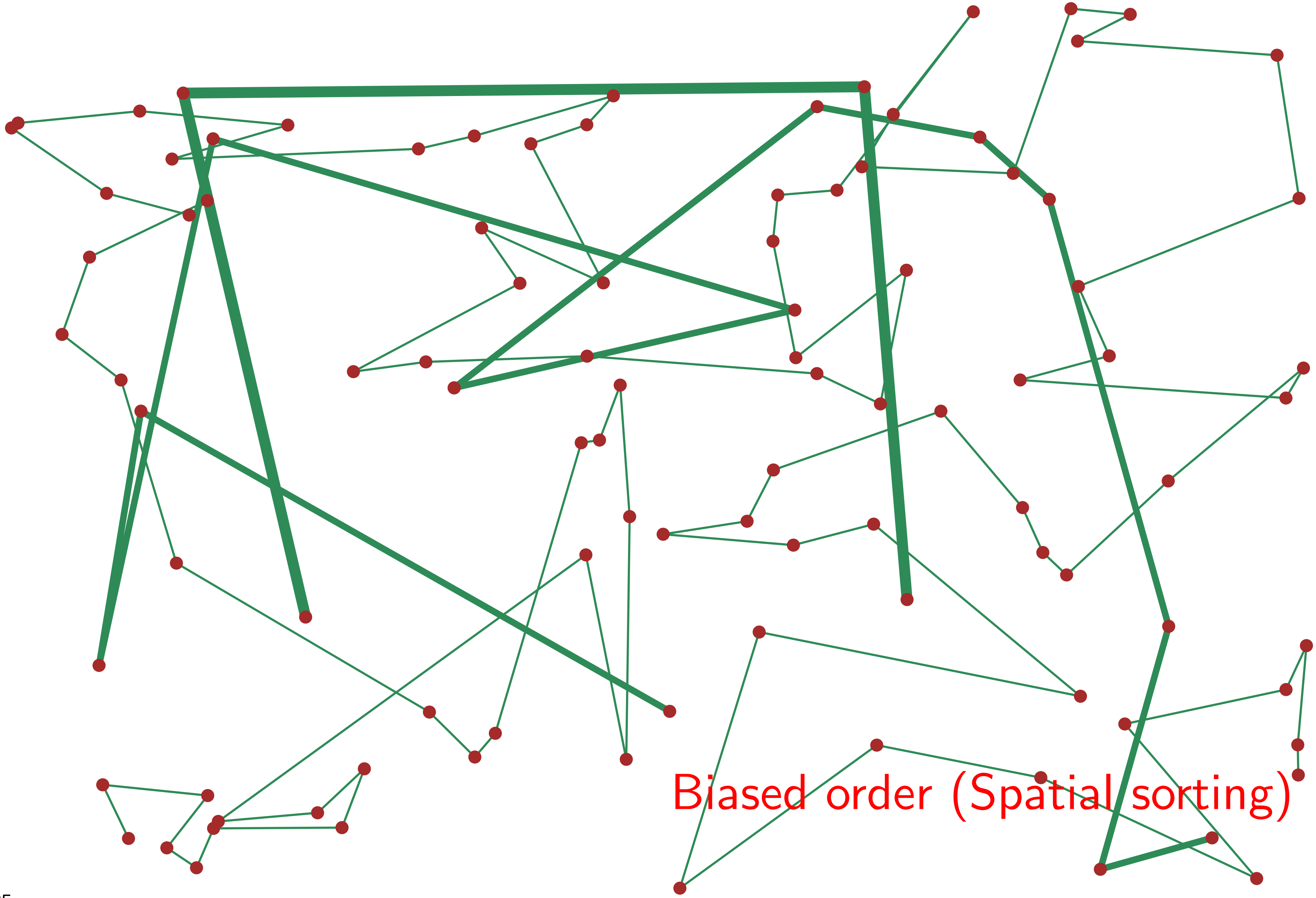
Data structures to locate - biased random insertion order



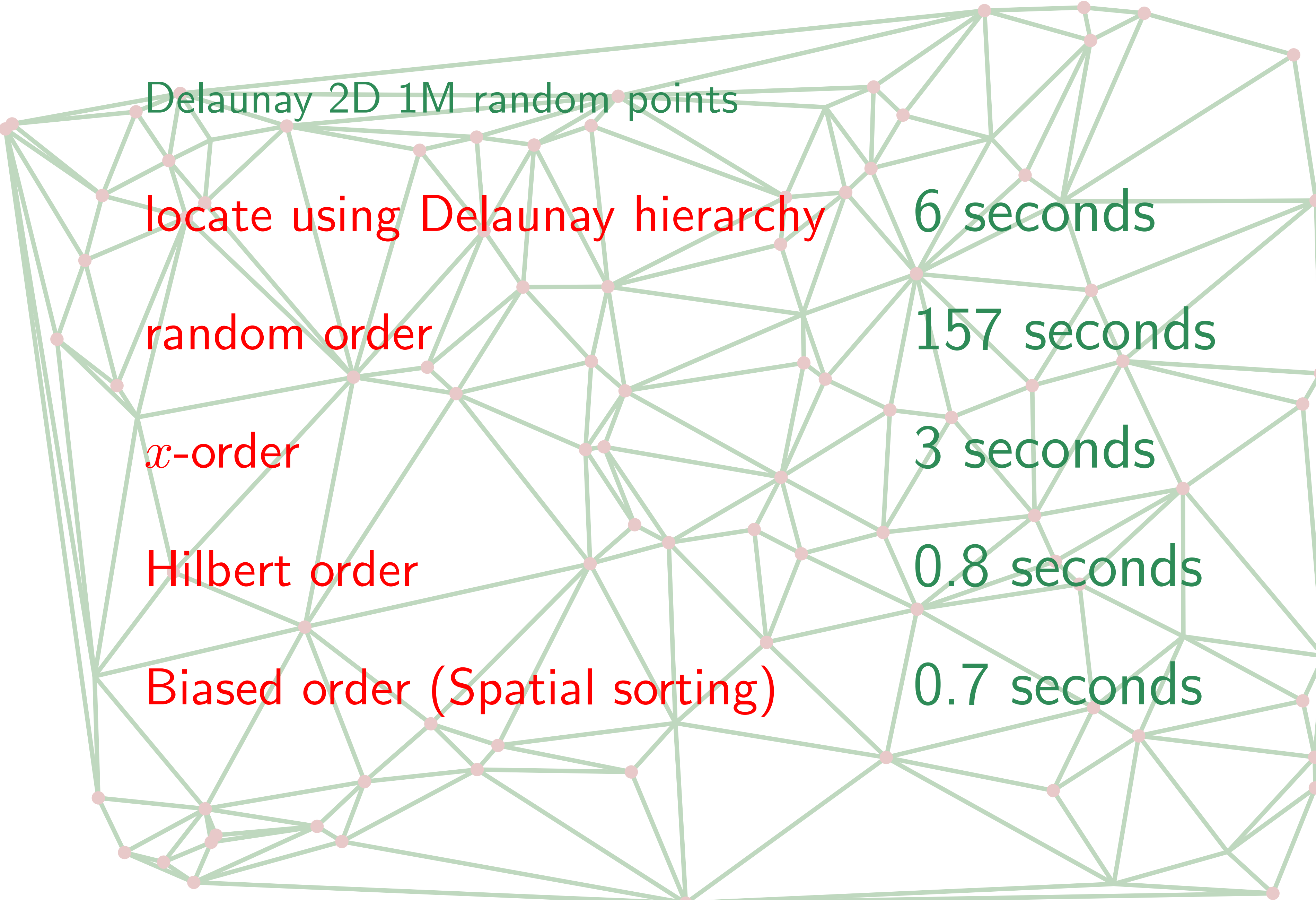
Data structures to locate - biased random insertion order



Data structures to locate - biased random insertion order



Data structures to locate - biased random insertion order



Delaunay 2D 1M random points

locate using Delaunay hierarchy	6 seconds
random order	157 seconds
x-order	3 seconds
Hilbert order	0.8 seconds
Biased order (Spatial sorting)	0.7 seconds

Data structures to locate - biased random insertion order

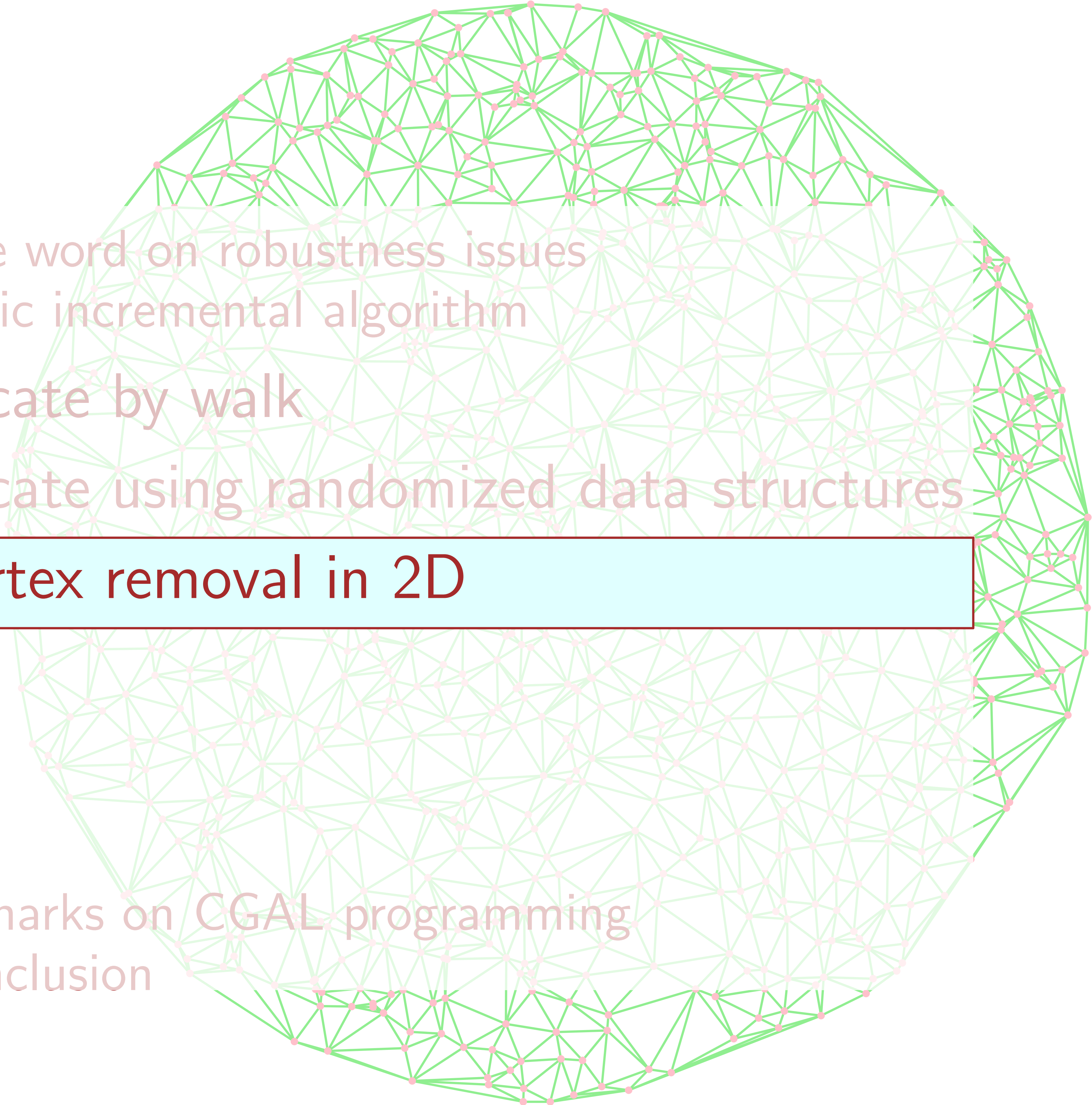
Delaunay 2D 100K parabola points

locate using Delaunay hierarchy	0.3 seconds
random order	128 seconds
x-order	632 seconds
Hilbert order	46 seconds
Biased order (Spatial sorting)	0.3 seconds

Data structures to locate

Construction of Delaunay 10 M random points

Delaunay tree	$\sim$ 10 mn (estimate)
Delaunay hierarchy	90 seconds
Biased random order	10.6 seconds



One word on robustness issues

Basic incremental algorithm

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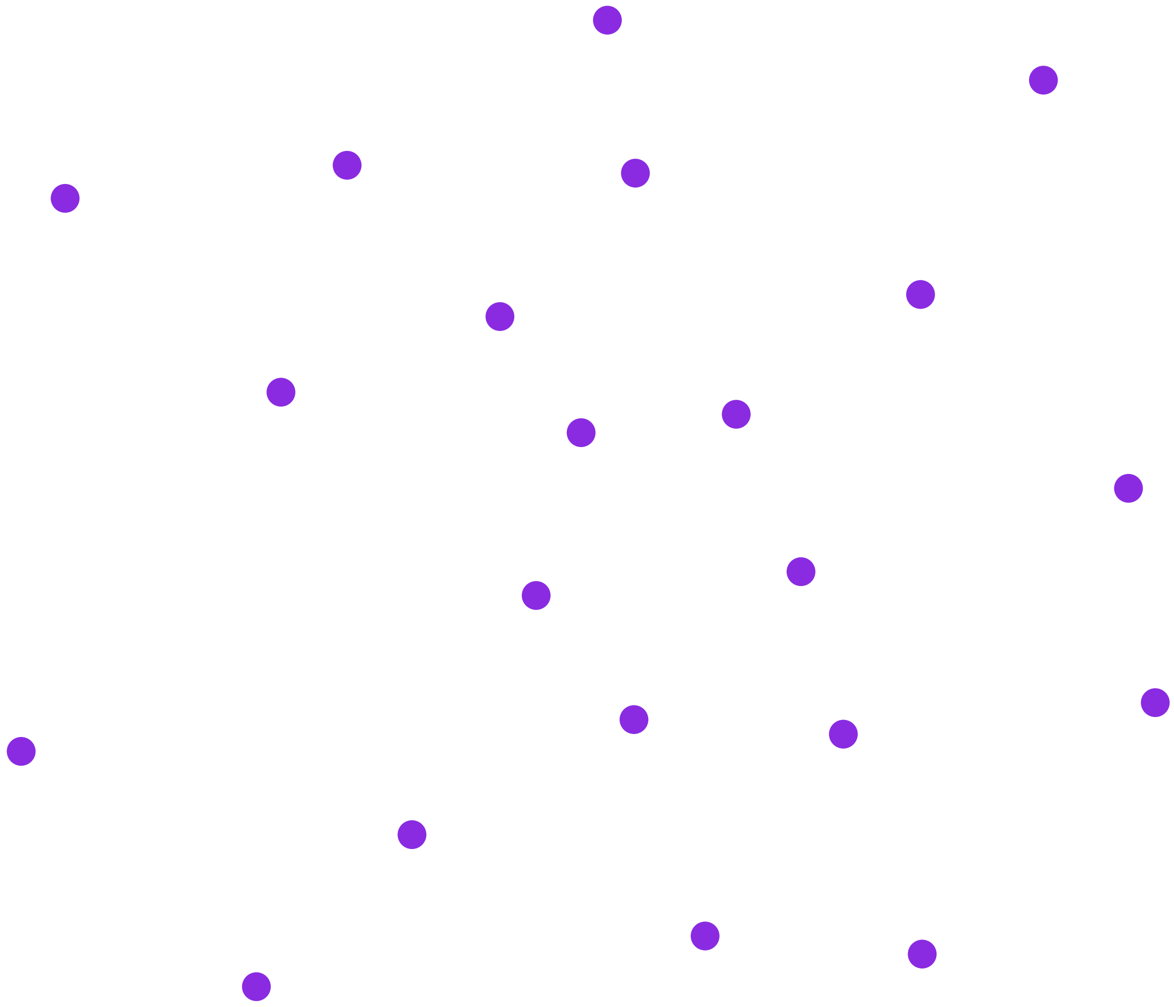
Vertex removal in 2D

