Pseudo-Triangulations & Applications

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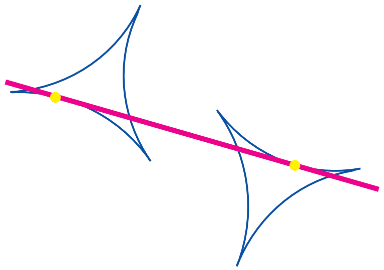
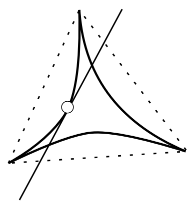


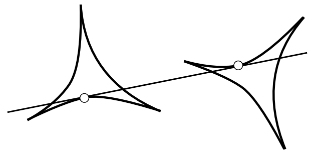
Obernai
Octobre 2002

Bibliographie

1. Ray-shooting (Geodesic triangulations) [CEGGHSS'94, GT97]
2. Visibility graphs [PV93,PV95,AP01]
3. Rigidity [Streinu'00, RSS'01]
4. Kinetic data structures [ABGHZ'01, KSS'00, Olaf A. Hall-Holt'02]
5. Art gallery theorems [B. Speckmann and C. D. Tóth'01],[PV99]
6. Enumeration [BKPS01] [S. Besptamyatnikh'02]
7. Stretchability [PV96]

Pseudotriangles, Pseudolines, and Stretchability

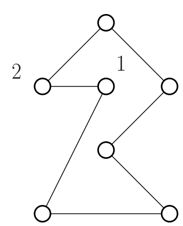


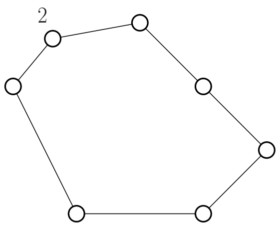


We say that an arrangement of n pseudolines is k-*stretchable* if it is isomorphic to an arrangement of pseudolines which satisfies the following property: its number of transversal intersection points with any (straight) line doesn't exceed $2k+n$. Clearly an arrangement of pseudolines is stretchable iff. it is 0-stretchable. Similarly an arrangement of pseudolines is 1-stretchable iff. it is realizable by pairwise disjoint pseudotriangles. Arrangements of pseudolines realizable by $k+1$ by $k+1$ disjoint pseudotriangles are examples of k-stretchable arrangements.

Problem 1. *Prove or disprove that arrangements of pseudolines are k-stretchable ($k<n$, $k=o(n)$,$k=O(1)$ and $k=1$).*

Rigidité





L'ensemble des polygones SIMPLES $p_1p_2p_3p_4p_5p_6p_7 \in \mathbb{R}^{14}$ tels que

$$\begin{aligned} p_1p_2 = p_6p_7/2 &= 1 \\ p_2p_3 = p_3p_4 = p_4p_5 = p_5p_6 &= \sqrt{2} \\ p_7p_1 &= \sqrt{5} \end{aligned}$$

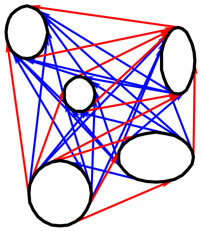
est-il connexe?

OUI, via le cône des dilatations infinitésimales, i.e., l'ensemble des $(v_1v_2\cdots v_7) \in \mathbb{R}^{14}$ tels que

$$\langle p_i - p_j, v_i - v_j\rangle \geq 0 \qquad (\forall ij)$$

avec égalité pour $j=i+1$. [Connelly, Demaine, Rote'00],[Streinu'00]

PT et Graphes de Visibilité



1. Une collection de n obstacles : $O_1, O_2 \ldots . O_n \subset \mathbb{R}^2$
2. Espace libre : $\mathbb{F} = \mathbb{R}^2 \setminus \bigcup_i \text{Interior}(O_i)$
3. Espace des rayons libres : $r = (o(r), \vec{r}) \in \mathbb{F} \times \mathbb{S}^1$
4. $$r \sim r' \iff \begin{cases} r \text{ et } r' \text{ ont même droite support} \\ [o(r), o(r')] \text{ est libre} \end{cases}$$
5. Complexe de Visibilité : $\mathbb{V} := \mathbb{F} \times \mathbb{S}^1 / \sim \longrightarrow \mathbb{S}^2$ [P. & Vegter 93]
6. Greedy Flip Algorithm
   (a) Pseudo-triangulations, Flip Property [PV93, PV95]
   (b) 'Sum of Squares' Theorem [Angelier & P. 01]
   (c) $O(n, k + n \log n)$ basé uniquement sur un prédicat 'matroidal' [AP01]
   (d) Code Cgal pour des polygones et des cercles (centre, rayon) [AP01]
   (e) Retour aux points, Questions ouvertes
7. Pseudo-triangulations de points ...... Théorie de la Rigidité [Streinu 00]

