Semicomputable geometry

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April 19, 2018

Abstract

Computability and semicomputability of compact subsets of the Euclidean spaces are important notions, that have been investigated for many classes of sets including fractals (Julia sets, Mandelbrot set) and objects with geometrical or topological constraints (embedding of a sphere). In this paper we investigate one of the simplest classes, namely the filled triangles in the plane. We study the properties of the parameters of semicomputable triangles, such as the coordinates of their vertices. This problem is surprisingly rich. We introduce and develop a notion of semicomputability of points of the plane which is a generalization in dimension 2 of the left-c.e. and right-c.e. numbers. We relate this notion to Solovay reducibility. We show that semicomputable triangles admit no finite parametrization, for some notion of parametrization.

1 Introduction

The notions of computable and computably enumerable sets of discrete objects such as $\mathbb{N}$ have been extended to sets of continuous objects such as real numbers. Arguably the most successful notions are defined for closed subsets of $\mathbb{R}^n$, especially $\mathbb{R}^2$ where they have a graphical interpretation. A computable subset of $\mathbb{R}^2$ corresponds to the intuitive notion of a set that can be drawn on a screen with arbitrary resolution by a single program. The computability of famous sets have been investigated in many articles. Whether the Mandelbrot set is computable is an open problem [Her05], related to a conjecture in complex dynamics. It has been shown that filled Julia sets are computable, while their boundaries are not always computable [BY08]. The computability of the Lorenz attractor has been addressed in [GRZ17] and is still an open problem.

While the computability of such sets is usually a difficult question, the mathematical definitions of these sets immediately enable one to *semicompute* them, in the same way as one can only semicompute the halting problem: if a pixel does not intersect the set then this can be recognized in finite time, but if it does not then one may never know. For instance, the set of fixed-points of a computable function is semicomputable: if $x \neq f(x)$ then it can be eventually discovered by computing $f(x)$ with sufficient precision, but if $x = f(x)$ then we will never know.

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Several studies have shown that topological or geometrical constraints on a semicomputable set make it computable [Mil02, Ilj11, BI14].

In this paper, we study one of the simplest family of geometrical objects, namely filled triangles in $\mathbb{R}^2$. Part of the study extends to other classes of compact convex subsets of $\mathbb{R}^2$. While a filled triangle is computable if and only if the parameters defining it (coordinates, lengths, angles, etc.) are computable, the case of semicomputable triangles is less clear and leads us to several investigations.

We give a first characterization of semicomputable triangles. We introduce the notion of a semicomputable point, which is essentially a point that can be computably approximated from a limited set of directions. We show that determining whether a triangle is semicomputable reduces to identifying the semicomputability ranges of its vertices. We then study the properties of the semicomputability range and develop tools to help determining it, notably the quantitative version of Solovay reducibility which was independently introduced and studied in [BL17, Mil17].

We study the (non-)computability of several parameters associated to triangles by investigating the properties of generic semicomputable triangles, which are in a sense the most typical ones and are far from being computable.

We end this paper with a slightly different viewpoint, by showing that the problem is inherently complex in that the semicomputability of a triangle cannot be reduced to the semicomputability of its parameters, for any *finite* parametrization. This result is proved for a particular notion of parametrization, but other notions are possible and should be studied in the future.

1.1 Background

A real number x is ***computable*** if there is a computable sequence of rationals q_i such that $|x - q_i| < 2^{-i}$. A real number x is ***left-c.e.*** if there is a computable increasing sequence of rationals converging to x, and ***right-c.e.*** if there is a computable decreasing sequence converging to x. A real number is ***difference-c.e.*** or ***d-c.e.*** if it is a difference of two left-c.e. numbers.

A rational box $B \subseteq \mathbb{R}^n$ is a product of n open intervals with rational endpoints. Let $(B_i)_{i\in\mathbb{N}}$ be a canonical enumeration of the rational boxes. A set $U \subseteq \mathbb{R}^d$ is an ***effective open set*** if it is a the union of a computable sequence of rational boxes. A ***semicomputable set*** is the complement of an effective open set. An ***effectively compact set*** is a compact set $K \subseteq \mathbb{R}^d$ such that the set $\{\langle i_1, \dots, i_k\rangle : K \subseteq B_{i_1} \cup \dots \cup B_{i_k}\}$ is c.e. ($\langle . \rangle : \mathbb{N}^* \to \mathbb{N}$ is a computable bijection). Equivalently, K is effectively compact if and only if K is bounded and semicomputable.

A function $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^e$ is ***computable*** if the sets $f^{-1}(R_i)$ are uniformly effective open sets on A, i.e. if there exist uniformly effective open sets $U_i \subseteq \mathbb{R}^d$ such that $f^{-1}(R_i) = U_i \cap A$. A function $f : A \subseteq \mathbb{R}^d \to \mathbb{R}$ is ***left-c.e.*** if the sets $f^{-1}(q_i, +\infty)$ are uniformly effective open sets on A. Every bounded left-c.e. function $f : A \subseteq \mathbb{R}^d \to \mathbb{R}$ has a left-c.e. extension $\hat{f} : \mathbb{R}^d \to \mathbb{R}$.

Let $f : \mathbb{R}^d \times \mathbb{R}^e \to \mathbb{R}$ be left-c.e.

- If $K \subseteq \mathbb{R}^e$ is a non-empty effectively compact set then $f_{\min} : \mathbb{R}^d \to \mathbb{R}$ defined by $f_{\min}(x) = \min_{y\in K} f(x,y)$ is left-c.e.
- If $U \subseteq \mathbb{R}^e$ is an effective open set then $f_{\sup} : \mathbb{R}^d \to \mathbb{R}$ defined by $f_{\sup}(x) = \sup_{y\in U} f(x,y)$ is left-c.e.

2 Semicomputability of convex sets

In dimension 1, a compact convex set is simply a closed interval. Such a set $[a,b]$ is semicomputable exactly when a is left-c.e. and b is right-c.e., i.e. when the extremal points of the set have computable approximations oriented inwards the set. It can be generalized to certain compact convex sets of the plane. While in $\mathbb{R}$ there are only two possible directions, in $\mathbb{R}^2$ there are infinitely many ones, represented by angles.

Let $A = (x,y)$ be a point of the plane. For $\theta \in \mathbb{R}$, the θ-coordinate of A is $A_\theta = x\cos\theta + y\sin\theta = (OA, u_\theta)$, i.e., the inner product of the vector $OA = (x,y)$ with $u_\theta = (\cos\theta, \sin\theta)$ ($O = (0,0)$ is the origin). Observe that the computability properties of A_θ do not depend on the choice of the origin, as long as it is computable.

Definition 2.1. If θ is computable then we say that A is ***θ*-c.e.** if A_θ is left-c.e. For a closed interval $I = [a,b]$, we say that A is ***I*-c.e.** if the function mapping $\theta \in I$ to A_θ is left-c.e.

For a non-empty compact convex set S and $\theta \in \mathbb{R}$, define $S_\theta = \min_{X\in S} X_\theta$, and for an extremal point V of S let $J_V^S = \{\theta \in \mathbb{R} : S_\theta = V_\theta\}$. J_V^S is a closed interval modulo 2π.

Proposition 1. *A non-empty compact convex set S is semicomputable iff the function mapping θ to S_θ is left-c.e. iff for $\theta \in \mathbb{Q}$, S_θ is uniformly left-c.e.*

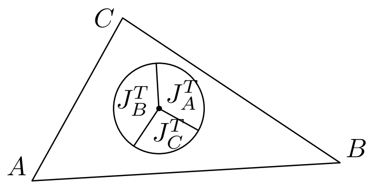
Proof. Assume that S is semicomputable, or equivalently effectively compact. The function $(A,\theta) \mapsto A_\theta$ is computable so the function $\theta \mapsto \min_{A\in S} A_\theta$ is left-c.e.

Conversely, assume that the function $\theta \mapsto S_\theta$ is left-c.e. For each θ let H_θ be the closed half-plane defined by $H_\theta = \{P \in \mathbb{R}^2 : P_\theta \geq S_\theta\}$. H_θ is semicomputable relative to and uniformly in θ, so $S = \bigcap_{\theta\in[0,2\pi]} H_\theta$ is semicomputable as $[0,2\pi]$ is effectively compact.

The function $\theta \mapsto S_\theta$ is L-Lipschitz for some L, so if for all $q \in \mathbb{Q}$, the number S_q is uniformly left-c.e. then the function is left-c.e. as $S_\theta = \sup\{S_q - L|q-\theta| : q \in \mathbb{Q}\}$. □

For a triangle, and more generally a convex polygon, the number of extremal points is finite and Proposition 1 can be improved as follows.