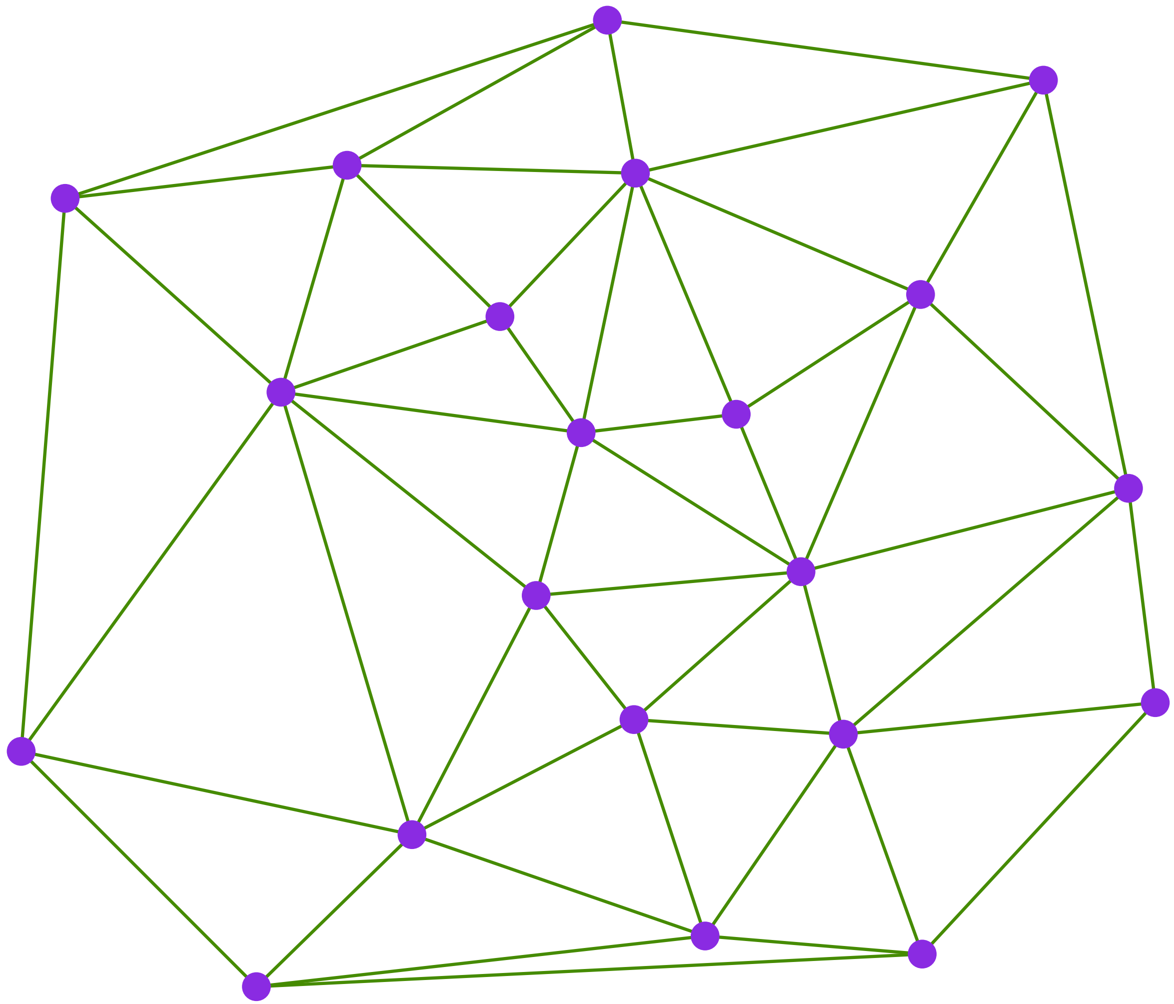
Remarks on CGAL programming

Conclusion

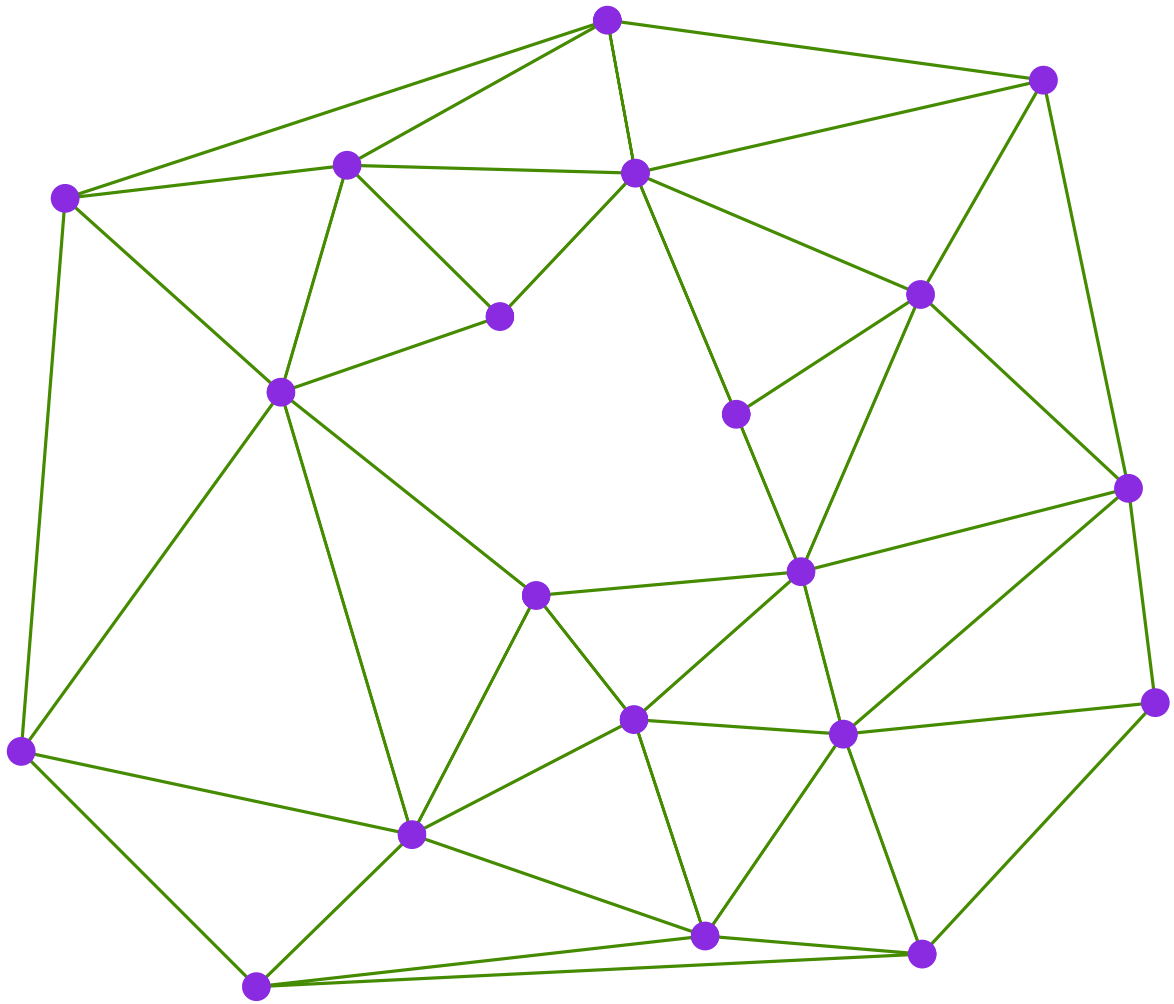
Vertex removal



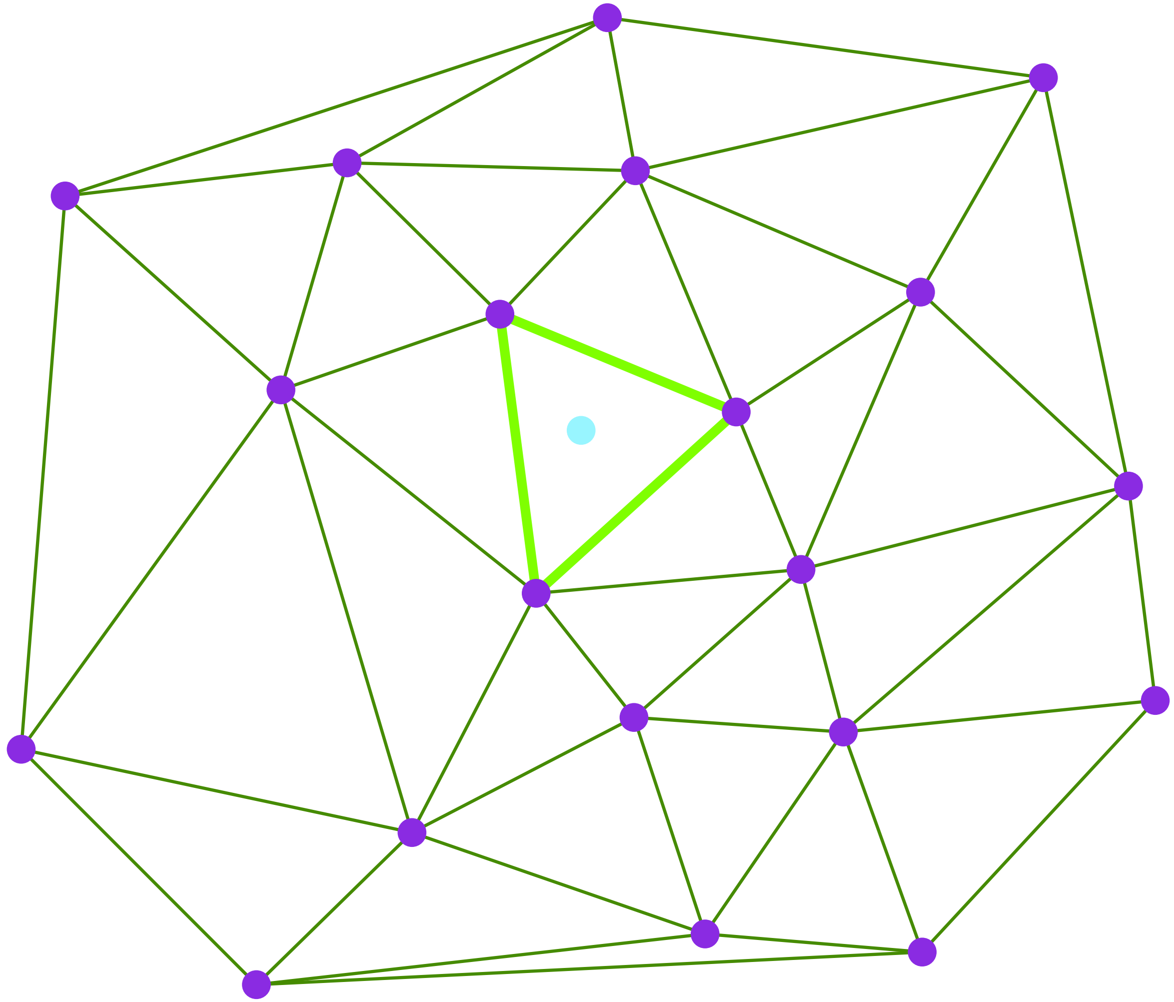
Vertex removal

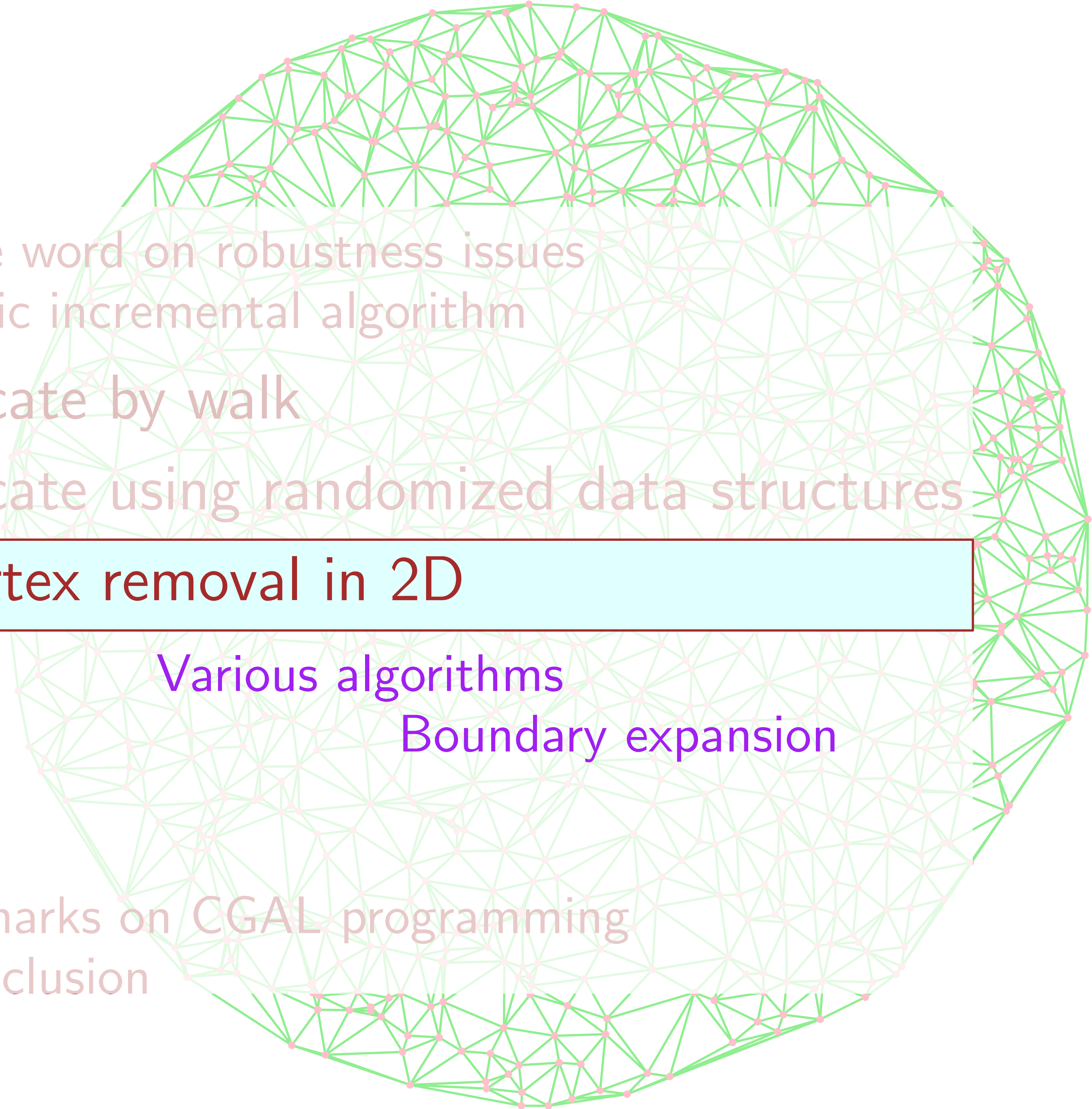


Vertex removal



Vertex removal





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Various algorithms

Boundary expansion

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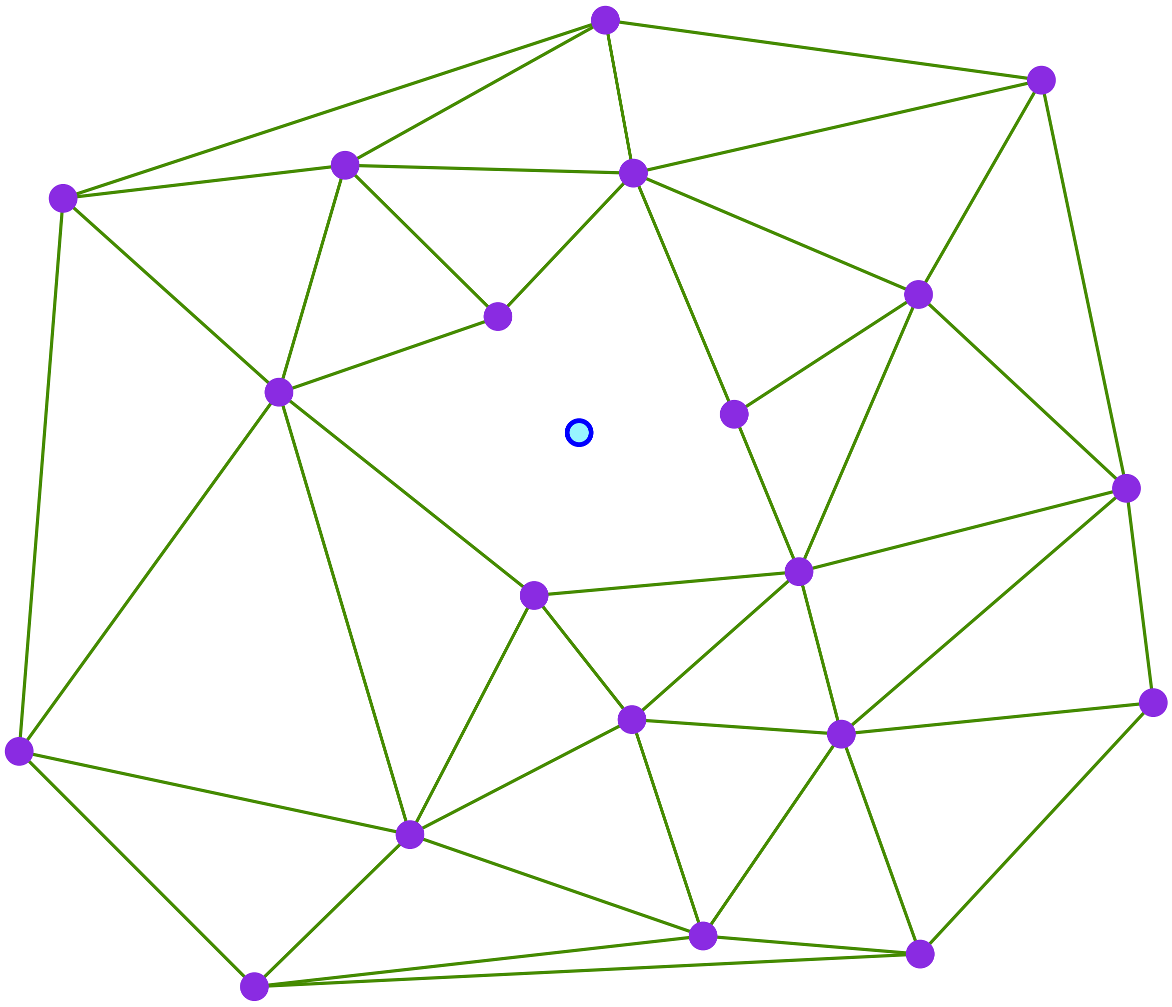
Conclusion

