Semicomputable points in Euclidean spaces

Mathieu Hoyrup
Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
mathieu.hoyrup@inria.fr

Donald M. Stull
Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
donald.stull@inria.fr

Abstract

We introduce the notion of a semicomputable point in $\mathbb{R}^n$, defined as a point having left-c.e. projections. We study the range of such a point, which is the set of directions on which its projections are left-c.e., and is a convex cone. We provide a thorough study of these notions, proving along the way new results on the computability of convex sets. We prove realization results, by identifying computability properties of convex cones that make them ranges of semicomputable points. We give two applications of the theory. The first one provides a better understanding of the Solovay derivatives. The second one is the investigation of left-c.e. quadratic polynomials. We show that this is, in fact, a particular case of the general theory of semicomputable points.

2012 ACM Subject Classification Theory of Computation → Computability

Keywords and phrases Semicomputable point – Left-c.e. real – Convex cone – Solovay reducibility – Genericity

Digital Object Identifier 10.4230/LIPIcs...

Funding *Donald M. Stull*: Supported by Inria Nancy Grand-Est.

1 Introduction

The general goal of this paper is to improve our understanding of weak notions of computability in computable analysis. Usually these notions are more difficult to understand than plain computability, and have a rich theory. For instance we mention the notions of computably enumerable (c.e.) subsets of $\mathbb{N}$, left-c.e. reals numbers, left-c.e. real functions, c.e. closed subsets of $\mathbb{R}^n$, co-c.e. closed sets, etc.

A closed subset of $\mathbb{R}^n$ is co-c.e. if its complement is a computable union of rational balls. When a closed set can be described by a few parameters, such as a simple geometrical figure, what properties must these parameters satisfy to make it a co-c.e. closed set? The case of filled triangles has been studied in [5], but the case of disks is more challenging.

A function $f : \mathbb{R} \to \mathbb{R}$ is left-c.e. if there is a program that takes x as input and outputs better and better approximations of $f(x)$ from the left (with no assumption on the speed of convergence to $f(x)$). When a function is described by a few parameters, such as a polynomial, what properties must these parameters satisfy to make it a left-c.e. function?

The cases of co-c.e. disks and left-c.e. polynomials are surprisingly two instances of a common framework that we investigate in this paper. In both cases, the object can be identified with a point in some Euclidean space (for instance, a polynomial is a vector of coefficients) and the computability notion can be expressed as the point having uniformly left-c.e. projections in some set of directions (the directions $(1, X, X^2)$ in the case of quadratic polynomials). This observation leads us to introduce and study the notion of a *semicomputable* point in Euclidean spaces. It is an extension to several dimensions of a notion introduced in [5] in the plane. In particular we define the semicomputability range of a point as the set of directions in which it is left-c.e., and investigate the possible sets that can be obtained as ranges of semicomputable points.

The extension from the plane to higher-dimensional Euclidean spaces is not a straightforward generalization because many subtleties appear in $\mathbb{R}^3$. For instance, the range of a point is a convex cone, so it is determined by two angles in $\mathbb{R}^2$ but can have many different shapes in $\mathbb{R}^3$. Another example is that the operation of taking the conical hull of two convex cones, while simple in $\mathbb{R}^2$, is not as simple in $\mathbb{R}^3$ in terms of computability.

The main results of the paper are realizations results: given a convex cone in $\mathbb{R}^n$ with some computability property, there exists a semicomputable point in $\mathbb{R}^n$ whose range is exactly that cone:

- Theorem 4.6: every Σ_2^0 cone is the range of some semicomputable point. If its closure is not Π_2^0 then the point is *non-uniformly* left-c.e. in the directions of the cone.
- Theorem 4.8: every salient Π_2^0 convex cone is the range of some semicomputable point. Moreover, that point is *uniformly* left-c.e. in the directions of the cone.

In Section 2.4 we give results about computability of convex sets and convex cones. In Section 3 we define semicomputable points of $\mathbb{R}^n$ and develop a thorough study of this notion. In particular we define the range of a semicommputable point, which is the set of directions in which its projections are left-c.e. In Section 4 we prove the main results of the paper, in which we identify classes of convex cones that can be realized as ranges of semicomputable points. In Section 5 we use these results to study Solovay derivatives and precisely identify the possible shapes of the functions $\overline{S}(aX+b,c)$ and $\underline{S}(aX+b,c)$ when a,b,c are fixed and X varies over the computable reals. In Section 6 we investigate the left-c.e. quadratic polynomials, which can be identified with semicomputable points with a certain range.

2 Background

2.1 Computability in Euclidean spaces

We endow $\mathbb{R}^n$ with the inner product $\langle x,y\rangle=\sum_{i=1}^n x_iy_i$, where $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$, the associated norm $\|x\|=\sqrt{\langle x,x\rangle}$ and the distance $d(x,y)=\|x-y\|$. An open set $U\subseteq\mathbb{R}^n$ is ***effectively open*** if it is the union of a computable sequence of rational open balls (centered at rational points with rational radii). Let $A\subseteq\mathbb{R}^n$ be a closed set. A is ***c.e. closed*** if A contains a dense computable sequence, or equivalently the function $x\mapsto d(x,A)=\min_{y\in A}d(x,y)$ is right-c.e. A is ***co-c.e. closed*** if the complement of A is effectively open, or equivalently the function $x\mapsto d(x,A)$ is left-c.e. A closed set is ***computably closed*** if it is both c.e. closed and co-c.e. closed. A compact set K is ***effectively compact*** if the set of finite lists of rational balls covering K is c.e. A compact set is effectively compact if and only if it is co-c.e. closed. More details on these notions can be found in [3].

A real number is ***left-c.e.*** if it is the limit of a computable increasing sequence of rational numbers. A real number x is ***right-c.e.*** if $-x$ is left-c.e. It is computable if it is both left-c.e. and right-c.e. If $D\subseteq\mathbb{R}^n$ then a function $f:D\to[-\infty,+\infty]$ is left-c.e. if there exist uniformly effective open sets $(U_q)_{q\in\mathbb{Q}}$ such that for all $q\in\mathbb{Q}$, $f^{-1}(q,+\infty)=D\cap U_q$. f is right-c.e. if $-f$ is left-c.e.

Let $f:\mathbb{R}^m\times\mathbb{R}^n$ be left-c.e.. If $K\subseteq\mathbb{R}^n$ is effectively compact then $f_{\min}:\mathbb{R}^m\to\mathbb{R}$ defined by $f_{\min}(x)=\min_{y\in K}f(x,y)$ is left-c.e. If $A\subseteq\mathbb{R}^n$ is c.e. closed then $f_{\sup}:\mathbb{R}^m\to\mathbb{R}$ defined by $f_{\sup}(x)=\sup_{y\in A}f(x,y)$ is left-c.e.

Each of these computability notions can be relativized to any oracle. We will be particularly interested in their relativization to the halting set, denoted by $\emptyset'$. For instance, a real is $\emptyset'$-left-c.e. if it is left-c.e. relative to $\emptyset'$.