Theorem 1. *A filled triangle $T = ABC$ is semicomputable iff each vertex $V \in \{A,B,C\}$ is J_V^T-c.e.*



In order to prove the theorem, we need the following Lemma.

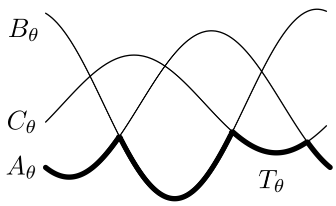
Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be left-c.e. and such that there exists $\epsilon > 0$ such that f is non-increasing on $[a, a+\epsilon)$ and non-decreasing on $(b-\epsilon, b]$. There exists a left-c.e. extension $\hat{f} : \mathbb{R} \to \mathbb{R}$ of f that is non-increasing on $(a-\epsilon, a+\epsilon)$ and non-decreasing on $(b-\epsilon, b+\epsilon)$ and $\hat{f} = +\infty$ outside $(a-\epsilon, b+\epsilon)$.*

Proof. Let f_0 be a left-c.e. extension of f. Let $q, q', r, r' \in \mathbb{Q}$ satisfy $q < a < q' < r' < b < r$, $q' - q < \epsilon$ and $r - r' < \epsilon$. Define $\hat{f}(x) = f(x)$ if $x \in [q', r']$, $\hat{f}(x) = \sup_{[x,q']} f_0$ if $x \in [q, q']$, $\hat{f}(x) = \sup_{[r',x]} f_0$ if $x \in [r', r]$, $\hat{f}(x) = +\infty$ if $x < q$ or $x > r$. $\square$

Proof of Theorem 1. If T is semicomputable then the function $\theta \mapsto T_\theta$ is left-c.e. It coincides with the function $\theta \mapsto V_\theta$ on J_V^T, so V is J_V^T-c.e.

Conversely assume that each vertex $V \in \{A, B, C\}$ is J_V^T-c.e. We show that the function mapping θ to T_θ is left-c.e. We know that it is left-c.e. on each J_V^T, and $\bigcup_{V\in\{A,B,C\}} J_V^T = \mathbb{R}$, but we must show how to merge the three algorithms. Let us assume that the origin of the Euclidean plane lies inside the triangle. If it is not the case, then one can translate the triangle by a rational vector, which preserves all the computability properties of T and its vertices.

If the origin is inside the triangle then for each vertex $V \in \{A, B, C\}$, if $J_V^T = [a, b]$ then there exists $\epsilon > 0$ such that the function V_θ is non-increasing on $(a-\epsilon, a+\epsilon)$ and non-decreasing on $(b-\epsilon, b+\epsilon)$, so by Lemma 1 there is a left-c.e. function $\hat{V}_\theta$ that coincides with V_θ on J_V^T, is non-increasing on $(a-\epsilon, a+\epsilon)$, non-decreasing on $(b-\epsilon, b+\epsilon)$ and $\hat{V}_\theta = +\infty$ for θ outside $(a-\epsilon, b+\epsilon)$. As a result, $T_\theta = \min\{A_\theta, B_\theta, C_\theta\} = \min\{\hat{A}_\theta, \hat{B}_\theta, \hat{C}_\theta\}$. $\square$



So the semicomputability of the triangle can be decomposed in terms of the properties of the vertices treated separately, which leads us to investigate the properties of a single point.

3 Semicomputable point

The following is a generalization of left-c.e. and right-c.e. reals to points of the plane.

Definition 3.1. A point A is ***semicomputable*** if there exist $\theta, \theta' \in \mathbb{Q}$ such that $\theta \neq \theta'$ mod π and A_θ and $A_{\theta'}$ are left-c.e.

Note that for a point, being semicomputable does not mean that the set $\{A\}$ is semicomputable. The latter is equivalent to saying that A is computable.

The vertices of a (non-degenerate) semicomputable triangle are necessarily semicomputable. We need tools to understand the directions in which the point is left-c.e.

Proposition 2. *Let θ_1, θ_2 be computable such that $\theta_1 < \theta_2 < \theta_1 + \pi$. A point A is $[\theta_1, \theta_2]$-c.e. iff A_{θ_1} and A_{θ_2} are left-c.e.*

Proof. For $\theta \in [\theta_1, \theta_2]$, $A_\theta = \frac{\sin(\theta_2-\theta)}{\sin(\theta_2-\theta_1)} A_{\theta_1} + \frac{\sin(\theta-\theta_1)}{\sin(\theta_2-\theta_1)} A_{\theta_2} = \alpha(\theta) A_{\theta_1} + \beta(\theta) A_{\theta_2}$ where $\alpha(\theta)$ and $\beta(\theta)$ are nonnegative computable functions. $\square$

Proposition 3. *Let $I = [\alpha, \beta]$ with $\alpha < \beta$. A is I-c.e. iff A_θ is left-c.e. uniformly in $\theta \in (\alpha, \beta) \cap \mathbb{Q}$.*

Proof. The forward direction is straightforward. Let us prove the other direction. Assume that A_θ is left-c.e. uniformly in $\theta \in (\alpha, \beta) \cap \mathbb{Q}$. The function $\theta \mapsto A_\theta$ is L-Lipschitz for some L, so it is computable on the closure of $(\alpha, \beta) \cap \mathbb{Q}$, which is $[\alpha, \beta]$. Indeed, given $\theta \in [\alpha, \beta]$, take a sequence of rationals $\theta_i \in (\alpha, \beta)$ such that $|\theta - \theta_i| < 2^{-i}$, then $A_\theta = \sup_i A_{\theta_i} - L2^{-i}$. The sequence θ_i can be computed as follows: fix some rational $q \in (\alpha, \beta)$ and some k such that $\beta - q > 2^{-k}$ and $q - \alpha > 2^{-k}$, start from some rational sequence θ'_i such that $|\theta - \theta'_i| < 2^{-i}$ and define, for $i \geq k$, $\theta_i = \theta'_{i+1} + 2^{-i-1}$ if $\theta'_{i+1} \leq q$, $\theta_i = \theta'_{i+1} - 2^{-i-1}$ if $\theta'_{i+1} > q$, and $\theta_i = \theta_k$ for $i < k$. $\square$

Definition 3.2. To a semicomputable point A we associate its ***semicomputability range*** I_A, which is the union of the sets $[\alpha, \beta] \mod 2\pi$, for all $\alpha < \beta$ such that A is $[\alpha, \beta]$-c.e.

The range I_A is a connected subset of $\mathbb{R}/2\pi\mathbb{Z}$, i.e. is the set of equivalence classes of all the reals in an interval of $\mathbb{R}$. By abuse of notation we will often act as if I_A was a subset of $\mathbb{R}$. For instance if $\theta \in \mathbb{R}$ then when we write $\theta \in I_A$ we mean that the equivalence class of θ belongs to I_A. By $I_A = [\alpha, \beta]$ we mean that $I_a = [\alpha, \beta] \mod 2\pi$. By $\inf I_A$ we mean the equivalence class of $\inf I$ where $I \subseteq \mathbb{R}$ is any interval such that $I_A = I \mod 2\pi$.

The length of I_A is at most π, unless A is computable.

Proposition 4. *A is computable $\iff I_A = [0, 2\pi] \iff |I_A| > \pi$.*

Proof. We prove that if $|I_A| > \pi$ then A is computable, the other implications are obvious. Take $\theta, \theta' \in \mathbb{Q}$ such that $\theta, \theta + \pi, \theta', \theta' + \pi$ are pairwise distinct modulo 2π and all belong to I_A. One has $A_\theta = -A_{\theta+\pi}$ and $A_{\theta'} = -A_{\theta'+\pi}$ so all these numbers are computable, and the coordinates of A are linear combinations with computable coefficients of these numbers, so they are computable. $\square$

For a computable angle θ, A_θ is left-c.e. $\iff \theta \in I_A$. The uniformity in θ depends on whether the interval I_A is closed or open at each endpoint.

- If I_A is closed at an endpoint, A_θ is *uniformly* left-c.e. for θ around that endpoint,

- If I_A is open at an endpoint, A_θ is *non-uniformly* left-c.e. for θ around that endpoint,
- In particular, I_A is closed iff A is I_A-c.e. iff for $\theta \in I_A \cap \mathbb{Q}$, A_θ is left-c.e. *uniformly* in θ.