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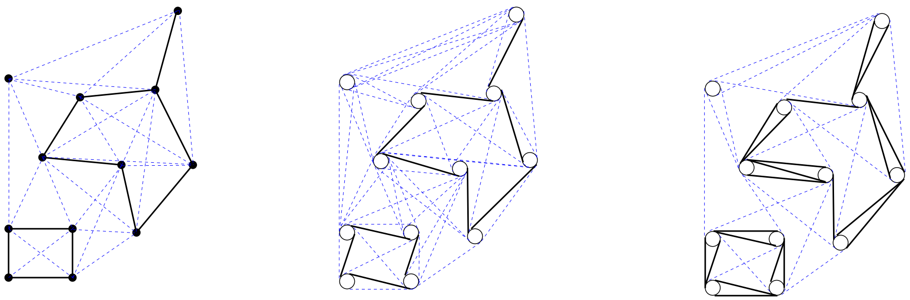
Polygonal Obstacles



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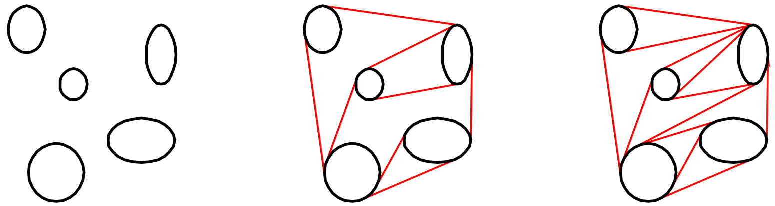
Pseudo-Triangulations of Disks



1. O a set of n pairwise disjoint 2-dimensional smooth convex sets (disks for short)
2. B the set of $k \leq 2n(n-1)$ undirected free bitangents
3. A pseudo-triangulation G is a maximal (for $\subseteq$) planar subset of B
4. $C \subset G$, $|G| = 3n - 3$
5. $(\forall\, G)\,(\forall\, b \in G \setminus C)\,(\exists!\, G' \neq G)\ \ (G \setminus G' = \{b\})$.

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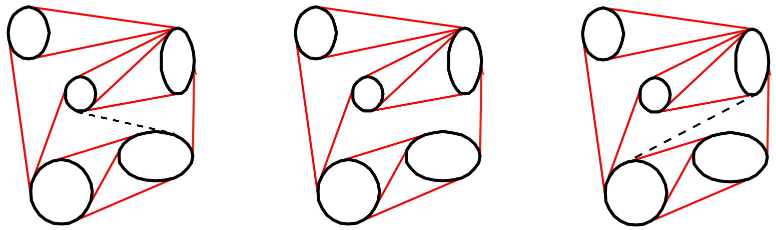
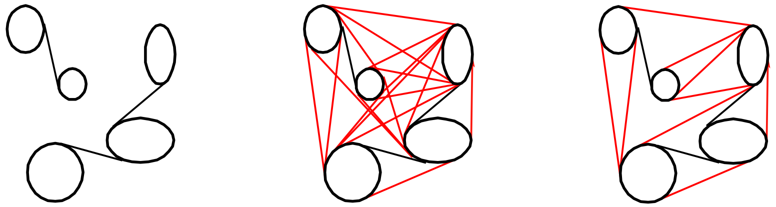
Flip Graph of Pseudo-Triangulations



1. G a pseudo-triangulation
2. **[Size]** $C \subset G$, $|G| = 3n - 3$
3. **[Flips]** $(\forall\, G)\,(\forall\, b \in G \setminus C)\,(\exists!\, G' \neq G)\ \ (G \setminus G' = \{b\})$ $\quad G \xrightarrow{b} G'$.
4. **[Connectivity]** Le graphe d'adjacence des pseudo-triangulations est 'fortement' connexe.
   $$\begin{cases} G = G_0 \xrightarrow{b_1} G_1 \cdots \xrightarrow{b_k} G_k = G' \\ G \cap G' \subseteq G_i \qquad (i = 0, \ldots, k) \end{cases}$$
5. $\{H \mid C \subseteq H, H \text{ planaire}\}$ ordonné par inclusion est un polytope abstrait de dimension $3n - 3 - |C|$.