2.2 Solovay derivatives

The quantitative study of Solovay reducibility was initiated in [1] and continued in [7] and [5]. We briefly recall that if a, b are real numbers such that b is left-c.e., then we define

$$\overline{S}(a,b) = \inf\{q \in \mathbb{Q} : qb - a \text{ is left-c.e.}\},$$
$$\underline{S}(a,b) = \sup\{q \in \mathbb{Q} : qb - a \text{ is right-c.e.}\}.$$

We say that a is ***Solovay reducible*** to b if $\overline{S}(a,b) < +\infty$ and $\underline{S}(a,b) > -\infty$. Intuitively, it means that a is easier to approximate than b in the following sense: if $\overline{S}(a,b) < q$ and $\underline{S}(a,b) > r$, then there exist computable sequences a_i, b_i converging to a, b such that $r \leq \frac{a-a_i}{b-b_i} \leq q$.

Some left-c.e. real numbers are ***Solovay complete***, meaning that each left-c.e. number is reducible to them, and it is proved in [1] that if b is Solovay complete, then $\overline{S}(a,b) = \underline{S}(a,b)$.

2.3 Background on convex cones

We give the minimal amount of background on convex analysis and refer the reader to [2] for more details. A ***cone*** is a set $C \subseteq \mathbb{R}^n$ that is closed under multiplication by a nonnegative scalar. A ***convex cone*** is a cone that is convex, i.e. a set that is closed under addition and multiplication by a nonnegative scalar. The ***dual*** of a set C is the closed convex cone $C^* = \{x \in \mathbb{R}^n : \forall y \in C, \langle x, y\rangle \geq 0\}$. $(C^*)^*$ is the smallest closed convex cone containing C. In particular if C is a closed convex cone then $(C^*)^* = C$.

For $x \neq 0$, let $H_x = \{z : \langle x, z\rangle \geq 0\}$ be the half-space delimited by the hyperplane orthogonal to x, in the direction of x. One has $C^* = \bigcap_{x\in C} H_x$. As a result, $d(z, C^*) \geq \sup_{x\in C} d(z, H_x)$ and we show that equality holds. Observe that $d(z, H_x) = \max(-\frac{\langle z,x\rangle}{\|x\|}, 0)$.

▶ **Lemma 2.1.** *Let C be a convex set. One has $d(z, C^*) = \sup_{x\in C} d(z, H_x)$.*

For a convex cone C, let C_1 be the intersection of C with the unit sphere. In the previous lemma, one has $d(z, C^*) = \sup_{x\in C_1} d(z, H_x)$ if C is a convex cone.

A convex cone is ***flat*** if it contains some $x \neq 0$ and its opposite $-x$. It is called ***salient*** if it is not flat. C is salient if and only if C^* is full-dimensional if and only if there exist $\epsilon > 0$ and y such that $\langle x, y\rangle > \epsilon$ for all $x \in C_1$.

If $A \subseteq \mathbb{R}^n$ is a full-dimensional convex set, then $A \subseteq \overline{\mathrm{int}(A)}$ and $\mathrm{int}(\overline{A}) \subseteq A$. In particular, among the class of full-dimensional convex sets, every closed set is regular closed and every open set is regular open.

2.4 Computability of convex sets and cones

Computability of convex sets has been investigated in [6] and [9]. Here we present new results that are used to prove the results of the paper and are of independent interest.

▶ **Proposition 2.2.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex cone.*
- *C is co-c.e. closed $\iff$ C^* is c.e. closed,*
- *C is c.e. closed $\iff$ C^* is co-c.e. closed,*
- *C is computably closed $\iff$ C^* is computably closed.*

Proof. If C is c.e. closed then let $(x_i)_{i\in\mathbb{N}}$ be a dense computable sequence in C. One has $C^* = \bigcap_i H_{x_i}$ which is therefore co-c.e. closed.

If C is co-c.e. then the intersection C_1 of C with the unit sphere is effectively compact. By Lemma 2.1 one has $d(z, C^*) = \max_{x\in C} d(z, H_x) = \max_{x\in C_1} d(z, H_x)$ which is a right-c.e. function of z, so C^* is c.e. closed.

The other implications can be obtained by observing that $(C^*)^* = C$. ◀

Observe that these results relativize to any oracle. The first equivalence in the next result was proved by Ziegler [9].

▶ **Proposition 2.3.** *Let $C \subseteq \mathbb{R}^n$ be a full-dimensional closed convex set.*

- *C is co-c.e. closed $\iff$ the set of rational points outside C is c.e.,*
- *C is c.e. closed $\iff$ its interior is effectively open.*

Proof. If C is co-c.e. closed then the set of rational points outside C is obviously c.e. Conversely, assume that the set of rational points outside C is c.e. Let $C_0 \subseteq C$ be any fixed full-dimensional rational polytope. Given $z \in \mathbb{R}^n$, one can compute the convex hull of $C_0 \cup \{z\}$ and in particular enumerate its interior U_z. As U_z is dense in that convex hull, one has $z \notin C \iff U_z$ contains a rational point outside C. It gives a procedure that given z, halts exactly when $z \notin C$, showing that C is co-c.e.

If the interior of C is effectively open then one can enumerate the rational points in the interior, which are dense in C. Conversely, if C is c.e. closed then let $(x_i)_{i\in\mathbb{N}}$ be a dense computable sequence in C. A point z belongs to the interior of C iff there exist $n+1$ points in the sequence such that z belongs to the interior of their convex hull, which gives a procedure that halts exactly when $z \in \mathrm{int}(C)$. ◀

The assumption that C is full-dimensional is needed. For the first item, if C contains no rational point then no information about C can be obtained from an enumeration of the rationals ouside C (i.e., all the rational points). For the second item, C needs to have a non-empty interior.

▶ **Lemma 2.4.** *Let $A, B \subseteq \mathbb{R}^n$ be c.e. closed convex sets.*

- *If $A \cap B$ has non-empty interior then $A \cap B$ is c.e. closed.*
- *$A \cap B$ is $\emptyset'$-co-c.e. closed. There exist $A, B \subseteq \mathbb{R}^3$ such that $A \cap B$ is not $\emptyset'$-c.e. closed.*

Proof. The interiors of A and B are effectively open and dense in A and B respectively. Their intersection is effectively open and dense in $A \cap B$, which is then c.e. closed.

In general, if A, B are c.e. closed then they are $\emptyset'$-computable and in particular $\emptyset'$-co-c.e. closed and so is their intersection.

There exists a right-c.e. convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(0)$ is not $\emptyset'$-c.e. closed. Let $\alpha > 0$ be $\emptyset'$-right-c.e. but not $\emptyset'$-left-c.e. There exists a sequence of uniformly left-c.e. reals $\alpha_i > 0$ such that $\alpha = \inf_i \alpha_i$. Let $f_i(x) = \max(0, x - \alpha_i)$ and $f = \sum_i 2^{-i} f_i$. The functions f_i are uniformly right-c.e. so f is right-c.e., and $f^{-1}(0) = [0, \alpha]$ is not $\emptyset'$-c.e. closed because α is not $\emptyset'$-left-c.e.

Let $A = \{(x, y) : y \geq f(x)\}$ be the epigraph of f and $B = \{(x, y) : y \leq 0\}$. A and B are c.e. closed but $A \cap B = \{(x, 0) : f(x) = 0\}$ is not $\emptyset'$-c.e. closed. ◀

▶ **Proposition 2.5.** *Let $C \subseteq \mathbb{R}^n$ be a full-dimensional closed convex set.*

- *C is $\emptyset'$-co-c.e. closed $\iff$ the set of rational points outside C is $\emptyset'$-c.e. $\iff$ C is Π^0_2,*
- *C is $\emptyset'$-c.e. closed $\iff$ its interior is $\emptyset'$-effectively open $\iff$ C contains a dense Σ^0_2-set.*

Proof. Several equivalences are obtained by relativizing Proposition 2.3, we prove the others. Any set that is $\emptyset'$-co-c.e. is Π^0_2, and if C is Π^0_2 then the set of rational points ouside C is obviously $\emptyset'$-c.e.