Using the last property and Definition 3.2, Theorem 1 can be reformulated as follows:

Corollary 3.1. *A filled triangle $T = ABC$ is semicomputable iff each vertex $V \in \{A, B, C\}$ is semicomputable and $J_V^T \subseteq I_V$.*

In particular, if the filled triangle ABC is semicomputable then $|I_A| + |I_B| + |I_C| \geq 2\pi$ and $I_A \cup I_B \cup I_C = [0, 2\pi]$. This condition is not sufficient, as the intervals I_V must have the right orientations.

3.1 Semicomputable points and converging sequences

The intervals I for which a point A is I-c.e. are related to the regions containing computable sequences of points converging to A. However this is not an exact correspondence.

A two-dimensional cone with endpoint at A and delimited by the semi-lines starting at A with angles α, β, $\alpha \leq \beta < \alpha + \pi$, is denoted by $C(A, \alpha, \beta)$ and can be formally defined as $\{P \in \mathbb{R}^2 : (PA)_{\alpha-\pi/2} \geq 0 \text{ and } (PA)_{\beta+\pi/2} \geq 0\}$, where $(PA)_\theta = A_\theta - P_\theta$. Observe that this definition depends on the equivalence classes of α and β modulo 2π, so strictly speaking we do not need $\alpha \leq \beta < \alpha + \pi$ but $\alpha \leq \beta + 2k\pi < \alpha + \pi$ for some $k \in \mathbb{Z}$.

Definition 3.3. If A is semicomputable then we define its ***Solovay cone*** as $C_A = C(A, \beta + \pi/2, \alpha - \pi/2)$ where $\alpha = \inf I_A$ and $\beta = \sup I_A$.

The name will be explained in Section 4.

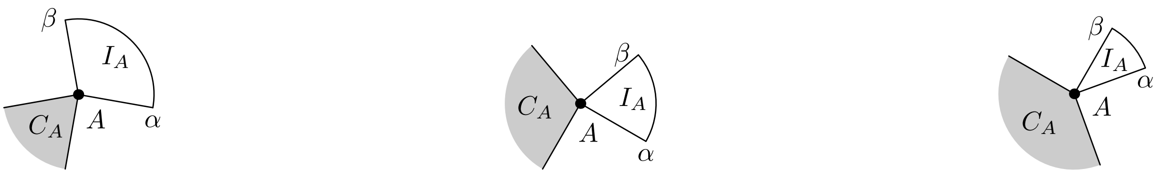


Figure 1: The semicomputability range (in white) and the Solovay cone (in gray) of the point $A = (x, y)$. (i) x, y are left-c.e., (ii) only x is left-c.e., (iii) x, y are not left-c.e.

Proposition 5. *C_A is the intersection of all the cones containing computable sequences converging to A.*

Proof. Let $\alpha < \beta \leq \alpha + \pi$. We show that if there exists a computable sequence A_i converging to A in the cone $C(A, \beta + \pi/2, \alpha - \pi/2)$ then I_A contains $[\alpha, \beta]$.

Observe that if $\alpha' = \beta + \pi/2$ and $\beta' = \alpha - \pi/2$ then $\alpha' \leq \beta' + 2\pi < \alpha' + \pi$.

Let A_i be a computable sequence converging to A in that cone. By definition of the cone one has $A_\alpha \geq (A_i)_\alpha$ and $A_\beta \geq (A_i)_\beta$ so for every rational $q \in (\alpha, \beta)$, $A_q \geq (A_i)_q$ so $A_q = \sup_i (A_i)_q$ is left-c.e. uniformly in q. $\square$

However there is not necessarily a computable sequence converging to A contained in C_A (an example will be given in Theorem 5).

Proposition 6. *Let $I = [\alpha, \beta]$ with $\alpha < \beta \leq \alpha + \pi$ and A be I-c.e.*

- *If α left-c.e. and β right-c.e. then there exists a computable sequence A_i converging to A in the cone $C(A, \beta + \pi/2, \alpha - \pi/2)$.*
- *If α is $\emptyset'$-right-c.e. and β is $\emptyset'$-left-c.e. then there exists a computable sequence A_i converging to A and converging to the cone $C(A, \beta + \pi/2, \alpha - \pi/2)$, i.e., eventually contained in $C(A, \beta + \pi/2 - \epsilon, \alpha - \pi/2 + \epsilon)$ for every $\epsilon > 0$.*

Proof. Assume α is left-c.e. and β is right-c.e. We show that the interior of the cone $C(A, \beta + \pi/2, \alpha - \pi/2)$ is an effective open set. Indeed,

$$P \in C \iff P_\beta < A_\beta \text{ and } P_\alpha < A_\alpha \iff \forall \theta \in I, P_\theta < A_\theta \iff \min_{\theta \in I}(A_\theta - P_\theta) > 0$$

which is a Σ^0_1 relation as I is effectively compact. Let $\theta \in I$ be rational and $(q_i)_{i\in\mathbb{N}}$ an increasing computable sequence of rationals converging to A_θ. For each i, one can compute some rational point A_i in C such that $(A_i)_\theta > q_i$. One easily shows that A_i converges to A. Indeed, $(A_iA)_\theta = \|A_iA\| \cos(\theta_i - \theta)$ where θ_i is the angle such that $A_iA = \|A_iA\| \, u_{\theta_i}$. One has $\theta_i \in (\beta + \pi/2, \alpha + 3\pi/2)$ and $\theta \in (\alpha, \beta)$ so $\theta_i - \theta \in (\beta + \pi/2 - \theta, \alpha + 3\pi/2 - \theta) \subseteq (\pi/2 + \epsilon, 3\pi/2 - \epsilon)$ for some $\epsilon > 0$, so $\cos(\theta_i - \theta) > \delta$ for some constant $\delta > 0$. As $(A_iA)_\theta \to 0$, $\|A_iA\| \to 0$.

We now assume that α is $\emptyset'$-right-c.e. and β is $\emptyset'$-left-c.e. One has $\alpha = \inf \alpha_i$ and $\beta = \sup \beta_i$ where $\alpha_i \searrow \alpha$ are uniformly left-c.e. and $\beta_i \nearrow \beta$ are uniformly right-c.e. For each i, we apply the first item of Proposition 6 to obtain a computable sequence $(A^i_j)_{j\in\mathbb{N}}$ in the cone $C(A, \beta_i + \pi/2, \alpha_i - \pi/2)$. Let again $\theta \in I$ be rational and $q_i \nearrow A_\theta$ be a computable sequence. We extract a sequence: for each i let $A_i \in \{A^i_j : j \in \mathbb{N}\}$ be such that $(A_i)_\theta > q_i$. □

We now identify the numbers α, β which can be endpoints of I_A for semicomputable A, when I_A is closed at these endpoints.

Theorem 2. *For a real number α, the following are equivalent:*

- *α is $\emptyset'$-left-c.e.,*
- *$\alpha = \min I_A$ for some semicomputable point A.*

Symmetrically, β is $\emptyset'$-right-c.e. iff $\beta = \max I_A$ for some semicomputable point A.

Proof. Assume that A is semicomputable and $\alpha = \min I_A$ is defined. Let $r \in I_A$ be rational. The left-c.e. function $\theta \mapsto A_\theta$ defined on $[\alpha, r]$ has a left-c.e. extension f. Let $q < \alpha$ be rational, with $r - \pi < q < \alpha < r$. For any rational $\theta \in (q, r)$, $\theta \geq \alpha$ iff A_θ is left-c.e. iff $A_\theta = f(\theta)$, so one has has $\alpha = \sup\{\theta \in \mathbb{Q} : r < \theta < q \text{ and } f(\theta) \neq A_\theta\}$. The point A

is $\emptyset'$-computable and $f(\theta)$ is $\emptyset'$-computable for $\theta \in \mathbb{Q}$, so α is $\emptyset'$-left-c.e. The argument for β is similar.

Conversely, let α be $\emptyset'$-left-c.e. We assume that $-\pi/2 < \alpha < 0$, the other cases can be obtained by applying a computable rotation. The number $r = \tan(\alpha + \pi/2) > 0$ is $\emptyset'$-left-c.e. We build $A = (b, a)$ with a, b left-c.e., and with computable sequences a_i, b_i such that $r = \lim \frac{a-a_i}{b-b_i} = \sup \frac{a-a_i}{b-b_i}$. It implies that $I_A = [\alpha, \alpha + \pi]$ or $I_A = [\alpha, \alpha + \pi)$ (if α is not $\emptyset'$-right-c.e. then the latter is the correct one).

There exists a computable sequence of rational numbers $r_i > 0$ such that $r = \liminf r_i$. For the moment, let a be any left-c.e. real number and let a_i be a computable increasing sequence of rationals converging to a. We define a computable sequence b_i as follows. Let $b_0 = 0$. Assume that $b_0, \ldots, b_i$ have been defined. Let $j \leq i$ be such that the quantity $b_j - \frac{a_j}{r_{i+1}}$ is maximal. Define $b_{i+1} = b_j + \frac{a_{i+1}-a_j}{r_{i+1}}$.