Greedy Pseudo-Triangulations



1. $\kappa \subset \mathcal{B}$ planar
2. $\mathcal{B}_\kappa = \{v \in \mathcal{B} \mid \kappa \cup \{v\} \text{ is planar}\}$, $\Theta(v)$, $\mathbb{B}_\kappa = \mathcal{B}_\kappa \times \mathbb{Z}$, $\Theta(v,k) = \Theta(v) + k\pi$
3. We define a partial order $\prec_\kappa$ on $\mathbb{B}_\kappa$ as follows : $v \prec_\kappa v'$ if there exists a sequence of bitangents $v = v_1, v_2, \ldots, v_k = v'$ such that v_i and v_{i+1} touch the same directed disk and $\Theta(v_i) < \Theta(v_{i+1})$
4. For any filter I of the poset $(\mathbb{B}_\kappa, \prec_\kappa)$, we define a maximal planar set of bitangents $\mathrm{G}(I) = \{v_1, v_2, \ldots, v_k\}$ recursively: v_1 is minimal in I, and, for $i > 1$, v_{i+1} is minimal in the set of bitangents $v \in I \setminus \{v_1, \ldots, v_i\}$ such that $\kappa \cup \{v_1, \ldots, v_i, v\}$ is planar—note that this set is empty after $3n - 3$ steps.

Greedy Flip Property

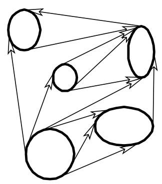
Theorem 1 (PV95,AP01). *Let v be a minimal element of the filter I of the poset $(\mathbb{B}_\kappa, \prec_\kappa)$. Then $\mathrm{G}(I \setminus \{v\})$ is obtained from $\mathrm{G}(I)$ by flipping v.*

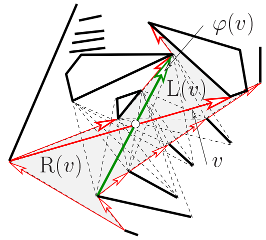
1. The mapping that associates with the bitangent $v \in \mathbb{B}_\kappa$ the bitangent $v' \in \mathbb{B}_\kappa$ defined by $\{v'\} = \mathrm{G}(I \setminus \{v\}) \setminus \mathrm{G}(I)$ where v is a minimal element of I is well-defined, one-to-one, and onto; the bitangent v' is denoted $\varphi(v;\kappa)$.
2. $\varphi(v;\kappa)$ is the smallest bitangent of $\mathbb{B}_\kappa$ crossing v and larger than v.
3. Greedy Flip Algorithm

Greedy Flip Algorithm

Algorithm Greedy Flip Algorithm

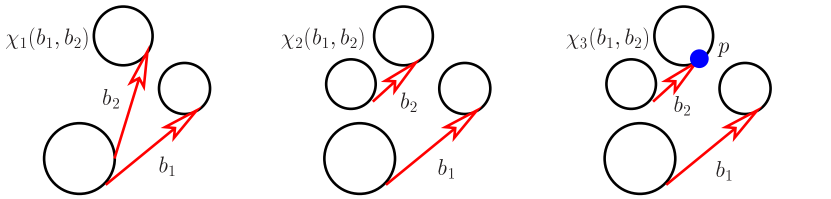
compute the greedy pseudo-triangulation $G := \mathrm{G}(I(0))$;
repeat
    select a minimal bitangent v in G with angle less than π;
    flip v; (i.e., replace v by $\varphi(v)$)
until there are no more bitangents of angle less than π.





The Flip Operation (version 95)

Theorem 2 (PV95). *The flip operation of the greedy flip algorithm can be implemented in constant amortized time using the χ_2 and χ_3 predicates for disks. Furthermore the data structures used are linked structures to represent the incidence relations in the sweep structure S and its associated pseudotriangulations $\mathrm{G}(S)$ and splittable queues to represent the boundaries of the pseudotriangles of $\mathrm{G}(S)$ that support the enqueue, dequeue, and split operations in constant amortized time.*



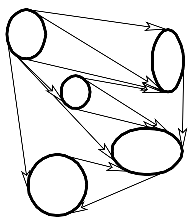
$\det(b_1,b_2) > 0$ $\det(b_1,b_2) > 0$ p is on the left side of b_1

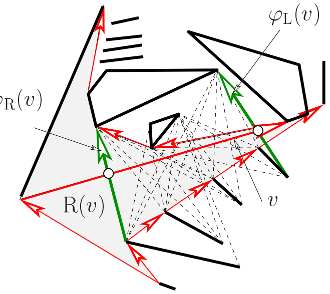
The Flip Operation (version 01)