Any $\emptyset'$-effectively open set is a Σ^0_2-set, and $\mathrm{int}(C)$ is dense in C. If C contains a dense Σ^0_2-set, then, with $\emptyset'$ as oracle, one can compute a dense sequence in that set, so C is $\emptyset'$-c.e. closed. ◀

Again the full dimension assumption is needed. For the first item, there exists a Π^0_2-singleton whose unique element is not computable relative to $\emptyset'$ (even relative to any $\emptyset^{(n)}$, $n \in \mathbb{N}$, see Proposition 1.8.62 in [8]). For the second item, if x is $\emptyset'$-computable but not computable, then $\{x\}$ is $\emptyset'$-c.e. closed convex but does not contain any non-empty Σ^0_2-set.

3 Semicomputable point

The notions of left-c.e. and right-c.e. real number can be extended to higher dimensions. A first extension to points of the plane has been introduced in [5]. We pursue this extension to $\mathbb{R}^n$ for any $n \geq 1$. Although the definition extends immediately to this more general setting, the results are more involved because higher dimensions allow richer behaviors. For instance, a convex cone in $\mathbb{R}^2$ is delimited by two directions only, while a convex cone in $\mathbb{R}^3$ is a delimited by an uncountable set of directions.

▶ **Definition 3.1.** *A point $x \in \mathbb{R}^n$ is* **semicomputable** *if there exist n linearly independent rational vectors $v_1, \dots, v_n$ such that each $\langle v_i, x\rangle$ is left-c.e., $1 \leq i \leq n$.*

▶ **Definition 3.2.** *Let $D \subseteq \mathbb{R}^n$ be a closed convex cone. We say that $x \in \mathbb{R}^n$ is* **D-c.e.** *if the mapping $d \in D \mapsto \langle d, x\rangle$ is left-c.e.*

Observe that this notion really makes sense when D is full-dimensional (or full-dimensional in some computable subspace), otherwise x could be D-c.e. only because the elements of D encode information about x. For instance, if $\|x\|$ is left-c.e. and $D = \{\lambda x : \lambda \geq 0\}$ then the mapping $d \in D \mapsto \langle d, x\rangle = \|d\| \cdot \|x\|$ is left-c.e., which should not be interpreted as "x is left-c.e. in some direction".

The closedness condition on D is justified by the next observation: D can always be assumed to be closed.

▶ **Proposition 3.3.** *Let $x \in \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ be a full-dimensional convex cone. If the mapping $d \in D \mapsto \langle d, x\rangle$ is left-c.e. then x is $\overline{D}$-c.e.*

Proof. Let $d_0 \in D$ be a rational vector in the interior of D. Let $d \in \overline{D}$ be given as oracle. The vectors $d_n = (1 - 2^{-n})d + 2^{-n}d_0$ are uniformly computable in d and belong to D. The number $\langle d, x\rangle$ is the effective limit of $\langle d_n, x\rangle$, which is left-c.e. uniformly in n and d, so $\langle d, x\rangle$ is left-c.e. uniformly in d. ◀

Being C^*-c.e. can be dually expressed in terms of C.

▶ **Proposition 3.4.** *Let $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ be a closed convex cone.*

- *When C is co-c.e. closed, x is C^*-c.e. $\iff$ $x + C$ is co-c.e. closed,*
- *When C is c.e. closed and full-dimensional, x is C^*-c.e. $\iff$ $x - C$ is c.e. closed.*

Proof.

- If x is C^*-c.e. then the complement of $x + C$ is effectively open. Indeed, $y \notin x + C \iff y - x \notin C \iff \inf_{d\in C^*} \langle d, y - x\rangle < 0$ which is a c.e. condition as C^* is c.e. closed and $\langle d, y - x\rangle$ is right c.e. in d, y.
  If $x + C$ is co-c.e. then let $K \in \mathbb{N}$ be an upper bound on $\|x\|$ and $A = (x + C) \cap \overline{B}(0, K)$. A is effectively compact, contains x and for $d \in C^*$, $\langle d, x\rangle = \min_{z\in A} \langle d, z\rangle$ which is a left-c.e. function of d.
- If x is C^*-c.e. then $\mathrm{int}(x - C)$ is effectively open. Indeed $y \in \mathrm{int}(x - C) \iff x - y \in \mathrm{int}(C) \iff \min_{d\in C^*_1} \langle d, x - y\rangle > 0$ which is a c.e. condition as C^*_1 is effectively compact and $\langle d, x - y\rangle$ is left-c.e. in d, y.
  If $x - C$ is c.e. closed and $(x_i)_{i\in\mathbb{N}}$ is a dense computable sequence in $x - C$, then for $d \in C^*$ one has $\langle d, x\rangle = \sup_i \langle d, x_i\rangle$ which is left-c.e. uniformly in d. ◀

▶ **Proposition 3.5.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex cone.*

- *If $x + C$ is co-c.e. closed for some $x \in \mathbb{R}^n$, then C is co-c.e. closed.*
- *If $x + C$ is c.e. closed for some $x \in \mathbb{R}^n$, then C is c.e. closed.*

Proof. Let $E \subseteq \mathbb{R}^n$ be a set such that 0 belongs to the convex hull of E. One has $C^* = \bigcup_{e\in E}(C+e)^*$. Indeed, if $y \in C^*$ then there exists $e \in E$ such that $\langle e, y\rangle \geq 0$, so $y \in (C+e)^*$. Conversely, if $y \in (C+e)^*$ and $c \in C$ then $\langle y, c\rangle = \lim_{n\to\infty} \frac{1}{n}\langle y, e+nc\rangle \geq 0$ so $y \in C^*$.

Given x, there exists a finite set E of rational points such that the convex hull of $x+E$ contains 0. As a result, $C^* = \bigcup_{e\in E}(C+x+e)^*$. If $C+x$ is co-c.e. closed then each $(C+x+e)^*$ is c.e. closed so C^* is c.e. closed, hence C is co-c.e. closed. If $C+x$ is c.e. closed then each $(C+x+e)^*$ is co-c.e. closed so C^* is co-c.e. closed, hence C is c.e. closed. ◀

It was proved in [7] and [5] that if f is computable and differentiable at c then $\underline{S}(f(c), c) = \overline{S}(f(c), c) = f'(c)$. If $x = (c, f(c))$ and $v = (1, f'(c))$ then it means that $\langle d, x\rangle$ is left-c.e. for all rational directions d such that $\langle d, v\rangle > 0$. We now investigate when this is uniform in d, i.e. when x is $\{v\}^*$-c.e.

