From ordered monoids to quantales and Petri nets: revisited completeness results in ILL

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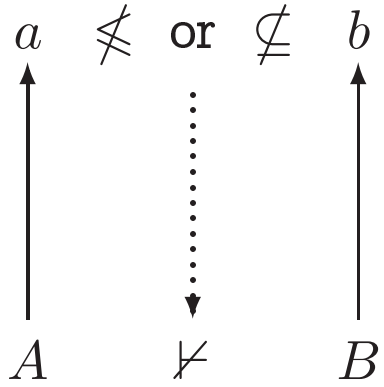
Overview

- Unified analysis of different semantics of ILL
  - relationship between the notions of order and closure
  - general construction of quantales from ordered monoids
- revisited completeness results
  - OM semantics complete w.r.t. ILL
  - the finite model property
- semantical approach of provability

Intuitionnistic Linear Logic

- Intuitionnistic Linear Logic (ILL): $\Gamma \vdash A$;
- Correspondence between ILL and Petri nets;
- Semantical approach to prove or disprove properties



Algebraic semantics

Two equivalent (complete) semantics of ILL

Phases spaces (Girard) $\longleftrightarrow$ Quantales (Yetter)

A (commutative) quantale (Qa) is a triplet $\mathcal{M} = (M, \bullet, \leqslant)$ such that

1. $\mathcal{M}$ is a commutative ordered monoid;
2. $(M, \leqslant)$ is a complete lattice;
3. $\forall a, b_i \in M \quad a \bullet \left(\bigvee_i b_i\right) = \bigvee_i (a \bullet b_i)$

The framework

- Unified analysis of different semantics: phases, Petri nets, ...

$$\text{Ordered monoid} \xrightarrow{\text{closure}} \text{Quantale}$$

- The closure has to satisfy two properties:
  1. compatibility with the order relation;
  2. compatibility with the monoidal operator.
- Study of the relation between the (pre)order and the closure.

Phase semantics

- A phase space is (commutative) monoid $(M, \bullet)$ together with a closure operator $(\cdot)^\star$ such that for all $X, Y \subseteq M$:

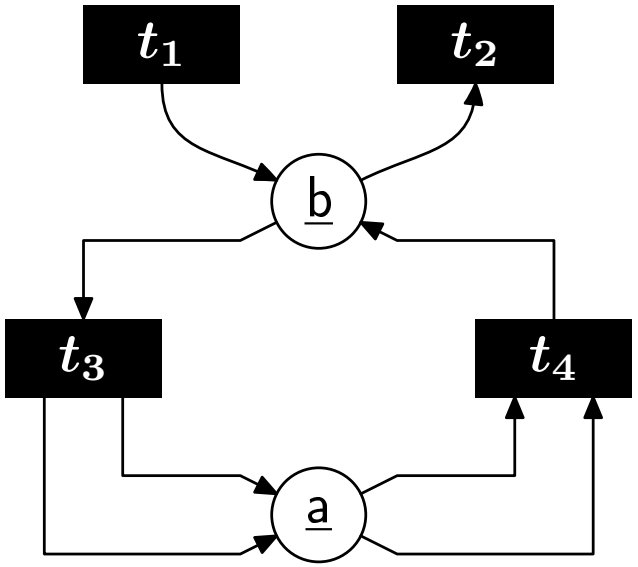
  (1) $X \subseteq Y^\star \Leftrightarrow X^\star \subseteq Y^\star$ (2) $X \bullet Y^\star \subseteq (X \bullet Y)^\star$

- For completeness theorem:
  - M monoid of contexts
  - $(\cdot)^\star$ greatest closure s.t. $\{A\}^\star = \{\Gamma \mid \Gamma \vdash A\} = {\downarrow}A$, i.e. the MacNeille closure.
- Closure obtained from a preorder relation on context:
  - provability relation;
  - in quantale: order $=$ provability.

Petri net semantic

- A Petri net is a graphical representation of a (pre)ordered monoid;
- Reachability ($\rightsquigarrow$) orders markings;
- The smallest closure s.t.

$$\{m\}^\star = \{n \mid n \rightsquigarrow m\} = \downarrow m$$



$\underline{a} \rightsquigarrow (2\mathbb{N}+1)\underline{a} + \mathbb{N}\underline{b}$

- Initial segment closure: $X^\star = \{n \mid \exists m \in X, n \rightsquigarrow m\}$;
- $X \bullet Y^\star \subseteq (X \bullet Y)^\star$ i.e. $n \rightsquigarrow m$ implies $n + x \rightsquigarrow m + x$.

Summary

- In both cases

$$\text{Ordered monoid} \xrightarrow{\text{closure}} \text{Quantale}$$

- The differences
  1. In phase semantics, a particular preordered monoid with $\multimap$;
  2. In Petri net semantics, no completeness because of the least closure.
- Choice of the order is central: either implicit or explicit.

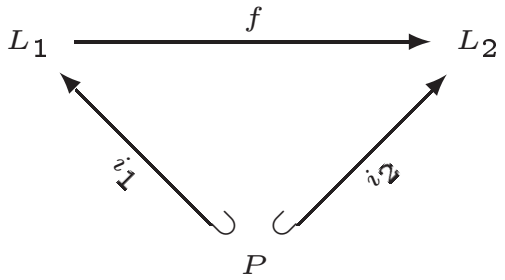
How to obtain a general construction of quantale from any OM ?

