Computability of invariant measures

Mathieu Hoyrup

Inria, Nancy (France)

Computability of in ergodic theory

Given a dynamical system,

- how to compute its invariant measures? the ergodic ones?
- how to compute the ergodic components of an invariant measure?
- how to compute the speed of convergence of Birkhoff averages?

Computability in ergodic theory

Given a dynamical system,

- **can** we compute its invariant measures? the ergodic ones?
- **can** we compute the ergodic components of an invariant measure?
- **can** we compute the speed of convergence of Birkhoff averages?

Ergodic decomposition

Simplex of invariant measures

Birkhoff's ergodic theorem

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map and μ a computable shift-invariant measure.

$$f^{(n)} = \frac{f + f \circ \sigma + \ldots + f \circ \sigma^{n-1}}{n} \underset{n\to\infty}{\longrightarrow} f^* \quad (L^1(\mu) \text{ and a.s.})$$

Theorem (V'yugin, 1997)

Let $f(x_0 x_1 x_2 \ldots) = x_0$. There exists a computable shift-invariant measure μ such that the speed of convergence of $f^{(n)}$ to f^* is not computable.

Birkhoff's ergodic theorem

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map and μ a computable shift-invariant measure.

$$f^{(n)} = \frac{f + f \circ \sigma + \ldots + f \circ \sigma^{n-1}}{n} \underset{n\to\infty}{\longrightarrow} f^* \quad (L^1(\mu) \text{ and a.s.})$$

Theorem (V'yugin, 1997)

Let $f(x_0 x_1 x_2 \ldots) = x_0$. There exists a computable shift-invariant measure μ such that the speed of convergence of $f^{(n)}$ to f^* is not computable.

Theorem (Avigad, Gerhardy & Towsner, 2010)

If μ is ergodic then the speed is computable.

Question

V'yugin builds a **countable** combination of ergodic measures. What about **finite** ones?

Ergodic decomposition

Let $\mu = \dfrac{\mu_1 + \mu_2}{2}$ with μ_1, μ_2 ergodic.

The speed of convergence in Birkhoff's ergodic theorem is computable

$$\Longleftrightarrow$$

the measures μ_1, μ_2 are computable.

Question

If $\mu = \dfrac{\mu_1 + \mu_2}{2}$ is computable, are μ_1, μ_2 computable?

Ergodic decomposition

Theorem (H., 2012)

There exist shift-invariant ergodic measures μ_1, μ_2 such that:

- μ_1, μ_2 *are not computable,*
- $\dfrac{\mu_1 + \mu_2}{2}$ *is computable.*

Ergodic decomposition

Theorem (H., 2012)

There exist shift-invariant ergodic measures μ_1, μ_2 *such that:*

- μ_1, μ_2 *are not computable,*
- $\dfrac{\mu_1 + \mu_2}{2}$ *is computable.*

Proof.

Game between a Player and infinitely many Opponents (the programs).

- The Player describes $\frac{\mu_1+\mu_2}{2}$,
- Each Opponent tries to guess μ_1.

Ergodic decomposition

Theorem (H., 2012)

There exist shift-invariant ergodic measures μ_1, μ_2 such that:

- μ_1, μ_2 *are not computable,*
- $\frac{\mu_1 + \mu_2}{2}$ *is computable.*

Proof.

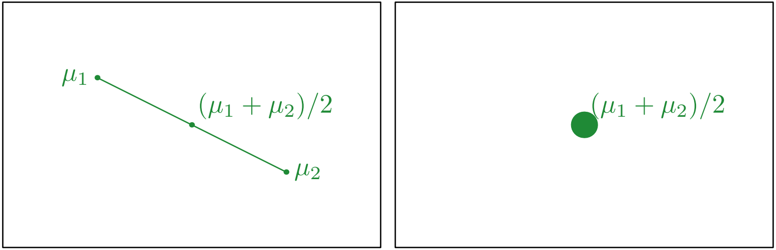
Game between a Player and infinitely many Opponents (the programs).

- The Player describes $\frac{\mu_1+\mu_2}{2}$,
- Each Opponent tries to guess μ_1.

We show that the Player has a computable winning strategy.

Strategy against one opponent

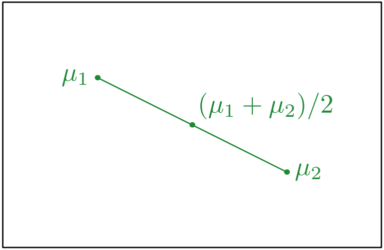
Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.

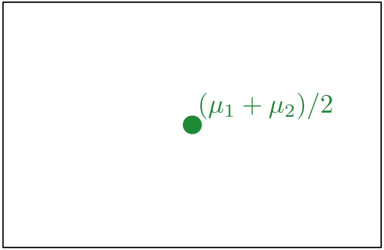


Knowledge of the Player Knowledge of the Opponent

Strategy against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.

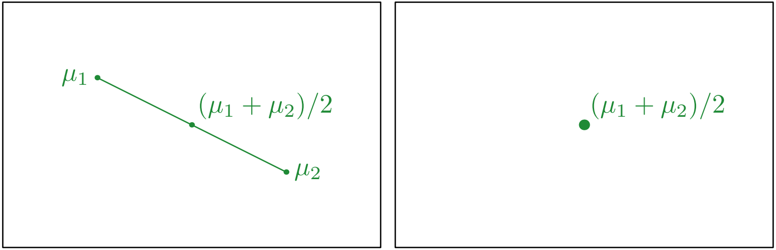




Knowledge of the Player Knowledge of the Opponent

Strategy against one opponent

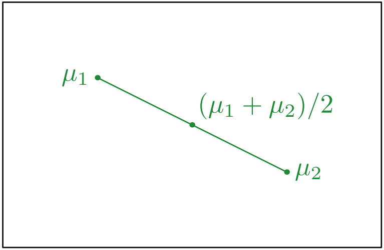
Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



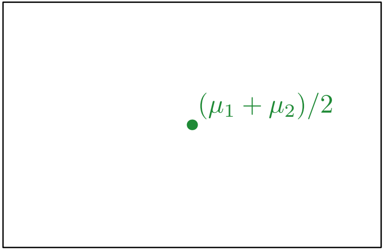
Knowledge of the Player Knowledge of the Opponent

Strategy against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



Knowledge of the Player



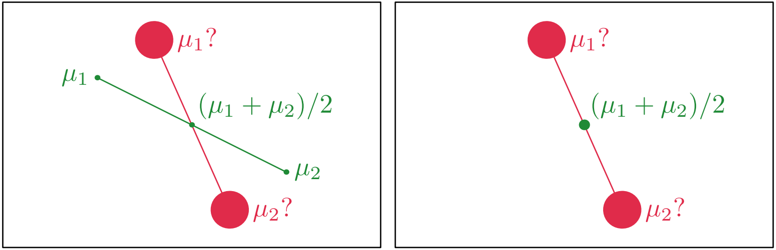
Knowledge of the Opponent

Three cases:

1. The Opponent remains silent forever: change nothing.

Strategy against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



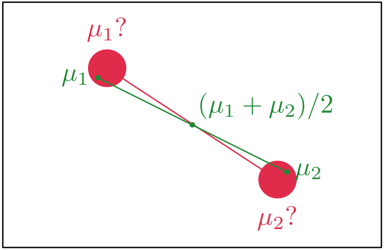
Knowledge of the Player Knowledge of the Opponent

Three cases:

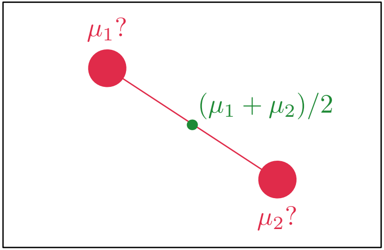
1. The Opponent remains silent forever: change nothing.
2. The Opponent eventually makes a wrong guess: change nothing.

Strategy against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



Knowledge of the Player



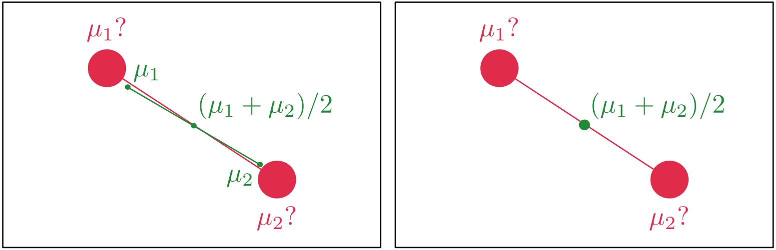
Knowledge of the Opponent

Three cases:

1. The Opponent remains silent forever: change nothing.
2. The Opponent eventually makes a wrong guess: change nothing.
3. The Opponent eventually makes a correct guess:

Strategy against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



Knowledge of the Player Knowledge of the Opponent

Three cases:

1. The Opponent remains silent forever: change nothing.
2. The Opponent eventually makes a wrong guess: change nothing.
3. The Opponent eventually makes a correct guess: move μ_1 and μ_2 much but $\mu_1 + \mu_2$ very little.