▶ **Proposition 3.6** (Positive case)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be computable and convex or concave. If $c \in \mathbb{R}$ is left-c.e. and $x = (c, f(c))$ and $v = (1, f'_-(c))$ where $f'_-(c)$ is the left-derivative of f at c, then x is $\{v\}^*$-c.e.*

Proof. Assume that f is convex, the other case is obtained by considering $-f$. Let c_i be a computable increasing sequence converging to x. Let q, r be rational numbers such that $r < f'_-(c) < q$. Compute i such that $\frac{f(c_{i+1})-f(c_i)}{c_{i+1}-c_i} > r$. For $j \geq i$ one has $r < \frac{f(c)-f(c_j)}{c-c_j} \leq f'(c) < q$, so $rc - f(c) = \inf_{j\geq i} rc_j - f(c_j)$ and $qc - f(c) = \sup_{j\geq i} qc_j - f(c_j)$ are respectively right-c.e. and left-c.e., uniformly in r and q. ◀

▶ **Proposition 3.7** (Negative case)**.** *Let $c \in \mathbb{R}$ be left-c.e. and $f : \mathbb{R} \to \mathbb{R}$ be computable and such that $f'(c) = 0$ and $f(c)$ is not right-c.e. Let $x = (c, f(c))$ and $v = (1, f'(c)) = (1, 0)$. x is not $\{v\}^*$-c.e.*

Proof. Simply take $d = (0, -1) \in \{v\}^*$. d is computable but $\langle d, x\rangle = -f(c)$ is not left-c.e. ◀

Said differently, in that case $qc - f(c)$ is non-uniformly left-c.e. for rationals $q > 0$.

3.1 Converging sequences

We may equivalently define semicomputable points to be those points which are the limit of a computable sequence converging in some restricted region of the space, namely a salient convex cone. There is a precise relation between the cones where such sequences can live and the cones of directions in which the point is left-c.e.

The first observation is straightforward.

▶ **Proposition 3.8.** *Let $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ be a convex cone. If there exists a computable sequence x_i converging to x in $x - C$, then x is C^*-c.e.*

Proof. If $d \in C^*$ then $\langle d, x - x_i\rangle \geq 0$ so $\langle d, x\rangle = \sup_i \langle d, x_i\rangle$ is left-c.e. uniformly in d. ◀

In general it is not an equivalence. However when C is c.e. closed, or equivalently when C^* is co-c.e. closed, the equivalence holds.

▶ **Proposition 3.9.** *Let $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ be a salient c.e. closed convex cone. x is C^*-c.e. $\iff$ there exists a computable sequence converging to x in $x - C$.*

Proof. Assume that x is C^*-c.e. The interior of $x - C$ is an effective open set. Indeed, y belongs to that set iff $\min_{d \in C_1^*} \langle d, x - y\rangle > 0$, which is a c.e. condition as C_1^* is effectively compact. As a result, there is a computable enumeration $(y_i)_{i\in\mathbb{N}}$ of the rational vectors in that set. Define a computable sequence $(x_i)_{i\in\mathbb{N}}$ as follows: take $x_{i+1} \in \int(x - C)$ such that $y_0, \dots, y_i \prec x_{i+1}$.

As C is salient, the growing sequence x_i converges to a point in $x - C$. As it eventually exceeds each y_i, the limit must be x. ◀

3.2 Taking unions of convex cones

In $\mathbb{R}^2$, let P, Q be full-dimensional closed convex cones and R be the conical hull of $P \cup Q$. If $x \in \mathbb{R}^2$ is P-c.e. and Q-c.e., then x is R-c.e. However we will see below (Theorem 3.12) that this property fails in higher dimensions. We first show that it can be recovered under computability assumptions on P, Q.

▶ **Proposition 3.10.** *Let $P, Q \subseteq \mathbb{R}^n$ be closed convex cones, R be the conical hull of $P \cup Q$ and $x \in \mathbb{R}^n$ be P-c.e. and Q-c.e.*

- *If P and Q are c.e. closed then R is c.e. closed and x is R-c.e.,*
- *If P and Q are co-c.e. closed and R is salient, then R is co-c.e. closed and x is R-c.e.*

In the second statement, the condition that R is salient is needed otherwise the complexity of R can increase, as we now show.

▶ **Proposition 3.11.** *If $P, Q \subseteq \mathbb{R}^n$ are co-c.e. closed convex cones and R is the convex cone induced by $P \cup Q$, then R is $\emptyset'$-c.e. closed. In dimension $n \geq 3$ one can take P, Q so that R is not $\emptyset'$-co-c.e. closed.*

The proof essentially uses Lemma 2.4. Indeed, P^* and Q^* are c.e. closed and $R^* = P^* \cap Q^*$. One can embed the convex sets A, B from Lemma 2.4 in $\mathbb{R}^3$ and take their conical hulls.

We will see that the second item fails when R is not salient (Corollary 4.7). We now build a simpler example without the co-c.e. assumption about P and Q.

▶ **Theorem 3.12.** *In dimension $n \geq 3$, there exist closed convex cones $P, Q \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ that is P-c.e. and Q-c.e. but not R-c.e., where R is the conical hull of $P \cup Q$.*

The idea of the proof is to build a, b, c such that $qc - a$ and $rc - b$ are *uniformly* left-c.e. for $q > \overline{S}(a, c)$ and $r > \overline{S}(b, c)$, but $sc - (a+b)$ is *non-uniformly* left-c.e. for $q > \overline{S}(a+b, c)$. To do this, we take $a = f(c)$ ang $b = g(c)$ where f, g satisfy the conditions of Proposition 3.6 but $f + g$ satisfies the conditions of Proposition 3.7.

3.3 Semicomputability range of a point

▶ **Definition 3.13.** *If $x \in \mathbb{R}^n$ then its* **semicomputability range***, or simply* **range***, is the set of computable $d \in \mathbb{R}^n$ such that $\langle d, x\rangle$ is left-c.e., and is denoted by* $\text{range}(x)$.

One of the main goals of the paper is to investigate the following problem.

▷ Problem 1. What sets can be realized as $\text{range}(x)$ for some x?

Let $\mathbb{R}_c$ be the field of computable real numbers. From the definition we see that $\text{range}(x)$ contains computable points from $\mathbb{R}_c^n$ only. By abuse of language, when we write $A \subseteq \text{range}(x)$ we mean $A \cap \mathbb{R}_c^n \subseteq \text{range}(x)$, and similarly, $\text{range}(x) = A$ means $\text{range}(x) = A \cap \mathbb{R}_c^n$. The interior of $\text{range}(x)$ is meant to be the interior of $\text{range}(x)$ in the subspace topology on $\mathbb{R}_c^n$.

▶ **Proposition 3.14.** *Let $x \in \mathbb{R}^n$:*

1. $\mathrm{range}(x)$ *is a convex cone over the field* $\mathbb{R}_c$.
2. *If $D \subseteq \mathbb{R}^n$ is a closed polygonal convex cone with computable coordinates, then x is D-c.e. $\iff$ $D \subseteq \mathrm{range}(x)$.*
3. *If $D \subseteq \mathbb{R}^n$ be a closed convex cone contained in the interior of* $\mathrm{range}(x)$*, then x is D-c.e.*

Proof. **1.** Straightforward.

2. x is D-c.e. $\iff$ x is d-c.e. for each extreme direction $d \in D$ $\iff$ each such direction belongs to $\mathrm{range}(x)$.
3. There exists a rational polygonal convex cone E containing D and contained in $\mathrm{range}(x)$. By 2., x is E-c.e. hence D-c.e. ◀

We will see that $\mathrm{range}(x)$ is not necessarily closed (in the subspace $\mathbb{R}^n_c$), and that the third item sometimes fails when D is just contained in $\mathrm{range}(x)$ (Theorem 4.6).