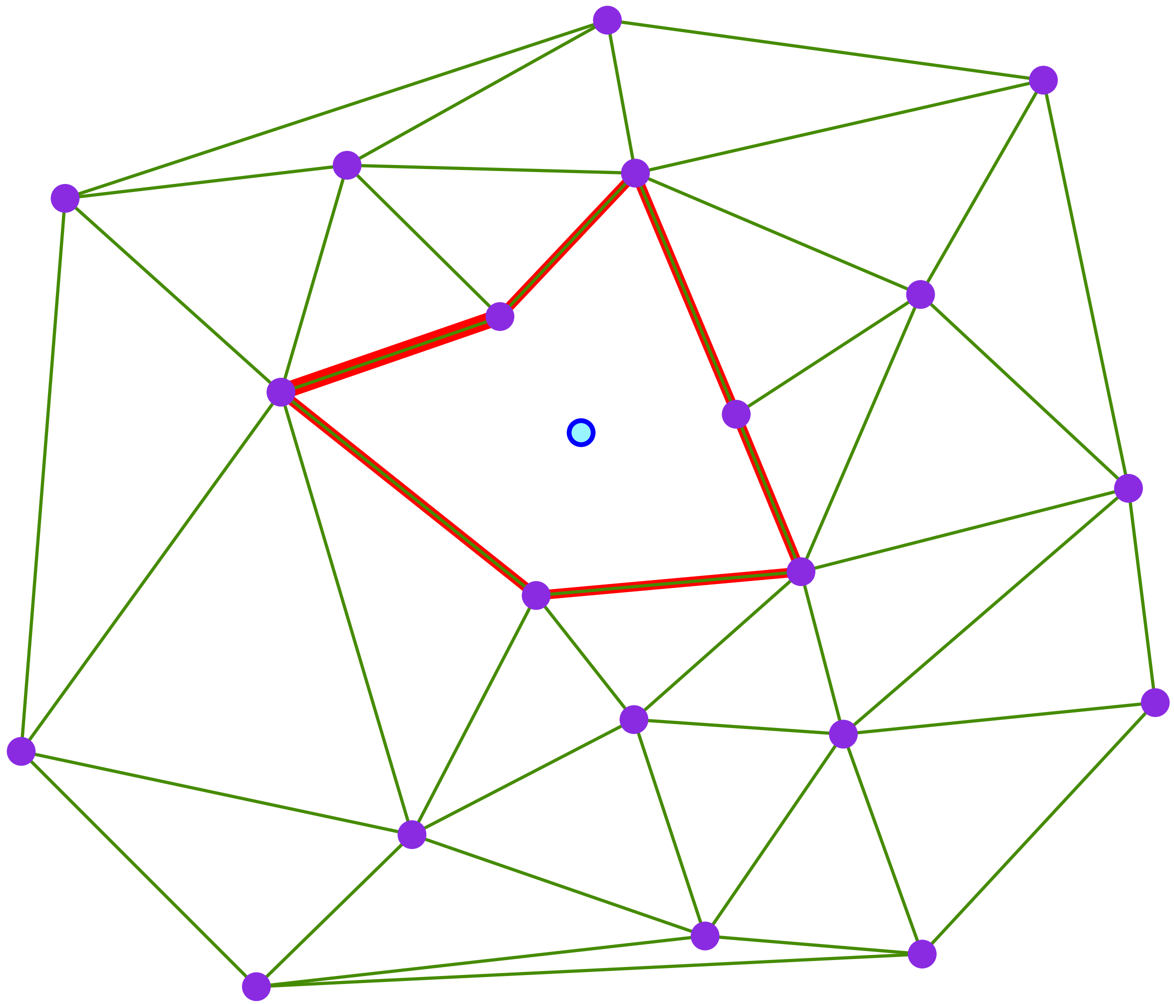
Vertex removal - boundary expansion

release 3.5, 2D implementation



Vertex removal - boundary expansion

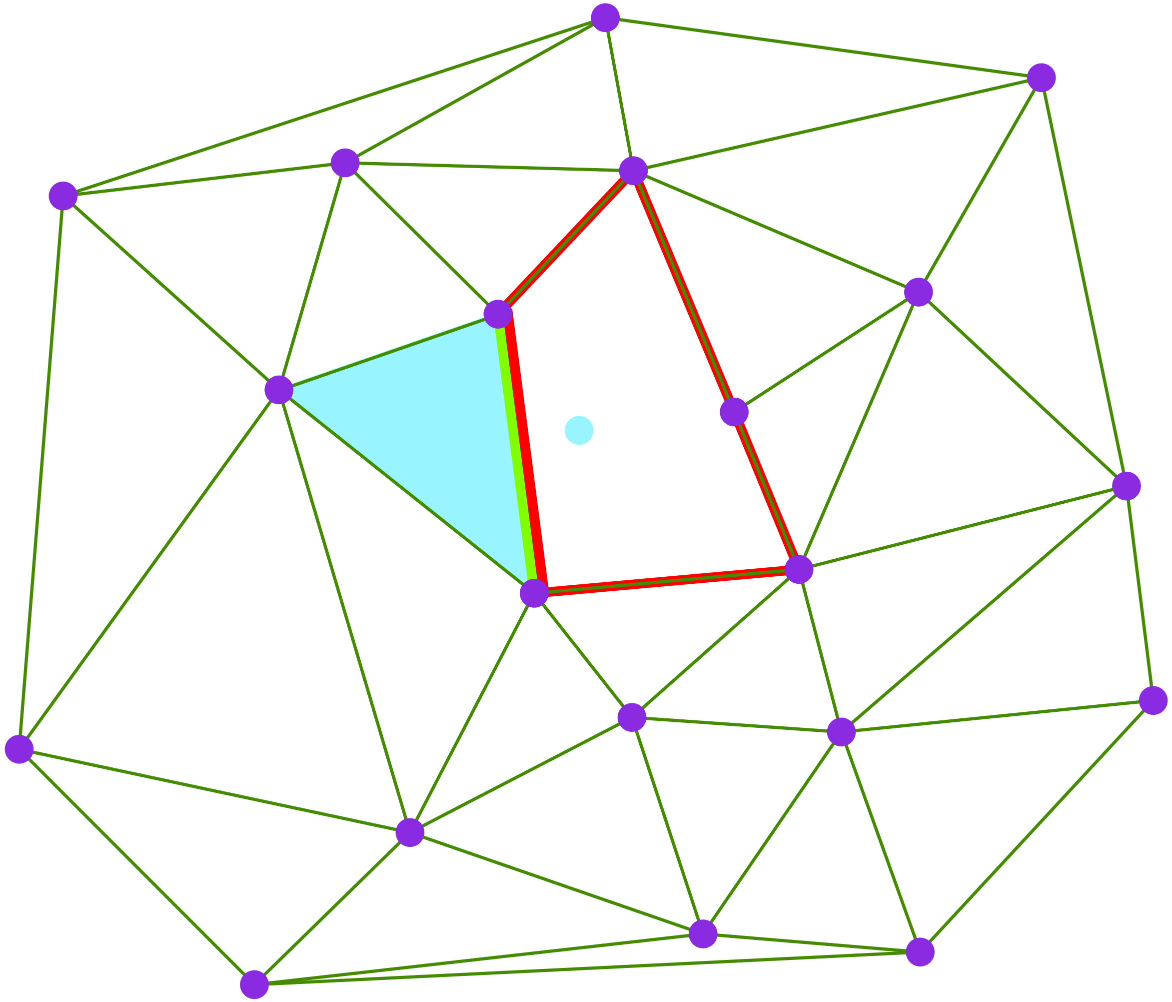
release 3.5, 2D implementation



hole boundary = queue

Vertex removal - boundary expansion

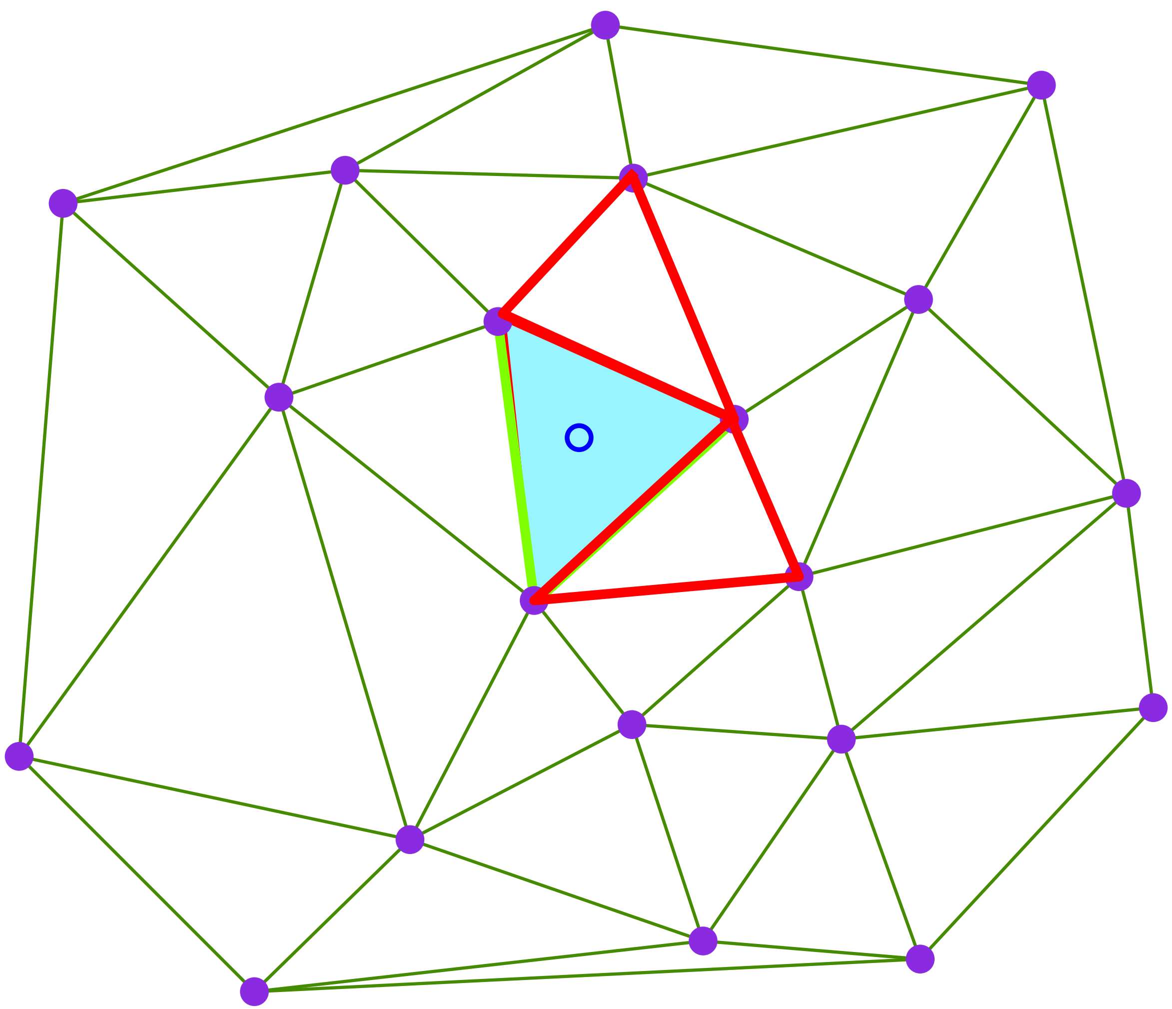
release 3.5, 2D implementation



find new incident triangle in linear time

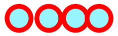
Vertex removal - boundary expansion

release 3.5, 2D implementation



Vertex removal - boundary expansion

release 3.5, 2D implementation



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Various algorithms

Boundary expansion

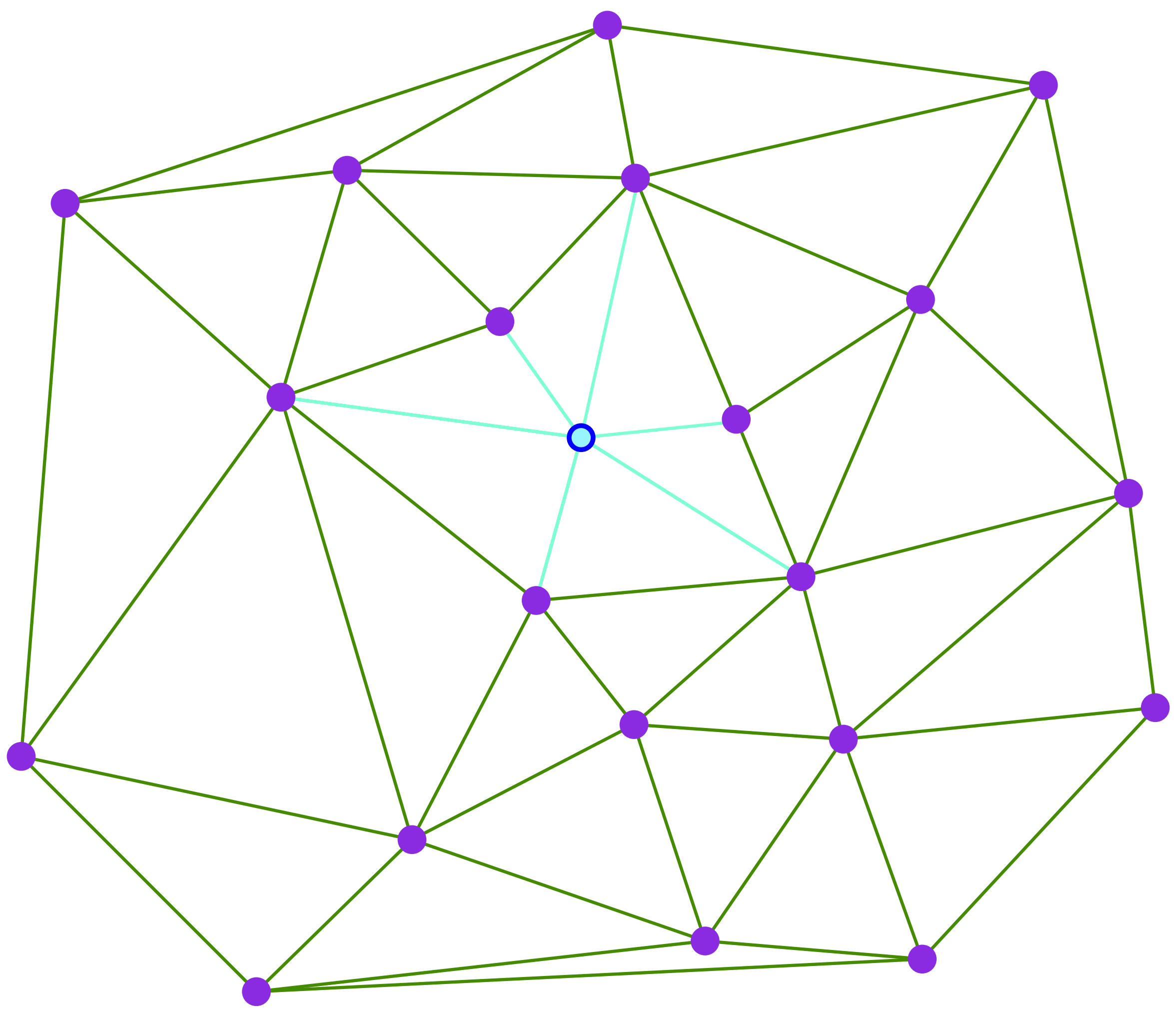
Triangulate and sew

Remarks on CGAL programming

Conclusion

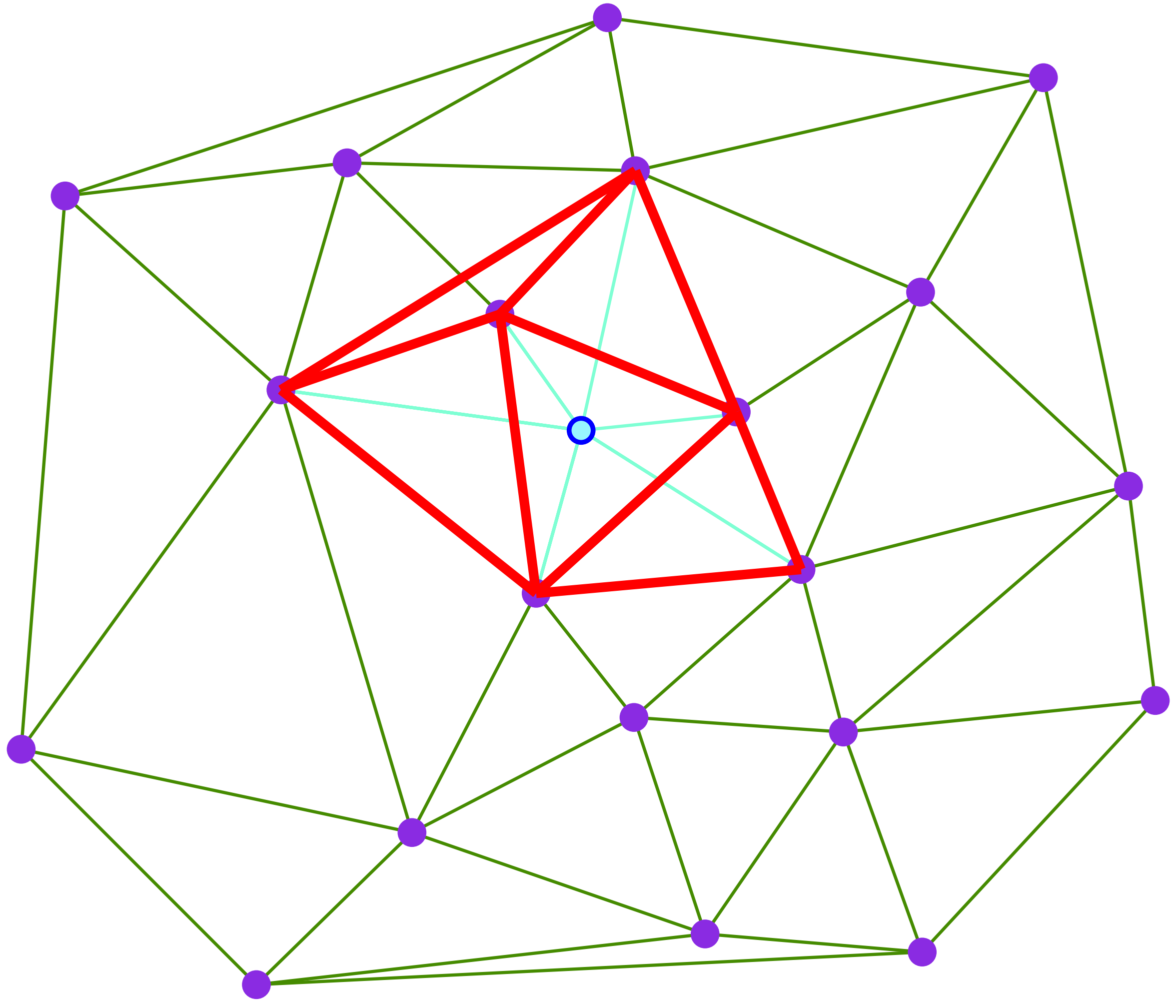
Vertex removal - triangulate and sew

current CGAL 3D implementation



Vertex removal - triangulate and sew

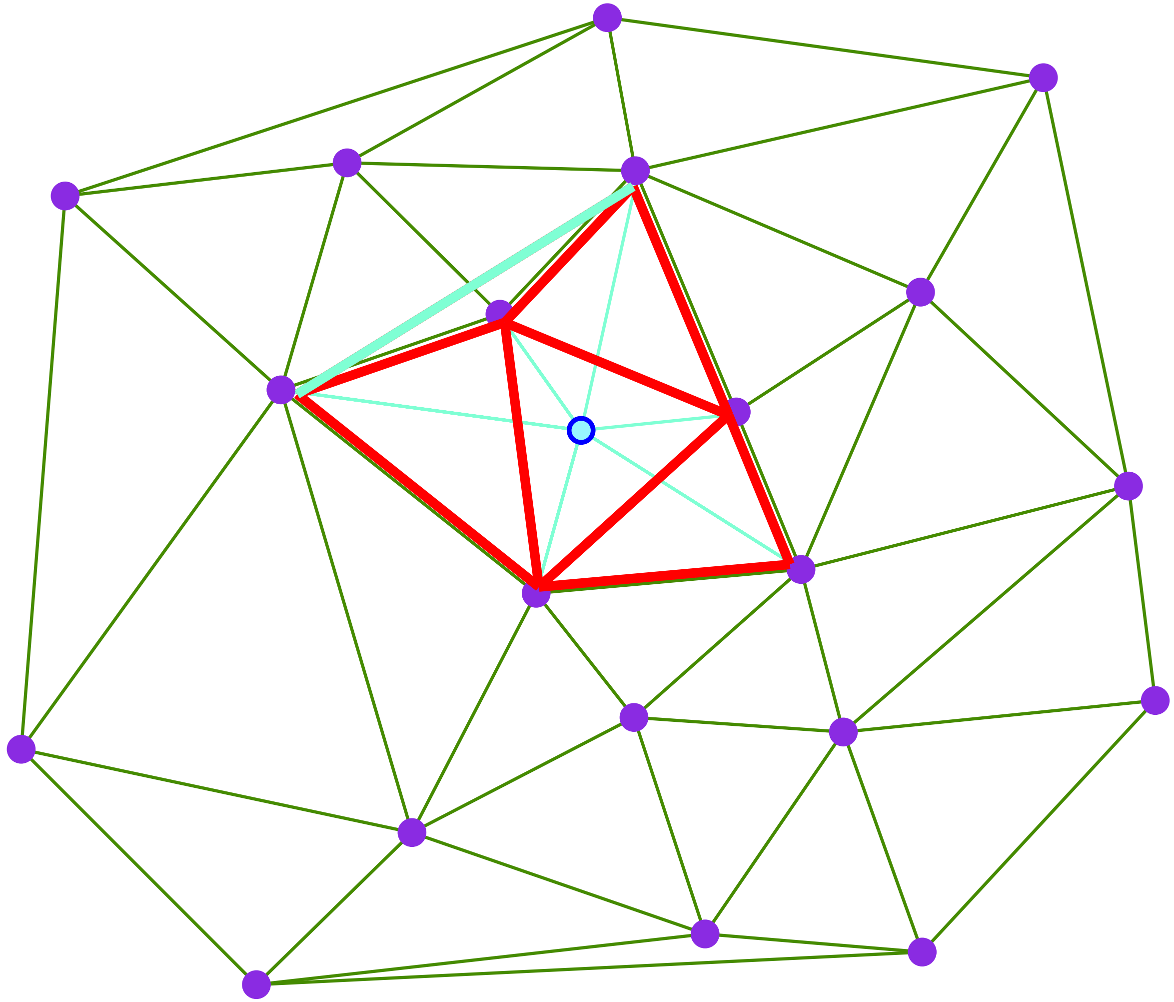
current CGAL 3D implementation



Delaunay of neighbors

Vertex removal - triangulate and sew

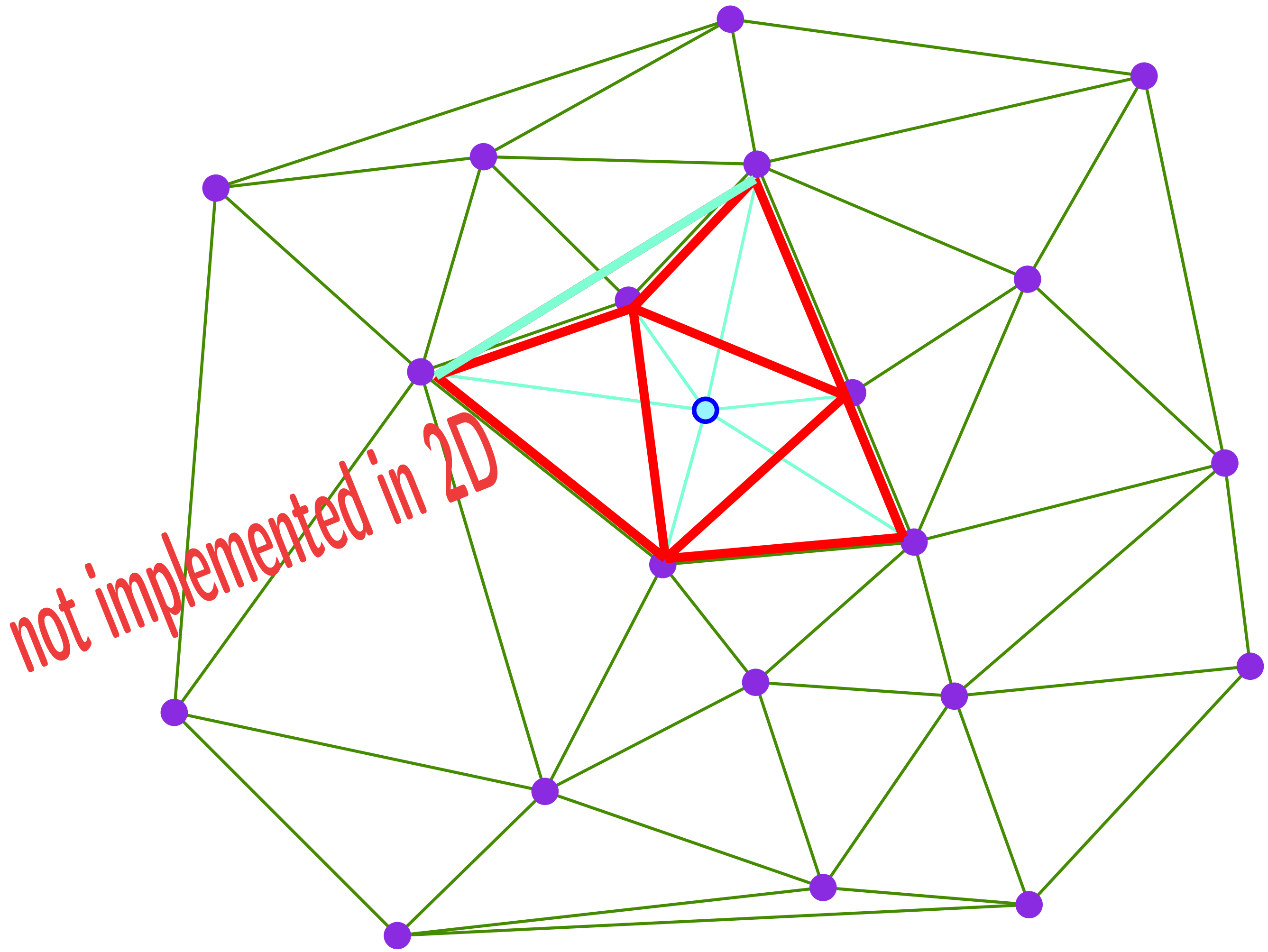
current CGAL 3D implementation



delete extra triangles and sew

Vertex removal - triangulate and sew

current CGAL 3D implementation



delete extra triangles and sew

One word on robustness issues

Basic incremental algorithm

Locate by walk

Locate using randomized data structures

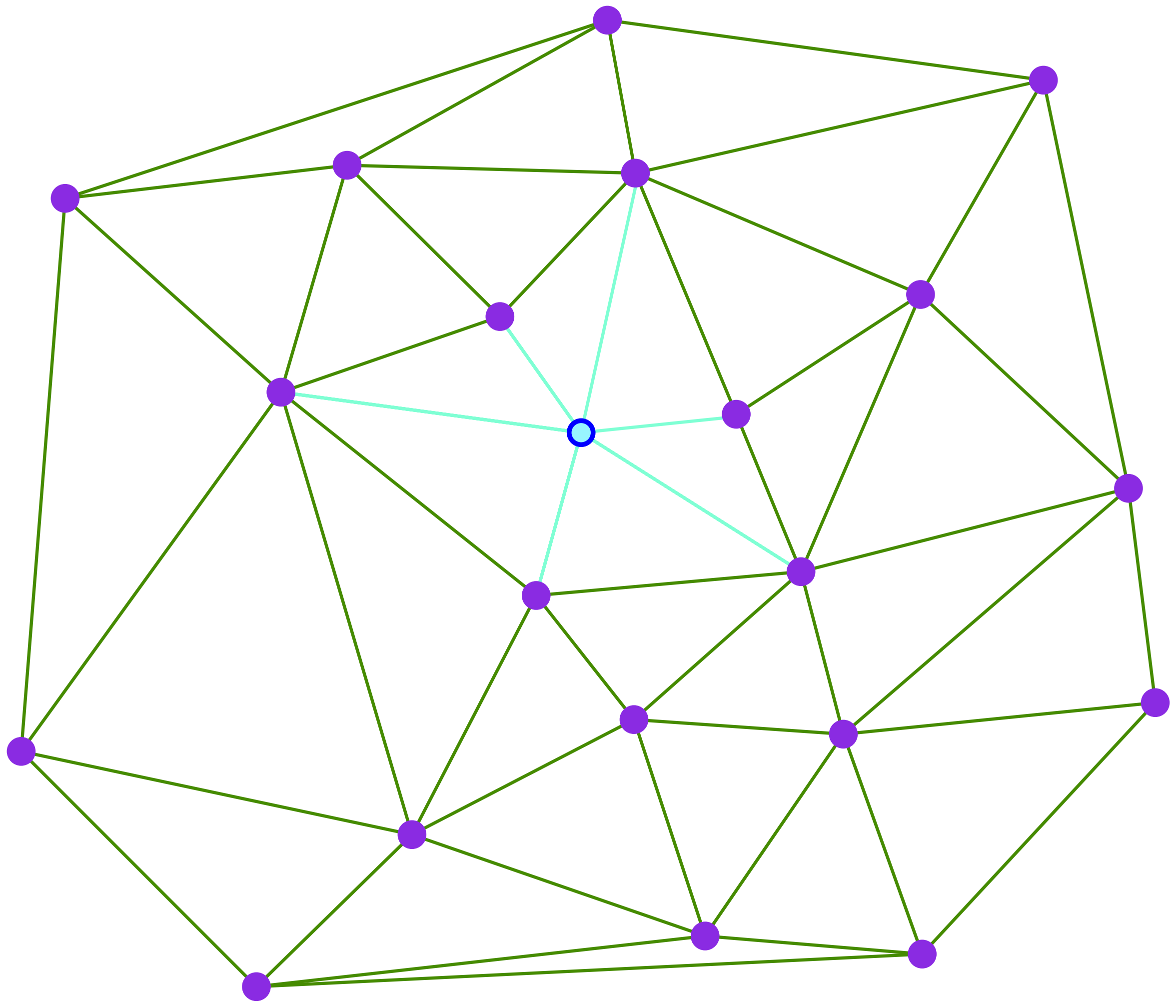
Vertex removal in 2D

- Various algorithms
  - Boundary expansion
  - Triangulate and sew
  - Flip the hole