Strategy against infinitely many opponents

End of the proof.

- This strategy can be applied **everywhere** and **at every scale**,
- Hence it can be applied “in parallel” against infinitely many opponents. □

Ergodic decomposition

Simplex of invariant measures

Simplex of invariant measures

Given a computable system (X, T), how computable is the set $\mathcal{M}_T(X)$ of T-invariant probability measures?

Computability of a set

Let $A \subseteq \mathbb{R}^2$ be closed.

Definition

A is **computable** if there is a program that draws A on a screen at any resolution.

Examples and counter-examples

- Hertling, 2005: if the hyperbolicity conjecture holds then the **Mandelbrot set** is computable.
- Braverman, Yampolsky, 2008: for each $c \in \mathbb{C}$, the **filled Julia set** K_c is computable from c (non uniformly).
- Braverman, Yampolsky, 2007: however, there exists a computable $c \in \mathbb{C}$ such that the **Julia set** J_c is not computable.
- Graça, Rojas, Zhong, 2017: the geometrical **Lorenz attractor** is computable.

Computability of a set

Counter-examples

- The **subshift** induced by a computable set of forbidden patterns is not always computable.
- The set of zeroes of a computable function $f : [0, 1] \to \mathbb{R}$ is not always computable.

Computability of a set

Counter-examples

- The **subshift** induced by a computable set of forbidden patterns is not always computable.
- The set of zeroes of a computable function $f : [0, 1] \to \mathbb{R}$ is not always computable.

But these sets are always **semicomputable**.

Definition

A is **semicomputable** if there is a program that recognizes whether $x \notin A$.

Computability of a set

Proposition

A *is computable* $\iff$ A *is semicomputable and contains a dense computable sequence.*

Sometimes equivalent

- $\{x\}$ is computable $\iff$ $\{x\}$ is semicomputable $\iff$ x is computable.
- Kůrka, 1999: for a minimal subshift, computable $\iff$ semicomputable (effective).
- Miller, 2002: in $\mathbb{R}^n$, if $A \cong$ a sphere then A computable $\iff$ A semicomputable.

Simplex of invariant measures

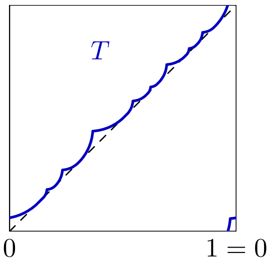
Given a computable system (X, T),

- $\mathcal{M}_T(X)$ is semicomputable: given $\mu \in \mathcal{M}_1(X)$, one can eventually see that $T\mu \neq \mu$,
- Is $\mathcal{M}_T(X)$ computable?
- How uncomputable can $\mathcal{M}_T(X)$ be?

Simplex of invariant measures

Proposition (Galatolo, H. & Rojas, 2009)

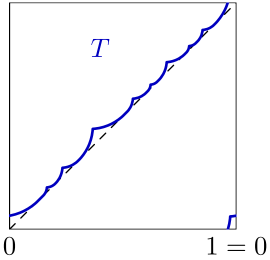
There exists a computable dynamical system $T : \mathscr{S}^1 \to \mathscr{S}^1$ *with no computable invariant measure.*



Simplex of invariant measures

Proposition (Galatolo, H. & Rojas, 2009)

There exists a computable dynamical system $T : \mathscr{S}^1 \to \mathscr{S}^1$ with no computable invariant measure.



Proposition

If a computable dynamical system is uniquely ergodic then its ergodic measure is computable.

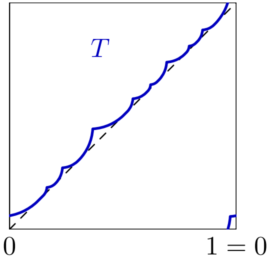
Proof.

$\mathcal{M}_T(X)$ is a semicomputable singleton. $\square$

Simplex of invariant measures

Proposition (Galatolo, H. & Rojas, 2009)

There exists a computable dynamical system $T : \mathscr{S}^1 \to \mathscr{S}^1$ with no computable invariant measure.



Proposition

If a computable dynamical system is uniquely ergodic then its ergodic measure is computable.

Proof.

$\mathcal{M}_T(X)$ is a semicomputable singleton. □

Question

What about the finitely ergodic case?

Finitely ergodic system

Proposition

If the system is finitely ergodic, then it has a computable invariant measure.

Proof.

$\mathcal{M}_T(X)$ is a semicomputable convex set of dimension k. Intersect with a rational hyperplane to produce a semicomputable convex set of dimension $k-1$, and apply induction. □

Finitely ergodic system

Proposition

If the system is finitely ergodic, then it has a computable invariant measure.

Proof.

$\mathcal{M}_T(X)$ is a semicomputable convex set of dimension k. Intersect with a rational hyperplane to produce a semicomputable convex set of dimension $k-1$, and apply induction. □

Question

Does it have a computable **ergodic** invariant measure?

Finitely ergodic system

Theorem (Coronel, Frank, H., Rojas, 2017)

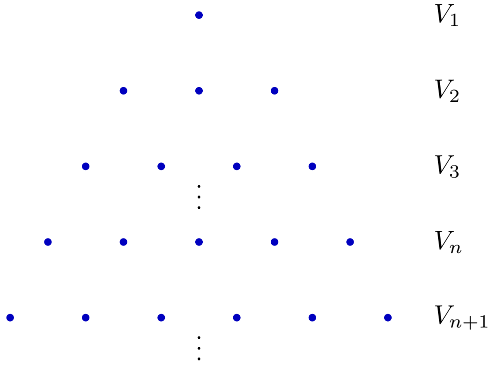
There exists a computable system with two ergodic measures, none of which is computable.

Bratteli diagram

- Bratteli diagrams give a way to build Cantor minimal systems.
- Every Cantor minimal system is conjugate to a Bratteli diagram.
- In particular every Choquet simplex is affine homeomorphic to the set of invariant measures of a Bratteli diagram.

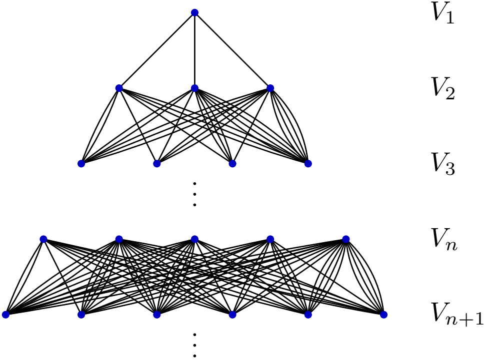
Bratteli diagram

An infinite graph with finite set of vertices V_n at each level,



Bratteli diagram

An infinite graph with finite set of vertices V_n at each level,
and edges between V_n and V_{n+1}.

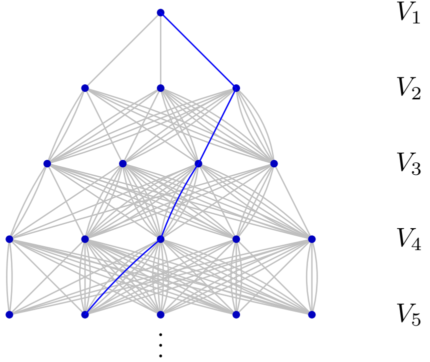


Bratteli diagram and Vershik map

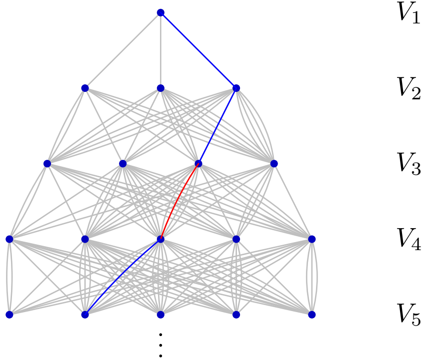
Dynamical system

- Space X: the set of paths $(e_1, e_2, \ldots)$ with e_n between V_n and V_{n+1},
- Vershik map $T : X \to X$.