Claim. If $k \leq i$ then $b_{i+1} - b_k \geq \frac{a_{i+1}-a_k}{r_{i+1}}$.

Indeed, let j come from the definition of b_{i+1}. One has $b_{i+1} = b_j + \frac{a_{i+1}-a_j}{r_{i+1}} \geq b_k + \frac{a_{i+1}-a_k}{r_{i+1}}$.

In particular, $b_{i+1} > b_i$ so the sequence b_i is increasing. It implies that if $k \leq i$ then $\frac{a_{i+1}-a_k}{b_{i+1}-b_k} \leq r_{i+1}$. As a result, for every $k \in \mathbb{N}$ one has $\frac{a-a_k}{b-b_k} = \lim_i \frac{a_{i+1}-a_k}{b_{i+1}-b_k} \leq \liminf_i r_{i+1} = r$.

Claim. One has $\sup_k \frac{a-a_k}{b-b_k} = r$.

For each $q < r$, we show $\sup_k \frac{a-a_k}{b-b_k} \geq q$.

Let $\epsilon > 0$ be such that $q + \epsilon < r$. Let k be such that $r_i \geq q + \epsilon$ for all $i > k$ and i be such that $\frac{a_i-a_l}{b_i-b_l} - \frac{a-a_l}{b-b_l} \leq \epsilon$ for all $l \leq k$. We define by induction a decreasing sequence $(j_s)_{0 \leq s \leq S+1}$ where $j_0 = i$ and $j_{s+1} < j_s$ is the number j used in the definition of b_{j_s}, as long as $j_s > k$. Let S be such that $j_{S-1} > k$ and $j_S \leq k$.

Now we have

$$\begin{aligned}
\frac{a_i - a_{j_S}}{b_i - b_{j_S}} = \frac{a_{j_0} - a_{j_S}}{b_{j_0} - b_{j_S}} &= \frac{(a_{j_0} - a_{j_1}) + \ldots + (a_{j_{S-1}} - a_{j_S})}{(b_{j_0} - b_{j_1}) + \ldots + (b_{j_{S-1}} - b_{j_S})} \\
&\geq \min\left(\frac{a_{j_0} - a_{j_1}}{b_{j_0} - b_{j_1}}, \ldots, \frac{a_{j_{S-1}} - a_{j_S}}{b_{j_{S-1}} - b_{j_S}}\right) \\
&= \min(r_{j_0}, \ldots, r_{j_S}) \\
&\geq q + \epsilon
\end{aligned}$$

so $\frac{a-a_{j_S}}{b-b_{j_S}} \geq \frac{a_i-a_{j_S}}{b_i-b_{j_S}} - \epsilon \geq q$, so $\sup_k \frac{a-a_k}{b-b_k} \geq q$.

Claim. If r is not right-c.e. then $\limsup_k \frac{a-a_k}{b-b_k} = r$.

If $\limsup_k \frac{a-a_k}{b-b_k} < r$ then $r = \frac{a-a_k}{b-b_k}$ for some k. Let $q \in \mathbb{Q}$ be in between. The number $qb - a$ is left-c.e. so $r - q = \frac{(a-a_k)-q(b-b_k)}{b-b_k}$ is the quotient of a positive right-c.e. number by a positive left-c.e. number, so it is right-c.e.

Now assume that a is random, which implies by [BL17] that $\frac{a-a_k}{b-b_k}$ converges, hence its limit is r, i.e. $\underline{S}(a,b) = \overline{S}(a,b) = r$. □

4 Solovay derivatives

We have seen that the semicomputability of a triangle can be reduced to the semicomputability of its vertices and more precisely to their semicomputability ranges. Therefore we need tools to determine the range of a semicomputable point. This can be done using Solovay reducibility and its quantitative versions.

The coordinates of a semicomputable $A = (x, y)$ are d-c.e. and might not be either left-c.e. nor right-c.e. However, there is always a rotation with a rational angle mapping A to a semicomputable point $A' = (x', y')$ whose range $I_{A'}$ contains 0 i.e. such that x' is left-c.e. If $|I_A| > \pi/2$ then one can even take $I_{A'}$ containing 0 and $\pi/2$, i.e. one can take both x' and y' left-c.e. Hence in the study of semicomputable points one can restrict for simplicity to points (x, y) where x is left-c.e.

We first recall Solovay's notion of reduction between left-c.e. real numbers. We then define its quantitative version and study it. It has been independently introduced and studied in [BL17, Mil17], but the overlap is small.

4.1 Solovay derivatives

More on Solovay reducibility can be found in [Nie09, DH10]. It was originally defined for left-c.e. reals and has been extended to arbitrary reals in [ZR04, RZ05].

Let $b_i \nearrow b$ denote that the sequence b_i is increasingly converging to b.

Definition 4.1. Let b be left-c.e. We say that a is ***Solovay reducible*** to b if there exists a constant q and computable sequences $a_i \to a, b_i \nearrow b$ such that $|a - a_i| \leq q(b - b_i)$ for all i.

It is denoted by $a \leq_{\mathrm{S}} b$. Equivalently, $a \leq_{\mathrm{S}} b$ if there exists $q \in \mathbb{Q}$ such that $qb - a$ is left-c.e. and $-qb - a$ is right-c.e., which implies that a is d-c.e. We are interested in the optimal constants q and r such that $qb - a$ is left-c.e. and $rb - a$ is right-c.e.

Let b be left-c.e. If q is rational and $qb-a$ is left-c.e. then for every rational $q' > q$, $q'b-a$ is left-c.e. as well. In other words, the set $\{q \in \mathbb{Q} : qb - a \text{ is left-c.e.}\}$ is closed upwards. Similarly, the set $\{q \in \mathbb{Q} : qb-a \text{ is right-c.e.}\}$ is closed downwards. The following quantities have also been defined in [BL17].

Definition 4.2. Let b be left-c.e. We define the ***upper and lower Solovay derivatives*** of a w.r.t. b as, respectively,

$$\overline{S}(a, b) = \inf\{q \in \mathbb{Q} : qb - a \text{ is left-c.e.}\},$$
$$\underline{S}(a, b) = \sup\{q \in \mathbb{Q} : qb - a \text{ is right-c.e.}\}.$$

The use of the word *derivative* will be justified in the sequel. By definition, $a \leq_{\mathrm{S}} b \iff \overline{S}(a, b) < +\infty$ and $\underline{S}(b, a) > -\infty$. When $\underline{S}(a, b) = \overline{S}(a, b)$, we denote this value by $S(a, b)$. For instance it was proved in [BL17] and generalized in [Mil17] that when b is Solovay complete $\underline{S}(a, b) = \overline{S}(a, b)$.

4.2 Basic properties

Here we investigate the possible values of $\underline{S}(a,b)$ and $\overline{S}(a,b)$ and their relationship. When a and b are both computable, $\underline{S}(a,b) = +\infty$ and $\overline{S}(a,b) = -\infty$.

Proposition 7. *Let b be left-c.e. The following conditions are equivalent:*

1. $\overline{S}(a,b) < \underline{S}(a,b)$,
2. $\overline{S}(a,b) = -\infty$ *and* $\underline{S}(a,b) = +\infty$,
3. a, b *are computable.*

Proof. $3 \Rightarrow 2 \Rightarrow 1$ is direct. We prove $1 \Rightarrow 3$. If $\overline{S}(a,b) < \underline{S}(a,b)$ then for rationals $q < r$ in between, $qb - a$ is left-c.e. and $rb - a$ is right-c.e. which implies, by performing linear combinations, that a and b are computable. □

We consider this case as degenerate. When a, b are not both computable, one has $\underline{S}(a,b) \leq \overline{S}(a,b)$. The possible values of $(\underline{S}(a,b), \overline{S}(a,b))$ are:

	b computable	b left-c.e. not computable
a computable	$(+\infty, -\infty)$	$\underline{S}(a,b) = \overline{S}(a,b) = 0$
a left-c.e. not computable	$(+\infty, +\infty)$	$0 \leq \underline{S}(a,b) \leq \overline{S}(a,b)$
a right-c.e. not computable	$(-\infty, -\infty)$	$\underline{S}(a,b) \leq \overline{S}(a,b) \leq 0$
a d-c.e. not left/right-c.e.	$(-\infty, +\infty)$	$\underline{S}(a,b) \leq 0 \leq \overline{S}(a,b)$

The name "Solovay derivative" is partly justified by the next property which relates the quantities $\underline{S}(a,b)$ and $\overline{S}(a,b)$ to the difference quotient when approximating a and b computably. We will see later a strong connexion with the usual notion of derivative.