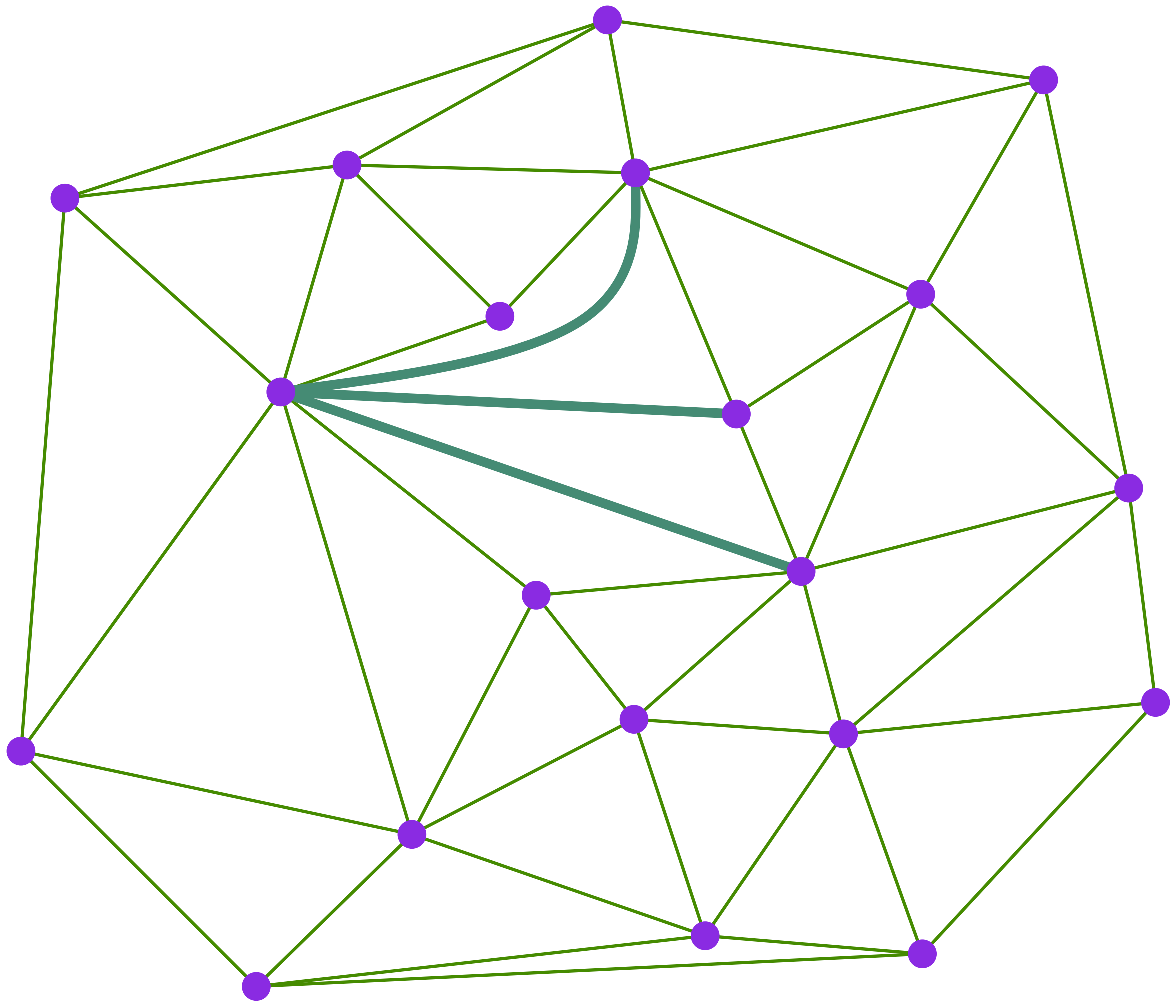
Remarks on CGAL programming

Conclusion

Vertex removal - flip the hole

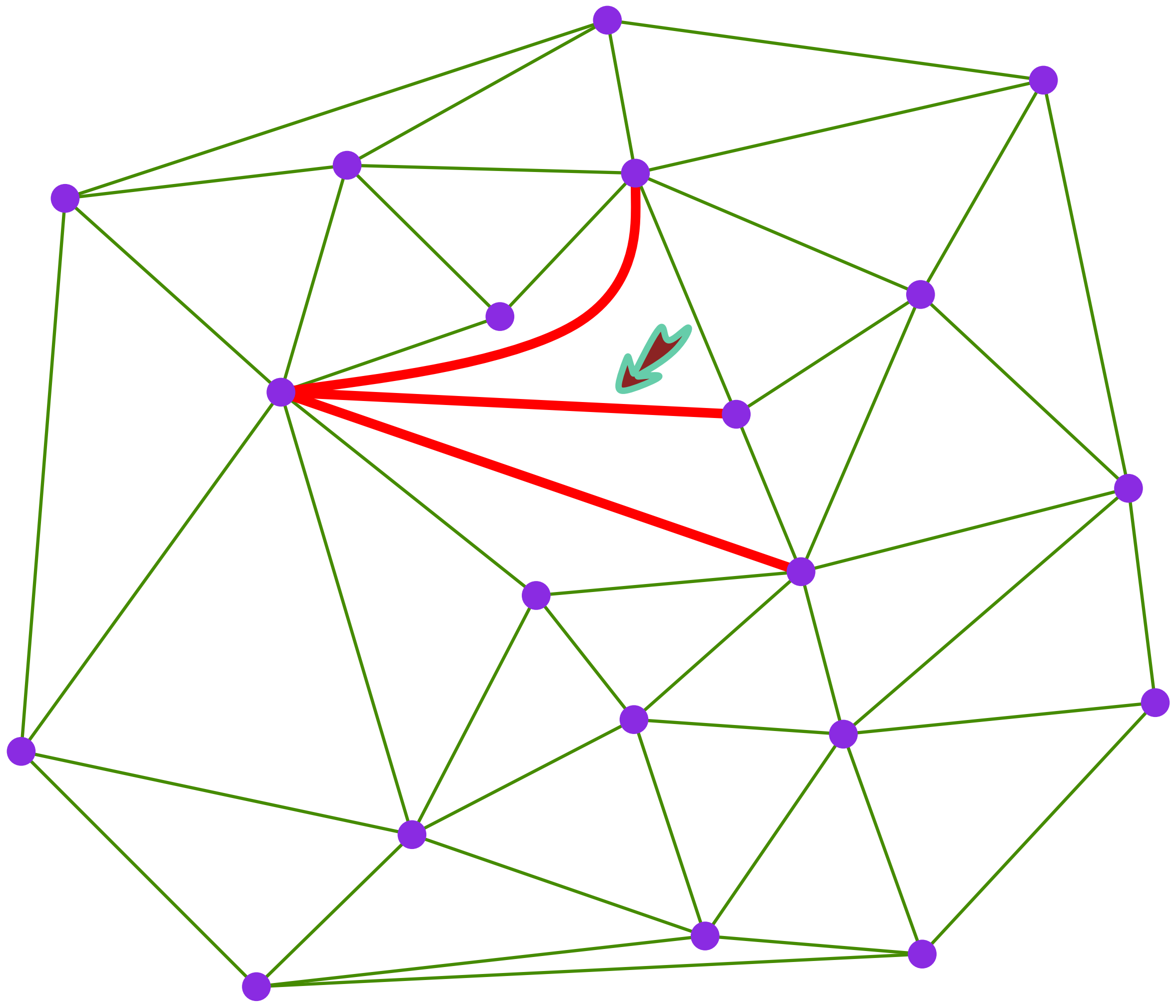


Vertex removal - flip the hole



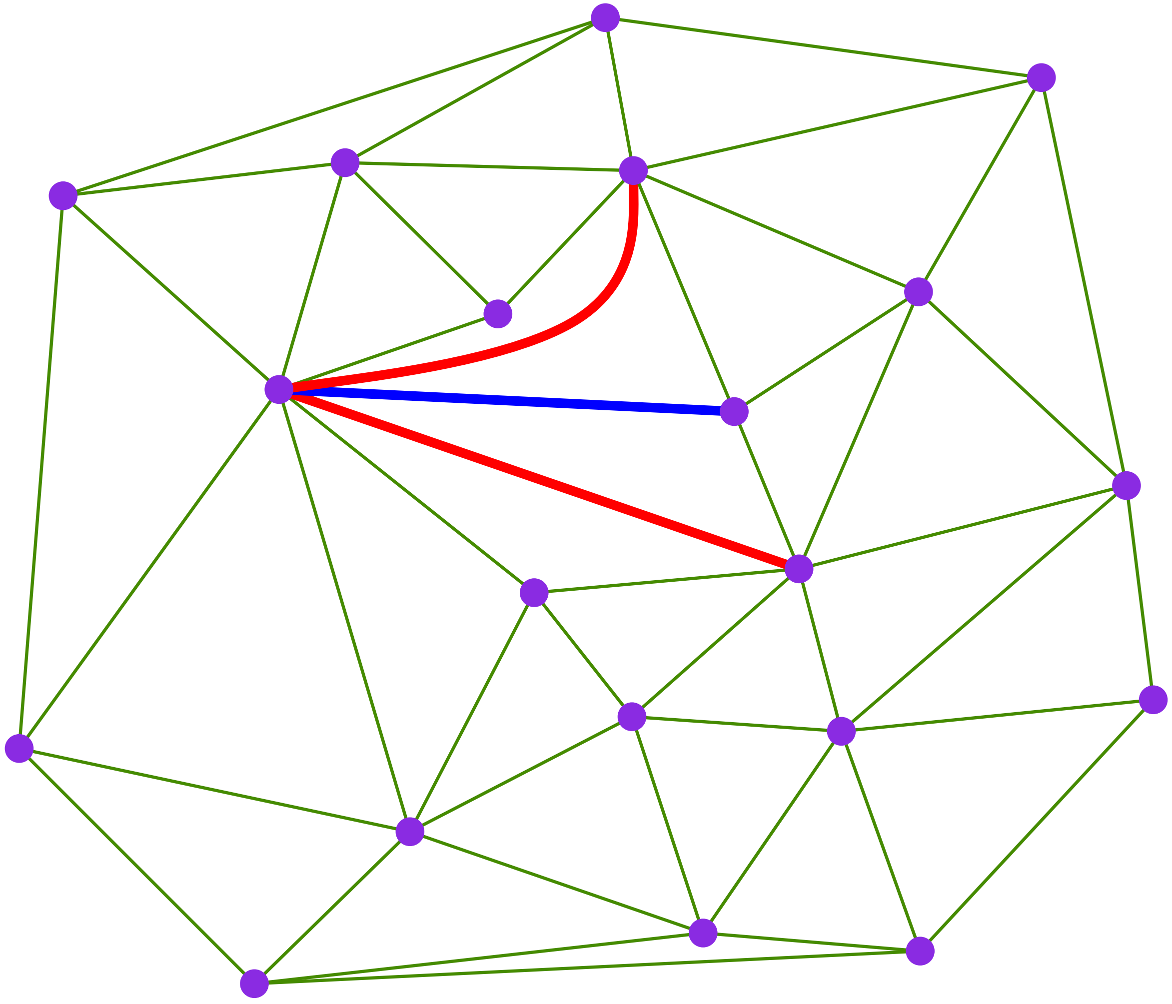
triangulate from any vertex

Vertex removal - flip the hole

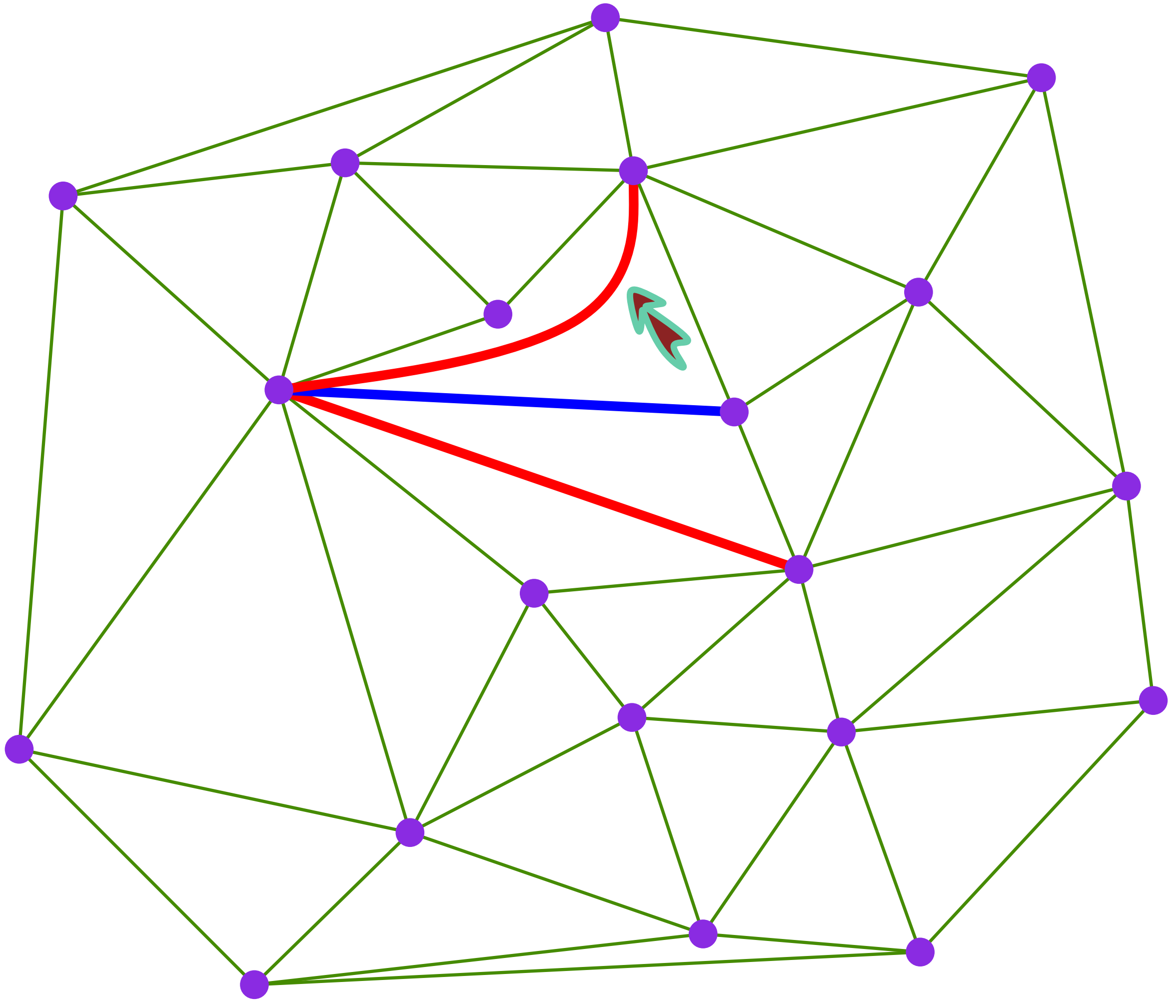


queue of edges to be checked

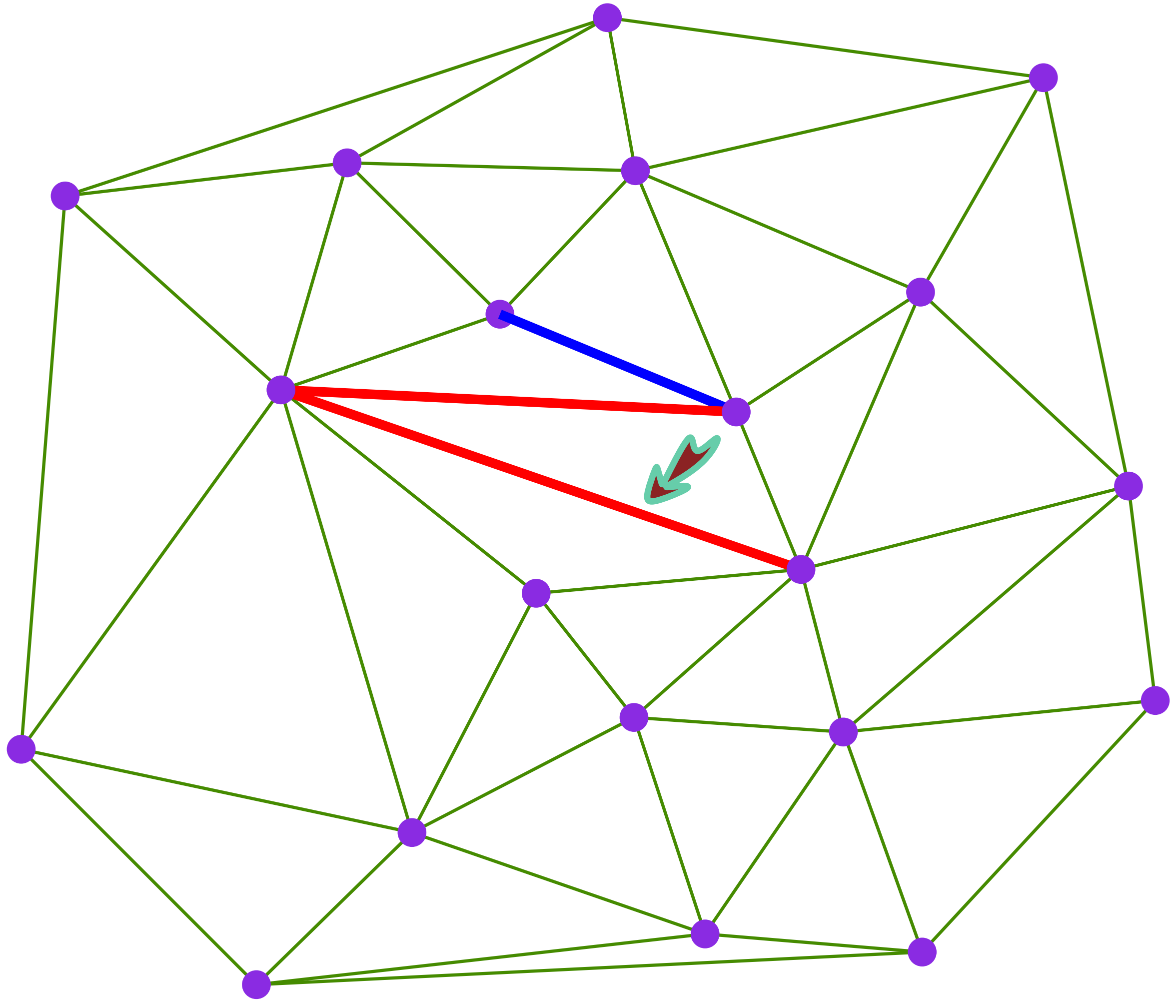
Vertex removal - flip the hole



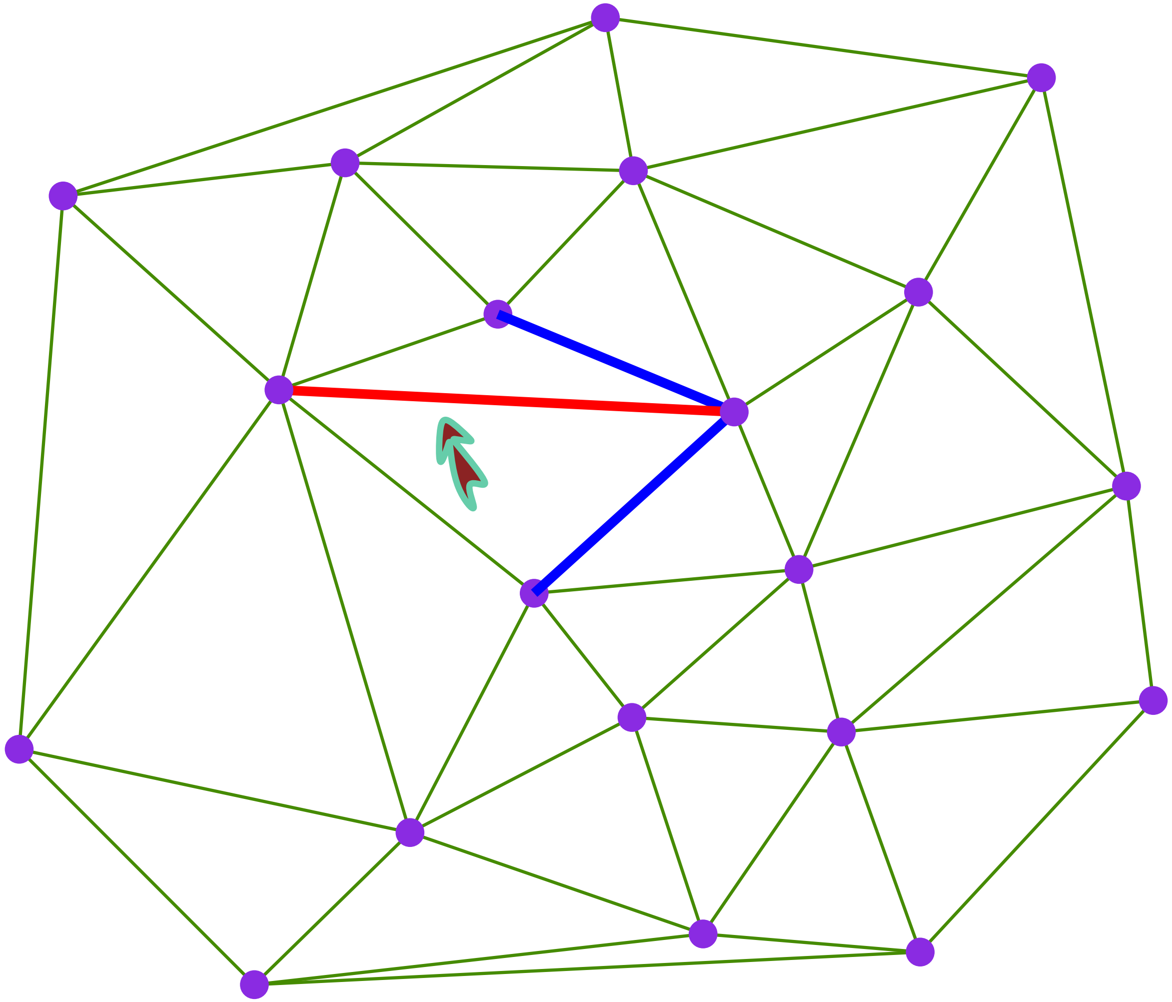
Vertex removal - flip the hole



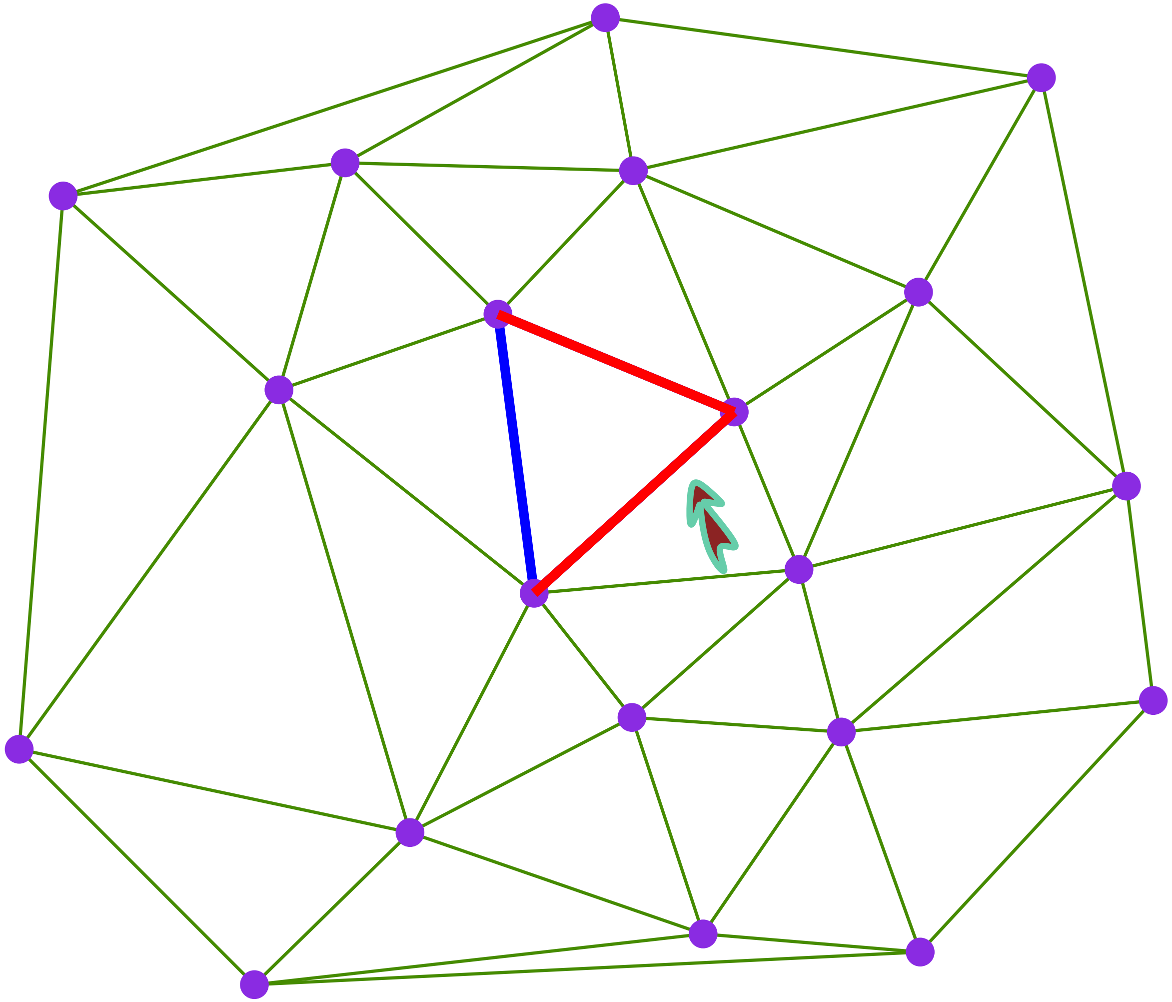
Vertex removal - flip the hole



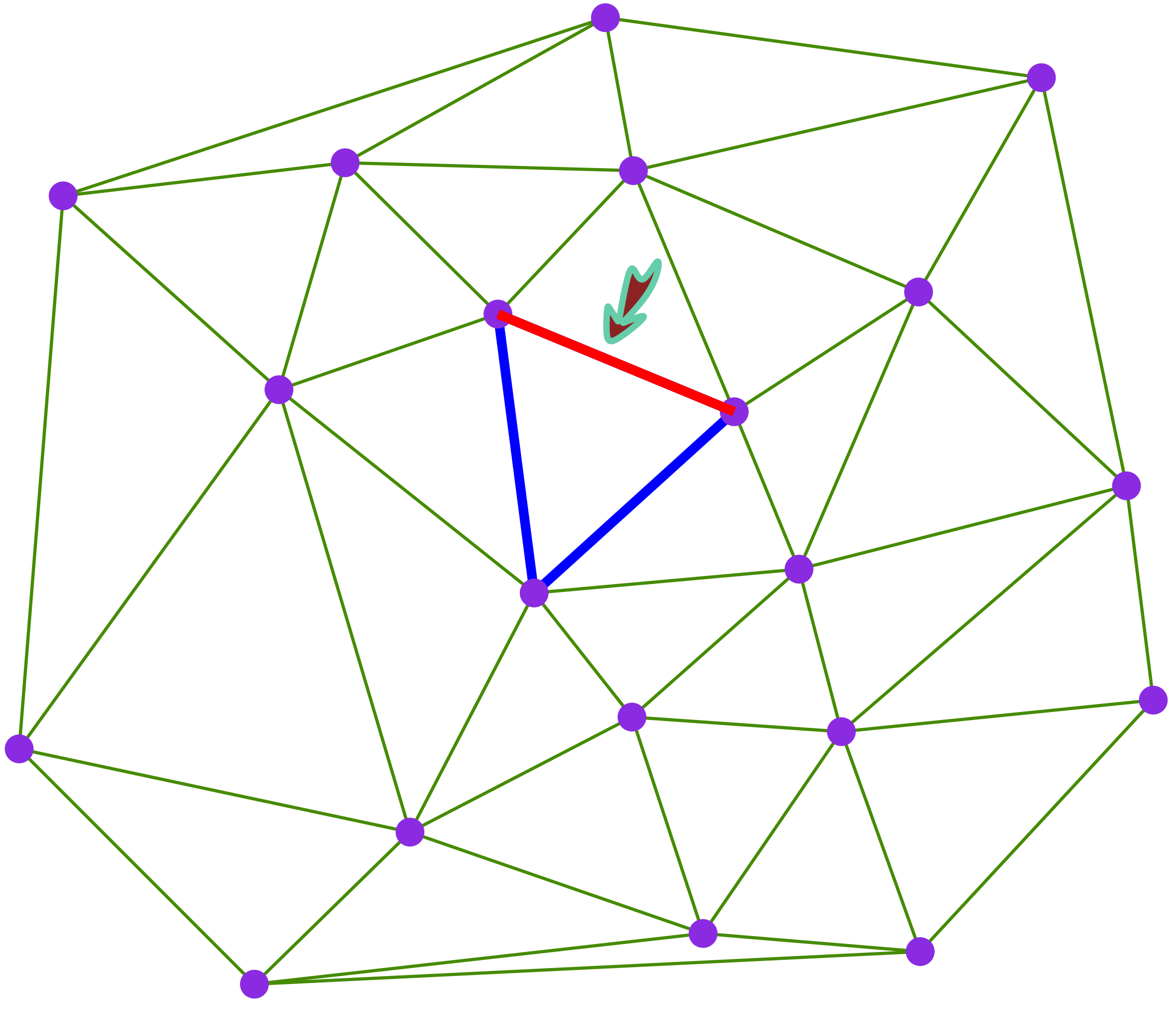
Vertex removal - flip the hole



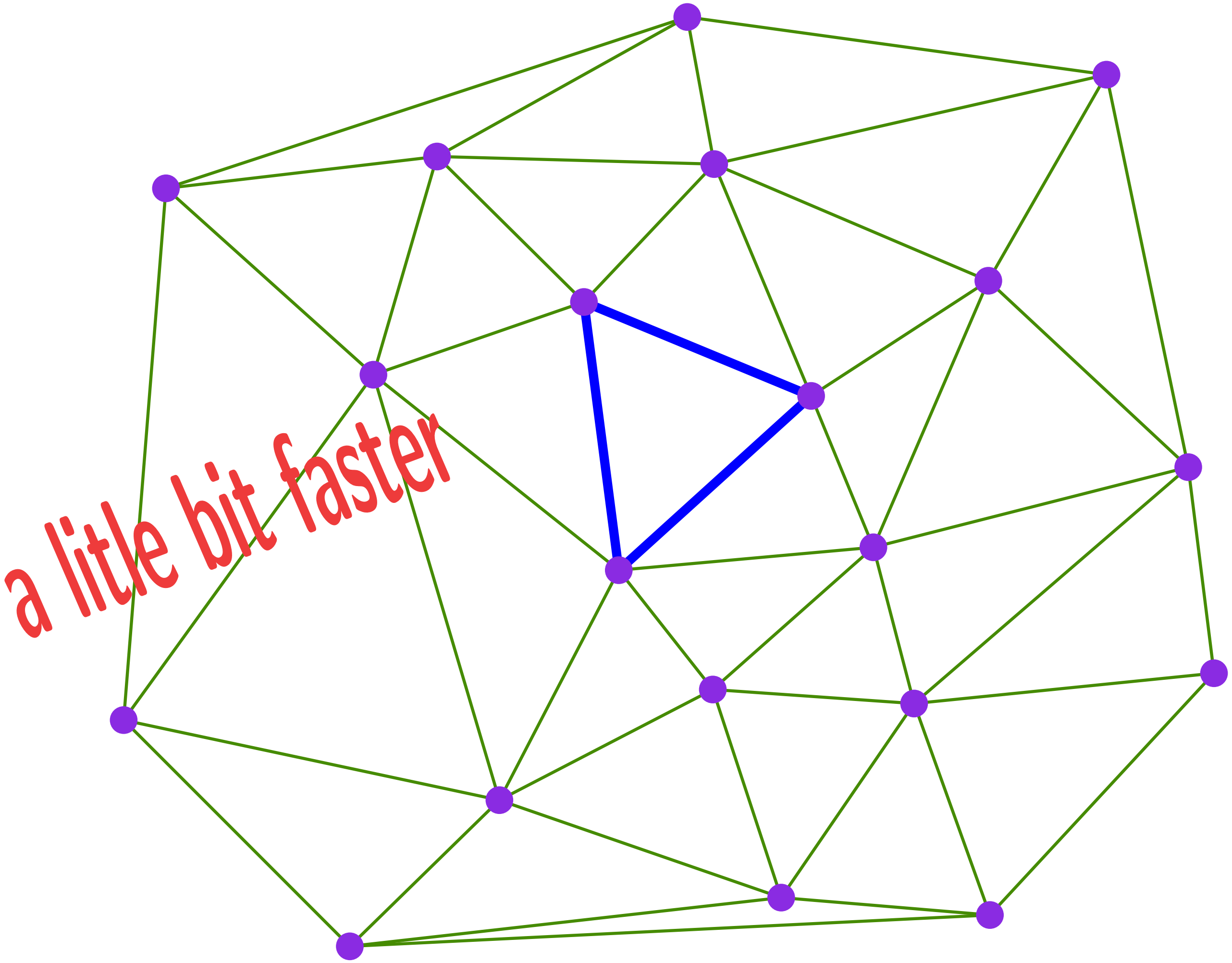
Vertex removal - flip the hole



Vertex removal - flip the hole



Vertex removal - flip the hole



One word on robustness issues

Basic incremental algorithm

Locate by walk

Locate using randomized data structures

Vertex removal in 2D

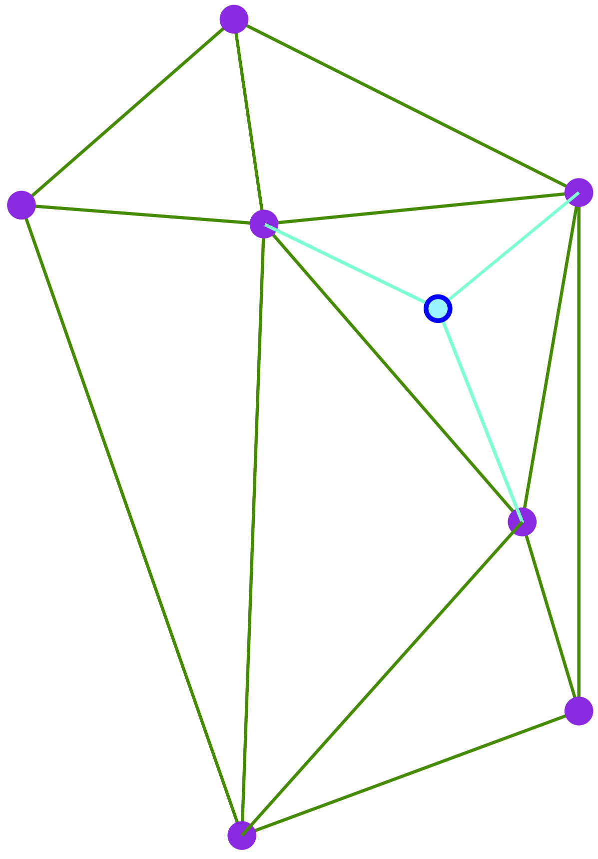
- Various algorithms
- Low degree optimization