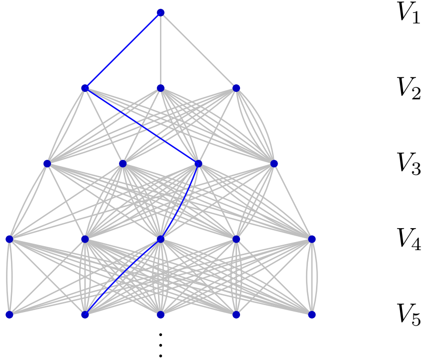
Bratteli diagram and Vershik map



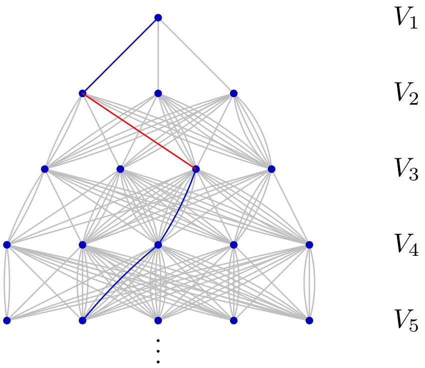
Bratteli diagram and Vershik map



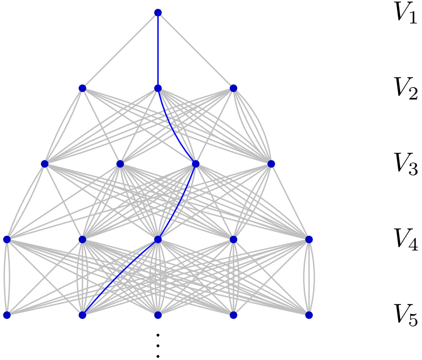
Bratteli diagram and Vershik map



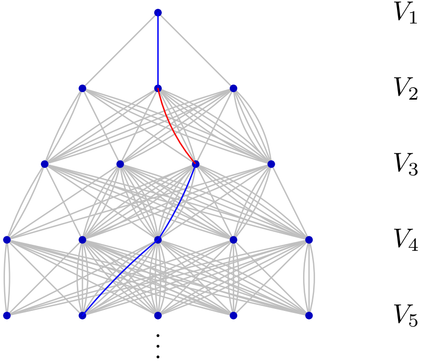
Bratteli diagram and Vershik map



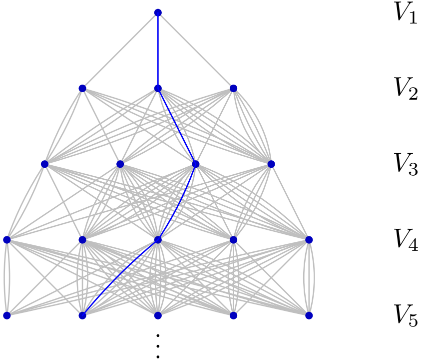
Bratteli diagram and Vershik map



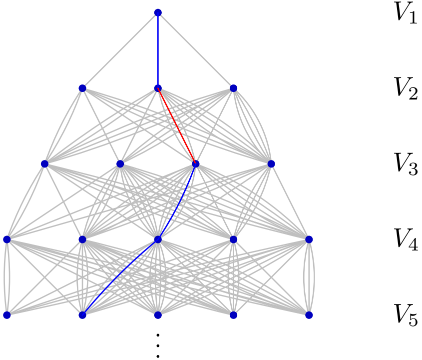
Bratteli diagram and Vershik map



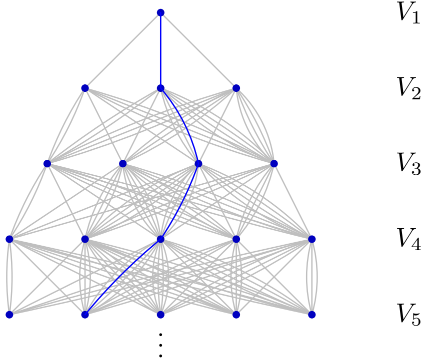
Bratteli diagram and Vershik map



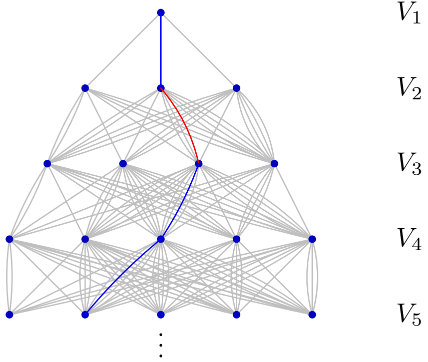
Bratteli diagram and Vershik map



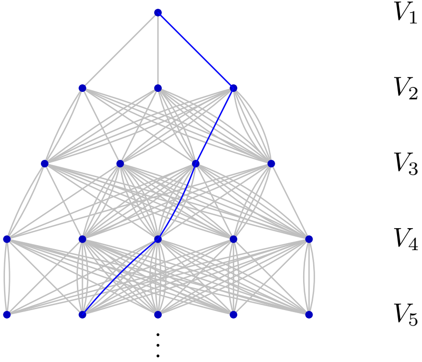
Bratteli diagram and Vershik map



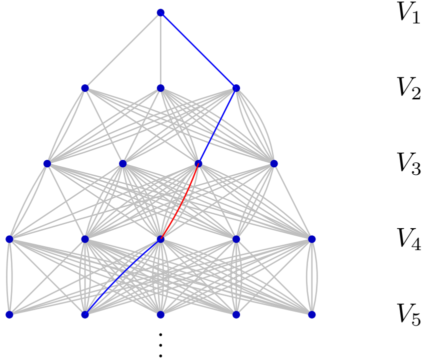
Bratteli diagram and Vershik map



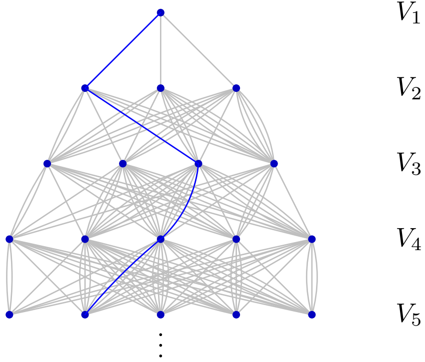
Bratteli diagram and Vershik map



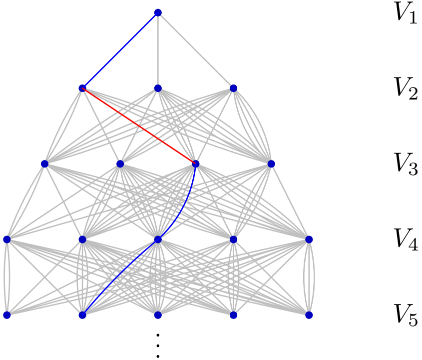
Bratteli diagram and Vershik map



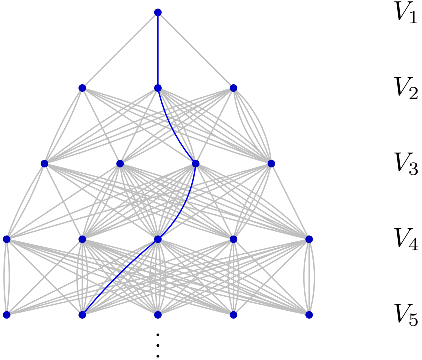
Bratteli diagram and Vershik map



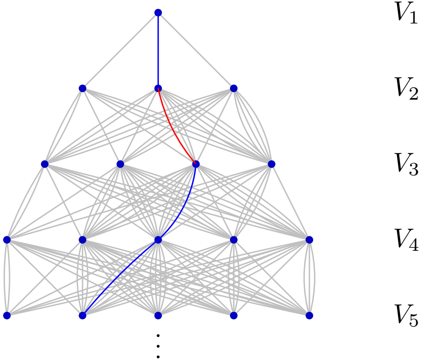
Bratteli diagram and Vershik map



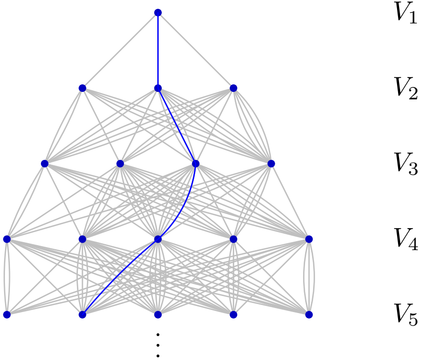
Bratteli diagram and Vershik map