▶ **Proposition 3.15.** *Let $x \in \mathbb{R}^n$ be a semicomputable point. Let $D \subseteq \mathbb{R}^n$ be a closed convex cone such that x is D-c.e. and* $\mathrm{range}(x) = D$*. Then D is $\emptyset'$-co-c.e. closed.*

Proof. Let M be a machine that given a rational point $d \in D$ approximates $\langle d, x\rangle$ from the left. With $\emptyset'$ as oracle, given a rational point d, one can compute x, $\langle d, x\rangle$ and $M(d)$ and eventually see whether $M(d) \neq \langle d, x\rangle$, which means that $d \notin D$. As a result, the set of rational points outside D is c.e. relative to $\emptyset'$ so D is Π^0_2 by Proposition 2.5. ◀

We will see that this is tight: every $\emptyset'$-co-c.e. closed convex cone can be obtained (Theorem 4.8).

3.4 Solovay complete coordinates

When one of the coordinates of $x \in \mathbb{R}^n$ is Solovay complete, the range of x is easily described.

▶ **Proposition 3.16.** *Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ where x_1 is Solovay complete. Let $v = (1, S(x_2, x_1), \dots, S(x_n, x_1))$. One has* $\overline{\mathrm{range}(x)} = \{v\}^*$.

For a closed convex cone C,

$$v \in \mathrm{int}(C) \implies x \text{ is } C^*\text{-c.e.} \implies v \in C.$$

Proof. A computable sequence x_i converging to x must asymptotically converge along the direction v, for each rational d such that $\langle d, v\rangle > 0$, one eventually has $\langle d, x - x_i\rangle > 0$, so $\langle d, x\rangle$ is left-c.e. The set of such vectors d is dense in $\{v\}^*$.

If v belongs to the interior of C then C^* is contained in the interior of $\{v\}^* = \overline{\mathrm{range}(x)}$ so x is C^*-c.e. by Proposition 3.14 item 3. If x is C^*-c.e. then $C^* \subseteq \mathrm{range}(x) \subseteq \{v\}^*$, i.e. $v \in C$. ◀

In particular, if d is a computable vector such that $\langle d, v\rangle > 0$, then $\langle d, x\rangle$ is left-c.e.

4 Realizing convex cones

In this section we investigate the possible ranges of semicomputable points. In order to realize a given convex cone D, we build a point that is left-c.e. along each computable direction in D, and no more. To do so, we make the point *generic* in some sense. Let us briefly recall from [4] the notion of genericity that we need.

4.1 Genericity

▶ **Definition 4.1.** *Let $A \subseteq \mathbb{R}^n$. A point $x \in A$ is* **generic inside** *A if for every effective open set $U \subseteq \mathbb{R}^n$, either $x \in U$ or there exists a neighborhood B of x such that $B \cap U \cap A = \emptyset$.*

▶ **Example 4.2.**
- Taking $A = X$, being generic inside X amounts to being 1-generic,
- Every x is obviously generic inside $\{x\}$,
- In the space of real numbers with the Euclidean topology, a real number $x \in (0, 1)$ is said to be right-generic if x is generic inside $[x, 1]$,

The last example is a particular instance of the following general situation.

If τ' is a weaker topology on X then we define $S(x)$ as the intersection of the τ'-open sets containing x. Equivalently, $S(x) = \{y \in X : x \leq_{\tau'} y\}$ where $\leq_{\tau'}$ is the specialization pre-order defined by $x \leq y$ iff every τ'-neighborhood of x contains y.

Let τ be the Euclidean topology on $\mathbb{R}^n$.

▶ **Theorem 4.3** (Theorem 4.1.1 in [4])**.** *Let τ' a topology that is effectively weaker than τ, such that emptiness of finite intersections of basic open sets in τ and τ' is decidable. There exists a point x that is computable in $(\mathbb{R}^n, \tau')$ and generic inside $S(x)$.*

For instance, the topology τ' generated by the semi-lines $(q, +\infty)$ is effectively weaker than τ, and its specialization pre-order is the natural ordering $\leq$ on $\mathbb{R}$. Theorem 4.3 implies the existence of right-generic left-c.e. reals.

▶ **Proposition 4.4.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex cone. If x is generic inside $x + C$ then $\mathrm{range}(x) \subseteq C^*$.*

Proof. Let $d \notin C^*$ be computable and assume that $\alpha := \langle d, x\rangle$ is left-c.e. The set $U = \{y : \langle d, y\rangle < \alpha\}$ is effectively open and $x \notin U$. As $d \notin C^*$ there exists $c \in C$ such that $\langle d, c\rangle < 0$. For $\epsilon > 0$, $\langle d, x + \epsilon c\rangle < \langle d, x\rangle = \alpha$ so $x + \epsilon c \in U \cap (x + C)$. As a result, x belongs to the closure of $U \cap (x + C)$ so x is not generic inside $x + C$. ◀

In particular, if x is C^*-c.e. and generic inside $x + C$ then $\mathrm{range}(x) = C^*$.

4.2 Realizing convex cones

Theorem 4.3 can now be applied to obtain a first class of cones realized as ranges of points.

▶ **Theorem 4.5** (C.e. closed cones)**.** *Let $C \subseteq \mathbb{R}^n$ be a co-c.e. closed convex cone. There exists x that is C^*-c.e. and generic inside $x + C$, hence $\mathrm{range}(x) = C^*$.*

Proof. C^* is c.e. closed, llet $d_i \in C^*$ be a computable dense sequence. Consider the topology τ' generated by the basic open sets $U_{i,j} = \{x : \langle d_i, x\rangle > q_j\}$, where $(q_j)_{j\in\mathbb{N}}$ is some computable enumeration of the positive rational numbers. One easily checks that emptiness of finite intersections of basic open sets in τ, τ' is decidable, so we can apply Theorem 4.3. We obtain a point x that is computable in $(\mathbb{R}^n, \tau')$, i.e. the numbers $\langle d_i, x\rangle$ are uniformly left-c.e. hence x is C^*-c.e. and $C^* \subseteq \mathrm{range}(x)$. Moreover x is generic inside $S(x) = x + C$, hence $\mathrm{range}(x) \subseteq C^*$ by Proposition 4.4. ◀

Therefore, any c.e. closed convex cone can be realized as the range of a point. We can extend the result to other classes of closed convex cones. To do so, we need to refine the construction techniques.

▶ **Theorem 4.6** (Σ^0_2 cones)**.** *Let* $(D_k)_{k\in\mathbb{N}}$ *be a growing sequence of uniformly co-c.e. closed convex cones in* $\mathbb{R}^n$*. There exists* x *such that for any co-c.e. closed convex cone* K*,* x *is* K*-c.e.* $\iff$ K *is contained in some* D_k*. In particular,* $\mathrm{range}(x) = \bigcup_k D_k$*.*

In particular, any $\emptyset'$-effectively open convex cone is the range of a point.

We can use this result to give a counter-example to Proposition 3.10 when the cone is not salient.

