Refutations as finite resource counter-models

Dominique Larchey and Didier Galmiche

LORIA UMR 7503 - CNRS & UHP

Campus Scientifique, BP 239

Vandœuvre-lès-Nancy, France

Overview

- Propositionnal Intuitionistic Linear Logic (ILL)
- Existing semantics:
  - $\rightarrow$ Quantales
  - $\rightarrow$ Phase spaces
  - $\rightarrow$ Petri nets
- New semantics:
  - $\rightarrow$ Partially ordered monoids
  - $\rightarrow$ Petri nets
  - $\rightarrow$ Resource models

Results

- Properties:
  - → Completeness
  - → Semantical proofs of cut-elimination
  - → Finite model property, finite counter models
- Important notions:
  - → Greatest stable closure
  - → Syntactic resource models
  - → Refutations

Intuitionistic Linear Logic

$$\frac{}{A \vdash A}\ [\mathrm{id}] \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ [\mathrm{cut}]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ [\otimes_L] \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ [\otimes_R]$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ [\multimap_L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ [\multimap_R]$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C}\ [\&^1_L] \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\ [\&^2_L] \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ [\&_R]$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ [\oplus_L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ [\oplus^1_R] \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ [\oplus^2_R]$$

$$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\ [1_L] \qquad \frac{}{\vdash 1}\ [1_R] \qquad \frac{}{\Gamma, \mathsf{F} \vdash A}\ [\mathsf{F}_L] \qquad \frac{}{\Gamma \vdash \mathsf{T}}\ [\mathsf{T}_R]$$

Algebraic semantic

- Commutative quantale $(\mathcal{M}, \bullet, \leqslant)$

  1. $(\mathcal{M}, \bullet)$ commutative monoid
  2. $(\mathcal{M}, \leqslant)$ complete lattice
  3. $a \bullet \bigvee_i b_i = \bigvee_i (a \bullet b_i)$

- Example of quantales

  $\rightarrow$ Stable closure over a monoid (phase space)

  $\rightarrow$ Completion of a preordered monoid (pretopology)

  $\rightarrow$ Petri nets (Winskel & Engberg)

  $\rightarrow$ Resource models

Interpretation of formulae of ILL

- By the following

$\otimes \mapsto \bullet$	$\multimap \mapsto -\!\!\bullet$
$\& \mapsto \wedge$	$\oplus \mapsto \vee$

- Example: $[\![A \otimes B]\!] = [\![A]\!] \bullet [\![B]\!]$
- Soundness: if $\vdash A$ provable then $\mathrm{e} \leqslant [\![A]\!]$
- Completeness by Lindenbaum construction or phase semantics
- Finite model property? (CLL, Lafont)

Phase semantic

- A commutative monoid $(\mathcal{M}, \bullet)$
- A stable closure $(\cdot)^\star$: $X^\star \bullet Y^\star \subseteq (X \bullet Y)^\star$
- A quantale structure on closed subsets:

$$\begin{array}{l|l} X \bullet Y = (X \bullet Y)^\star & X \multimap Y = \{x \mid x \bullet X \subseteq Y\} \\ X \vee Y = (X \cup Y)^\star & X \wedge Y = X \cap Y \end{array}$$

- Soundness condition: if $\vdash A$ provable then $\mathsf{e} \in [\![A]\!]$

Preordered monoid semantic

$$\textit{(Pre)ordered monoid} \xrightarrow{\textit{closure } (\cdot)^\star} \textit{Quantale}$$

- Pretopology $(\cdot)^\star$: $\begin{cases} \textit{compatible} & \downarrow\{x\} = \{x\}^\star \\ \textit{stable} & X^\star \bullet Y^\star \subseteq (X \bullet Y)^\star \end{cases}$
- Pretopologies form a complete lattice

  Smallest $\quad x \in \downarrow X \Leftrightarrow \exists y \in X,\ x \leqslant y$