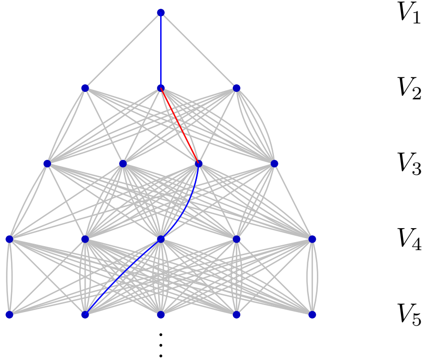
Bratteli diagram and Vershik map



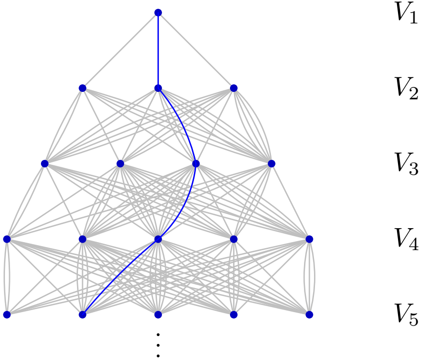
Bratteli diagram and Vershik map



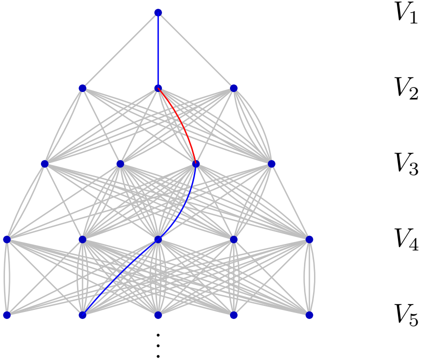
Bratteli diagram and Vershik map



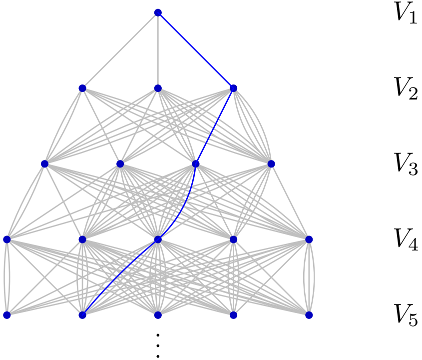
Bratteli diagram and Vershik map



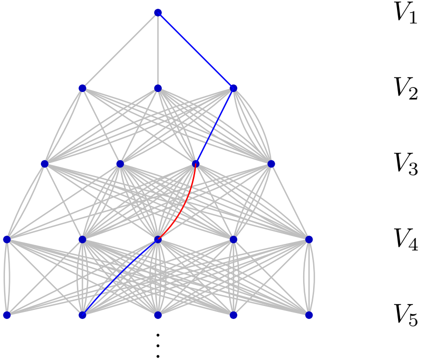
Bratteli diagram and Vershik map



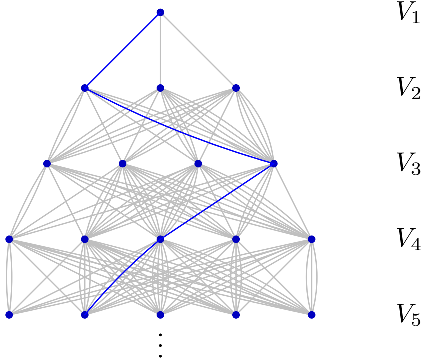
Bratteli diagram and Vershik map



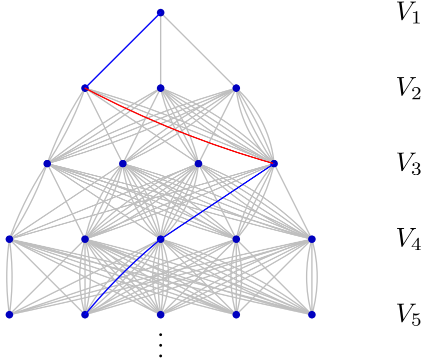
Bratteli diagram and Vershik map



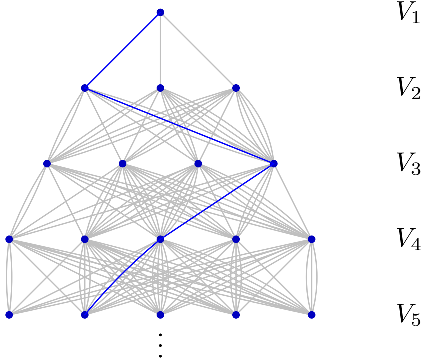
Bratteli diagram and Vershik map



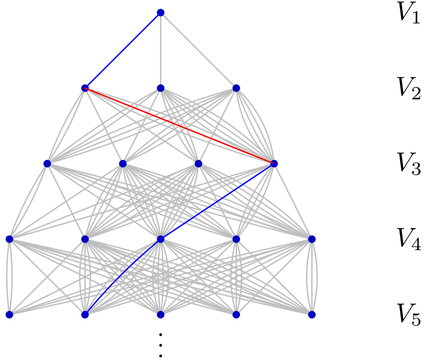
Bratteli diagram and Vershik map



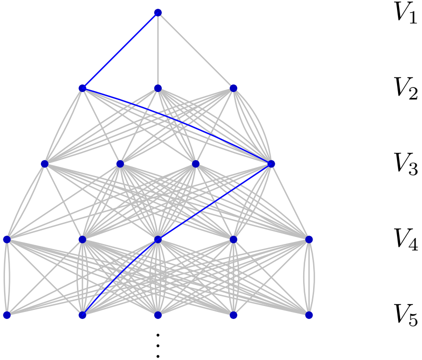
Bratteli diagram and Vershik map



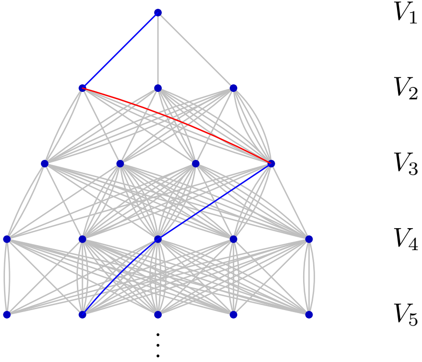
Bratteli diagram and Vershik map



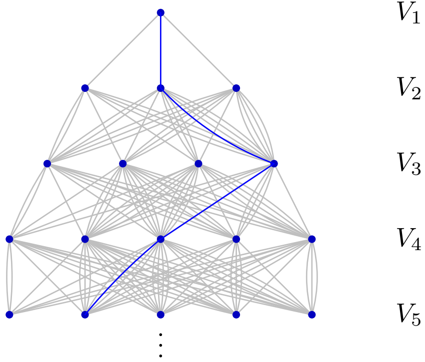
Bratteli diagram and Vershik map



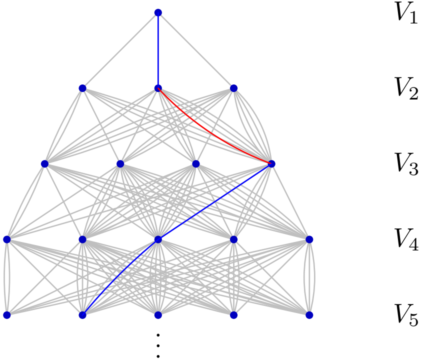
Bratteli diagram and Vershik map



Bratteli diagram and Vershik map

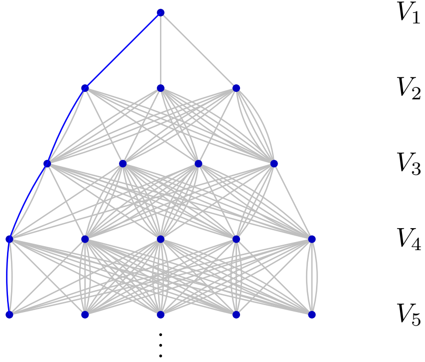


Bratteli diagram and Vershik map

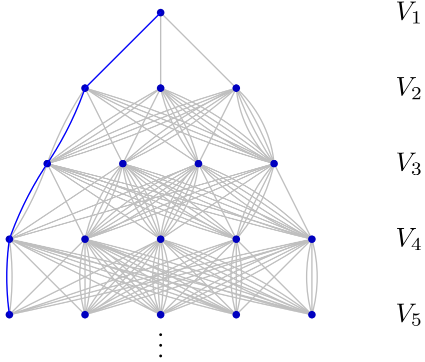


etc.