Remarks on CGAL programming

Conclusion

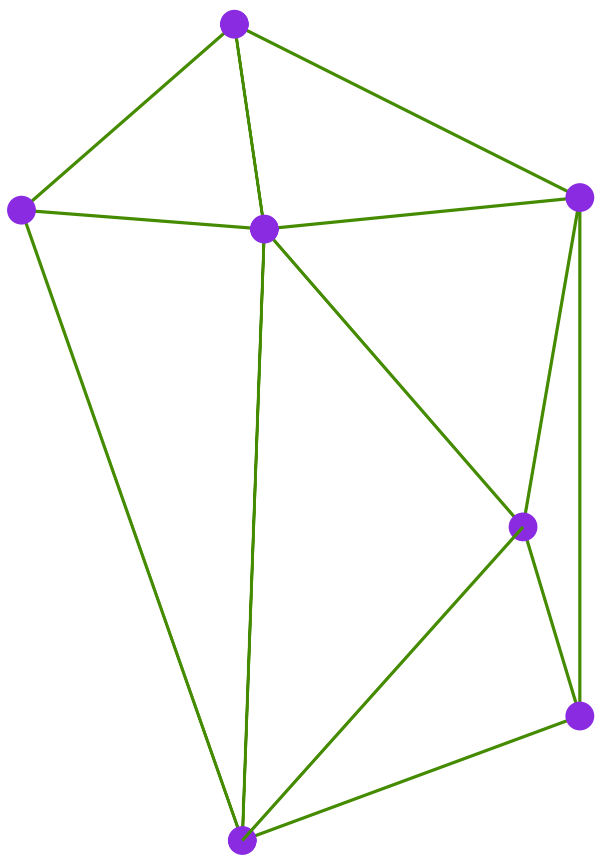
Vertex removal - low degree optimization

degree 3



Vertex removal - low degree optimization

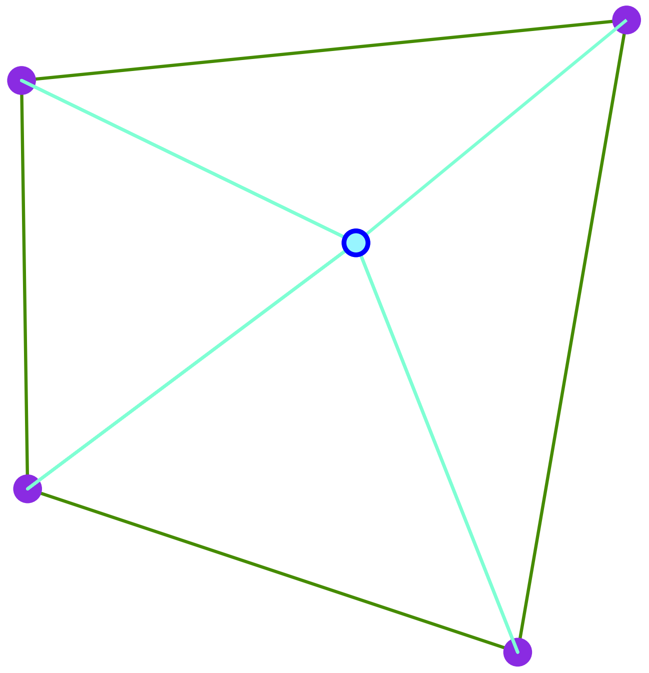
degree 3



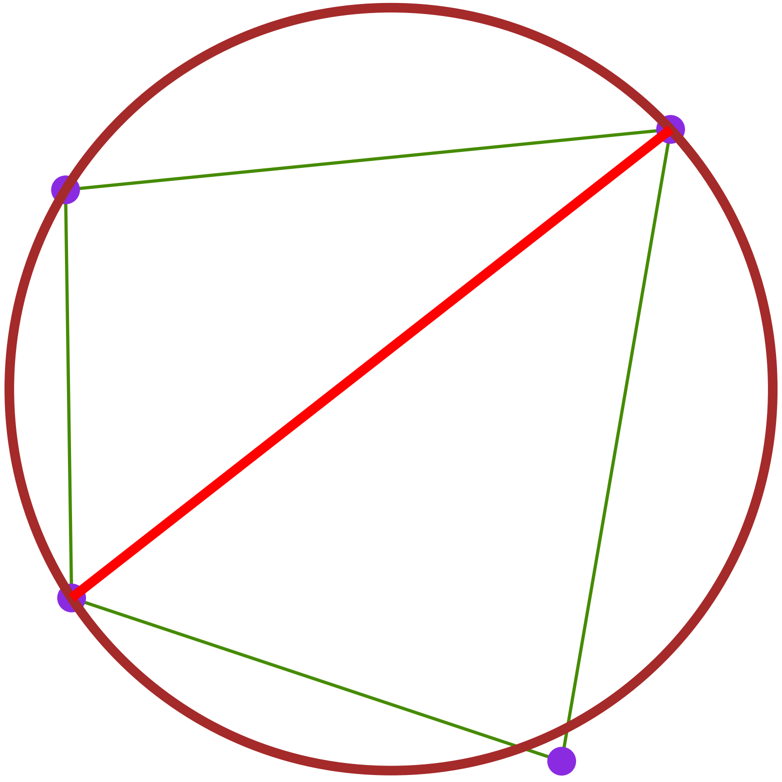
almost nothing to do

Vertex removal - low degree optimization

degree 4



Vertex removal - low degree optimization

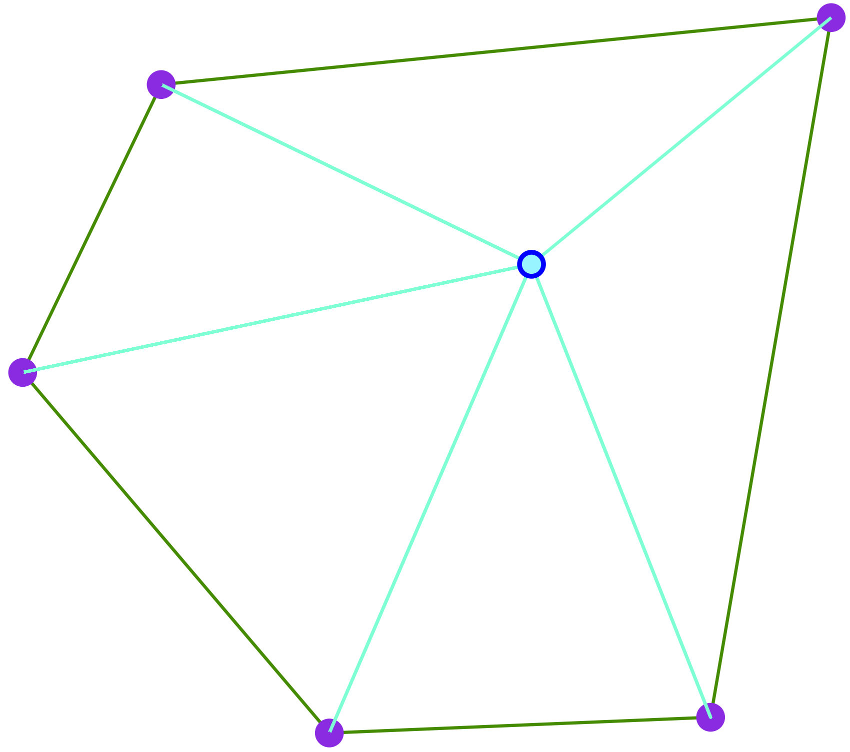


degree 4

just one incircle test to decide

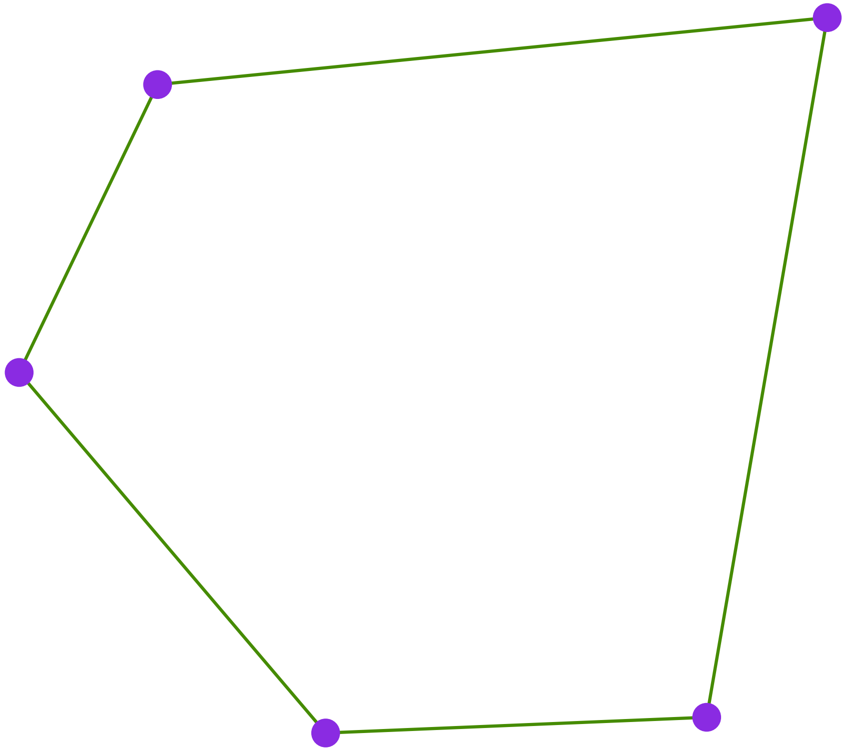
Vertex removal - low degree optimization

degree 5



Vertex removal - low degree optimization

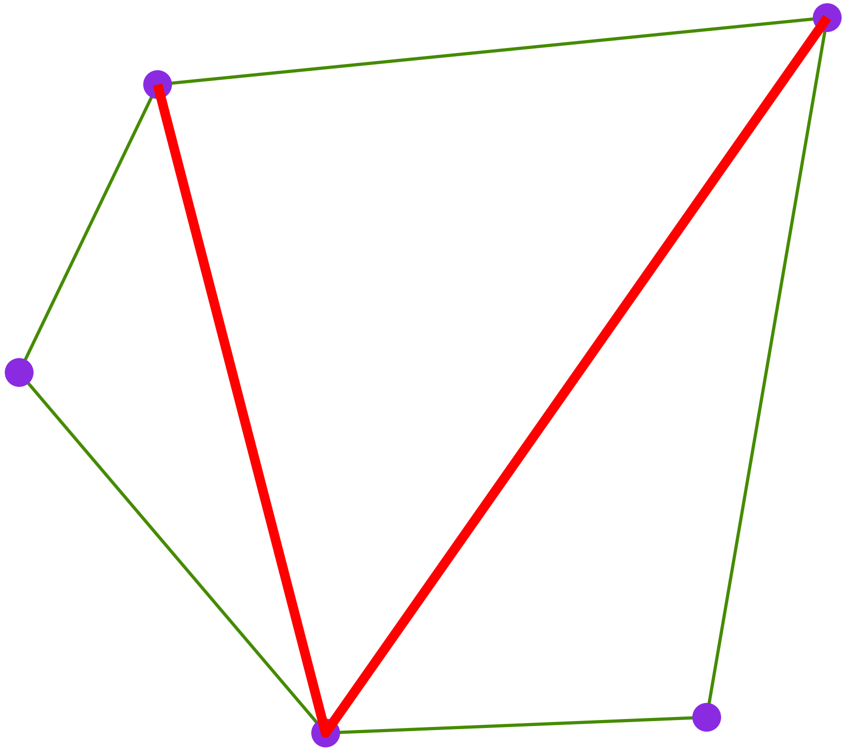
degree 5



"star" the pentagon from the right vertex

Vertex removal - low degree optimization

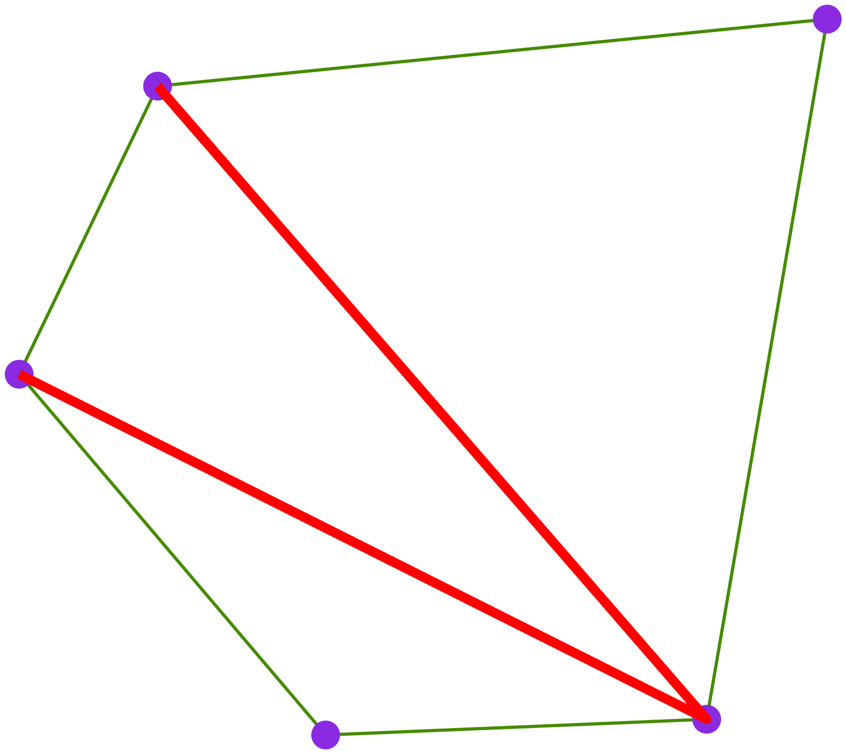
degree 5



”star” the pentagon from the right vertex

Vertex removal - low degree optimization

degree 5



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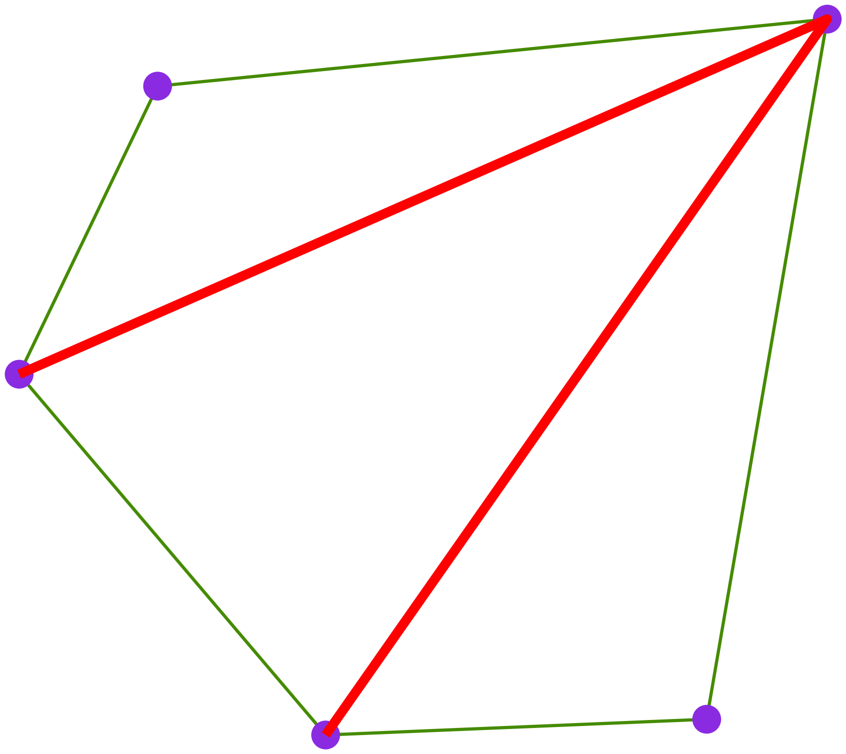
Vertex removal - low degree optimization

