Provability in Intuitionistic Linear Logic from a new interpretation on Petri nets

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Overview

We propose a new Petri net semantic of Intuitionnistic Linear Logic (ILL) based on:

- the links between **ordered monoids** (OM) and **Petri nets**;
- **closures** as a basis for embedding OM into **quantales**;
- **completeness** of ILL w.r.t. the class of quantales.

We otain the following results:

- a full embedding of OM into quantales;
- completeness for a new semantic of ILL.
- the finite model property for the !-free ILL.

Petri nets and ILL

Petri nets are models of ILL.

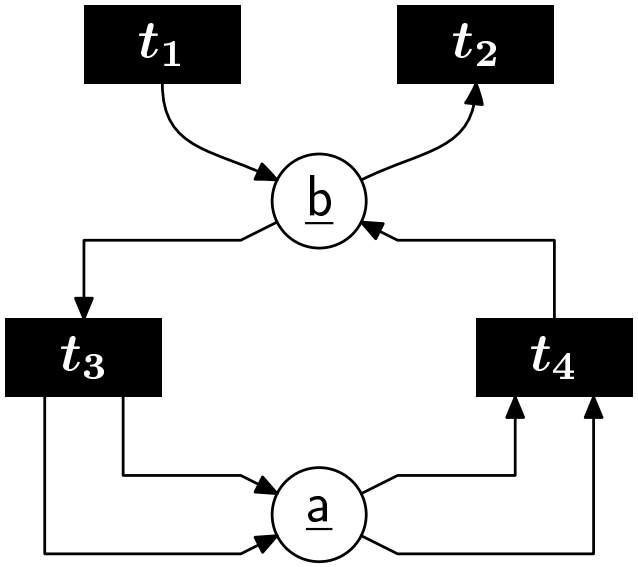
Applications are:

- **Refutation** techniques by providing counter-models;
- Expressing and proving **properties** of Petri nets using ILL.

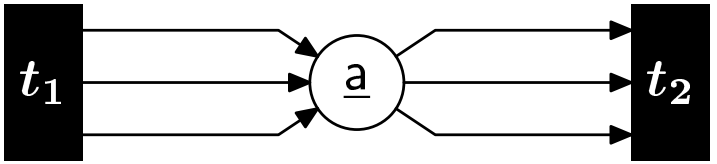
Completeness questions of this semantic of ILL:

1. partially answered in the paper of Winskel and Engberg;
2. from the algebraic point of view of **quantales**.

Petri Nets



$\underline{\mathsf{a}} \rightsquigarrow (2\mathbb{N}+1)\underline{\mathsf{a}} + \mathbb{N}\underline{\mathsf{b}}$



$\underline{\mathsf{a}} \rightsquigarrow (3\mathbb{N}+1)\underline{\mathsf{a}}$

Petri nets semantic

- X is downward closed if $b \in X$ and $a \rightsquigarrow b$ implies $a \in X$;
- formulae are interpreted by downward closed sets of markings.

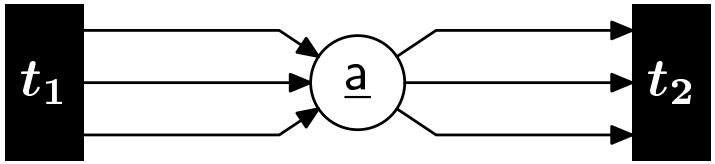
$$[\![\top]\!] = \mathcal{M}$$
$$[\![\bot]\!] = \emptyset$$
$$[\![1]\!] = \{m \mid m \rightsquigarrow 0\}$$

$$[\![A \oplus B]\!] = [\![A]\!] \cup [\![B]\!]$$
$$[\![A \,\&\, B]\!] = [\![A]\!] \cap [\![B]\!]$$

$$[\![A \otimes B]\!] = \{m \mid \exists a \in [\![A]\!], \exists b \in [\![B]\!],\ m \rightsquigarrow a + b\}$$
$$[\![A \multimap B]\!] = \{m \mid \forall a \in [\![A]\!],\ a + m \in [\![B]\!]\}$$