Proposition 8. *Let a, b be d-c.e. and left-c.e. respectively, not both computable. If $a_i \to a$ and $b_i \nearrow b$ are computable sequences then*

$$\liminf \frac{a - a_i}{b - b_i} \leq \underline{S}(a,b) \leq \overline{S}(a,b) \leq \limsup \frac{a - a_i}{b - b_i}.$$

Proof. If $\limsup \frac{a-a_i}{b-b_i} < q$ then $a - a_i < q(b - b_i)$ for sufficiently large i, so $\overline{S}(a,b) \leq q$. Similarly, if $\liminf \frac{a-a_i}{b-b_i} > q$ then $a - a_i > q(b - b_i)$ for sufficiently large i, so $\underline{S}(a,b) \geq q$. □

In particular, if there are computable sequences $a_i \to a$ and $b_i \nearrow b$ such that $\frac{a-a_i}{b-b_i}$ has a limit s, then $\underline{S}(a,b) = \overline{S}(a,b) = s$.

Question 1. Are there always computable sequences $a_i \to a$ and $b_i \nearrow b$ such that

$$\liminf \frac{a - a_i}{b - b_i} = \underline{S}(a,b) \leq \overline{S}(a,b) = \limsup \frac{a - a_i}{b - b_i}?$$

4.3 Calculation of the Solovay derivatives

We give formulas to derive the values of $\underline{S}(a,b)$ and $\overline{S}(a,b)$ in several situations.

Proposition 9 (Properties)**.** *1. (Reflexivity) $\underline{S}(b,b) = \overline{S}(b,b) = 1$ if b is left-c.e. not computable.*

2. When both a and b are left-c.e., one has $\underline{S}(a,b) = 1/\overline{S}(b,a)$.

3. (Transitivity) For all d-c.e. real a and left-c.e. reals b,c such that $a \leq_S b \leq_S c$,

- *If $\overline{S}(a,b) \geq 0$ then $\overline{S}(a,c) \leq \overline{S}(a,b)\overline{S}(b,c)$, otherwise $\overline{S}(a,c) \leq \overline{S}(a,b)\underline{S}(b,c)$.*
- *If $\underline{S}(a,b) \geq 0$ then $\underline{S}(a,c) \geq \underline{S}(a,b)\underline{S}(b,c)$, otherwise $\underline{S}(a,c) \geq \underline{S}(a,b)\overline{S}(b,c)$.*

4. In some cases we can also derive equalities. For all d-c.e. real a and left-c.e. reals b,c such that $a \leq_S b \leq_S c$ and $\underline{S}(a,b) = \overline{S}(a,b) =: S(a,b)$,

- *If $S(a,b) \geq 0$ then $\overline{S}(a,c) = S(a,b)\overline{S}(b,c)$ and $\underline{S}(a,c) = S(a,b)\underline{S}(b,c)$.*
- *If $S(a,b) \leq 0$ then $\overline{S}(a,c) = S(a,b)\underline{S}(b,c)$ and $\underline{S}(a,c) = S(a,b)\overline{S}(b,c)$.*

Proof. 1. Apply Proposition 8 to $b = a$ and $b_i = a_i$.

2. Observe that for a positive rational q, $qa - b$ is left-c.e. if and only if $b/q - a$ is right-c.e.

3. If $q > \overline{S}(a,b) \geq 0$ and $r > \overline{S}(b,c)$ then $qrc - a = q(rc-b) + qb - a$ is left-c.e. If $0 > q > \overline{S}(a,b)$ and $r < \underline{S}(b,c)$ then $qrc - a = q(rc-b) + qb - a$ is left-c.e. If $0 < q < \underline{S}(a,b)$ and $r < \underline{S}(b,c)$ then $qrc - a = q(rc-b) + rb - a$ is right-c.e. If $q < \underline{S}(a,b) \leq 0$ and $r > \overline{S}(b,c)$ then $qrc - a = q(rc-b) + rb - a$ is right-c.e.

4. We prove the case $S(a,b) \geq 0$, the other one is similar. By the preceding inequalities one has $\overline{S}(a,c) \leq S(a,b)\overline{S}(b,c)$ and $\underline{S}(a,c) \geq S(a,b)\underline{S}(b,c)$. If $S(a,b) = 0$ then necessarily $\underline{S}(a,c) = \overline{S}(a,c) = 0$ and we get the equalities. If $S(a,b) > 0$ then a is left-c.e. and $b \leq_S a$ so the preceding inequalities applied to b,a,c give $\overline{S}(b,c) \leq S(b,a)\overline{S}(a,c)$ and $\underline{S}(b,c) \geq S(b,a)\underline{S}(a,c)$ so $\overline{S}(a,c) \geq S(a,b)\overline{S}(b,c)$ and $\underline{S}(a,c) \leq S(a,b)\underline{S}(b,c)$ as $S(b,a) = 1/S(a,b)$ is positive. ☐

In [DHN02] it is proved that $a \leq_S b$ iff there exists a computable sequence $b_n \geq 0$ such that $b = \sum_n b_n$ and a computable bounded sequence $\epsilon_n \geq 0$ such that $a = \sum_n \epsilon_n b_n$. One easily gets a quantitative version.

Proposition 10. *Let $b = \sum_n b_n$ where the sequence $b_n \geq 0$ is computable. Let $\epsilon_n \geq 0$ be a bounded computable sequence and $a = \sum_n \epsilon_n b_n$. One has*

$$\liminf \epsilon_n \leq \underline{S}(a,b) \leq \overline{S}(a,b) \leq \limsup \epsilon_n.$$

In particular if ϵ_n converges to ϵ then $\underline{S}(a,b) = \overline{S}(a,b) = \epsilon$.

Proof. Let $a_i = \sum_{j \leq i} \epsilon_j b_j$ and $B_i = \sum_{j \leq i} b - j$. Let $u_i = a - a_i = \sum_{n > i} \epsilon_i b_i$ and $v_i = b - B_i = \sum_{n > i} b_i$. If $r < \liminf \epsilon_n \leq \limsup \epsilon_n < s$ then u_i is eventually between rv_i and sv_i, so $r \leq \underline{S}(a,b) \leq \overline{S}(a,b) \leq s$. ☐

4.3.1 Differentiation

The name Solovay derivative is justified by the following result, also obtained in [Mil17] when b is Solovay complete.

Proposition 11. *Let b be a non-computable left-c.e. real. If f is computable and differentiable at b then*

$$\underline{S}(f(b), b) = \overline{S}(f(b), b) = f'(b).$$

Proof. It is a direct application of Proposition 8. Let $b_i \nearrow b$ be a computable sequence. The sequence $f(b_i)$ is computable and $\lim(f(b) - f(b_i))/(b - b_i) = f'(b)$. □

It also implies that if f, g are computable, differentiable and $f'(b)$ and $g'(b)$ are positive then

$$S(f(a), g(b)) = \frac{f'(a)}{g'(b)} S(a, b).$$

This is proved by applying two times Proposition 9, item 4.

Example 1. For instance, if b is not computable then $S(2b, b) = 2$, $S(b^2, b) = 2b$, $\overline{S}(\log(a), \log(b)) = b\overline{S}(a, b)/a$ and $\underline{S}(\log(a), \log(b)) = a\underline{S}(a, b)/a$.

In particular, for $a, b > 0$, $\overline{S}(a, b) = \frac{a}{b} \inf\{q \in \mathbb{Q} : \frac{b^q}{a} \text{ is left-c.e.}\}$.

Proposition 11 can be extended to bivariate differentiable functions.

Theorem 3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be totally differentiable and computable. Let y be left-c.e. and assume that x, y are not both computable.*

- *If $\frac{\partial f}{\partial x}(x, y) > 0$, then* $\begin{cases} \underline{S}(f(x, y), y) = \underline{S}(x, y)\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y), \\ \overline{S}(f(x, y), y) = \overline{S}(x, y)\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y). \end{cases}$

- *If $\frac{\partial f}{\partial x}(x, y) < 0$, then* $\begin{cases} \underline{S}(f(x, y), y) = \overline{S}(x, y)\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y), \\ \overline{S}(f(x, y), y) = \underline{S}(x, y)\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y). \end{cases}$

- *If $\frac{\partial f}{\partial x}(x, y) = 0$ and $x \leq_S y$ then $\underline{S}(f(x, y), y) = \overline{S}(f(x, y), y) = \frac{\partial f}{\partial y}(x, y)$.*

In particular, if $\frac{\partial f}{\partial x}(x, y) \neq 0$ then $f(x, y) \leq_S y$ implies $x \leq_S y$.