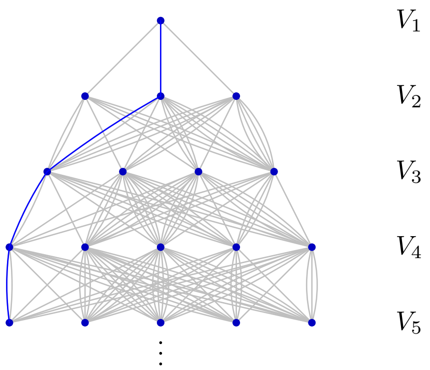
Bratteli diagram and Vershik map



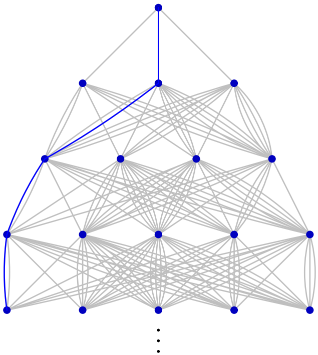
Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



V_1

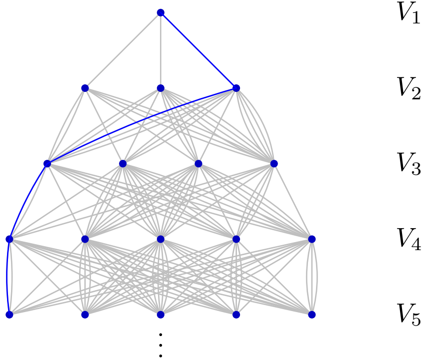
V_2

V_3

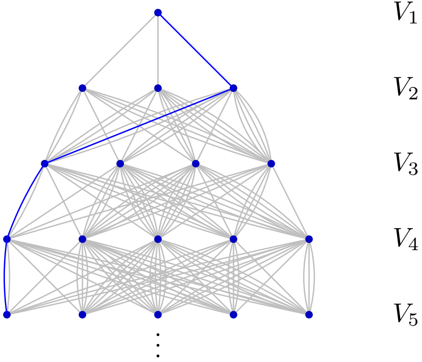
V_4

V_5

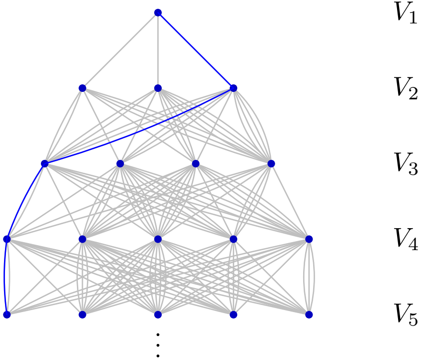
Bratteli diagram and Vershik map



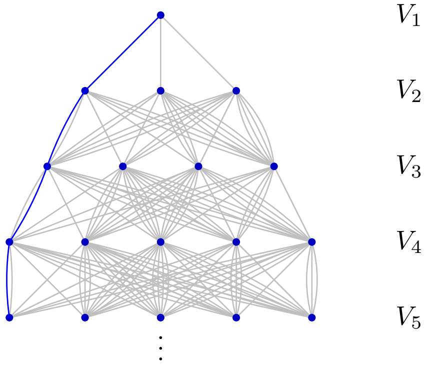
Bratteli diagram and Vershik map



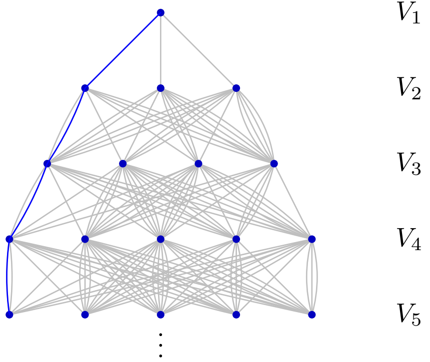
Bratteli diagram and Vershik map



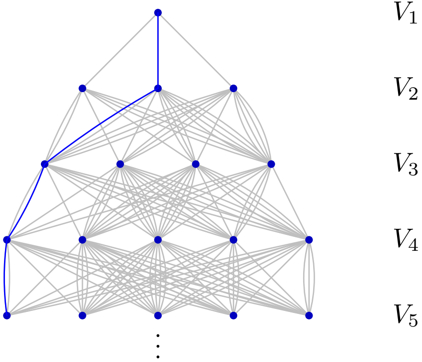
Bratteli diagram and Vershik map



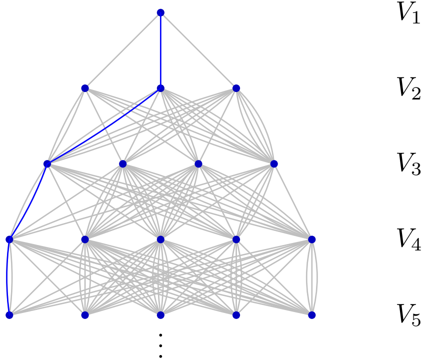
Bratteli diagram and Vershik map



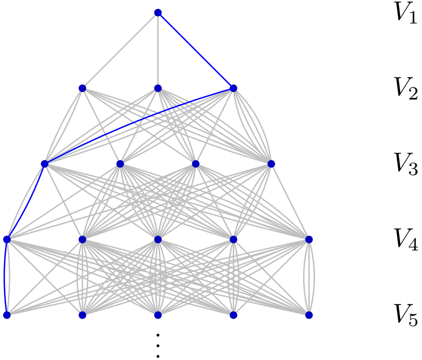
Bratteli diagram and Vershik map



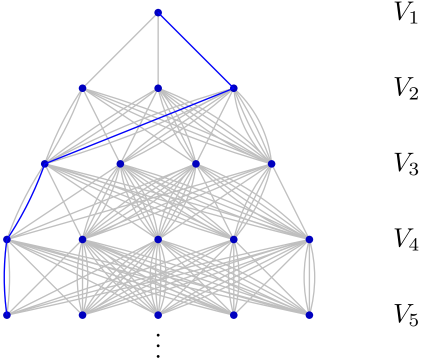
Bratteli diagram and Vershik map



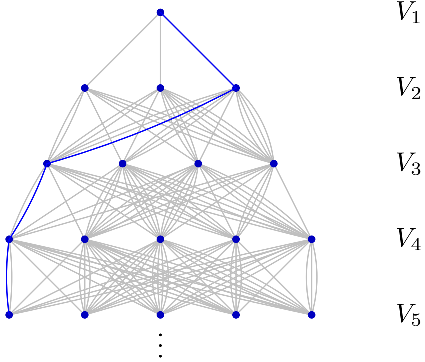
Bratteli diagram and Vershik map



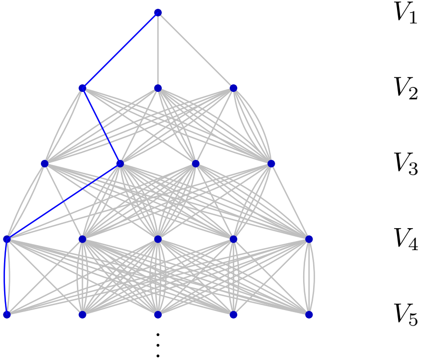
Bratteli diagram and Vershik map



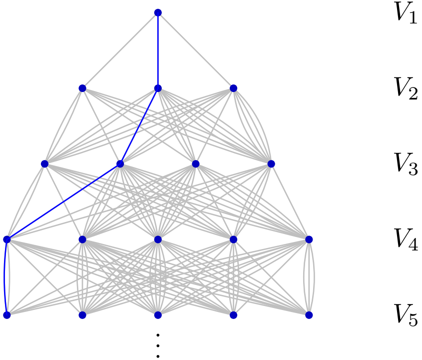
Bratteli diagram and Vershik map



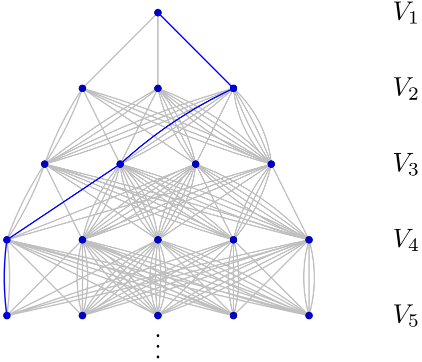
Bratteli diagram and Vershik map



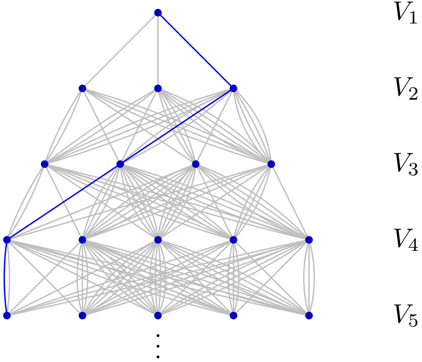
Bratteli diagram and Vershik map



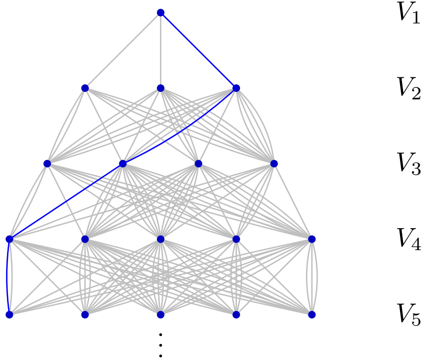
Bratteli diagram and Vershik map



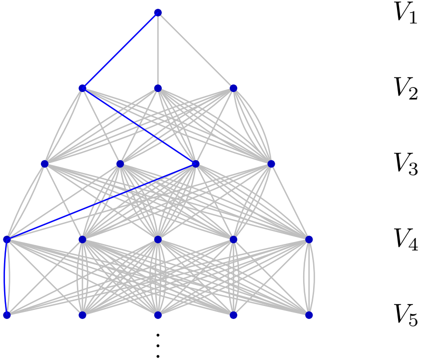
Bratteli diagram and Vershik map