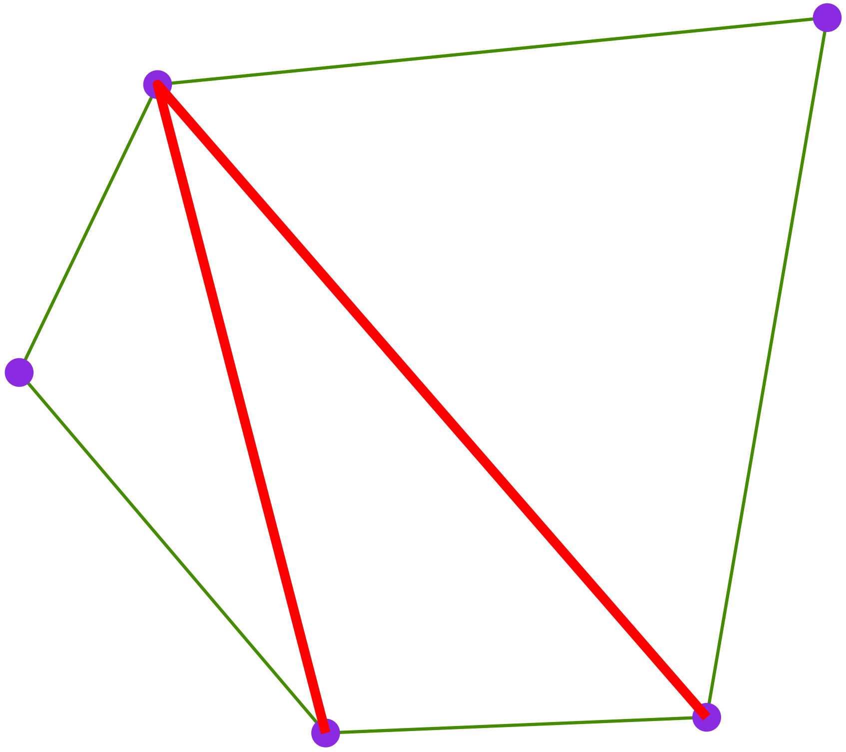
degree 5



"star" the pentagon from the right vertex

Vertex removal - low degree optimization

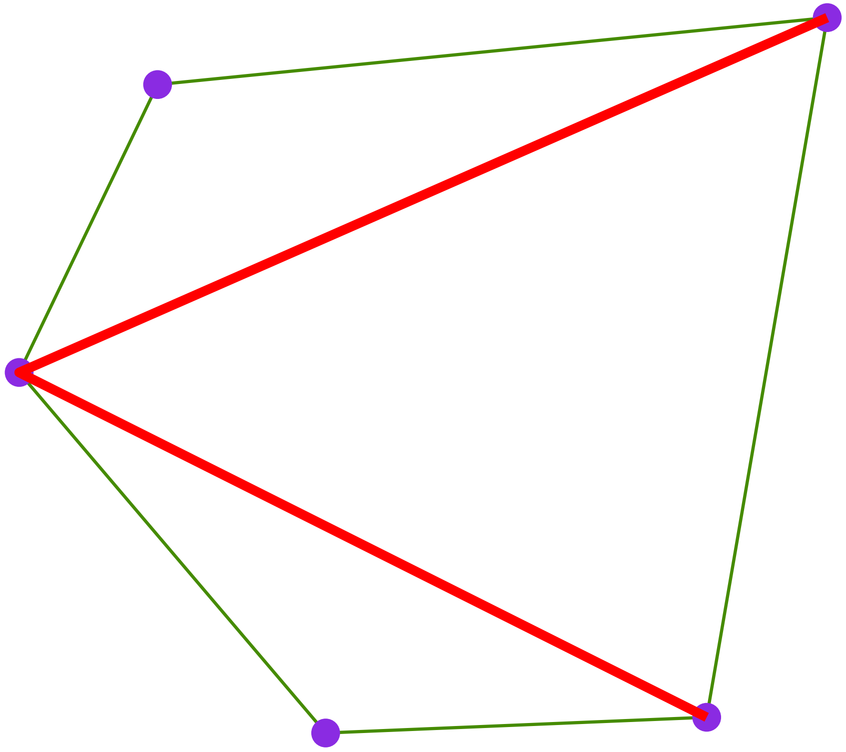
degree 5



"star" the pentagon from the right vertex

Vertex removal - low degree optimization

degree 5



"star" the pentagon from the right vertex

Vertex removal - low degree optimization

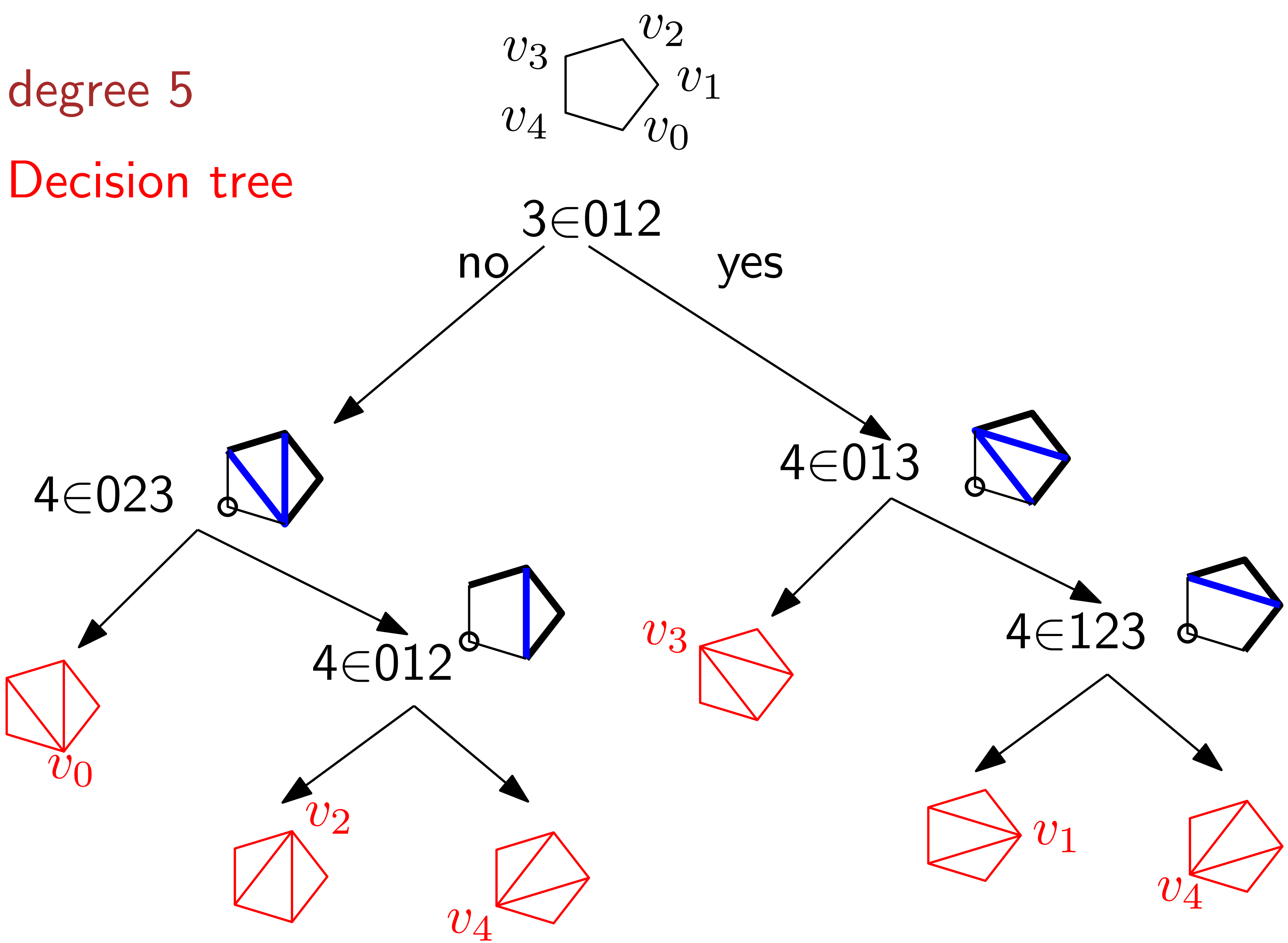
degree 5

Decision tree

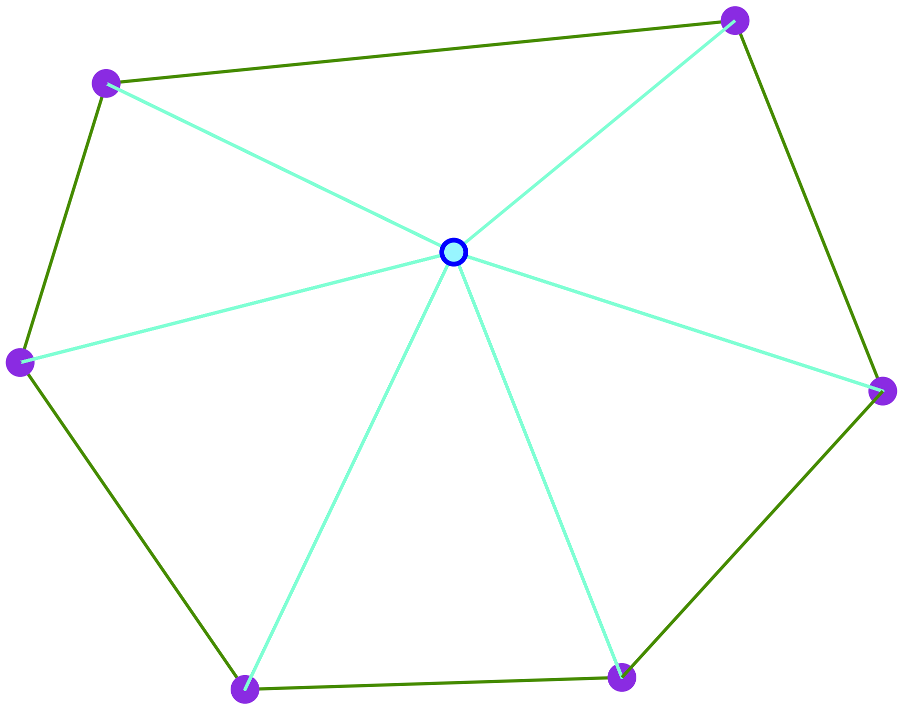
Vertex removal - low degree optimization

degree 5

Decision tree



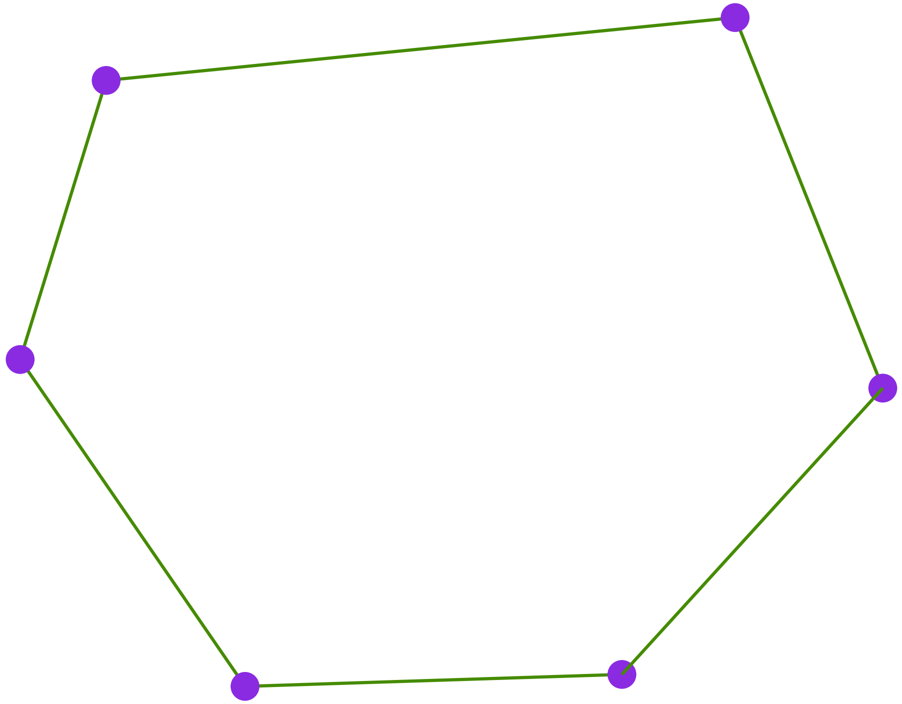
Vertex removal - low degree optimization



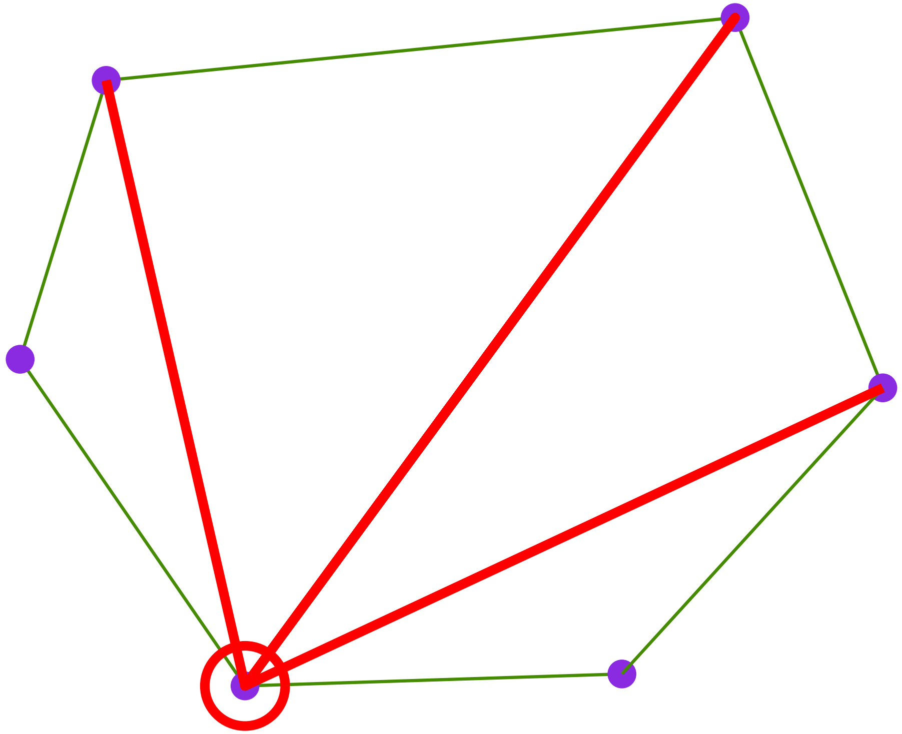
degree 6

Vertex removal - low degree optimization

degree 6



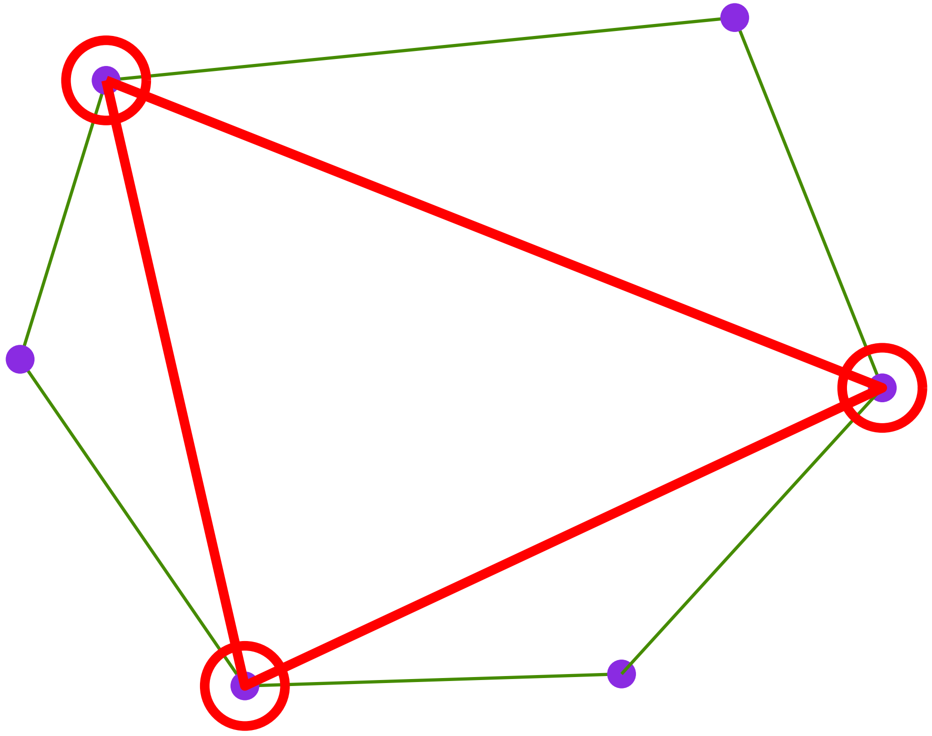
Vertex removal - low degree optimization



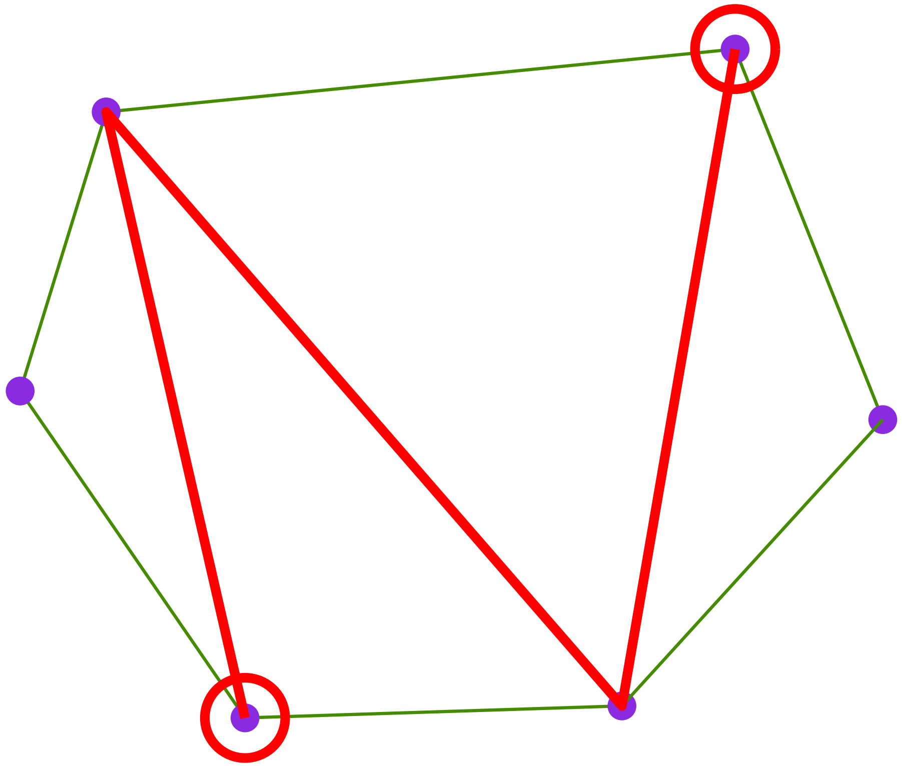
degree 6

Vertex removal - low degree optimization

degree 6



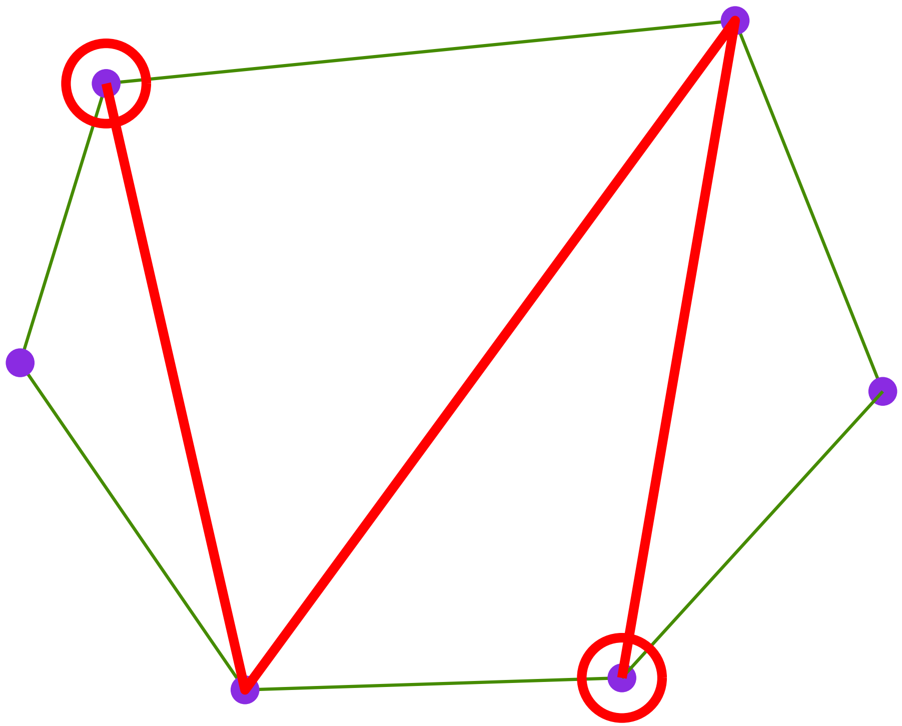
Vertex removal - low degree optimization



degree 6

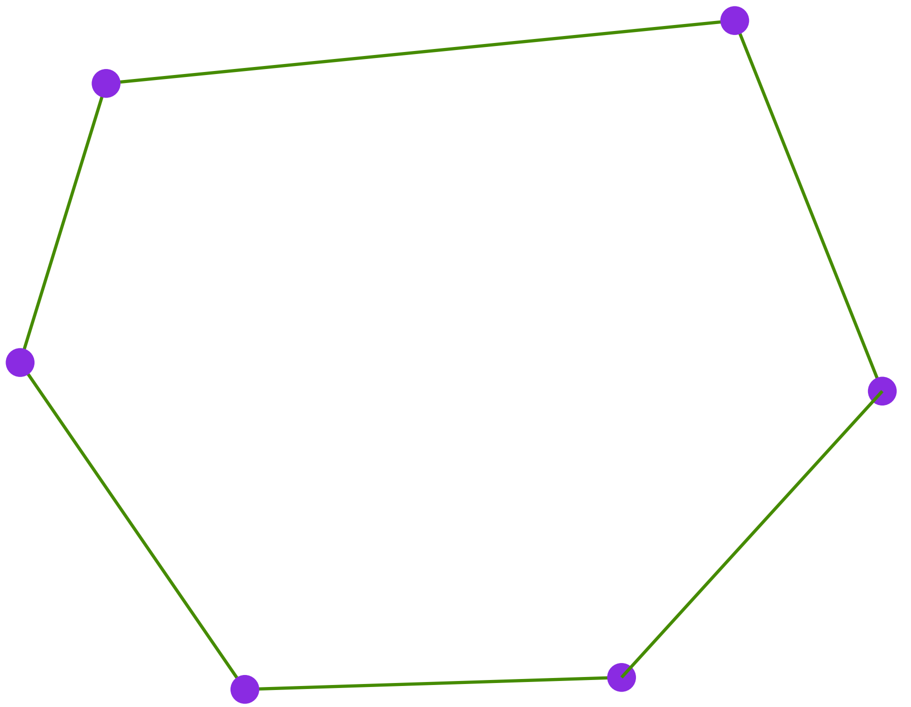
Vertex removal - low degree optimization