In order to prove the Theorem, we need the following lemma, stating that sequences witnessing the values of $\underline{S}(a, b)$ and $\overline{S}(a, b)$ can be extracted from given sequences.

Lemma 2. *Assume that b is not computable and $a \leq_S b$, and let $q, r \in \mathbb{Q}$ satisfy $q < \underline{S}(a, b) \leq \overline{S}(a, b) < r$.*

Let $a_i \to a$ and $b_i \nearrow b$ be computable sequences. There exists a computable increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $q(b - b_{\varphi(i)}) < a - a_{\varphi(i)} < r(b - b_{\varphi(i)})$ for all $i \in \mathbb{N}$.

Proof. One has $\liminf \frac{a-a_i}{b-b_i} \leq \underline{S}(a,b) \leq \overline{S}(a,b) < r$ so the set $E = \{i \in \mathbb{N} : a - a_i < r(b-b_i)\}$ is infinite. As $rb - a$ is left-c.e. E is a c.e. set so there exists a computable increasing function $\psi : \mathbb{N} \to \mathbb{N}$ whose range is contained in E. One has $q < \underline{S}(a,b) \leq \overline{S}(a,b) \leq \limsup \frac{a-a_{\psi(i)}}{b-b_{\psi(i)}}$ so the set $F := \{i \in \mathbb{N} : q(b-b_{\psi(i)}) < a - a_i\}$ is infinite. As $qb - a$ is right-c.e. F is a c.e. set so there exists a computable increasing function ψ' whose range is contained in F. Take $\varphi = \psi \circ \psi'$. $\square$

Now we can prove the Theorem.

Proof of Theorem 3. If y is computable then applying Proposition 11 to $g(x) = f(x,y)$ gives the result.

Now assume that y is not computable. Let A be the supremum over computable sequences $x_i \to x$ and $y_i \nearrow y$ of $\inf_i \frac{f(x,y)-f(x_i,y_i)}{y-y_i}$, and B the infimum over the same sequences of $\sup_i \frac{f(x,y)-f(x_i,y_i)}{y-y_i}$. One has $A \leq \underline{S}(f(x,y),y) \leq \overline{S}(f(x,y),y) \leq B$. It might happen that the outermost inequalities are strict, because one may need to consider computable sequences $z_i \to f(x,y)$ and $y_i \nearrow y$ that are not of the form $z_i = f(x_i,y_i)$ for some computable sequence x_i. But Lemma 2 rules out this possibility, because z_i can be extracted from $f(x_i,y_i)$, so $\underline{S}(f(x,y),y) = A$ and $\overline{S}(f(x,y),y) = B$.

Let us now calculate A and B. Let $x_i \to x$ and $y_i \nearrow y$ be computable sequences. One has $f(x_i,y_i) \to f(x,y)$ and

$$\begin{aligned}\frac{f(x,y)-f(x_i,y_i)}{y-y_i} &= \frac{Df_{x,y}(x-x_i,y-y_i)}{y-y_i} + \frac{o(|x-x_i|+|y-y_i|)}{y-y_i} \\ &= \frac{x-x_i}{y-y_i}\left(\frac{\partial f}{\partial x}(x,y)+o(1)\right) + \frac{\partial f}{\partial y}(x,y)+o(1).\end{aligned}$$

- If $\frac{\partial f}{\partial x}(x,y) \neq 0$ then the first term is equivalent to $\frac{x-x_i}{y-y_i}\frac{\partial f}{\partial x}(x,y)$. One easily derives $A = \underline{S}(x,y)\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)$ and $B = \overline{S}(x,y)\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y))$ or the converse, depending on the sign of $\frac{\partial f}{\partial x}(x,y)$.
- If $\frac{\partial f}{\partial x}(x,y) = 0$ and $x \leq_{\mathrm{S}} y$ then we can choose sequences such that $|\frac{x-x_i}{y-y_i}|$ is bounded, hence the first term converges to 0, so $A = B = \frac{\partial f}{\partial y}(x,y)$. $\square$

Remark 4.1. In the remaining case where $\frac{\partial f}{\partial x}(x,y) = 0$ and $x \not\leq_{\mathrm{S}} y$, the values of $\underline{S}(f(x,y),y)$ and $\overline{S}(f(x,y),y)$ cannot be expressed in terms of $\underline{S}(x,y)$, $\overline{S}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ only.

Example 2.
- One has $\overline{S}(a+b,a) = 1 + \overline{S}(b,a)$, $\underline{S}(a+b,a) = 1 + \underline{S}(b,a)$.
- One has $\overline{S}(ab,a) = b + a\overline{S}(ab,a)$ and $\underline{S}(ab,a) = b + a\underline{S}(ab,a)$.

4.4 Back to semicomputable points

We now relate the semicomputability range of a point $A = (x, y)$ to the quantities $\overline{S}(y, x)$ and $\underline{S}(y, x)$, when x is left-c.e.

Proposition 12. *Let $A = (x, y)$ be semicomputable but not computable with x left-c.e. and let $\alpha = \inf I_A$ and $\beta = \sup I_A$. One has $-\pi \le \alpha \le 0 \le \beta \le \pi$ and*

$$\alpha = \arctan(\overline{S}(y, x)) - \pi/2 \qquad \overline{S}(y, x) = \tan(\alpha + \pi/2)$$
$$\beta = \arctan(\underline{S}(y, x)) + \pi/2 \qquad \underline{S}(y, x) = \tan(\beta - \pi/2).$$

The functions tan and arctan are understood as functions between $[-\pi/2, \pi/2]$ and $[-\infty, +\infty]$.

Proof. As $0 \in I_A$ and $|I_A| \le \pi$, $-\pi \le \alpha \le 0 \le \beta \le \pi$. We recall that $A_\theta = b \cos\theta + a \sin\theta$.

For $0 \le \theta \le \pi$ rational, $\theta \le \beta \iff A_\theta = -x \sin(\theta - \pi/2) + y \cos(\theta - \pi/2)$ is left-c.e. $\iff x \tan(\theta - \pi/2) - y$ is right-c.e. $\iff \tan(\theta - \pi/2) \le \underline{S}(y, x)$. As a result, $\underline{S}(y, x) = \tan(\beta - \pi/2)$.

For $-\pi \le \theta \le 0$ rational, $\theta \ge \alpha \iff A_\theta = x \sin(\theta + \pi/2) - y \cos(\theta + \pi/2)$ is left-c.e. $\iff x \tan(\theta + \pi/2) - y$ is left-c.e. $\iff \tan(\theta + \pi/2) \ge \overline{S}(y, x)$. As a result, $\overline{S}(y, x) = \tan(\alpha + \pi/2)$. □

Therefore the slopes of the Solovay cone C_A are $\underline{S}(y, x)$ and $\overline{S}(y, x)$, which explains the name of the cone.

We now give examples of semicomputable points and calculate their ranges. Let $A = (x, y)$ with x left-c.e.

- If x, y are Solovay incomparable left-c.e. reals then $\underline{S}(y, x) = 0$ and $\overline{S}(y, x) = +\infty$. The point $A = (x, y)$ is semicomputable with $I_A = [0, \pi/2]$.
- Let $x = \Omega$ be some Solovay complete left-c.e. real.
  - If y is left-c.e. incomplete then $S(y, \Omega) = 0$ and $I_A = (-\pi/2, \pi/2]$,
  - If y is right-c.e. incomplete then $S(y, \Omega) = 0$ and $I_A = [-\pi/2, \pi/2)$,
  - If y is d-c.e., neither left-c.e. nor right-c.e. then $S(y, \Omega) = 0$ and $I_A = (-\pi/2, \pi/2)$,
  - If $y = \Omega$ then $S(y, \Omega) = 1$ and $I_A = [-\pi/4, \pi/4]$.
- Let $y = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is such that f' is computable and monotonic. One has $S(y, x) = f'(x)$ and $I_A = [\arctan(f'(x)) - \pi/2, \arctan(f'(x)) + \pi/2]$.

It is proved in [Mil17] that every $\emptyset'$-computable (or Δ^0_2) number can be obtained as $S(b, \Omega)$ for some d-c.e. b and Solovay complete Ω. The proof of Theorem 2 shows that every $\emptyset'$-left-c.e. can be obtained this way, and symmetrically every $\emptyset'$-right-c.e. hence every $\emptyset'$-d-c.e. It gives a partial answer to Question 2.7 in [Mil17].