▶ **Corollary 4.7.** *There exists co-c.e. closed convex cones* $P, Q \subseteq \mathbb{R}^3$ *and a point* x *that is* P*-c.e. and* Q*-c.e. but not* R*-c.e., where* R *is the convex cone induced by* $P \cup Q$*.*

Proof. Take P, Q from Proposition 3.11. The induced cone R is $\emptyset'$-c.e. closed but not $\emptyset'$-co-c.e. closed. By Proposition 2.3, R contains a dense Σ^0_2-set R', and we can assume that R' contains P and Q (otherwise replace R' with $R' \cup P \cup Q$). By Theorem 4.6 there exists x such that $\mathrm{range}(x) = R'$, x is P-c.e. and Q-c.e. But x is not R-c.e., otherwise R would be $\emptyset'$-co-c.e. closed by Proposition 3.15. ◀

▶ **Theorem 4.8** (Π^0_2 cones)**.** *Let* $D \subseteq \mathbb{R}^n$ *be a salient* Π^0_2 *convex cone. There exists* x *that is* D*-c.e. and such that* $\mathrm{range}(x) = D$*.*

4.3 Beyond linear maps

If x is C^*-c.e. and generic inside $x + C$ then we know for which computable linear maps $f : \mathbb{R}^n \to \mathbb{R}$ the number $f(x)$ is left-c.e.: exactly when $f \in C^*$ (f can be identified with the vector v such that $f(x) = \langle v, x\rangle$).

Genericity has also consequences on functions $f : \mathbb{R}^n \to \mathbb{R}$ that are not linear but totally differentiable. We recall that if f is ***totally differentiable*** at x then there exists a vector $\mathrm{grad} f(x)$ such that $f(x + h) = f(x) + \langle \mathrm{grad} f(x), h\rangle + o(h)$.

▶ **Proposition 4.9.** *Let* $C \subseteq \mathbb{R}^n$ *be a closed convex cone. Let* $x \in \mathbb{R}^n$ *be* C^**-c.e. and generic inside* $x + C$*. Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be computable and totally differentiable at* x*.*

- *If* $\mathrm{grad} f(x) \in \mathrm{int}(C^*)$ *then* $f(x)$ *is left-c.e.*
- *If* $\mathrm{grad} f(x) \notin C^*$ *then* $f(x)$ *is not left-c.e.*

Proof. Let $D^* \subseteq \mathrm{int}(C^*)$ be a computable polygonal convex salient cone containing $\mathrm{grad} f(x)$ in its interior. There exists $\delta > 0$ such that $\langle \mathrm{grad} f(x), d\rangle > \delta$ for all $d \in D_1$. As x is D^*-c.e., there exists a computable sequence $x_i \in x - D$ converging to x by Proposition 3.9. Therefore one has $f(x_i) = f(x) - \langle \mathrm{grad} f(x), (x - x_i)\rangle + o(x - x_i) < f(x) - \|x - x_i\|\, \delta + o(x - x_i) < f(x)$ for i larger than some i_0, so $f(x) = \sup_{i\geq i_0} f(x_i)$ is left-c.e.

Assume that $\mathrm{grad} f(x) \notin C^*$ and that $\alpha := f(x)$ is left-c.e. The set $U = \{y : f(y) < \alpha\}$ is effectively open. As $\mathrm{grad} f(x) \notin C^*$, there exists $c \in C$ such that $\langle \mathrm{grad} f(x), c\rangle < 0$. One has $f(x+\epsilon c) = f(x)+\epsilon\langle \mathrm{grad} f(x), c\rangle+o(\epsilon) < f(x)$ for sufficiently small ϵ, so $x+\epsilon c \in U\cap(x+C)$. Therefore $x \notin U$ and belongs to the closure of $(x + C) \cap U$, contradicting the assumption that x is generic inside $x + C$. ◀

5 Application to the Solovay derivatives

We pursue the study of the Solovay derivatives $\overline{S}(a, b)$ and $\underline{S}(a, b)$ started in [5] in the general case, i.e. without assuming that b is Solovay complete. The general goal is to find ways to calculate $\underline{S}(a, b)$ and $\overline{S}(a, b)$ for given a, b. Although formulae are available in some cases, we investigate one of the simplest situations in which no general formula exists:

▷ Problem 2. If $a, b, c \in \mathbb{R}$ are fixed, what can be the shapes of the functions $\overline{S}(aX + b, c)$ and $\underline{S}(aX + b, c)$, where X varies among the computable real numbers?

When c is Solovay complete, one has $S(x, c) := \overline{S}(x, c) = \underline{S}(x, c)$ for any d-c.e. x and

$$S(aX + b, c) = S(a, c)X + S(b, c).$$

However in general only inequalities can be derived (see [5]):

If $X \geq 0$,

$$\overline{S}(aX + b, c) \leq \overline{S}(a, c)X + \overline{S}(b, c)$$
$$\underline{S}(aX + b, c) \geq \underline{S}(a, c)X + \underline{S}(b, c).$$

If $X \leq 0$,

$$\overline{S}(aX + b, c) \leq \underline{S}(a, c)X + \overline{S}(b, c)$$
$$\underline{S}(aX + b, c)) \geq \overline{S}(a, c)X + \underline{S}(b, c).$$

It seems at first that these two functions of X should be very rigid because a, b, c are fixed, so their local shape should not depend too much on X. However, we will see that, up to some geometrical contraints, they can have a wide variety of possible shapes. Fortunately, we can use the notions and results of this paper to precisely identify the class of possible shapes of these two functions. The idea is geometrical: these functions can be read in some way from the convex cone range(x), where $x = (a, b, c)$. Therefore their shapes are precisely the shapes that can be obtained from arbitrary convex cones. Let $\overline{\mathbb{R}} = [-\infty, +\infty]$.

▶ **Definition 5.1.** *Let $\mathcal{F}$ be the family of pairs of functions (f, g) from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that:*

- *$f \geq g$,*
- *f is convex and g is concave (i.e., the epigraphs of f and $-g$ are convex sets),*
- *Every line segment joining the graph of f to the graph of g lies below the graph of f and above the graph of g.*

The third condition implies that $\lim_{x\to-\infty} f'(x) = \lim_{x\to+\infty} g'(x)$ and $\lim_{x\to+\infty} f'(x) = \lim_{x\to-\infty} g'(x)$. Examples of such pairs are: $f(X) = -g(X) = \sqrt{1 + X^2}$, or $f(X) = X^2$ and $g(X) = -\infty$.

The main result of this section is that $\mathcal{F}$ captures essentially the possible shapes of $(\overline{S}(aX + b, c), \underline{S}(aX + b, c))$, up to computability conditions.

▶ **Theorem 5.2.** *Let $a, b, c \in \mathbb{R}$ with c left-c.e. and non-computable. One has $(\overline{S}(aX + b, c), \underline{S}(aX + b, c)) \in \mathcal{F}$. Conversely,*

- *Any pair $(f, g) \in \mathcal{F}$ where f is $\emptyset'$-left-c.e. and g is $\emptyset'$-right-c.e. can be realized,*
- *Any pair $(f, g) \in \mathcal{F}$ where f is $\emptyset'$-right-c.e. and g is $\emptyset'$-left-c.e. can be realized,*

To prove this result we show that the pairs $(f, g) \in \mathcal{F}$ are exactly the functions that can be read on convex cones in $\mathbb{R}^3$ in the following way: given a cone C in $\mathbb{R}^3$, the intersection of C with the planes $y = \pm 1$ convex sets, and the curves delimiting them are exactly the pairs $(f, g) \in \mathcal{F}$.

Now, if $x = (a, b, c) \in \mathbb{R}^*$ then the pair $(\overline{S}(aX + b, c), \underline{S}(aX + b, c))$ is obtained in this way from the cone $C = \text{range}(x)$, so it belongs to $\mathcal{F}$. A pair $(f, g) \in \mathcal{F}$ can be realized by building a point whose range induces (f, g), which can be done by imposing computability conditions on f and g and applying the results from Section 4.