Algebraic analysis

- Relationship between extensions and closures;
- An algebraic analysis in two cases:
  1. Posets and complete lattices;
  2. Ordered monoids and quantales;
- Quantales viewed as generalisation of complete lattices;
- Completions characterised by closures.

Order completions (1)

- Category of complete lattices L;
- Morphisms f preserve lubs;
- Given an poset P, a completion by lubs is an embedding $i : P \longrightarrow L$ such that every point of L is a lub of some points of $i(P)$.



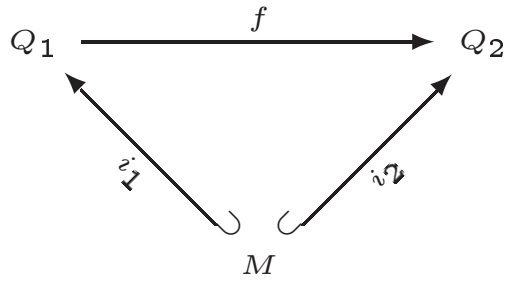
- Completions by lubs form a subcategory where morphisms are commutative diagrams.

Order completions (2)

- The subcategory of completions is equivalent to the poset of closure on P that satisfy $\{x\}^\star = \downarrow x$;
- The closures have a complete lattice structure;
- The least closure is $\downarrow X = \{x' \mid \exists x \in X,\ x' \leqslant x\}$;
- The greatest closure is $X^{\mathtt{n}} = \{x \mid \forall m,\ X \leqslant m \Rightarrow x \leqslant m\}$.

OM completions (1)

- Category of quantales Q;
- Morphisms f preserve $\bigvee$ and $\bullet$;
- Given an OM M, a completion by lubs is an embedding $i : M \longrightarrow Q$ such that every point of Q is a lub of some points of $i(M)$.



- Completions by lubs form a subcategory where morphisms are commutative diagrams.

OM completions (2)

- The subcategory of completions is equivalent to the poset of closure on P that satisfy $\{x\}^\star = {\downarrow}x$ and $X \bullet Y^\star \subseteq (X \bullet Y)^\star$;
- The closures have a complete lattice structure;
- The least closure is ${\downarrow}X = \{x' \mid \exists x \in X,\ x' \leqslant x\}$;
- But the Mac Neille construction does not lead to the greatest closure;
- We have a natural characterisation.

A new closure operator

- Let M be an ordered monoid.
- Search a closure s.t. $\{x\}^\star = {\downarrow}x$ and $X \bullet Y^\star \subseteq (X \bullet Y)^\star$.

  MacNeille closure $\quad X^{\mathtt{n}} = \{x \mid \forall m,\ X \leqslant m \Rightarrow x \leqslant m\}$

  New closure $\quad X^\circ = \{x \mid \forall k, m,\ k \bullet X \leqslant m \Rightarrow k \bullet x \leqslant m\}$

- $k \bullet X \leqslant m$ iff $X \leqslant k \multimap m$ then they are identical in case of $\multimap$.

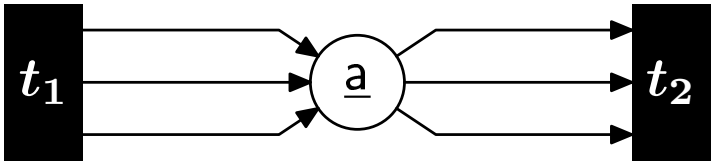
Results

- This embedding preserves $\bullet$ and existing $\wedge$, $\multimap$;
- This embedding preserves existing $\bigvee x_i$ s.t. $a \bullet \bigvee x_i = \bigvee a \bullet x_i$
- If M is already a quantale then $M^\star$ is isomorphic to M;
- $M^\star$ is finite if M is finite.

The OM-semantics is complete w.r.t. ILL and has the finite model property

Applications

- Algebraic semantics for the study of $\vdash A, A$ (Avron);
- Concrete quantales for (WC)-formulae (Piazza & Castellan);
- Petri nets (Winskel);
- Simple counter model for non-distributivity.



Definitions are on the lhs,
computations on the rhs:

$$[\![A]\!] = 3\mathbb{N}\underline{\mathsf{a}} \qquad [\![A \oplus B]\!] = \mathbb{N}\underline{\mathsf{a}}$$
$$[\![B]\!] = (3\mathbb{N}+1)\underline{\mathsf{a}} \qquad [\![A \oplus C]\!] = \mathbb{N}\underline{\mathsf{a}}$$
$$[\![C]\!] = (3\mathbb{N}+2)\underline{\mathsf{a}} \qquad [\![B \,\&\, C]\!] = \emptyset$$

$$[\![(A \oplus B) \,\&\, (A \oplus C)]\!] = \mathbb{N}\underline{\mathsf{a}}$$
$$[\![A \oplus (B \,\&\, C)]\!] = 3\mathbb{N}\underline{\mathsf{a}}$$

Conclusion

- A unified framework for different semantics of ILL;
- A completeness result and finite model property.

Perspectives

- The non-commutative case;
- The semantic of proofs;
- Automated generation of counter-models;
- Applications to specification and verification.