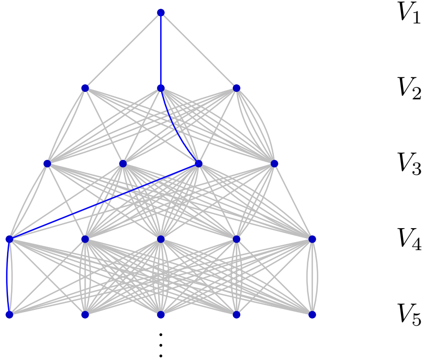
Bratteli diagram and Vershik map



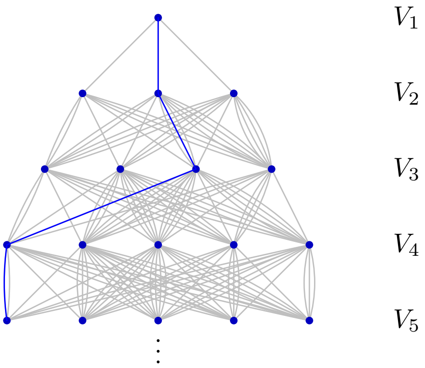
Bratteli diagram and Vershik map



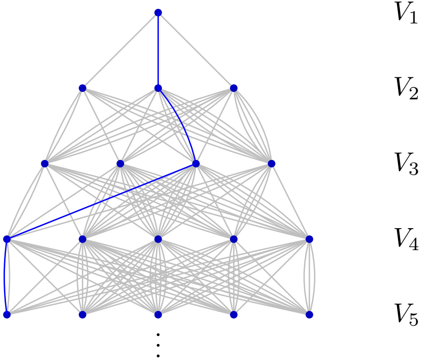
Bratteli diagram and Vershik map



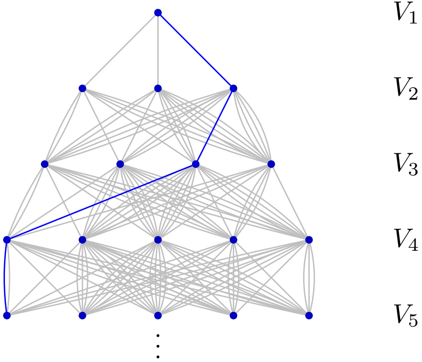
Bratteli diagram and Vershik map



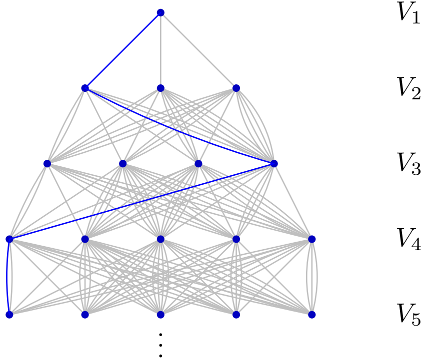
Bratteli diagram and Vershik map



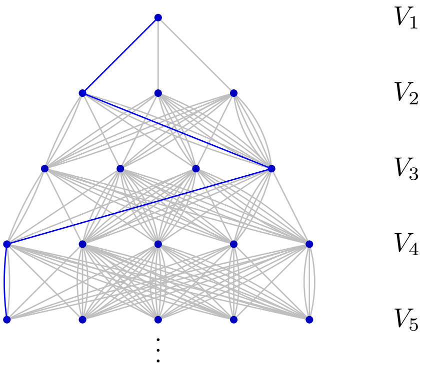
Bratteli diagram and Vershik map



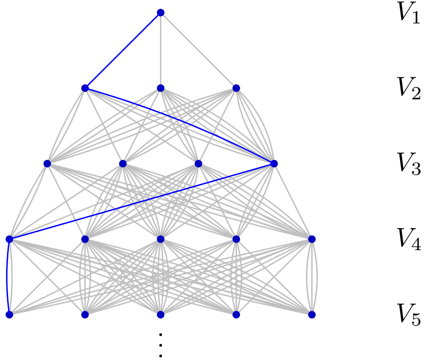
Bratteli diagram and Vershik map



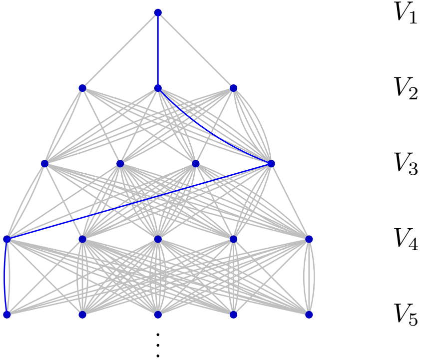
Bratteli diagram and Vershik map



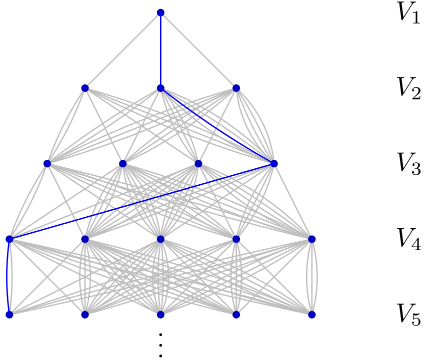
Bratteli diagram and Vershik map



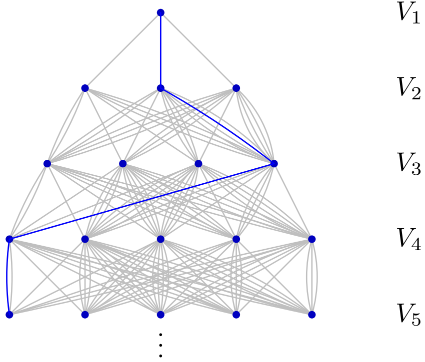
Bratteli diagram and Vershik map



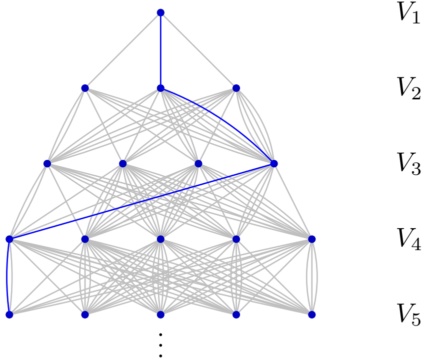
Bratteli diagram and Vershik map



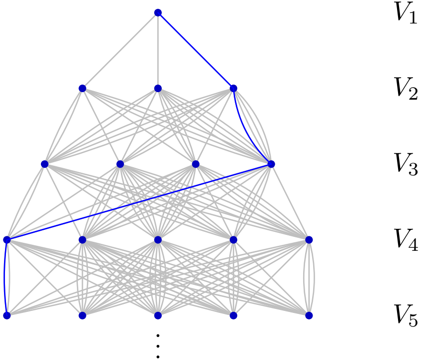
Bratteli diagram and Vershik map



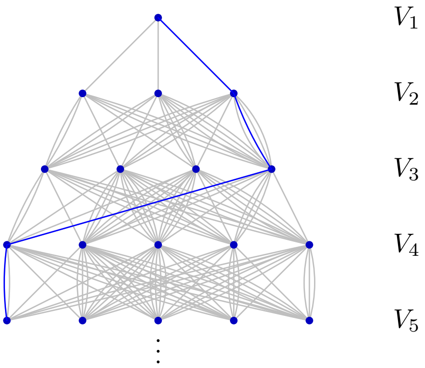
Bratteli diagram and Vershik map



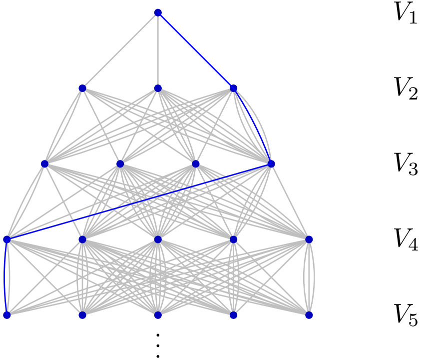
Bratteli diagram and Vershik map



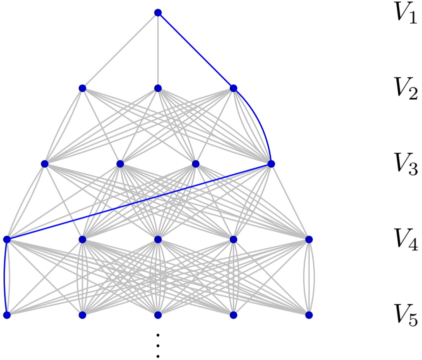
Bratteli diagram and Vershik map



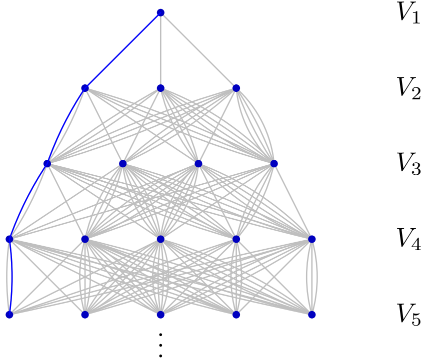
Bratteli diagram and Vershik map



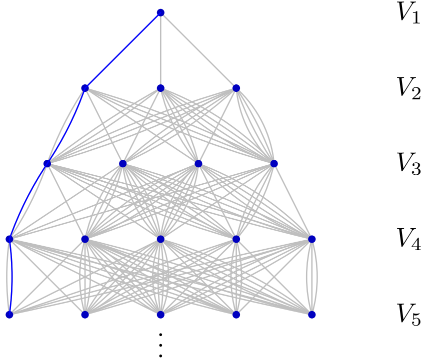
Bratteli diagram and Vershik map