6 Left-c.e. quadratic polynomials

In this section, we briefly investigate the quadratic real polynomials $P_{a,b,c}(X) = aX^2 + bX + c$ that are left-c.e. functions of X. Our main problem is the following:

▷ Problem 3. For which triples (a, b, c) is the polynomial $P_{a,b,c}$ left-c.e.?

The key observation is that $P_{a,b,c}(X)$ is linear in (a, b, c), which allows to think of a left-c.e. polynomial as a semicomputable point $(a, b, c) \in \mathbb{R}^3$. More precisely, the ordering $(a, b, c) \preceq (a', b', c')$ defined by $P_{a,b,c} \leq P_{a',b',c'}$ is a vector space ordering. Hence its positive cone is a convex cone $C = \{(a, b, c) \in \mathbb{R}^3 : P_{a,b,c} \geq 0\} = \{(a, b, c) \in \mathbb{R}^3 : a, c \geq 0 \text{ and } b^2 \leq 4ac\}$. Its dual is $C^* = \{(a, b, c) \in \mathbb{R}^3 : a, c \geq 0 \text{ and } b^2 \leq ac\}$ and is the closure of the conical hull of the vectors $(X^2, X, 1)$, with $X \in \mathbb{R}$.

Thus $P_{a,b,c}$ is left-c.e. if and only if (a, b, c) is C^*-c.e. This reformulation allows us to think geometrically about left-c.e. polynomials, and to apply the results of this paper to these objects. Let us list a few properties of left-c.e. polynomials, some of them being derived from the analysis developed in the paper:

1. There is a symmetry between a and c and between b and $-b$. More precisely, $P_{a,b,c}$ is left-c.e. $\iff$ $P_{a,b,c}, P_{c,b,a}, P_{a,-b,c}$ and $P_{c,-b,a}$ are left-c.e. for $X \geq 1$.
2. If $P_{a,b,c}$ is left-c.e. then a, c are left-c.e. and b is d-c.e. (b is a difference of left-c.e. numbers).
3. If a is Solovay complete left-c.e. then (Proposition 3.16)
$$S(b, a)^2 < 4S(c, a) \implies P_{a,b,c} \text{ is left-c.e.} \implies S(b, a)^2 \leq 4S(c, a).$$
4. Let $P_{a,b,c}$ be left-c.e. For computable $X > 0$,
$$-\frac{1}{\sqrt{X}} \leq \underline{S}(b, aX + c) \quad \text{and} \quad \overline{S}(b, aX + c) \leq \frac{1}{\sqrt{X}}.$$
Indeed, $aX^2 + bX + c$ is left-c.e. for all computable $X \in \mathbb{R}$ $\iff$ $\frac{1}{\sqrt{Y}}(aY + c) \pm b$ is left-c.e. for all computable $Y > 0$ (take $Y = X^2$).
5. Let $x = (a, b, c)$ be C^*-c.e. and generic inside $x + C$ (it exists as C^* is computable, see Theorem 4.3). $P_{a,b,c}$ is left-c.e. and for computable $X > 0$,
$$-\frac{1}{\sqrt{X}} = \underline{S}(b, aX + c) \quad \text{and} \quad \overline{S}(b, aX + c) = \frac{1}{\sqrt{X}}.$$
The second equality is obtained as follows: for a rational $q < \frac{1}{\sqrt{X}}$, $(qX, -1, q) \notin C^* = \text{range}(x)$ so $q(aX + c) - b$ is not left-c.e., hence $\overline{S}(b, aX + c) = \frac{1}{\sqrt{X}}$.
Although b is Solovay reducible to $aX + c$ for each computable $X > 0$, b is not reducible to neither a nor c and $\underline{S}(b, a) = \underline{S}(b, c) = -\infty$ and $\overline{S}(b, a) = \overline{S}(b, c) = +\infty$. Indeed, for $q \in \mathbb{Q}$, both $(q, \pm 1, 0)$ and $(0, \pm 1, q)$ are outside C^*.
6. The condition that $P_{a,b,c}$ is left-c.e. cannot be reduced to a finite number of linear combination of a, b, c being left-c.e. Indeed, such a condition would express that the point (a, b, c) is D-c.e. for some polygonal convex cone D, but the convex cone C^* is not polygonal (it is determined by infinitely many directions).
7. The condition that $P_{a,b,c}$ is left-c.e. cannot be characterized by simply considering the values of $\underline{S}(b, c)$, $\overline{S}(b, c)$, $\underline{S}(a, c)$, $\overline{S}(a, c)$, $\underline{S}(b, a)$, $\overline{S}(b, a)$. Indeed, these values only reflect the intersections of $\text{range}(a, b, c)$ with the three planes $z = 0$, $x = 0$ and $y = 0$, which do not determine completely $\text{range}(a, b, c)$.

We do not know if it is possible to better understand Problem 3, i.e. whether it is possible to reduce this property to more fundamental properties of a, b, c. The results presented above suggest a negative answer to that question.

We mention that a similar analysis can be made of co-c.e. disks in the plane. The disk centered at (a, b) with radius c is co-c.e. if and only if the point $(a, b, c) \in \mathbb{R}^3$ is C^*-c.e., where $C = C^* = \{(x, y, z) : z \leq 0, x^2 + y^2 \leq z^2\}$. Again C^* is not polygonal so one cannot reduce the condition that the disk is co-c.e. to a finite number of conditions.

References

1 George Barmpalias and Andrew Lewis-Pye. Differences of halting probabilities. *J. Comput. Syst. Sci.*, 89:349–360, 2017.

2 Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.

3 Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1-3):43–76, 2003.

4 Mathieu Hoyrup. Genericity of weakly computable objects. *Theory Comput. Syst.*, 60(3):396–420, 2017.

5 Mathieu Hoyrup, Diego Nava Saucedo, and Don M. Stull. Semicomputable geometry. In Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella, editors, *45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic*, volume 107 of *LIPIcs*, pages 129:1–129:13. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018.

6 Martin Kummer and Marcus Schäfer. Computability of convex sets. In Ernst W. Mayr and Claude Puech, editors, *STACS 95*, pages 550–561, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg.

7 Joseph S. Miller. On work of Barmpalias and Lewis-Pye: A derivation on the D.C.E. reals. In Adam R. Day, Michael R. Fellows, Noam Greenberg, Bakhadyr Khoussainov, Alexander G. Melnikov, and Frances A. Rosamond, editors, *Computability and Complexity - Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday*, volume 10010 of *Lecture Notes in Computer Science*, pages 644–659. Springer, 2017.

8 André Nies. *Computability and randomness*. Oxford logic guides. Oxford University Press, 2009.

9 Martin Ziegler. Computability on regular subsets of euclidean space. *Mathematical Logic Quarterly*, 48(S1):157–181, 2002.

A Proof of Lemma 2.1

If $z \in C^*$ then $d(z, C^*) = 0$ so equality is satisfied. Assume that $z \notin C^*$. Let $y_0 \in C^*$ be such that $\|y_0 - z\| = d(z, C^*)$ (such a point exists because C^* is closed, hence its intersection with a closed ball is compact). Let $x = y_0 - z$, we show that $d(z, C^*) = d(z, H_x)$.

▷ Claim A.1. x and y_0 are orthogonal.