Theorem 3 (AP01). *The flip operation of the greedy flip algorithm can be implemented in constant amortized time using only the left-turn predicate for disks. Furthermore the only necessary data structures are linked structures to represent the incidence relations in the sweep structure S and its associated pseudotriangulations $\mathrm{G}(S)$ and $\mathrm{G}_*(S)$, a list to store the minimal bitangents of $\mathrm{G}(S)$, and for each minimal bitangent v pointers to the arcs of $\mathrm{G}(S)$ that $\varphi_{\mathrm{R}}(v)$ and $\varphi_{\mathrm{L}}(v)$ leave and pointers to the arcs of $\mathrm{G}_*(S)$ that $\varphi_{\mathrm{R}}(v)$ and $\varphi_{\mathrm{L}}(v)$ enter.* □

1. dual order $\prec_*$
2. $\mathrm{G}_*(S)$ is the dual greedy pseudo-triangulation
3. $\varphi_{\mathrm{R}}(v)$ is the bitangent that leaves $\mathrm{R}(v)$ and enters $\mathrm{R}_*(\iota(v))$
4. amortized analysis using a 'sum of squares' like theorem for visibility graphs

GFA with χ_1





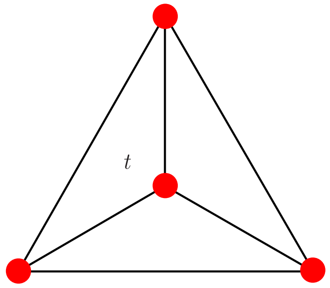
Procedure χ_1-Walk(C, C' : List of Arcs)
while $C \neq \emptyset$ and $C' \neq \emptyset$
$r \leftarrow \mathrm{Top}(C)$; $l \leftarrow \mathrm{Top}(C')$; $t \leftarrow \mathrm{bit}(r, l)$;
if $\chi_1(r^+, t)$ $C \leftarrow \mathrm{Pop}(C)$
elseif $\chi_1(l^+, t)$ $C' \leftarrow \mathrm{Pop}(C')$
else exit;

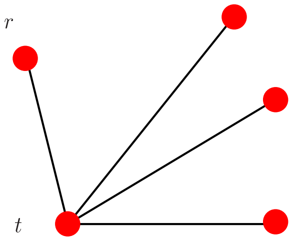
Chirotope

- $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$
- $\chi : (i, j, k) \mapsto |ijk|$ = orientation du triplet $(p_i, p_j, p_k) \in \{+, 0, -\}$
- #chirotopes $= 2^{\Theta(n \log n)}$
- Propriétés de χ sur au plus 5 points en position générale

Axiome 1 symétrie circulaire	$pqr \implies qrp.$
Axiome 2 antisymétrie	$pqr \implies \neg prq.$
Axiome 3 non dégénérescence	$pqr \vee prq.$
Axiome 4 intériorité	$tqr \wedge ptr \wedge pqt \implies pqr$
Axiome 5 transitivité	$tsp \wedge tsq \wedge tsr \wedge tpq \wedge tqr \implies tpr$
Axiome 5' transitivité duale	$stp \wedge stq \wedge str \wedge tpq \wedge tqr \implies tpr.$

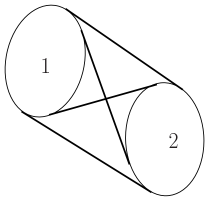
Chirotope

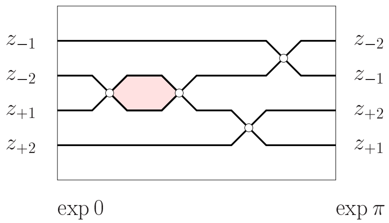




Axiome 4 intériorité	$tqr \wedge ptr \wedge pqt \implies pqr$
Axiome 5 transitivité	$tsp \wedge tsq \wedge tsr \wedge tpq \wedge tqr \implies tpr$