Question 2. Can every Δ^0_3 real be obtained as $S(b, \Omega)$ for some left-c.e./d-c.e. real b?

5 Generic triangles

All the classical parameters (like the angles or the coordinates of the centroid) of a semicomputable triangle are d-c.e. numbers, because the function mapping a triangle to a parameter is computable and Lipschitz. Some of them, like the sides lengths, the area or the perimeter, are always right-c.e.

In this section we show that these upper bounds on the effectiveness of the parameters are optimal. To do this we prove the existence of semicomputable triangles with prescribed properties. However instead of building them explicitly we use the existence of semicomputable triangles that are *generic* in some sense, and then investigate the properties of such triangles. We first give the minimal material needed, taken from [Hoy17].

Definition 5.1. Let X be an effective Polish space and $A \subseteq X$. A point $x \in A$ is ***generic inside*** A if for every effective open set $U \subseteq X$, either $x \in U$ or there exists a neighborhood B of x such that $B \cap U \cap A = \emptyset$.

Example 3.

- Taking $A = X$, being generic inside X amounts to being 1-generic,
- Every x is obviously generic inside $\{x\}$,
- In the space of real numbers with the Euclidean topology, a real number $x \in (0,1)$ is ***right-generic*** if x is generic inside $[x, 1]$,
- The space of filled triangles is a subspace of the space of non-empty compact subsets of $\mathbb{R}^2$ with the Hausdorff metric and is an effective Polish space. A triangle T is ***inner-generic*** if it is generic inside $S(T) := \{T' \in \mathcal{T}, T' \subseteq T\}$. In other words, for every effective open set $\mathcal{U} \subseteq \mathcal{T}$, if T contains arbitrarily close (in the Hausdorff metric) triangles $T' \in \mathcal{U}$, then $T \in \mathcal{U}$.

The latter two examples are particular instances of the following general situation.

If τ' is a weaker topology on X then we define $S(x)$ as the closure of x in the topology τ', which is the intersection of the τ'-open sets containing x. Equivalently, $S(x) = \{y \in X : x \leq_{\tau'} y\}$ where $\leq_{\tau'}$ is the specialization pre-order defined by $x \leq y$ iff every τ'-neighborhood of x contains y.

Theorem 4 (Theorem 4.1.1 in [Hoy17])**.** *Let (X, τ) be an effective Polish space and τ' an effectively weaker topology, such that emptiness of finite intersections of basic open sets in τ, τ' is decidable. There exists a point x that is computable in (X, τ') and generic inside $S(x)$.*

For instance, $\mathbb{R}$ with the Euclidean topology is effective Polish, the topology τ' generated by the semi-lines $(q, +\infty)$ is effectively weaker, and its specialization pre-order is the natural ordering $\leq$ on $\mathbb{R}$. Theorem 4 implies the existence of right-generic left-c.e. reals.

In the effective Polish space $\mathcal{T}$ of filled triangles, we take the topology τ' generated by the following open sets: given a finite union U of open metric balls in $\mathbb{R}^2$, the set of triangles contained in U is a basic open set of the topology τ'. The specialization ordering is the reversed inclusion. Theorem 4 implies the existence of inner-generic semicomputable triangles.

To investigate the properties of those triangles, we will use the following result.

Proposition 13. *Let $A \subseteq X$ and $f : A \to Y$ be computable such that $f : A \to f(A)$ is open. If $x \in A$ is generic inside A then $f(x)$ is generic inside $f(A)$.*

Proof. Let $x \in A$ be generic inside A. Let $U \subseteq Y$ be effectively open. If $f(x) \notin U$ then $x \notin f^{-1}(U)$ which is the intersection of an effectively open with A, so there exists a neighborhood B of x such that $B \cap f^{-1}(U) \cap A = \emptyset$, so $f(B \cap A) \cap U = f(B \cap f^{-1}(U) \cap A) = \emptyset$. As f is open at x, $f(B \cap A) = V \cap f(A)$ for some neighborhood V of $f(x)$, so $V \cap U \cap f(A) = \emptyset$ which shows that $f(x)$ is generic inside $f(A)$. $\square$

Actually one only needs that $f : A \to f(A)$ is open at x, i.e. that every $y \in f(A)$ sufficiently close $f(x)$ has a pre-image in A close to x.

Now we have the tools to prove the main result of this section.

Theorem 5. *Let $T = ABC$ be an inner-generic semicomputable triangle.*

- *Each vertex A, B, C is generic inside T,*
- *For each vertex $V \in \{A, B, C\}$, $I_V = J_V^T$,*
- *For each vertex V, there is no computable sequence converging to V in the cone C_V,*
- *The slopes of the sides of T are 1-generic d-c.e. reals,*
- *The angles of T are 1-generic d-c.e. reals,*
- *A is not computable relative to the pair (B, C) (idem for B and C),*
- *The area of T is a left-generic right-c.e. real,*
- *The centroid of T is a 1-generic point with d-c.e. coordinates.*

This list could of course be extended *ad nauseam.*

Proof. Technically, a triangle is a subset of $\mathbb{R}^2$ and not a triple of points (A, B, C), so it is not possible to distinguish A and obtain it as a continuous function of T defined on all $\mathcal{T}$. However it is possible if T is taken in some restricted open set (for instance the triangles having a vertex strictly to the left of the others). This is implicit in the proofs, so that the function mapping T to the triple (A, B, C) is computable and defined on a neighborhood of T, which is sufficient to investigate genericity.

All the results are applications of Proposition 13.

- Let $S(T)$ be the set of triangles contained in T. The function mapping T to A is computable and the image of $S(T)$ is T. Moreover, every $A' \in T$ close to A comes from a triangle $T' = (A'BC) \subseteq T$ close to T, so the restriction of the function to $S(T)$ is open at T.

- Let θ be a rational outside J_A^T and α be a left-c.e. real. We show that $A_\theta \neq \alpha$ which implies the result (indeed, α is arbitrary so A_θ is not left-c.e., i.e. $\theta \notin I_A$, and θ is arbitrary outside J_A^T, so $I_A \subseteq J_A^T$). Define the effective open set $U = \{P \in \mathbb{R}^2 : P_\theta < \alpha\}$. We know from the first item that A is generic inside T. Hence if $A_\theta = \alpha$ then $A \notin U$ so there exists a neighborhood V of A such that $V \cap U \cap T = \emptyset$. As $\theta \notin J_A^T$, $A_\theta > B_\theta$ or $A_\theta > C_\theta$. Assume that $A_\theta > B_\theta$, the other case is similar. Let A' be a point close to A, i.e. in V, lying on the edge AB. One has $A'_\theta < A_\theta = \alpha$ so $A' \in U$, and $A' \in T$, contradicting $V \cap U \cap T = \emptyset$. Hence $A_\theta \neq \alpha$.

- For a triangle $T = (ABC)$, the outer cone at A is the cone delimited by the semi-lines starting at A in the directions BA and CA. Let P_i be a computable sequence of points. Let $\mathcal{U} = \{T = (ABC) : \exists i, P_i \text{ is outside the outer cone at } A\}$. $\mathcal{U}$ is an effective open set. Let $T = (ABC)$ be inner-generic and assume that P_i converges to A. First, as $I_A = J_A^T$ one has $C_A = C_A^T$ so P_i is not contained in a cone smaller than C_A^T. One can find $T' \subseteq T$ close to T such that $T' \in \mathcal{U}$. Indeed, T' can be obtained by moving B or C slightly inwards the triangle. One has $C_A^{T'}$ smaller than C_A^T, so P_i is not contained in $C_A^{T'}$.

- The function mapping $T = (ABC)$ so the slope s_{AB} of AB is computable and if $T = (ABC)$ then the image of $S(T)$ contains an open interval around s_{AB} (the function is implicitly restricted to an open set of triangles, so we the image may not be $\mathbb{R}$). Moreover, every s' close to s_{AB} is the slope of a triangle $T' \subseteq T$ close to T, obtained by moving either A or B towards C (one of them makes the slope decrease, the other makes the slope increase). As a result, the restriction of the slope function to $S(T)$ is open at T.

- The function mapping $T = (ABC)$ to the angle θ_A at A is computable and the image of $S(T)$ contains an open interval around θ_A. Every θ' close to θ_A is the angle of a triangle $T' \subseteq T$ close to T, obtained by moving A inwards T to make the angle increase, or by moving B inwards T to make the angle decrease. Hence the restriction of the angle map to $S(T)$ is open at T.