  Greatest $\quad x \in X^\circ \Leftrightarrow \forall m, p,\ (m \bullet X \leqslant p \Rightarrow m \bullet x \leqslant p)$

- $\downarrow(\cdot)$ semantic incomplete ($\oplus$-& distributivity)
- $(\cdot)^\circ$ semantic is complete

Petri net semantics

$$\textit{Petri net} \quad \overset{\rightsquigarrow}{\longleftrightarrow} \quad \textit{Ordered monoid} \quad \xrightarrow{(\cdot)^\star} \quad \textit{Quantale}$$

- Petri net $\mathcal{R} = (\mathcal{P}, \mathcal{T}, {}^\bullet(\cdot), (\cdot)^\bullet)$
- Free monoid of *distributions* $\mathbb{M}_{\mathrm{f}}(\mathcal{P})$ of tokens
- *Accessibility* preorder $\rightsquigarrow$ between distributions
- (pre)ordered monoid $(\mathbb{M}_{\mathrm{f}}(\mathcal{P}), +, \rightsquigarrow)$
- Least pretopology $\downarrow(\cdot)$ semantic incomplete (Winskel & Engberg)
- Greatest pretopology $(\cdot)^\circ$ semantic is complete

Resource frames

- Resource frame $(\mathcal{M}, \mathcal{P}, \bullet, \succ)$:
  - $\rightarrow$ $(\mathcal{M}, \bullet)$ monoid of *ressources*
  - $\rightarrow$ $\mathcal{P}$ set of *products*
  - $\rightarrow$ $\succ \subseteq \mathcal{M} \times \mathcal{P}$ production *relation*
- Sections denoted by $\mathcal{M}_p$ or $\underline{p} \triangleq \{m \in \mathcal{M} \mid m \succ p\}$

$$x \in X^{\succ} \quad \Leftrightarrow \quad \forall m, p\ (m \bullet X \succ p \Rightarrow m \bullet x \succ p)$$

- $(\cdot)^{\succ}$ *greatest* stable closure st. sections are closed
- A quantale denoted $\mathcal{M}^{\succ}$ formed by $(\cdot)^{\succ}$-closed subsets
- $(X, Y) \mapsto (X \bullet Y)^{\succ}$ is the monoidal operation

Interpretation

$$[\![V]\!] \triangleq \{m \mid m \succ \sigma(V)\} \qquad [\![A \multimap B]\!] \triangleq [\![A]\!] \multimap\!\bullet\, [\![B]\!]$$

$$[\![\mathsf{T}]\!] \triangleq \mathcal{M} \qquad [\![A \,\&\, B]\!] \triangleq [\![A]\!] \cap [\![B]\!]$$

$$[\![\mathsf{F}]\!] \triangleq \{x \mid \forall m, p\ m \bullet x \succ p\} = \emptyset^{\succ}$$

$$[\![1]\!] \triangleq \{x \mid \forall m, p\ (m \succ p \Rightarrow m \bullet x \succ p)\} = \mathsf{e}^{\succ}$$

$$[\![A \otimes B]\!] \triangleq \{x \mid \forall m, p\ (m \bullet [\![A]\!] \bullet [\![B]\!] \succ p \Rightarrow m \bullet x \succ p)\}$$

$$[\![A \oplus B]\!] \triangleq \{x \mid \forall m, p\ (m \bullet [\![A]\!] \succ p \text{ and } m \bullet [\![B]\!] \succ p \Rightarrow m \bullet x \succ p)\}$$