degree 6



Vertex removal - low degree optimization

degree 6



14 results

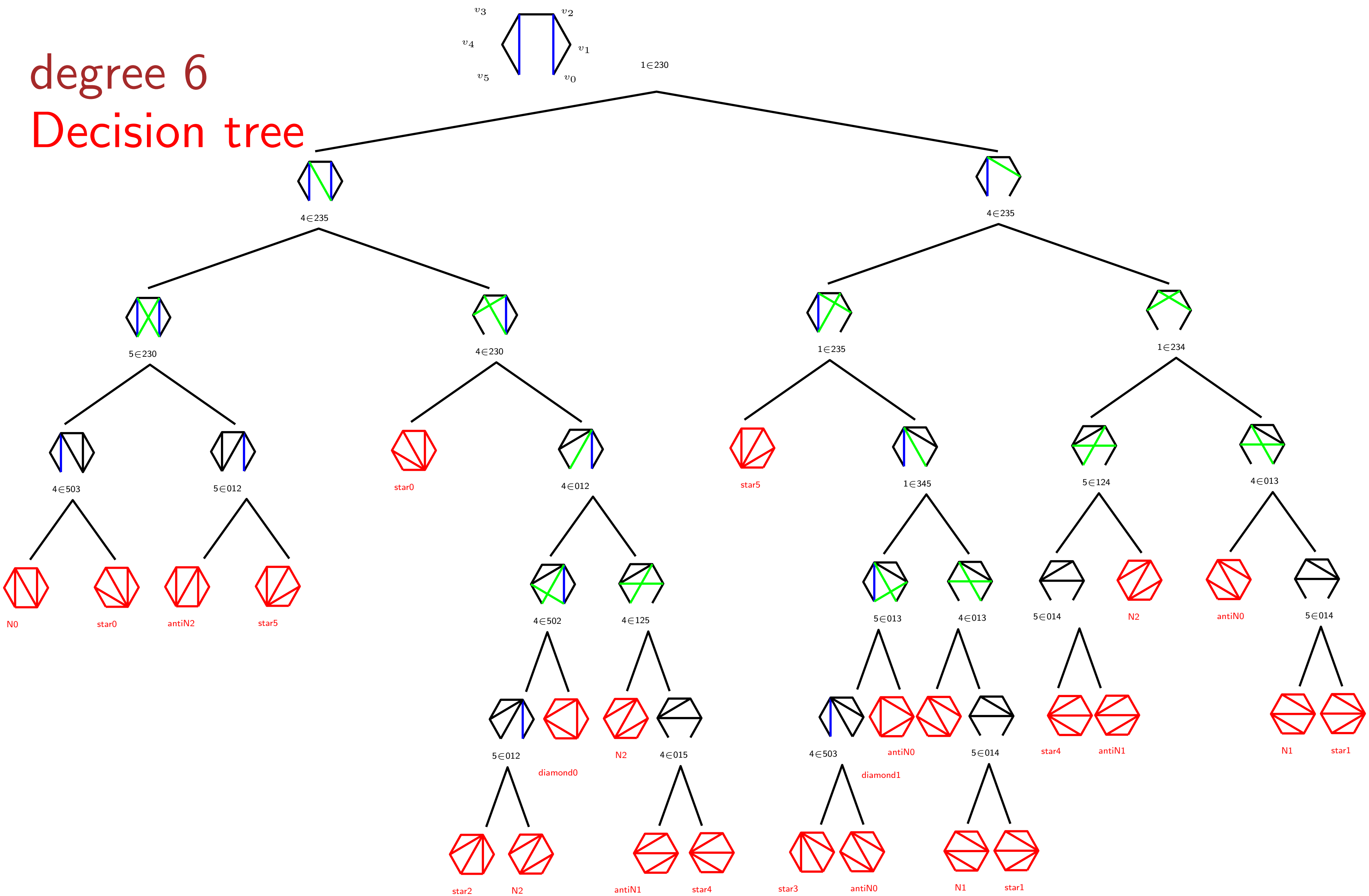
Vertex removal - low degree optimization

degree 6
Decision tree

Vertex removal - low degree optimization

degree 6

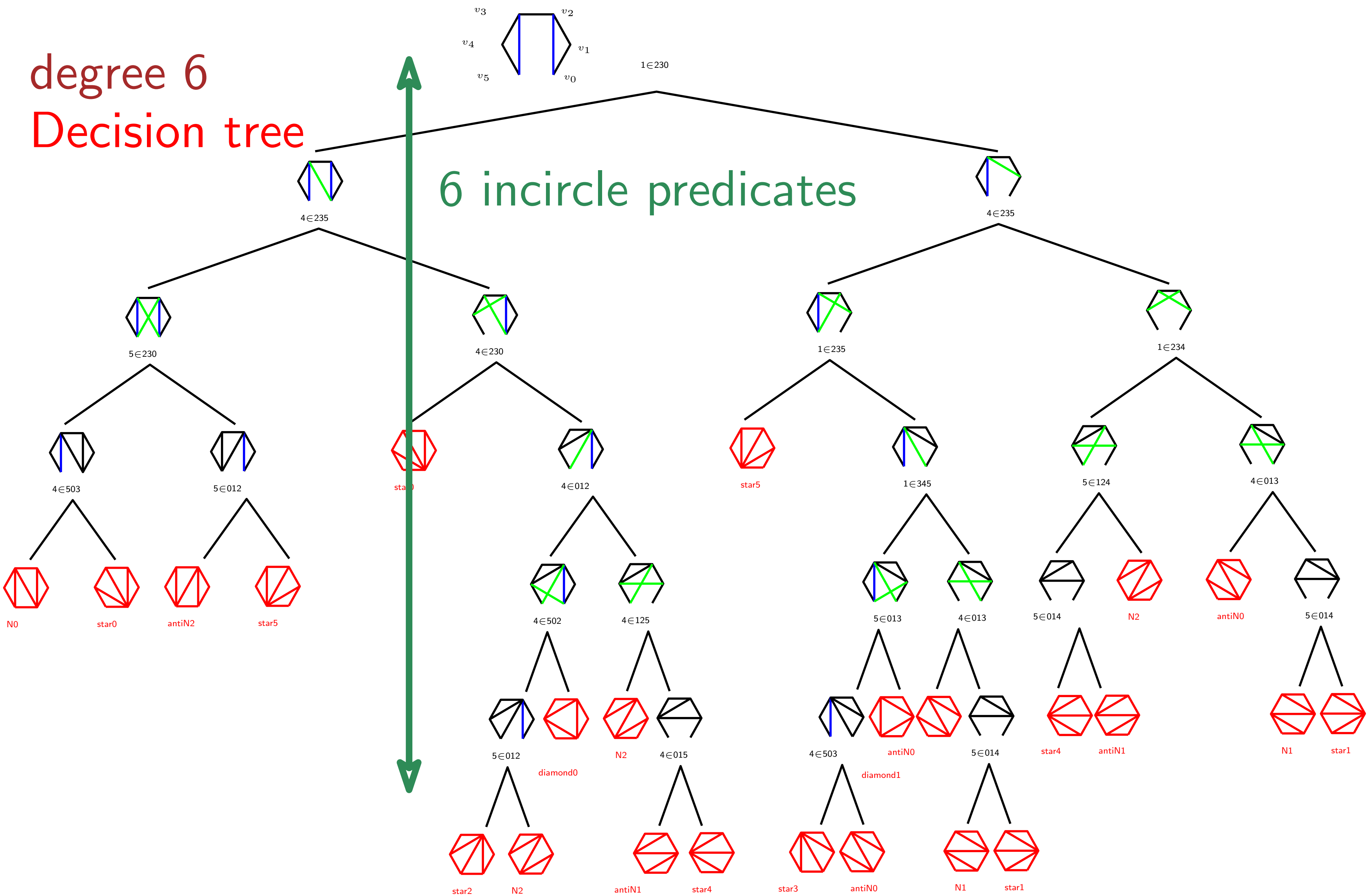
Decision tree



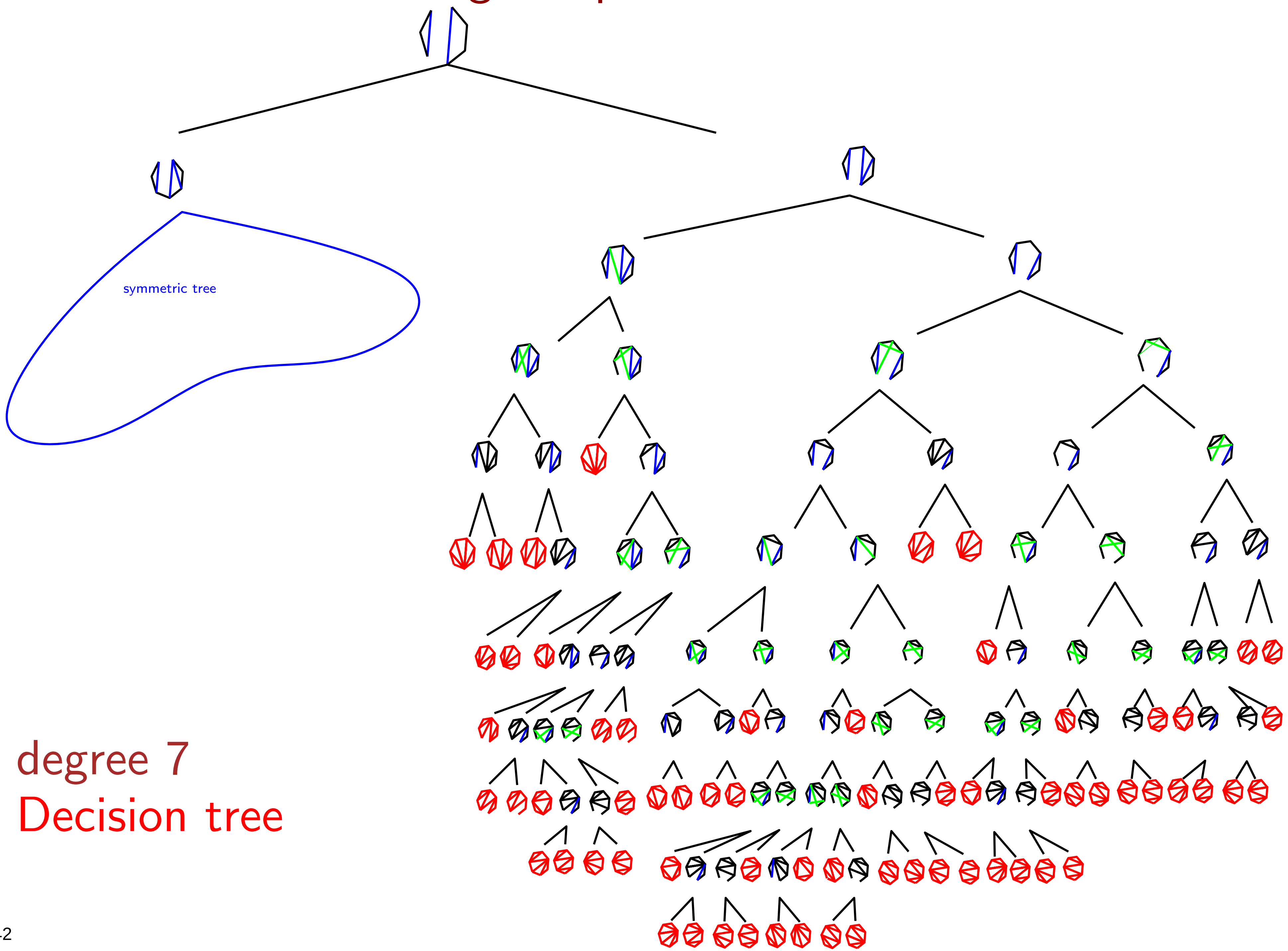
Vertex removal - low degree optimization

degree 6

Decision tree

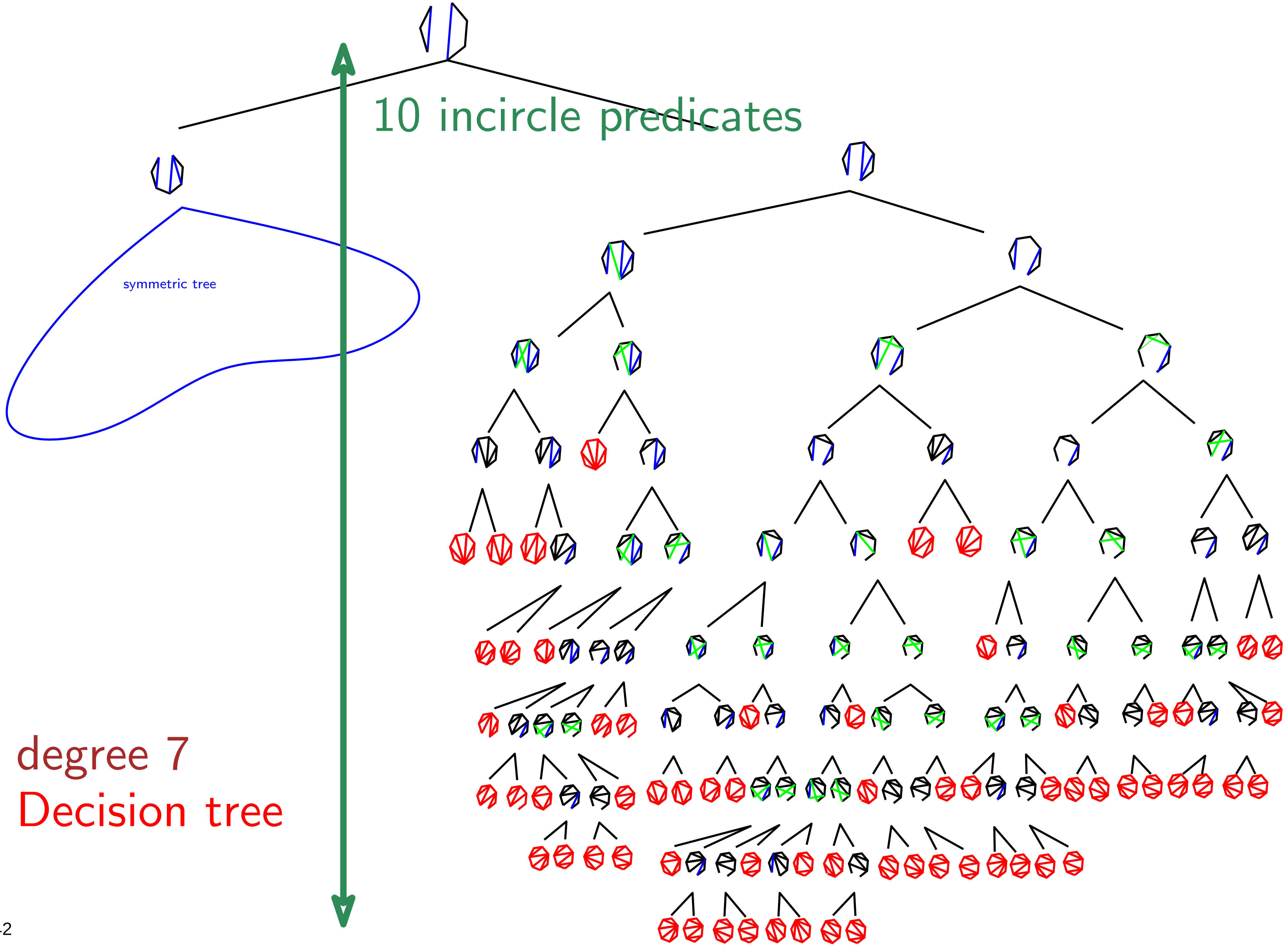


Vertex removal - low degree optimization



degree 7
Decision tree

Vertex removal - low degree optimization



degree 7
Decision tree

Vertex removal - low degree optimization

degree	3	4	5	6	7	8⋆	9
♯ results	1	2	5	14	42	132	429
♯ leaves	1	2	6	24	130	≃*500*	
$\lceil \log_2 \sharp results \rceil$	0	1	3	4	6	8	9
tree height	0	1	3	6	10	≃*14*	
♯ lines of code	30	40	90	280	700	≃*2500*	

⋆ not implemented. The sizes of the tree and the code are estimated

Vertex removal - low degree optimization

Remarks on implementation

limited memory allocation, use old faces "in place"

re-use as many neighbor links as possible

Vertex removal - low degree optimization

Remarks on implementation

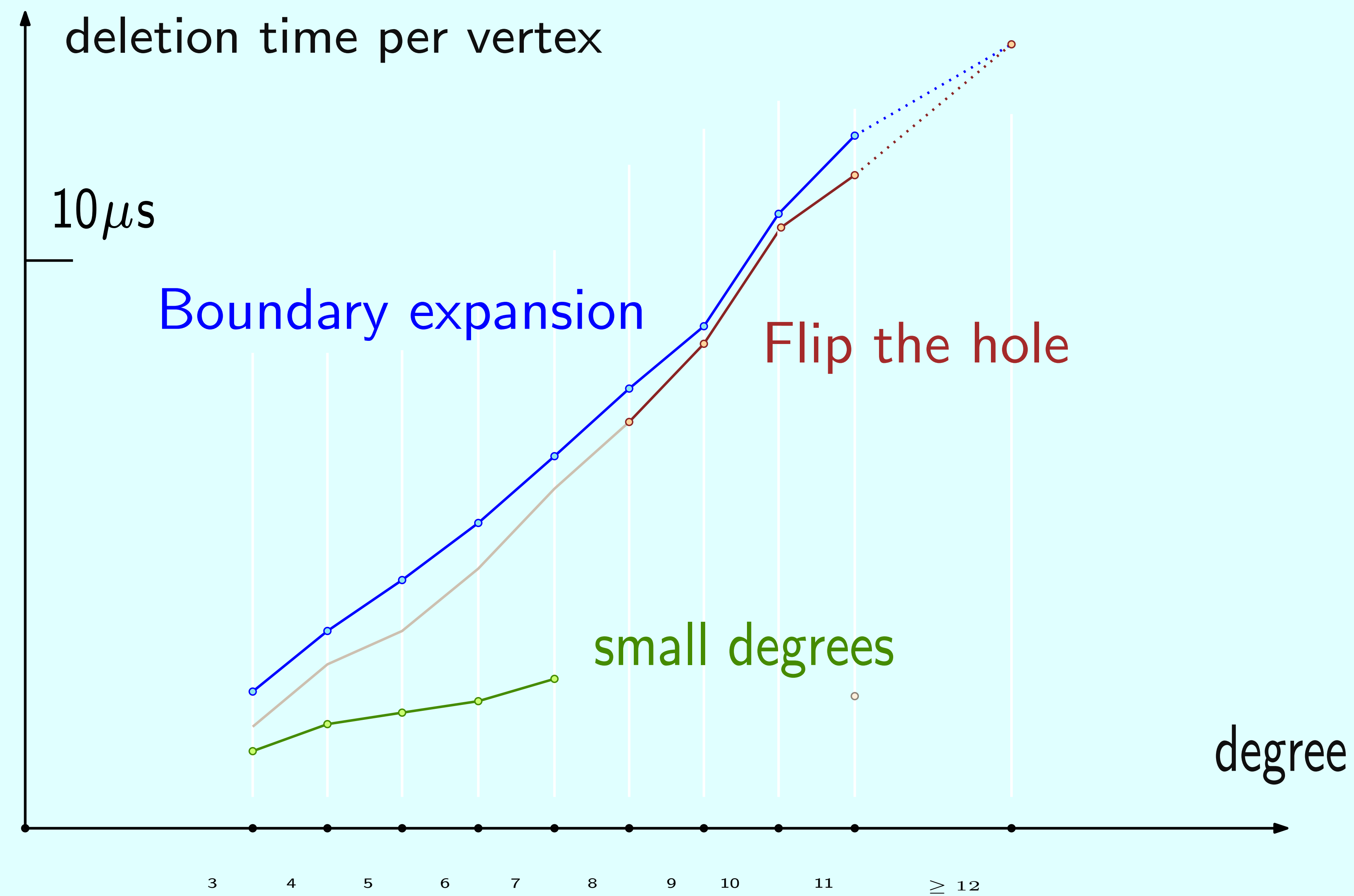
limited memory allocation, use old faces "in place"

re-use as many neighbor links as possible

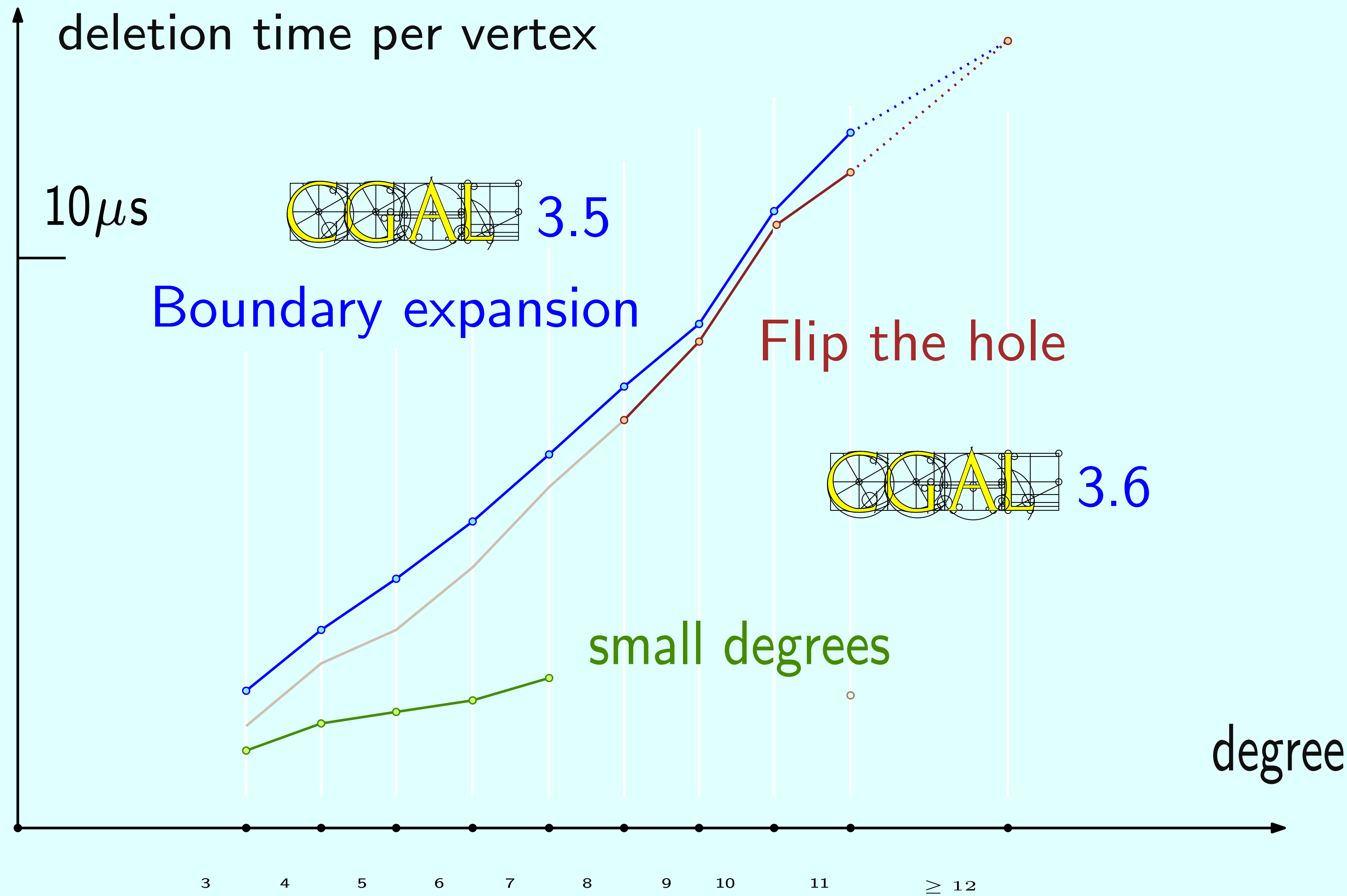
tree implementation

```
if incircle(...)
    if incircle(...)
        if incircle(...)  use_this_shape(face0,face1,face2...)
        else               use_other_shape(face2,face3,face4...)
......
```

