Combining proof search and linear counter-model construction

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Gödel-Dummett logic LC

- Intermediate logic: IL $\subset$ LC $\subset$ CL
- Syntactic characterization: IL $+ (X \supset Y) \vee (Y \supset X)$
- Semantic models:
  - Linear Kripke trees (no branching)
  - The lattice $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ with its natural order
- Complexity:
  - LC (and CL) are NP-complete
  - IL is PSPACE-complete

Deciding LC

- **Proof search and counter-models combined**
  - Strongly invertible rules to reduce sequents
  - Semantic fixpoint computation to decide irreducible sequents
- **Efficient** (duplication-free, loop-free) proof-search
  - IL (Dyckhoff & Hudelmair, Weich, Larchey & Galmiche)
  - Intermediate logics (Avellone et al. and Fiorino)
  - LC (Dyckhoff, Avron, Larchey)
- **Invertibility** and strong invertibility of logical rules
  - No backtracking in proof-search
  - Counter-model generation

The results

- Duplication-free proof search with bounded logical rules
  - Sequents $\rightarrow$ *flat sequents* (indexing)
  - Flat sequents $\rightarrow$ *pseudo-atomic sequents* (proof-search)
- Decision of pseudo-atomic sequent
  - Fixpoint computation
  - Either a **proof** (with a new proof rule)
  - Or a **counter-model**
- Graph based fixpoint computation

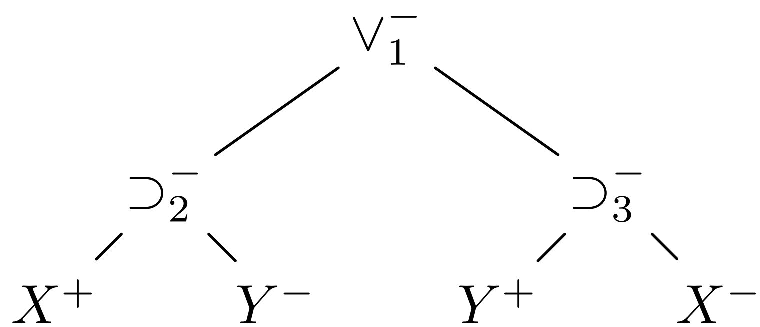
Flattening by indexing

- **Flat sequent**: flat and pseudo-atomic formulae.

$$X, X \supset Y, (X \otimes Y) \supset Z \text{ or } X \supset (Y \otimes Z) \vdash X \text{ or } X \supset Y$$

- Indexing result: $\boxed{\vdash D \quad \Leftrightarrow \quad \delta^-(D) \vdash X_D}$

- Example of indexing of $\vdash (X \supset Y) \vee (Y \supset X)$



$$(X_2 \vee X_3) \supset X_1, (X \supset Y) \supset X_2, (Y \supset X) \supset X_3 \vdash X_1$$

Proof-search (duplication free)

- Reduction of any flat sequent into **pseudo-atomic sequents**

$$\frac{\Gamma, A \supset C \vdash \Delta \qquad \Gamma, B \supset C \vdash \Delta}{\Gamma, (A \wedge B) \supset C \vdash \Delta}\ [\supset_2] \qquad \frac{\Gamma, A \supset B, A \supset C \vdash \Delta}{\Gamma, A \supset (B \wedge C) \vdash \Delta}\ [\supset_2']$$

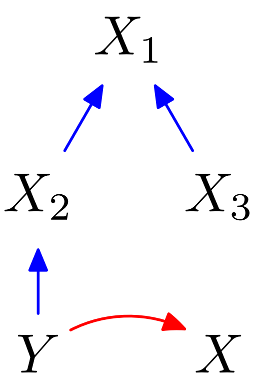
$$\frac{\Gamma, A \supset C, B \supset C \vdash \Delta}{\Gamma, (A \vee B) \supset C \vdash \Delta}\ [\supset_3] \qquad \frac{\Gamma, A \supset B \vdash \Delta \qquad \Gamma, A \supset C \vdash \Delta}{\Gamma, A \supset (B \vee C) \vdash \Delta}\ [\supset_3']$$

$$\frac{\Gamma, B \supset C \vdash \boxed{A \supset B}, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma, (A \supset B) \supset C \vdash \Delta}\ [\supset_4] \qquad \frac{\Gamma, A \supset C \vdash \Delta \qquad \Gamma, B \supset C \vdash \Delta}{\Gamma, A \supset (B \supset C) \vdash \Delta}\ [\supset_4']$$

- The connectors $\otimes$ of flat formulae (like $(X \otimes Y) \supset Z$)
  - occur has the internal nodes of the initial formula tree
  - are decomposed exactly once by proof-search branch
- All premises are strongly invertible and there is no duplication

An example of proof search branch

- Proof search as syntactic graph orientation



$$\dfrac{\dfrac{\dots}{X_2 \supset X_1, X_3 \supset X_1, Y \supset X_2, (Y \supset X) \supset X_3 \vdash X \supset Y, X_1}}{\dfrac{X_2 \supset X_1, X_3 \supset X_1, \boxed{X \supset Y} \supset X_2, (Y \supset X) \supset X_3 \vdash X_1}{\boxed{X_2 \vee X_3} \supset X_1, (X \supset Y) \supset X_2, (Y \supset X) \supset X_3 \vdash X_1}[\supset_3]}[\supset_4] \text{ left}$$

Counter-models by fixpoint computation

- Deciding the pseudo-atomic sequent:

$$\Gamma_a \vdash X_1 \supset Y_1, \ldots, X_n \supset Y_n \qquad (\Gamma_a \text{ atomic implications})$$