- Let M be a Turing machine. Let $\mathcal{U} = \{T = (ABC) : M(B,C) \perp A\}$ be the set of triangles such that the machine M on input (B,C) eventually halts and produces an output that is incompatible with A. If $M(B,C) = A$ then $T \notin \mathcal{U}$ so there exists a neighborhood $\mathcal{V}$ of T such that $\mathcal{V} \cap \mathcal{U} \cap S(T) = \emptyset$. But if $A' \in T$ is close to A then $T' = (A'BC) \in \mathcal{V}$, $T' \subseteq T$ and $M(B,C) = A \perp A'$, so $T' \in \mathcal{U}$, contradicting $\mathcal{V} \cap \mathcal{U} \cap S(T) = \emptyset$.

- The function mapping T to its area a_T is computable, the image of $S(T)$ is $[0, a_T]$. If $a' < a_T$ is close to a_T then a' can be obtained as the area of a triangle $T' \subseteq T$ close to T, so the area function restricted to $S(T)$ is open at T.

- The function mapping T to its centroid c_T is computable, the image of $S(T)$ is T. One easily see that if c' is close to c_T then c' can be obtained as the centroid of a triangle $T' \subseteq T$ close to T, so the centroid map, restricted to $S(T)$, is open at T.

□

6 Parametrizations

In the one-dimensional case, there is a simple parametrization of the semicomputable compact convex subsets of $\mathbb{R}$: they are exactly the closed intervals $[a, b]$ where a is left-c.e. and b is right-c.e. Apart from the fact that $a \leq b$, the two parameters a and b are independent. In this section we investigate the possibility of having a similar parametrization for classes of semicomputable compact convex subsets of $\mathbb{R}^2$, for instance the filled triangles. We show that for some definition of parametrization, no finite parametrization is possible.

A ***numbered set*** is a pair $\mathcal{S} = (S, \nu)$ where S is a countable set and $\nu_S : \mathrm{dom}(\nu) \subseteq \mathbb{N} \to S$ is surjective. If $\mathcal{S} = (S, \nu)$ is a numbered set then each $T \subseteq S$ has a canonical numbering, given by the restriction of ν to $\nu^{-1}(T)$. A ***morphism*** from $\mathcal{S} = (S, \nu)$ to $\mathcal{S}' = (S', \nu')$ is a function $\phi : S \to S'$ such that there exists a computable function $\varphi : \mathrm{dom}(\nu) \to \mathrm{dom}(\nu')$ such that $\nu' \circ \varphi = \phi \circ \nu$.

Definition 6.1. Let $\mathcal{S} = (S, \nu_S)$ and $\mathcal{P} = (P, \nu_P)$ be numbered sets. A ***$\mathcal{P}$-parametrization*** of $\mathcal{S}$ is an isomorphism between $\mathcal{S}$ and a subset of $\mathcal{P}$.

We are interested in the case where $\mathcal{S}$ is the class of semicomputable triangles and $\mathcal{P} = \mathbb{R}^d_{\mathrm{lce}}$ is the class of vectors of d left-c.e. numbers, both with their canonical numberings. Proposition 1 implies the existence of a $\mathbb{R}^{\mathbb{N}}_{\mathrm{lce}}$-parametrization of the semicomputable filled triangles, i.e. that each such triangle T can be represented by a sequence of uniformly left-c.e. real numbers T_{θ_i}, where $(\theta_i)_{i\in\mathbb{N}}$ is a canonical enumeration of the rational numbers. We prove that no finite parametrization exists.

Theorem 6. *For each $d \in \mathbb{N}$, there is no $\mathbb{R}^d_{\mathrm{lce}}$-parametrization of the semicomputable filled triangles.*

Proof. We first observe that an isomorphism between $\mathcal{T}$ and a subset of $\mathbb{R}^d_{\mathrm{lce}}$ would be order-preserving in both directions, where $\mathcal{T}$ is endowed with the reverse inclusion $\supseteq$ and $\mathbb{R}^d_{\mathrm{lce}}$ with the component-wise natural ordering $\leq$. This is a consequence of the generalization of the Myhill-Shepherdson theorem to effective continuous directed complete partial orders (dcpo's) [WD80]. It would imply that $(\mathcal{T}, \supseteq)$ embeds in $(\mathbb{R}^d, \leq)$, which we show is not possible. For this we use the order-theoretic notion of dimension and show that $(\mathcal{T}, \supseteq)$ is infinite-dimensional, while $(\mathbb{R}^d, \leq)$ is d-dimensional.

All the details about the dimension of partially ordered sets can be found in [Sch03], we only give the key notions. A partially ordered set (poset) $(P, \leq)$ has dimension k if there exist k linear extensions of $\leq$ whose intersection is $\leq$, and k is minimal with this property. The standard n-dimensional ordering is $S_n = \{a_1, \dots, a_n, A_1, \dots, A_n\}$ with $a_i < A_j$ if $i \neq j$. If a poset $(P, \leq_P)$ embeds into a poset $(Q, \leq_Q)$ then the dimension of $(P, \leq_P)$ is no more than the dimension of $(Q, \leq_Q)$. The poset $(\mathbb{R}^d, \leq)$ has dimension d and we show that $(\mathcal{T}, \supseteq)$ is not finite-dimensional by embedding the standard ordering S_d into $(\mathcal{T}, \supseteq)$, for each $d \in \mathbb{N}$.

For each i, $a_i \in S_d$ is mapped to a large triangle t_i and $A_i \in S_d$ is mapped to a small triangle T_i such that $t_i \supseteq T_j \iff i \neq j$. This is achieved by starting from a regular polygon with d vertices $v_1, \ldots, v_d$, taking for each i a large triangle t_i containing all the vertices except v_i, and a small triangle T_i containing v_i. We simply show a picture for $d = 5$, but it can be generalized to any $d \in \mathbb{N}$.

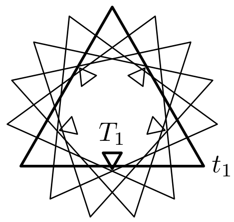


Figure 2: Embedding the standard 5-dimensional ordering in the poset of triangles. Note that T_1 is *not* contained in t_1.

One could relax the notion of parametrization in different ways:

- If one requires a morphism from a subset of $\mathbb{R}^d_{\text{lce}}$ onto $\mathcal{T}$ then there is a $\mathbb{R}^2_{\text{lce}}$-parametrization, essentially because all the elements of the anti-diagonal of $\mathbb{R}^d_{\text{lce}}$ are pairwise incomparable.
- If one requires a one-to-one morphism from $\mathcal{T}$ to $\mathbb{R}^d_{\text{lce}}$ then there is a $\mathbb{R}_{\text{lce}}$-parametrization because $\mathcal{T}$ embeds in $\mathbb{R}^{\mathbb{N}}_{\text{lce}}$ and there is a one-to-one morphism from $\mathbb{R}^{\mathbb{N}}_{\text{lce}}$ to $\mathbb{R}_{\text{lce}}$.

In both cases, the parametrizations are not satisfactory because they are not geometrically meaningful. Other variations on the definition of parametrizations should be investigated.

The argument in the proof of Theorem 6 is actually very general and can be extended to many classes of sets.

Theorem 7. *Let $\mathcal{F}$ be a class of compact semicomputable subsets of $\mathbb{R}^2$ that contains a set with non-empty interior and is closed under translations, scaling and rotations with rational parameters. There is no $\mathbb{R}^d$-parametrization of $\mathcal{F}$ for any $d \in \mathbb{N}$.*

Proof. We embed the standard d-dimensional ordering in $(\mathcal{F}, \supseteq)$.

Let $S \in \mathcal{F}$ be a set with non-empty interior. There exists a closed ball $\overline{B}(c, r)$ contained in S and intersecting the boundary ∂S of S in exactly one point. Indeed, take c_0 in the interior of S and $r_0 = d(c_0, \partial S)$. $\overline{B}(c_0, r_0)$ is contained in S and intersects ∂S in at least one point p. Let $c = (c_0 + p)/2$ and $r = r_0/2$. One easily checks that $\overline{B}(c, r)$ intersects ∂S in exactly one point.

Given $d \in \mathbb{N}$, let $(S_i)_{1 \leq i \leq d}$ be d distinct copies of S, rotated around c. The disk $\overline{B}(c, r)$ is contained in each S_i and intersects its boundary in exactly one point p_i. Therefore, for $i \neq j$, p_i belongs to the interior of S_j. For each i, let s_i be a small scaled copy of S containing p_i in its interior. As $p_i \in \partial S_i$, s_i is not contained in S_i. One can take s_i sufficiently small so that it is contained in each S_j, $j \neq i$. The family of sets S_i and s_i is an embedding of the standard d-dimensional ordering in $\mathcal{F}$. $\square$

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