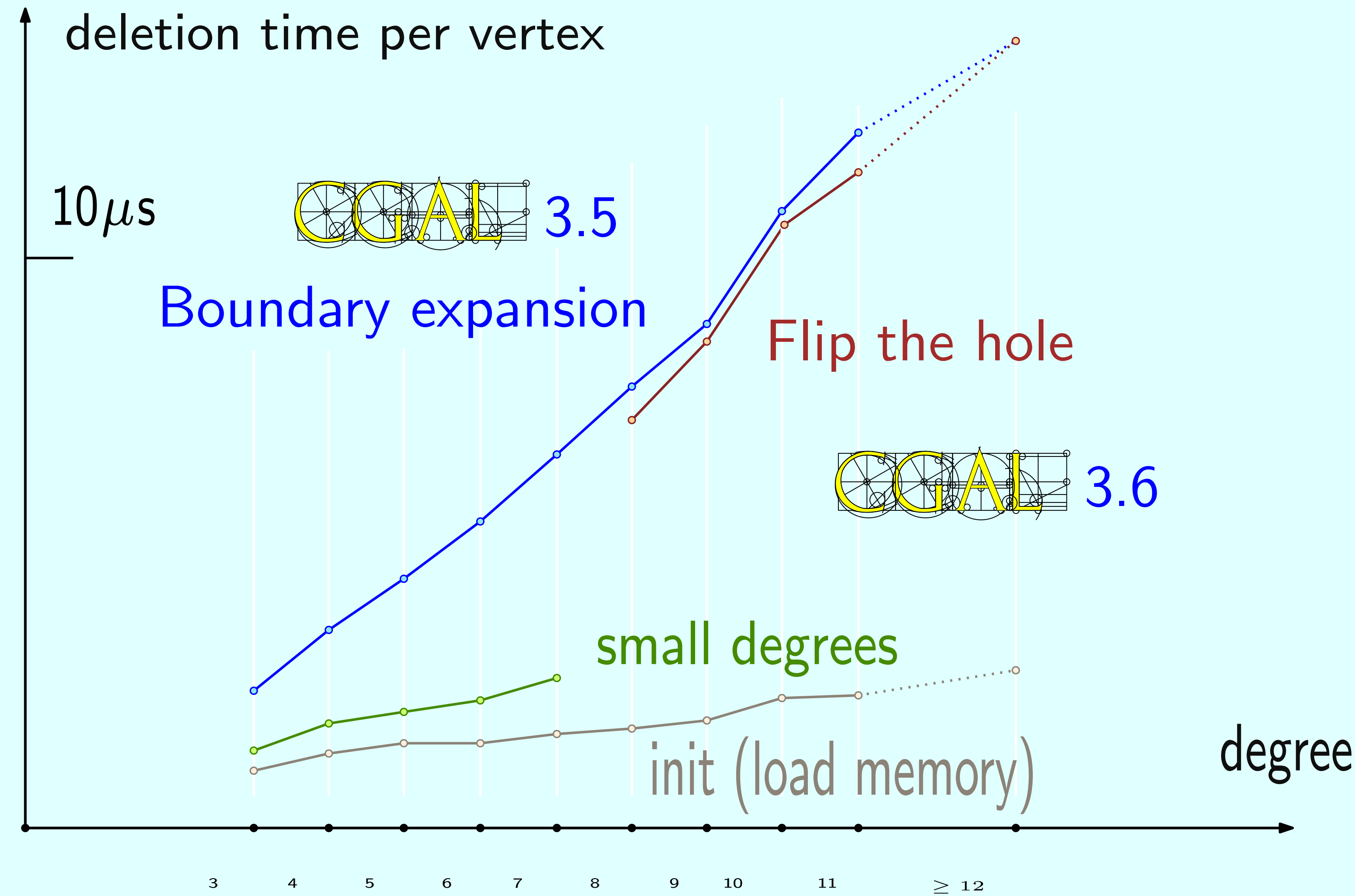
Vertex removal



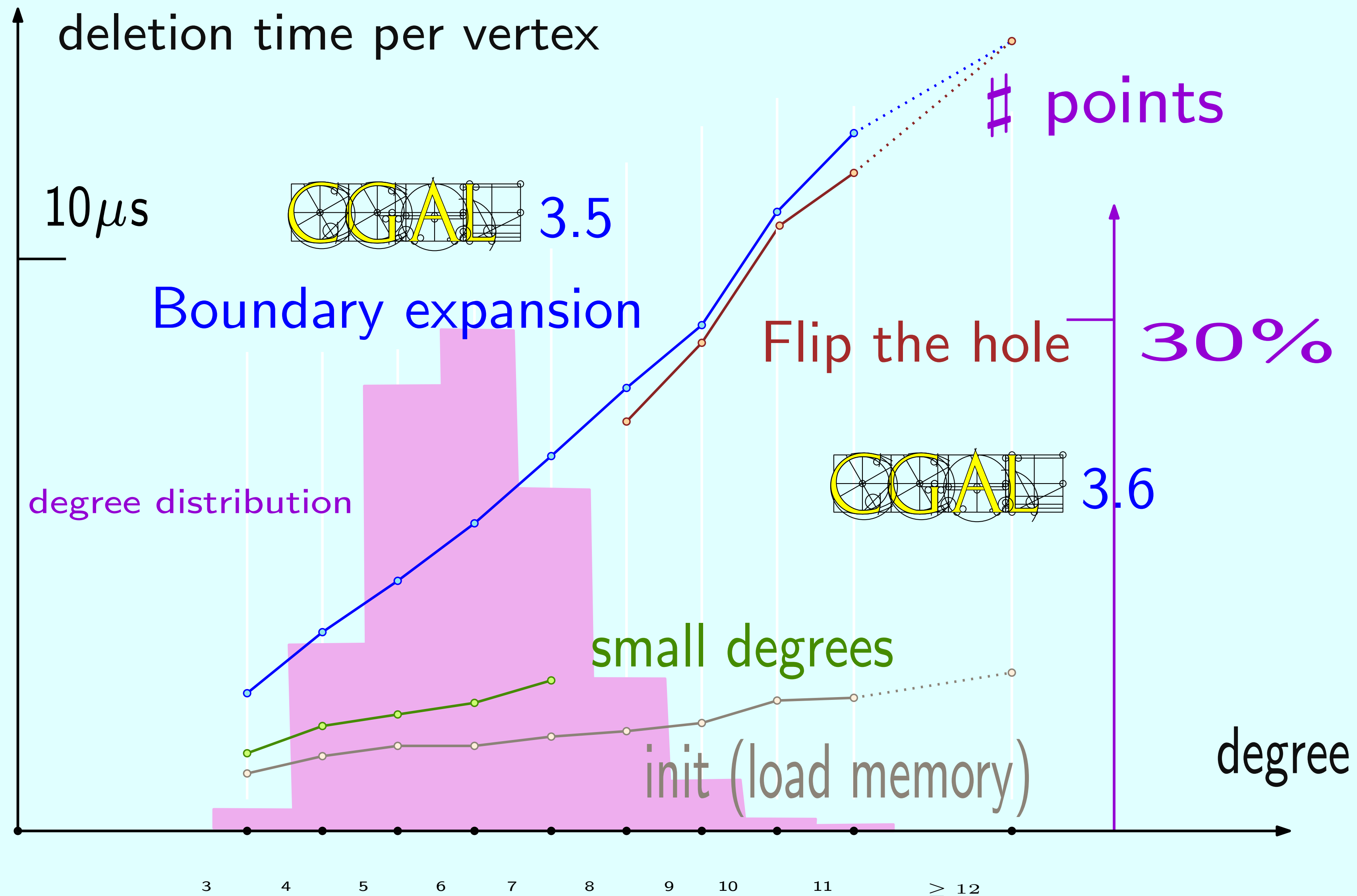
Vertex removal

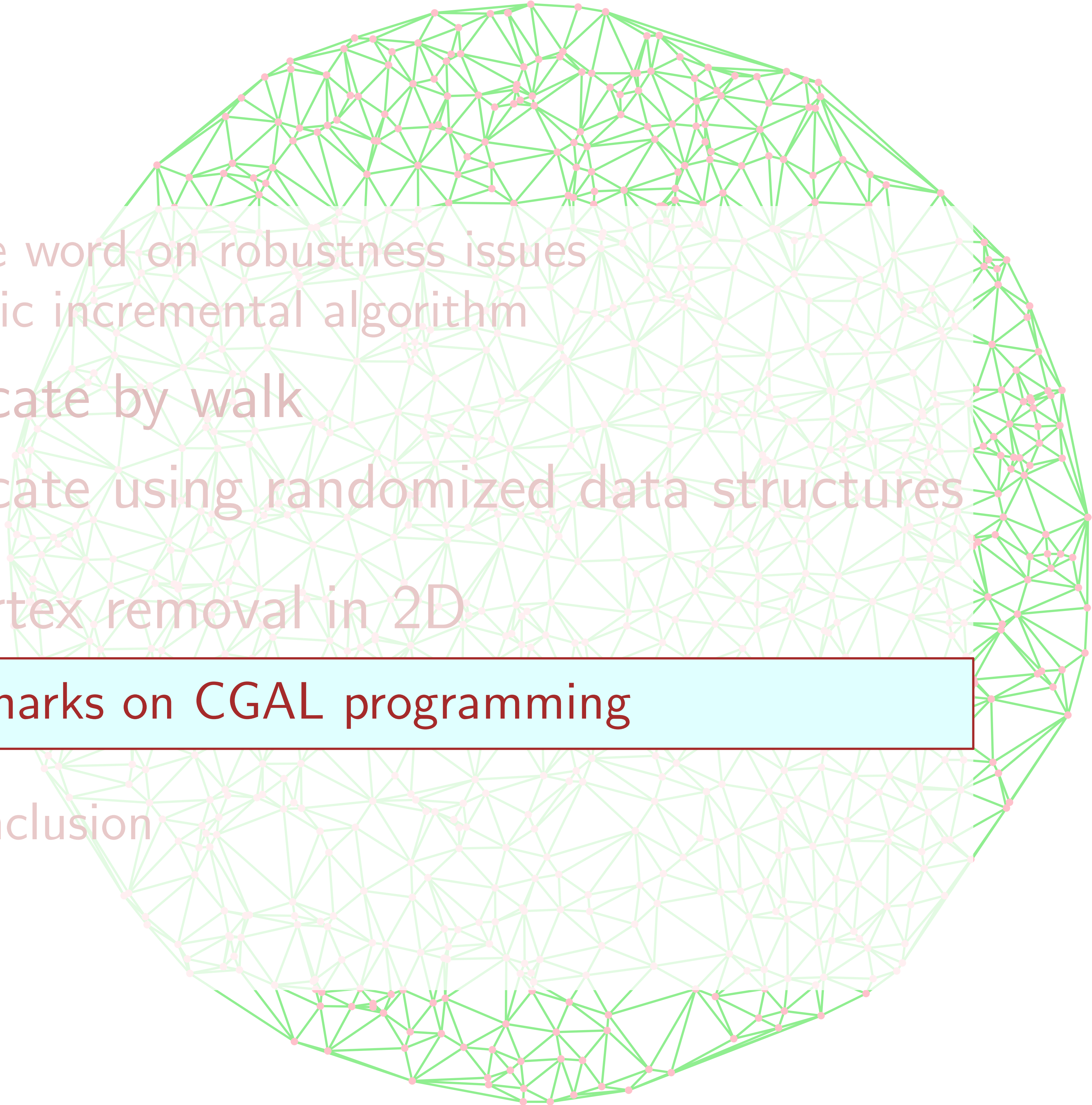


Vertex removal



Vertex removal





One word on robustness issues

Basic incremental algorithm

Locate by walk

Locate using randomized data structures

Vertex removal in 2D

Remarks on CGAL programming

Conclusion

Algorithmic choices

Theoretical efficiency

Algorithmic choices

Practical efficiency

Theoretical efficiency

Algorithmic choices

Practical efficiency

Theoretical efficiency

Algorithmic choices

Robustness issues

Practical efficiency

Theoretical efficiency

Modularity

- traits classes
  - data structures
  - geometry

Algorithmic choices

Robustness issues

Practical efficiency

Theoretical efficiency

Modularity

traits classes

data structures

geometry

Algorithmic choices

Robustness issues

Minimal requirements

e.g. do not use strange predicates

Geometric traits

Triangulation data structure

Delaunay triangulation

$$\mathbb{R}^2$$

Geometric traits

Triangulation data structure

Delaunay triangulation

$\mathbb{R}^3$ projection

Geometric traits

Triangulation data structure

Delaunay triangulation

$\mathbb{R}^2$ other metric

Geometric traits

Triangulation data structure

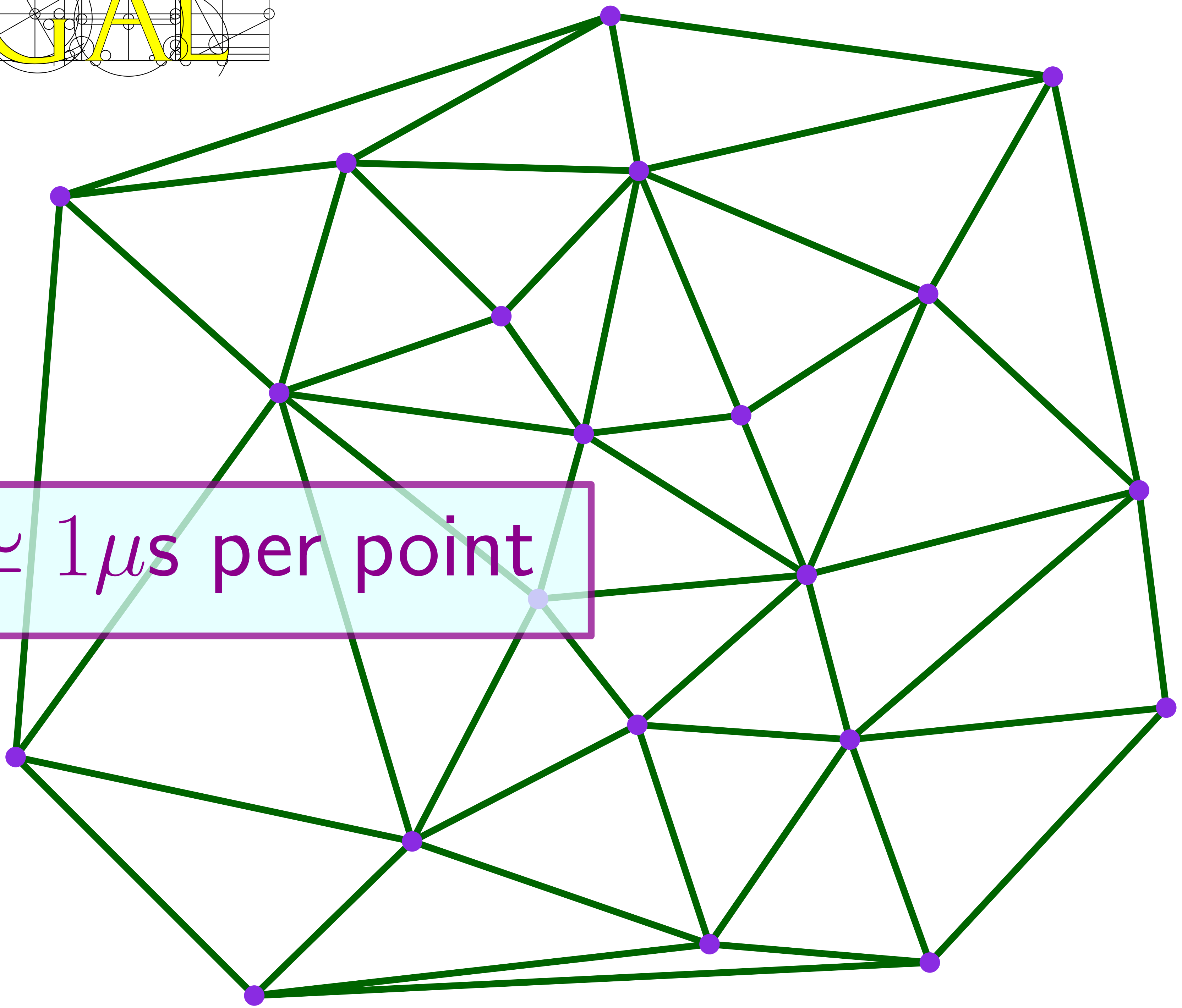
Delaunay triangulation

Geometric traits

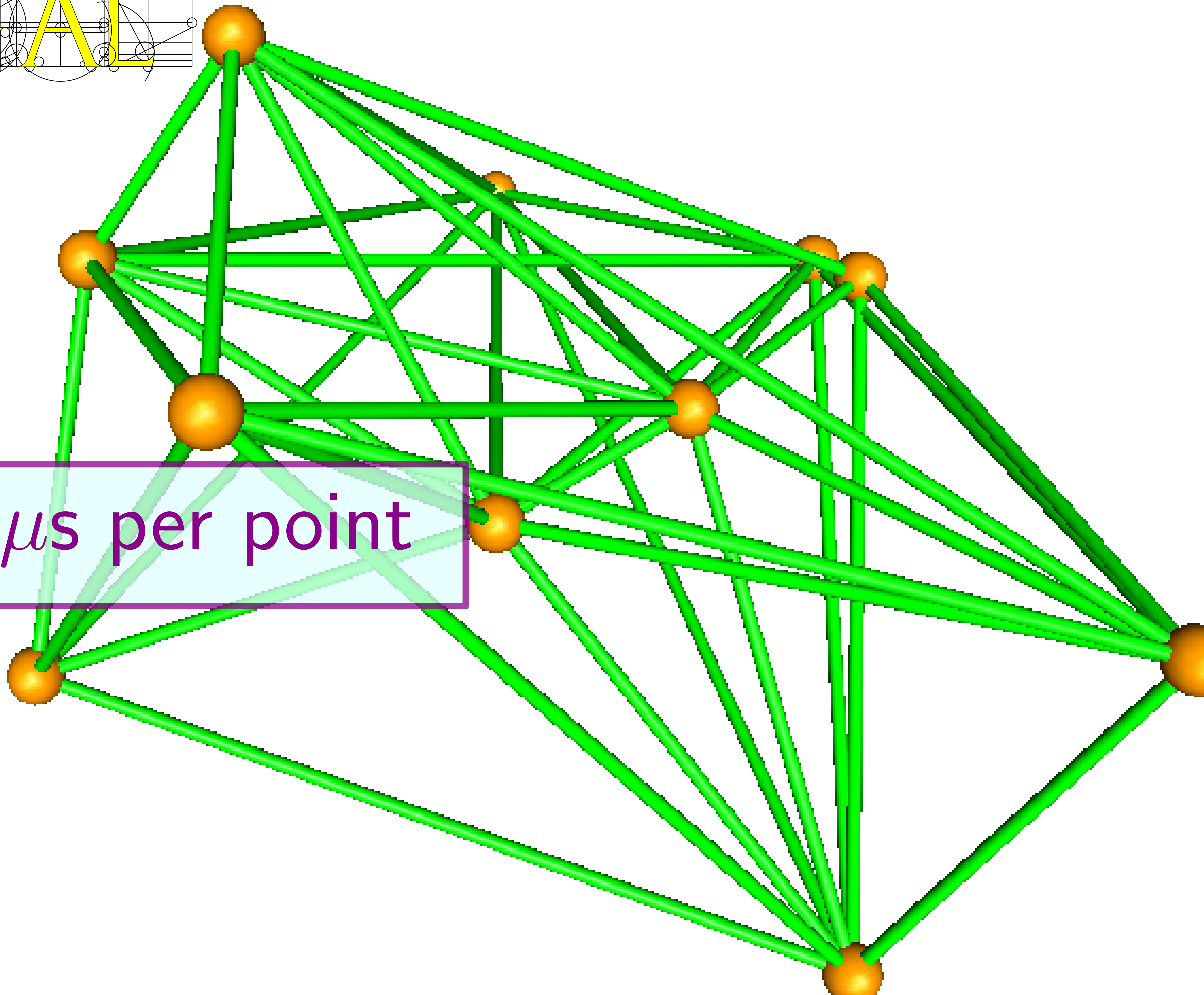
Triangulation data structure

Delaunay triangulation

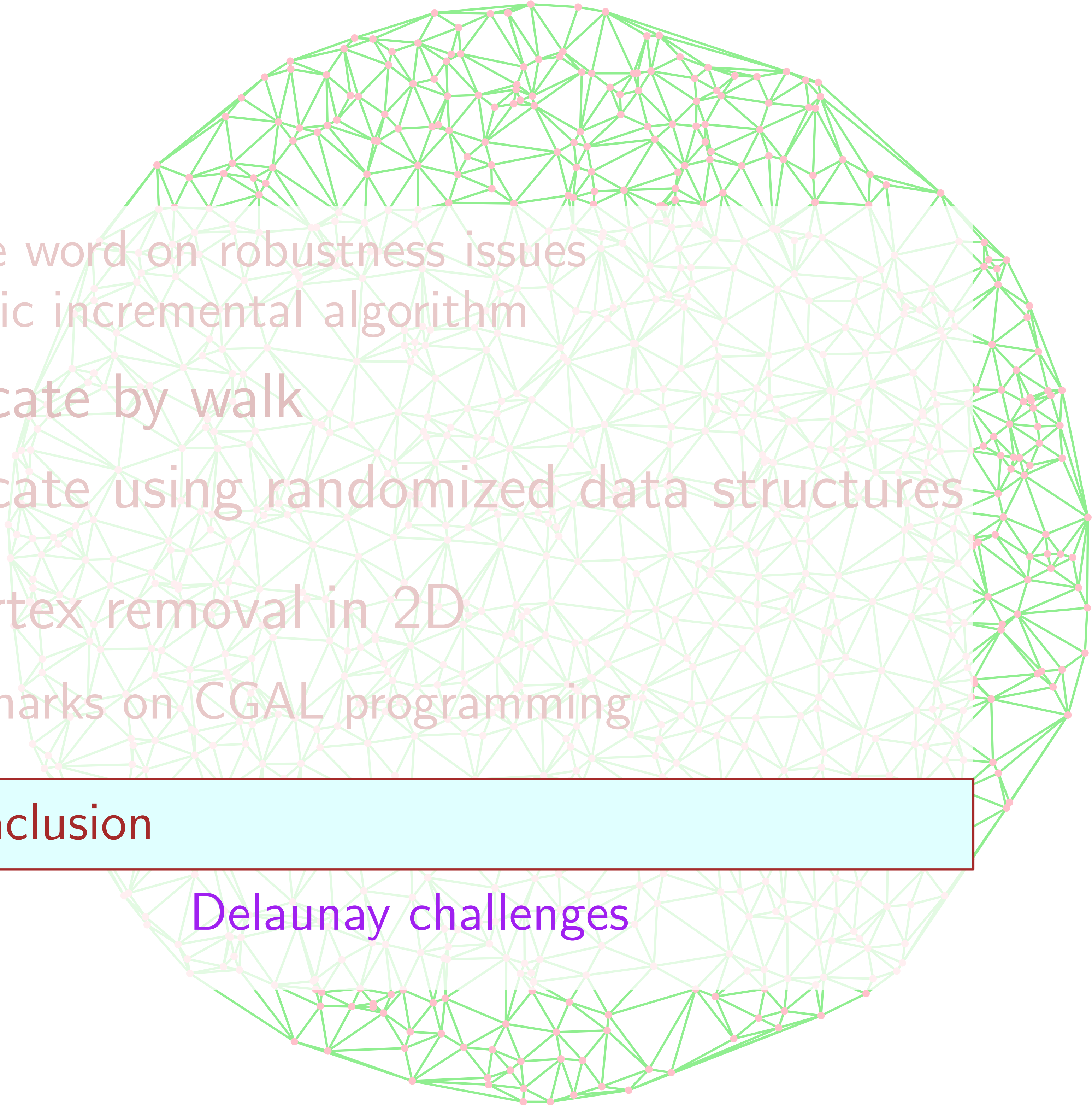
Periodic triangulation



$\simeq 1\mu$s per point



$\simeq 8\mu$s per point



Delaunay challenges

Practical vs worst case size of Delaunay 3D

Delaunay challenges

Practical vs worst case size of Delaunay 3D

Known results

$\Theta(n^2)$ worst case

$\Theta(n)$ random in ball

$\Omega(n) O(n \log n)$ random on polyhedron

$O(n \log n)$ good sample of smooth generic surface

$\Theta(n \log n)$ random on cylinder

Delaunay challenges

Practical vs worst case size of Delaunay 3D

Known results

$\Theta(n^2)$ worst case

Find good models of practical data

$\Theta(n)$ random in ball

(Smooth analysis)

$\Omega(n)O(n \log n)$ random on polyhedron

$O(n \log n)$ good sample of smooth generic surface

$\Theta(n \log n)$ random on cylinder

Delaunay challenges

Practical vs worst case size of Delaunay 3D

Better algorithm for 3D deletion

10 μs to insert

100 μs to delete

Delaunay challenges

Practical vs worst case size of Delaunay 3D

Better algorithm for 3D deletion

One billion points

Needs memory efficient algorithms

Cache effects are already important


Questions