Soundness

- If A is provable in ILL then $\mathsf{e} \in [\![A]\!]$

Example of resource frames

- $(\mathcal{M}, \bullet, (\cdot)^\star)$ phase space

  $\rightarrow$ $(\mathcal{M}, \mathcal{M}^\star, \bullet, \in)$ resource frame

  $\rightarrow$ Obtain phase semantics: $X^\in = X^\star$

- $(\mathcal{M}, \bullet, \leqslant)$ ordered monoid

  $\rightarrow$ $(\mathcal{M}, \mathcal{M}, \bullet, \leqslant)$ resource frame

  $\rightarrow$ Obtain preordered monoid semantics: $X^\leqslant = X^\circ$

- Resource semantic is a generalization

Syntatic resource frames

- Resources ($\mathcal{M}$) are *contexts* (Γ)
- Products ($\mathcal{P}$) are *formulae* (A)
- $\succ$ is a relation closed under some logical rules
- Superposition of contexts $\bullet$, empty context e
- Context Γ denoted by $\lfloor\Gamma\rfloor$, or $\lfloor A\rfloor$ for a single formula
- Validity condition (soundness): $\lfloor\emptyset\rfloor \in [\![A]\!]$

Logical rules as semantic equations

- Closure of $\succ$ under logical rules = semantic equations

$$\frac{}{A \succ A}\ [\mathrm{id}] \qquad \lfloor A \rfloor \in \underline{A}$$

$$\frac{\Gamma \succ A \qquad A, \Delta \succ B}{\Gamma, \Delta \succ B}\ [\mathrm{cut}] \qquad \underline{A} \subseteq \lfloor A \rfloor^{\succ}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ [\otimes_L] \qquad \lfloor A \otimes B \rfloor \in \lfloor A, B \rfloor^{\succ}$$

- Very natural conditions
- One to one match between ILL and the syntactic model

Logical rules as semantic equations (2)

[id]	$\lfloor A \rfloor \in \underline{A}$	[cut]	$\underline{A} \subseteq \lfloor A \rfloor^{\succ}$

$[1_L]$	$\lfloor 1 \rfloor \in \mathsf{e}^{\succ}$	$[1_R]$	$\mathsf{e} \in \underline{1}$
$[\mathsf{F}_L]$	$\lfloor \mathsf{F} \rfloor \in \emptyset^{\succ}$	$[\mathsf{T}_R]$	$\mathsf{Context} \subseteq \underline{\mathsf{T}}$
$[\otimes_L]$	$\lfloor A \otimes B \rfloor \in \lfloor A, B \rfloor^{\succ}$	$[\otimes_R]$	$\underline{A} \bullet \underline{B} \subseteq \underline{A \otimes B}$
$[\oplus_L]$	$\lfloor A \oplus B \rfloor \in \{\lfloor A \rfloor, \lfloor B \rfloor\}^{\succ}$	$[\oplus_R^{1,2}]$	$\underline{A} \cup \underline{B} \subseteq \underline{A \oplus B}$
$[\&_L^{1,2}]$	$\lfloor A \,\&\, B \rfloor \in \lfloor A \rfloor^{\succ} \cap \lfloor B \rfloor^{\succ}$	$[\&_R]$	$\underline{A} \cap \underline{B} \subseteq \underline{A \,\&\, B}$
$[\multimap_L]$	$\lfloor A \multimap B \rfloor \in \underline{A} \multimap\!\!\bullet \lfloor B \rfloor^{\succ}$	$[\multimap_R]$	$\{\lfloor A \rfloor\} \multimap\!\!\bullet \underline{B} \subseteq \underline{A \multimap B}$

Completeness & cut elimination

- Suppose $\succ$ closed under any rule except [cut]
- For example $\vdash_{\mathrm{ILL}}$ or $\vdash_{\mathsf{wc}}$
- By induction on A: $\boxed{\lfloor A \rfloor \in [\![A]\!] \subseteq \underline{A}}$
- Soundness condition on the $\vdash_{\mathrm{ILL}}$ syntactic model: *completeness theorem*

$$\lfloor \emptyset \rfloor \in [\![A]\!] \Rightarrow \lfloor \emptyset \rfloor \in \underline{A} \Rightarrow \vdash_{\mathrm{ILL}} A$$