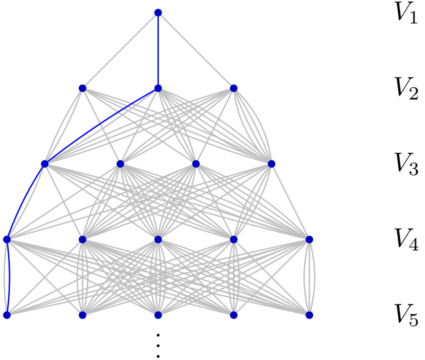
Bratteli diagram and Vershik map



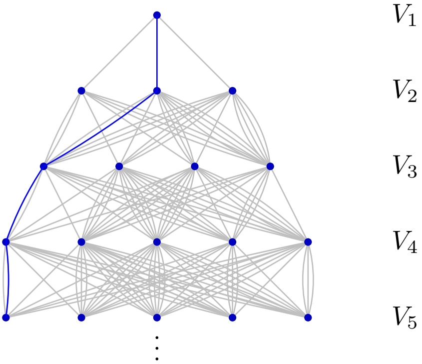
Bratteli diagram and Vershik map



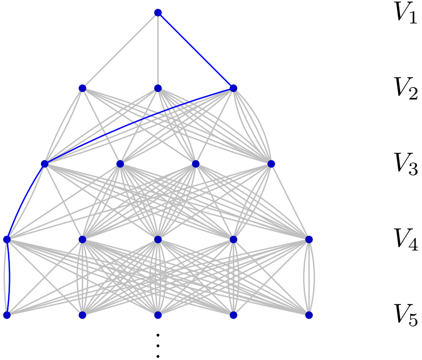
Bratteli diagram and Vershik map



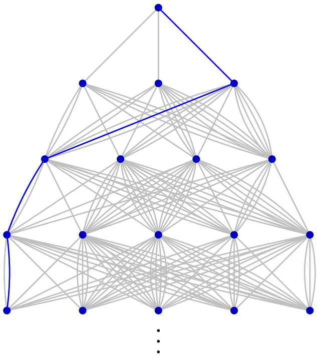
Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



V_1

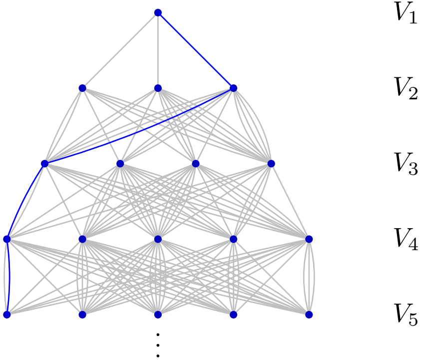
V_2

V_3

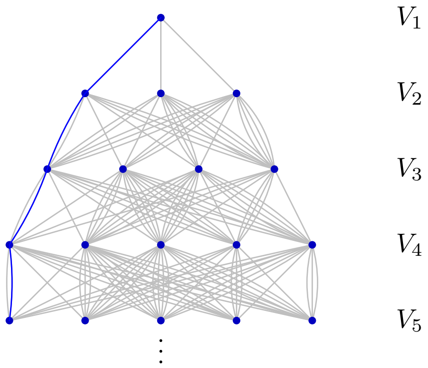
V_4

V_5

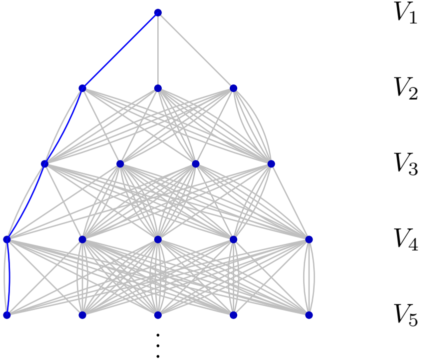
Bratteli diagram and Vershik map



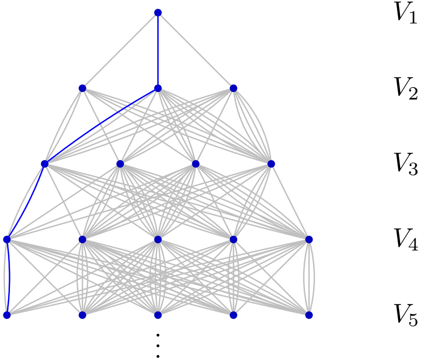
Bratteli diagram and Vershik map



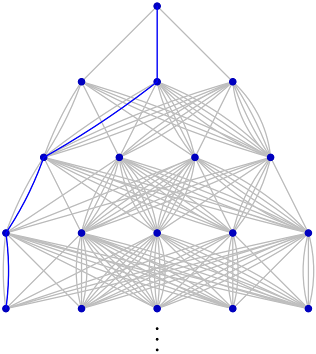
Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



V_1

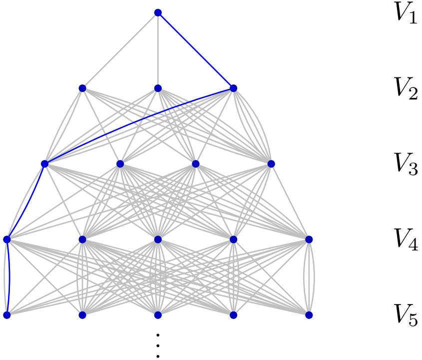
V_2

V_3

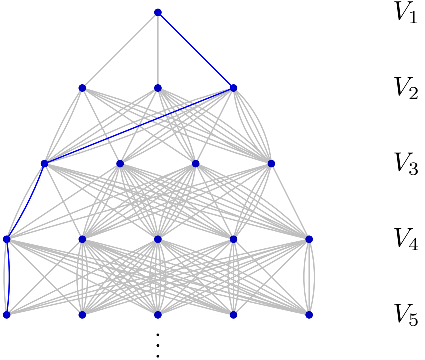
V_4

V_5

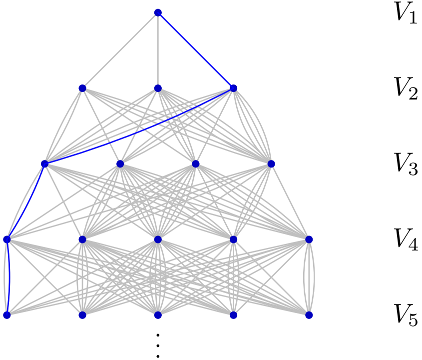
Bratteli diagram and Vershik map



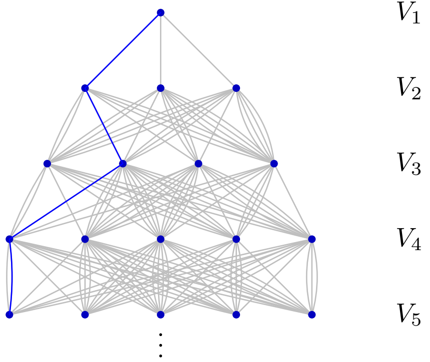
Bratteli diagram and Vershik map



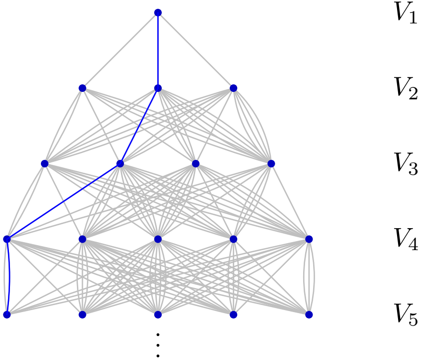
Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



Bratteli diagram and Vershik map



Bratteli diagram

- If the graph is sufficiently connected then the system is minimal and expansive.
- If $|V_n| \leq k$ for all n, then the system has at most k ergodic measures.
- If the graph is computable and the minimal path is unique then the Vershik map is computable.