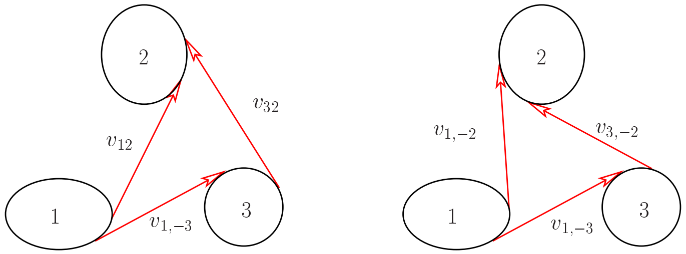
The Left-Turn Predicate





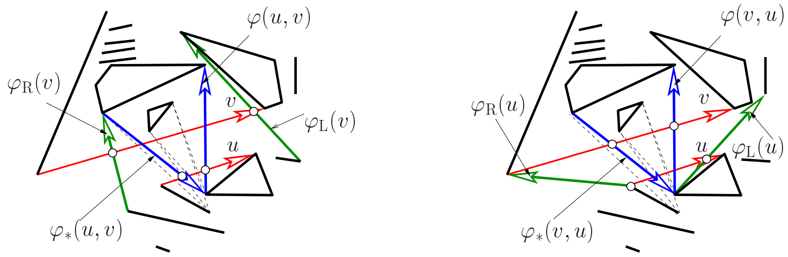
– Disque o_i, disques orientés ou cycles associés c_{-i} et c_{+i}

– v_{ij} la bitangente quittant le disque orienté o_i et entrant le disque orienté o_j, $(i, j \in [\pm n])$



Theorem 4 (AP01). *Consider three cycles c_i, c_j, and c_k whose underlying disks are distinct. Assume that the bitangents v_{ki}, v_{kj} and v_{-ij} are free and that $\chi_1(v_{ki}, v_{kj})$. Then* $\chi_1(v_{kj}, v_{-i,j})$ *and* $\chi_1(v_{-i,j}, v_{-i,-k})$

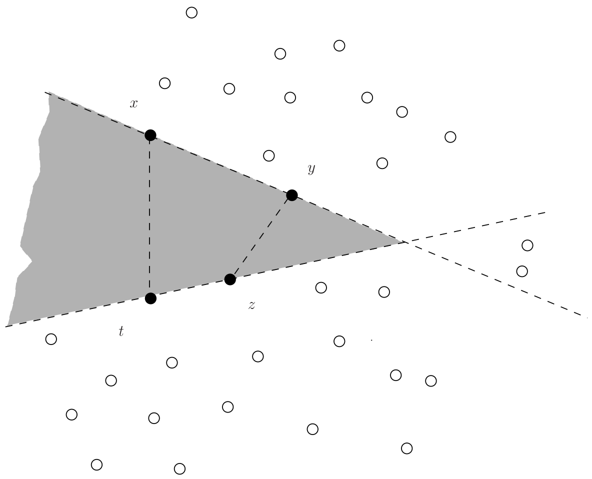
SoS Theorem



$\varphi(u, v) = \varphi(u; \{\varphi_{\mathrm{R}}(v), \varphi_{\mathrm{L}}(v)\})$.

Theorem 5 (AP01). *The number of pairs of free bitangents (u, v) such that $\varphi(u, v) = \varphi(v, u)$ and $\varphi_*(u, v) = \varphi_*(v, u)$ is bounded by a constant times the number of free bitangents.* □

SoS Theorem for points



Enumeration of pseudo-triangulations

Let I be a filter (for example the filter of bitangents with positive angle) of $(\mathbb{B}_\kappa, \prec_\kappa)$, and let $G_0 = G(I)$ be the associated greedy pseudotriangulation. For $H = (F, G)$, $F \cap G = \emptyset$, $\kappa \subset F \cup G \subset I$ a pseudotriangulation, we define $\omega(H)$ to be the pair $(F \cup \{v\}, G \setminus \{v\})$ and $\phi(H)$ to be the pair $(F, G \setminus \{v\} \cup \{v'\})$ where $v = \min_{<_\kappa} G$ and v' is the shadow of v in $F \cup G$. (Observe that ω and ϕ are defined only if $G \neq \emptyset$.) For $G \supseteq \kappa$ a pseudo-triangulation of the o_i's we define a sequence $H(G) = H_1, H_2, \ldots, H_i = (F_i, G_i)$ by setting $H_0 = (\emptyset, G_0)$, and

$$H_{i+1} = \begin{cases} \omega(H_i), & \text{if } \min_{<_\kappa} G_i \in G; \\ \phi(H_i), & \text{otherwise.} \end{cases} \tag{1}$$

Il est aisé de voir that the sequence $H(G)$ is finite, that its last term is $\overline{H}(G) = (G, \emptyset)$, and that $H(G) = H(G')$ iff. $G = G'$. This proves the strong connectivity of the flip graph of pseudotriangulations and, with some extra work, provides a linear space traversal algorithm of the flip graph whose complexity is linear amortized per node

❑ — Brönnimann et al'01, Bespamyatnikh'02

Bibliographie

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