Completeness: $A \vdash B$ provable iff $[\![A]\!] \subseteq [\![B]\!]$

Using Petri nets semantic



Definitions are on the left side, computations on the right side:

$$[\![A]\!] = \mathbb{N}\underline{a} \qquad [\![A \otimes B]\!] = \mathbb{N}\underline{a}$$

$$[\![B]\!] = (3\mathbb{N}+1)\underline{a} \qquad [\![A \otimes C]\!] = \mathbb{N}\underline{a}$$

$$[\![C]\!] = (3\mathbb{N}+2)\underline{a} \qquad [\![B \,\&\, C]\!] = \emptyset$$

$$[\![(A \otimes B) \,\&\, (A \otimes C)]\!] = \mathbb{N}\underline{a}$$

$$[\![A \otimes (B \,\&\, C)]\!] = \emptyset$$

$(A \otimes B) \,\&\, (A \otimes C) \vdash A \otimes (B \,\&\, C)$ has a counter model

Completeness problem: $[\![(A \,\&\, B) \oplus C]\!] = [\![(A \oplus B) \,\&\, (A \oplus C)]\!]$

Ordered monoids

An (commutative) ordered monoid (OM) is a triplet $\mathcal{M} = (M, \bullet, \leqslant)$ such that

1. $(M, \leqslant)$ is an ordered set;
2. $(M, \bullet)$ is commutative monoid;
3. $(x, y) \mapsto x \bullet y$ is increasing.

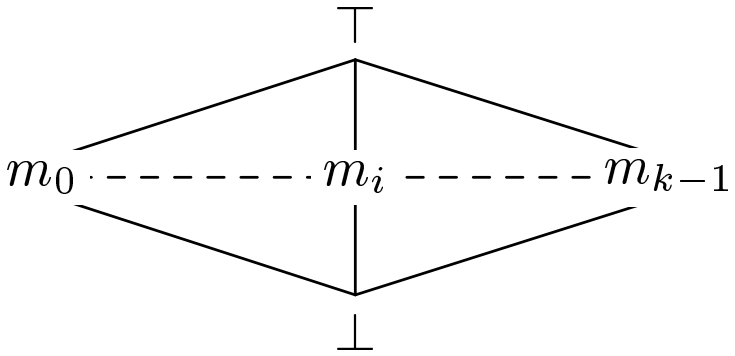
Quantales

A (commutative) quantale (Qa) is a triplet $\mathcal{M} = (M, \bullet, \leqslant)$ such that

1. $\mathcal{M}$ is already an ordered monoid;
2. $(M, \leqslant)$ is a complete lattice;
3. $a \bullet \left(\bigvee_i b_i\right) = \bigvee_i (a \bullet b_i)$

The class of quantales is complete w.r.t. ILL

Examples



$$\begin{aligned} \bot \bullet \top &= \bot \\ m_i \bullet \top &= \top \\ m_i \bullet \bot &= \bot \\ m_i \bullet m_j &= m_{i+j} \end{aligned}$$

- $\{m_0, \ldots, m_{k-1}\}$ forms a monoid for which we consider the flat order obtaining an OM;
- we can extend this OM into a quantale adding $\{\bot, \top\}$.

Analysing Winskel and Engberg's construction

First consider a Petri net and its reachability relation $\rightsquigarrow$. We claim:

Petri net = OM when you forget the operationnal aspect

Then build a quantale on the top of the OM using the completion by downward closed subsets.

OM + closure = quantale

But why cannot we obtain any quantale this way ?

Choose another closure !

Modified Mac Neille closure

A very "strong" closure is the MacNeille closure:

$$X^{\mathrm{n}} = \{z \in M \mid \forall m \in M,\ X \leqslant m \Rightarrow z \leqslant m\}$$

It does not work for an OM in general, i.e. you do not obtain a quantale. Here is the modified version:

$$X^{\circ} = \{z \in M \mid \forall a, m \in M,\ a \bullet X \leqslant m \Rightarrow a \bullet z \leqslant m\}.$$

- Given an OM $\mathcal{M}$, the set of closed subset of $\mathcal{M}$ forms a quantale.
- Up to isomorphism, quantales are preserved by the construction.