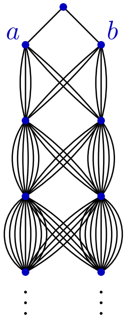
We now prove:

Theorem (Coronel, Frank, H., Rojas, 2017)

There exists a computable system with two ergodic measures, none of which is computable.

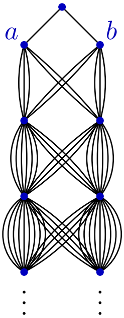
Two non-computable ergodic measures

- Take $|V_n| = 2$ for all $n \geq 2$ and adjacency matrices $M_n = \left(\begin{smallmatrix} X_n & Y_n \\ Y_n & X_n \end{smallmatrix}\right)$.



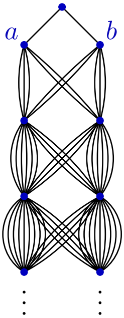
Two non-computable ergodic measures

- Take $|V_n| = 2$ for all $n \geq 2$ and adjacency matrices $M_n = \left(\begin{smallmatrix} X_n & Y_n \\ Y_n & X_n \end{smallmatrix}\right)$.
- Take computable sequences $X_n, Y_n \in \mathbb{N}$ such that $\sum_n \frac{Y_n}{X_n} < \infty$ is not computable.



Two non-computable ergodic measures

- Take $|V_n| = 2$ for all $n \geq 2$ and adjacency matrices $M_n = \left(\begin{smallmatrix} X_n & Y_n \\ Y_n & X_n \end{smallmatrix}\right)$.
- Take computable sequences $X_n, Y_n \in \mathbb{N}$ such that $\sum_n \frac{Y_n}{X_n} < \infty$ is not computable.
- The Vershik map has two ergodic measures, that are not computable.

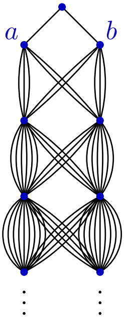


Two non-computable ergodic measures

- Take $|V_n| = 2$ for all $n \geq 2$ and adjacency matrices $M_n = \left(\begin{smallmatrix} X_n & Y_n \\ Y_n & X_n \end{smallmatrix}\right)$.
- Take computable sequences $X_n, Y_n \in \mathbb{N}$ such that $\sum_n \frac{Y_n}{X_n} < \infty$ is not computable.
- The Vershik map has two ergodic measures, that are not computable.
- The weight $\mu(a)$ takes all values in $[\frac{1-r}{2}, \frac{1+r}{2}]$, where

$$r = \prod_n \frac{X_n - Y_n}{X_n + Y_n} > 0,$$

which is not computable.



Bratteli diagram

Let $\Delta_k = \{(p_1, \ldots, p_k) \in [0,1]^k : \sum_i p_i = 1\}$.

Theorem (Coronel, Frank, H., Rojas, 2017)

Every semicomputable convex set $S \subseteq \Delta_n$ is the projection of the invariant measures of some computable dynamical system.

Bratteli diagram

What are the invariant measures of a given Bratteli diagram?

- Let $u \in V_n$ and $v \in V_{n+1}$. For every invariant measure μ, one has

$$\mu(u|v) = \frac{\text{number of paths to } v \text{ through } u}{\text{number of paths to } v},$$

  which **does not depend** on μ.
- The invariant measures are exactly the ones with those transition probabilities.

Bratteli diagram

Building a Bratteli diagram

Lemma

Given a Bratteli diagram up to V_n, a set V_{n+1} and rational probability transitions $A(u,v)$, one can choose the edges between V_n and V_{n+1} so that $\mu(u|v) = A(u,v)$.

Proof.

Let $h(u) =$ number of paths to u. Write $\frac{A(u,v)}{h(u)} = \frac{p(u,v)}{q(v)}$ with $p(u,v), q(v) \in \mathbb{N}$. Take $p(u,v)$ edges from u to v. Then $q(v) =$ number of paths to v. □

Bratteli diagram

Geometrical interpretation

- Let $\Delta_k = \{(p_1, \ldots, p_k) \in [0,1]^k : \sum_i p_i = 1\}$.
- The vector $\mu_n = (\mu(u))_{u \in V_n}$ is in $\Delta_{|V_n|}$.
- A stochastic matrix $A_n(u,v)$ corresponds to a linear map from $\Delta_{|V_{n+1}|}$ to $\Delta_{|V_n|}$.

Bratteli diagram

Geometrical interpretation

- Let $\Delta_k = \{(p_1, \dots, p_k) \in [0,1]^k : \sum_i p_i = 1\}$.
- The vector $\mu_n = (\mu(u))_{u \in V_n}$ is in $\Delta_{|V_n|}$.
- A stochastic matrix $A_n(u, v)$ corresponds to a linear map from $\Delta_{|V_{n+1}|}$ to $\Delta_{|V_n|}$.
- The set of invariant measures is the inverse limit

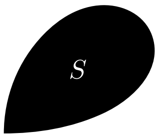
$$\Delta_{|V_1|} \xleftarrow{A_1} \Delta_{|V_2|} \xleftarrow{A_2} \Delta_{|V_3|} \xleftarrow{A_3} \dots$$

  i.e. the set of sequences $\mu_n \in \Delta_{|V_n|}$ such that $\mu_n = A_n \mu_{n+1}$.
- In particular, by compactness, its projection on $\Delta_{|V_1|}$ is

$$\bigcap_n A_1 A_2 \dots A_n(\Delta_{|V_{n+1}|}).$$

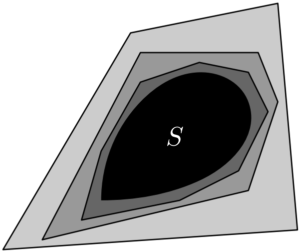
Bratteli diagram

- Let $S \subseteq \Delta_k$ be a semicomputable convex set.



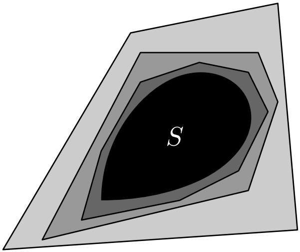
Bratteli diagram

- Let $S \subseteq \Delta_k$ be a semicomputable convex set.
- There exists a computable sequence of rational convex polytopes $P_{n+1} \subseteq P_n$ such that $S = \bigcap_n P_n$.



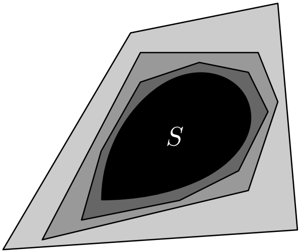
Bratteli diagram

- Let $S \subseteq \Delta_k$ be a semicomputable convex set.
- There exists a computable sequence of rational convex polytopes $P_{n+1} \subseteq P_n$ such that $S = \bigcap_n P_n$.
- Let $|V_n| =$ number of vertices of P_n.



Bratteli diagram

- Let $S \subseteq \Delta_k$ be a semicomputable convex set.
- There exists a computable sequence of rational convex polytopes $P_{n+1} \subseteq P_n$ such that $S = \bigcap_n P_n$.
- Let $|V_n| =$ number of vertices of P_n.



One can inductively build rational stochastic matrices A_n such that

$$\begin{aligned} A_2(\Delta_{|V_2|}) &= P_2, \\ A_2 A_3(\Delta_{|V_3|}) &= P_3, \\ A_2 A_3 A_4(\Delta_{|V_4|}) &= P_4, \\ &\dots \end{aligned}$$

so $\bigcap_n A_2 \dots A_n(\Delta_{|V_n|}) = S$.

Bratteli diagram

Theorem

Every semicomputable finite-dimensional Choquet simplex S can be realized as the set of invariant measures of a computable dynamical system.

Proof.

Apply the previous construction with constant $|V_n|$ and **one-to-one** A_n, so that the inverse limit

$$\Delta_{|V_1|} \xleftarrow{A_1} \Delta_{|V_2|} \xleftarrow{A_2} \Delta_{|V_3|} \xleftarrow{A_3} \dots$$

is computably affine homeomorphic to $\bigcap_n A_1 A_2 \dots A_n(\Delta_{|V_1|}) = S$. $\square$

Every Choquet simplex is affine homeomorphic to $\mathcal{M}_T(X)$ for some (X, T).

Open question

Is every **semicomputable** Choquet simplex **computably** affine homeomorphic to $\mathcal{M}_T(X)$ for some **computable** (X, T)?

Every Choquet simplex is affine homeomorphic to $\mathcal{M}_T(X)$ for some (X, T).

Open question

Is every **semicomputable** Choquet simplex **computably** affine homeomorphic to $\mathcal{M}_T(X)$ for some **computable** (X, T)?

Thank you!