- Define the following functor of subsets of $[1, n]$:

$$\varphi(I) = \{i \mid \Gamma_a, \mathcal{X}_I \Vdash Y_i\}$$

- Compute the greatest fixpoint sequence:

$$I_0 = [1, n] \supsetneq I_1 = \varphi([1, n]) \supsetneq \cdots \supsetneq I_p = \varphi^p([1, n]) = \mu_\varphi$$

- The sequent has a counter-model iff. $\boxed{\mu_\varphi = \emptyset}$
- Counter model extracted from the sequence $I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_p$

The fixpoint as a new proof-rule

- In the case $\mu_\varphi = \{i_1, \ldots, i_k\}$ is not empty
- The fixpoint property induces a new proof rule

$$\frac{\Gamma_a, X_{i_1}, \ldots, X_{i_k} \vdash Y_{i_1} \quad \ldots \quad \Gamma_a, X_{i_1}, \ldots, X_{i_k} \vdash Y_{i_k}}{\Gamma_a \vdash X_1 \supset Y_1, \ldots, X_n \supset Y_n} \ [\supset_N]$$

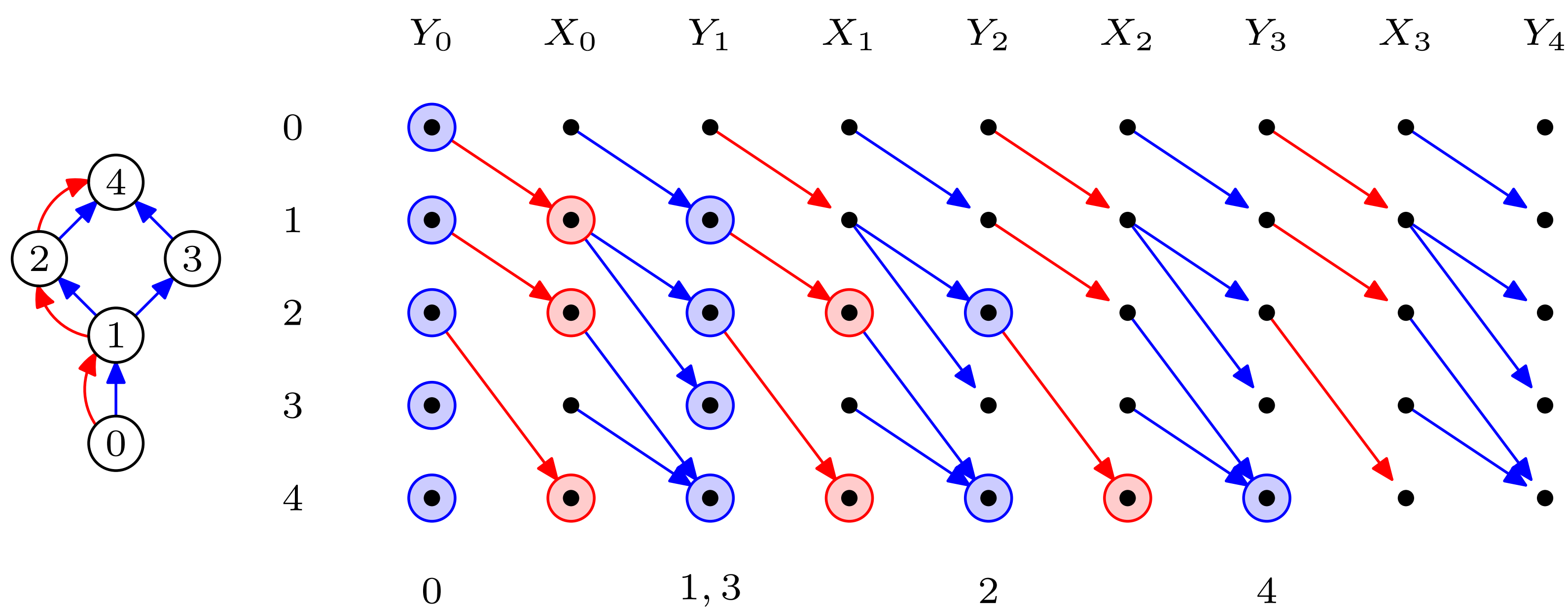
- All the premises are valid (fixpoint property)
- We obtain a one step proof (exponential with $[\supset_R]$ Dyckhoff)

The decision algorithm

- A **combination** of proof-search and counter-model generation
- **Indexing** of the sequent into a flat sequent
- **Reduction** to a set of pseudo-atomic sequents (proof-search)
- For $\Gamma_a \vdash X_1 \supset Y_1, \ldots, X_n \supset Y_n, Z_1, \ldots, Z_k$ (say $\mathcal{S}$)
- If one of the atomic $\Gamma_a \vdash Z_i$ is valid so is the sequent $\mathcal{S}$
- Or **compute the fixpoint** for $\Gamma_a \vdash X_1 \supset Y_1, \ldots, X_n \supset Y_n$
  - Case $\mu \neq \emptyset$, get a proof of the sequent $\mathcal{S}$ (weakening)
  - Case $\mu = \emptyset$, obtain a counter-model
  - This counter-model also holds for the sequent $\mathcal{S}$

Example of fixpoint computation

$$0 \supset 1, 1 \supset 2, 1 \supset 3, 2 \supset 4, 3 \supset 4 \vdash 2 \supset 1, 1 \supset 0, 4 \supset 2$$



$$[\![0]\!] = 0, [\![1]\!] = [\![3]\!] = 1, [\![2]\!] = 2, [\![4]\!] = 3$$

Conclusion and perspectives

- A new efficient graph based decision procedure for LC
- Linear time algorithm for fixpoint computation
- Sharing fixpoint computation among branches
  - On the fly fixpoint computation
- Extension to other intermediate logics