New Petri net semantic

From the quantale structure obtained by the modified Mac Neille closure, we derive the following semantic:

$$[\![\top]\!] = \mathcal{M} \qquad [\![\bot]\!] = \{m \mid \forall a,\ a + m \rightsquigarrow 0\}$$

$$[\![A \,\&\, B]\!] = [\![A]\!] \cap [\![B]\!] \qquad [\![1]\!] = \{m \mid m \rightsquigarrow 0\}$$

$$[\![A \multimap B]\!] = \{m \mid \forall a \in [\![A]\!],\ a + m \in [\![B]\!]\}$$

$$[\![A \oplus B]\!] = \{m \mid \forall k, n,\ k + ([\![A]\!] \cup [\![B]\!]) \rightsquigarrow n \Rightarrow k + m \rightsquigarrow n\}$$

$$[\![A \otimes B]\!] = \{m \mid \forall k, n,\ k + [\![A]\!] + [\![B]\!] \rightsquigarrow n \Rightarrow k + m \rightsquigarrow n\}$$

Completeness

We recall the main points:

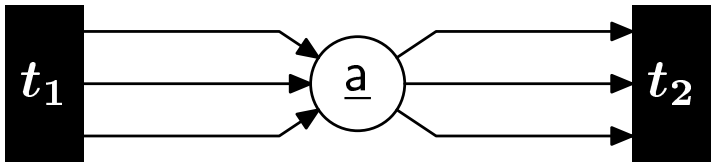
- Any OM can be obtained from a Petri net.
- Any quantale can be obtained from an OM using the new closure.
- ILL is complete with respect to quantales.

The new semantic is sound and complete with respect to ILL

Moreover, finiteness is preserved during the process

The !-free ILL has the fmp for the new interpretation

Example

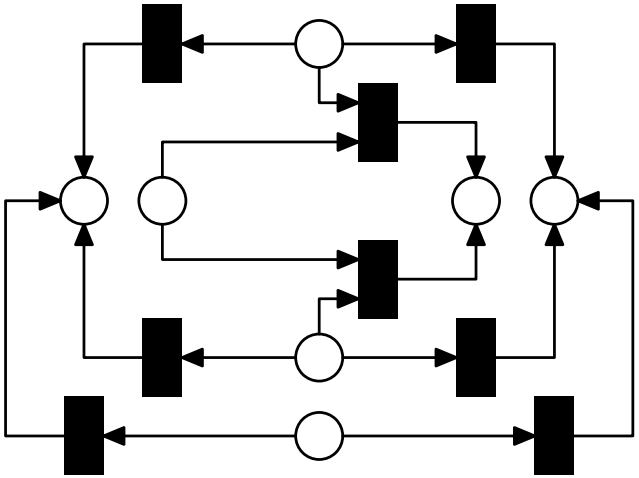


Definitions are on the left side, computations on the right side:

$$\begin{array}{l|l} [\![A]\!] = 3\mathbb{N}\underline{\mathsf{a}} & [\![A \oplus B]\!] = \mathbb{N}\underline{\mathsf{a}} \\ [\![B]\!] = (3\mathbb{N}+1)\underline{\mathsf{a}} & [\![A \oplus C]\!] = \mathbb{N}\underline{\mathsf{a}} \\ [\![C]\!] = (3\mathbb{N}+2)\underline{\mathsf{a}} & [\![B \,\&\, C]\!] = \emptyset \end{array}$$

$$[\![(A \oplus B) \,\&\, (A \oplus C)]\!] = \mathbb{N}\underline{\mathsf{a}}$$

$$[\![A \oplus (B \,\&\, C)]\!] = 3\mathbb{N}\underline{\mathsf{a}}$$

$(A \oplus B) \,\&\, (A \oplus C) \vdash A \oplus (B \,\&\, C)$ has a counter model

Why should we modify the Mac Neille closure ?