- Soundness of the $\vdash_{\mathsf{wc}}$ syntatic model: *cut elimination theorem*

$$\vdash_{\mathrm{ILL}} A \Rightarrow \lfloor \emptyset \rfloor \in [\![A]\!] \Rightarrow \lfloor \emptyset \rfloor \in \underline{A} \Rightarrow \vdash_{\mathsf{wc}} A$$

Refutation trees

- Refutation *dual* notion to that of proof
- Tree $\mathcal{R}$ where one chooses one premiss for each applicable rule

$$\frac{H_1^1 \ldots H_{i_1}^1 \ldots H_{p_1}^1}{C} \; , \ldots , \; \frac{H_1^n \ldots H_{i_n}^n \ldots H_{p_n}^n}{C}$$

$$\downarrow$$

$$\frac{H_{i_1}^1 \quad \ldots \quad H_{i_n}^n}{C}$$

- Relation $(\cdot) \notin \mathcal{R}$ is closed under the logical rules

Refutation trees (bis)

- Refutation tree in [cut] free ILL

$$\dfrac{\dfrac{\dfrac{}{A \vdash B}}{A \vdash A \,\&\, B} \quad \dfrac{\dfrac{}{B \vdash A}}{A \oplus B \vdash A}}{A \oplus B \vdash A \,\&\, B}$$
$$\vdash (A \oplus B) \multimap (A \,\&\, B)$$

- Proof search in [cut] free ILL

For any sequent, either a proof or a finite refutation

Finite model property

- $\mathcal{R}$ finite refutation tree for the invalid sequent $\vdash C$
- The relation $(\Gamma \vdash A) \notin \mathcal{R}$ is closed under the rules of [cut] free ILL
- The obtained syntatic resource frame is a counter model to every sequent in $\mathcal{R}$ and in particular to the sequent $\vdash C$
- Quotient by identifying sub-contexts outside $\mathcal{R}$
- This quotient preserves semantic invalidity
- A finite resource frame counter model to the sequent $\vdash C$

Example of finite counter model

$$X, Y \vdash X$$

$$\begin{aligned} \mathcal{M} &\triangleq \mathsf{Context}_{\mathcal{R}} \cup \{\Pi\} \\ &= \{0, 1, 2, 3, \Pi\} \end{aligned}$$

$\lfloor \emptyset \rfloor$	
$\lfloor X \rfloor$	
$\lfloor Y \rfloor$	
$\lfloor X, Y \rfloor$	

$\bullet$	0	1	2	3	Π
0	0	1	2	3	Π
1	1	Π	3	Π	Π
2	2	3	Π	Π	Π
3	3	Π	Π	Π	Π
Π	Π	Π	Π	Π	Π

$$\underline{X} = \{0, 1, 2, \Pi\} \text{ and } \underline{Y} = \mathcal{M} \text{ and } \mathcal{M}^{\succ} = \{K \mid \Pi \in K \subseteq \mathcal{M}\}$$

$$\left.\begin{array}{c} [\![X]\!] = \underline{X} = \mathcal{M} - \{3\} \\ [\![Y]\!] = \mathcal{M} \quad \text{and} \quad [\![X, Y]\!] = \mathcal{M} \end{array}\right\} \text{ then } [\![X, Y]\!] \not\subseteq [\![X]\!]$$

Applications

- Completeness and finite model property for
  - $\rightarrow$ Quantales
  - $\rightarrow$ Phase spaces
  - $\rightarrow$ Partially ordered monoids
  - $\rightarrow$ Petri nets (new)
- Refutations as a tool to build counter models

Conclusion

- Resource frames as promising semantic
- Completeness & finite model property
- Semantic proof of [cut] elimination
- Proof search and refutations as a semantic tool

Perspectives

- Automated counter model construction
- Cut elimination:
  - $\rightarrow$ The intuitionistic case (non-linear)
  - $\rightarrow$ The second order case (linear and non-linear)