For every $y \in C^*$, $\langle x, y\rangle \geq \langle x, y_0\rangle$. Indeed, let $y \in C^*$ and for $t \in [0,1]$ let $y_t = y_0 + t(y - y_0)$. The function $t \mapsto \|y_t - z\|$ is minimal at $t = 0$ so its derivative at 0, which is $2\langle x, y - y_0\rangle$, is nonnegative. Now, applying the inequality to $y = 0$ and $y = 2y_0$, one derives that $\langle x, y_0\rangle = 0$. As a result, hence $x \in (C^*)^*$.

Now, $d(z, H_x) = -\frac{\langle x,z\rangle}{\|x\|} = \frac{\langle x,y_0\rangle - \langle x,z\rangle}{\|x\|} = \frac{\langle x,y_0-z\rangle}{\|x\|} = \|x\| = d(z, C^*)$.

To finish, $(C^*)^*$ is the topological closure of $\mathbb{R}_+C = \{\alpha c : \alpha \geq 0, c \in C\}$ and $H_{\alpha x} = H_x$ for $\alpha > 0$. For every $\epsilon > 0$, x is ϵ-close to some $\alpha x'$, $x' \in C$ so $d(z, H_x)$ is close to $d(z, H_{x'})$ by $\|z\|\,\epsilon$.

B Proof of Proposition 3.10

Assume first that P, Q are c.e. closed, let $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ be computable dense sequences in P and Q respectively. The computable sequence $(p_i + q_j)_{i,j\in\mathbb{N}}$ is dense in R, which is then c.e. closed. We show that x is R-c.e. Let $K \in \mathbb{N}$ be an upper bound on $\|x\|$. Given $r \in R$ and $\epsilon > 0$, one can compute $i, j \in \mathbb{N}$ such that $d(r, p_i + q_j) < \epsilon$. One has $|\langle r, x\rangle - \langle p_i + q_j, x\rangle| < \epsilon K$ and $\langle p_i + q_j, x\rangle = \langle p_i, x\rangle + \langle q_j, x\rangle$ is left-c.e. Everything is uniform in r, ϵ, so $\langle r, x\rangle$ is left-c.e. uniformly in r.

Now assume that P, Q are co-c.e. closed and R is salient. The dual cones P^*, Q^* are c.e. closed. As R is salient, its dual $R^* = P^* \cap Q^*$ is full-dimensional. By Lemma 2.4, R^* is c.e. closed hence R is co-c.e. closed. We show that x is R-c.e. by applying Proposition 3.4. As x is P-c.e. and Q-c.e., $x - P^*$ and $x - Q^*$ are c.e. closed hence $x - R^* = (x - P^*) \cap (x - Q^*)$ is c.e. closed, so x is R-c.e.

C Proof of Proposition 3.11

Relative to $\emptyset'$, P and Q are c.e. closed so R is c.e. closed by relativizing the first item in Proposition 3.10.

We first build a right-c.e. convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(0)$ is not $\emptyset'$-c.e. closed. Let $\alpha > 0$ be $\emptyset'$-right-c.e. but not $\emptyset'$-left-c.e. There exists a sequence of uniformly left-c.e. reals $\alpha_i > 0$ such that $\alpha = \inf_i \alpha_i$. Let $f_i(x) = \max(0, x - \alpha_i)$ and $f = \sum_i 2^{-i} f_i$. The functions f_i are uniformly right-c.e. so f is right-c.e., and $f^{-1}(0) = [0, \alpha]$ is not $\emptyset'$-c.e. closed because α is not $\emptyset'$-left-c.e.

Let P, Q be the closed convex cones such that P^* is induced by $\{(x, y, 1) : x \in \mathbb{R}, y \geq f(x)\}$ and $Q^* = \{(x, y, z) : y \leq 0\}$. Both are computably closed, but $R^* = P^* \cap Q^* = \{(x, 0, z) : f(x) = 0, z \geq 0\}$ is not $\emptyset'$-c.e. closed so R is not $\emptyset'$-co-c.e. closed.

D Proof of Theorem 3.12

▶ **Lemma D.1.** *There exists a left-c.e. number c and differentiable non-decreasing functions f, g satisfying the following conditions:*

- *f' is non-decreasing and g' is non-increasing and both are computable,*

- *$f'(c) + g'(c) = 0$ and $f(c) + g(c)$ is not right-c.e.*

Proof. Let $K \subseteq [0,1]$ be a non-empty co-c.e. closed set containing no computable real, and let $c = \min K$. Let (a_i, b_i) be a computable covering of its complement. Let F_i, G_i be the piecewise affine functions such that $F_i(x) = 0$ for $x \leq a_i$, $F_i(x) = b_i - a_i$ for $x \geq \frac{a_i+b_i}{2}$ and $G_i(x) = 0$ for $x \leq \frac{a_i+b_i}{2}$ and $G_i(x) = -(b_i - a_i)$ for $x \geq b_i$. These functions are uniformly computable. Let $F = \sum_i 2^{-i} F_i$ and $G = \sum_i 2^{-i} G_i$. F and G are computable and $F + G$ is non-negative and null exactly on K, in particular $F(c) + G(c) = 0$.

Let f, g be the antiderivatives of F, G respectively such that $f(0) = g(0) = 0$, and $h = f+g$. Observe that h is non-decreasing and computable so $h(c)$ is left-c.e. If $h(c)$ is computable then $h^{-1}(h(c))$ is a co-c.e. closed interval containing c. As c is not right-c.e., this interval is not a singleton. As a result, h is constant hence $h' = 0$ on that interval, which is not possible as h' is null on K only, and K contains no interval. ◀

Proof of Theorem 3.12. Let $x = (c, f(c), g(c))$, $v_f = (1, f'(c), 0)$ and $v_g = (1, 0, g'(c))$. Let

$$P_1 = \{v_f\}^* \qquad P_2 = \{(0,0,-1)\}^* \qquad Q_1 = \{v_g\}^* \qquad Q_2 = \{(0,1,0)\}^*.$$

Observe that x is P_1-c.e. and Q_1-c.e. by Proposition 3.7, and P_2-c.e. and Q_2-c.e. because $f(c)$ is left-c.e. and $g(c)$ is right-c.e. Therefore, if $P = P_1 \cap P_2$ and $Q = Q_1 \cap Q_2$ then x is P-c.e. and Q-c.e.

Let R be the conical hull of $P \cup Q$. One has $R = \{(1, f'(c), g'(c))\}^*$ and x is not R-c.e. as $d := (0,-1,-1)$ is a computable point of R but $\langle d, x\rangle = -f(c) - g(c)$ is not left-c.e. ◀

E Proof of Theorem 4.6

E.1 Background on computability of sets

For compact sets X, Y, let $d(x,Y) = \inf_{y\in Y} d(x,y)$, $d(X,Y) = \sup_{x\in X} d(x,Y)$ and $d_H(X,Y) = \max\{d(X,Y), d(Y,X)\}$ be the Hausdorff metric.

Let X be compact. Let $\mathcal{S}$ be any sequence of uniformly computable compact sets that is dense for d_H. The following conditions are equivalent:

- X is co-c.e. closed,
- The function $x \mapsto d(x, X)$ is left-c.e.,
- The function $S \in \mathcal{S} \mapsto d(S, X)$ is left-c.e.,
- The function $S \in \mathcal{S} \mapsto d(X, S)$ is right-c.e.

Let X be compact. The following conditions are equivalent:

- X is c.e. closed,
- The function $x \mapsto d(x, X)$ is right-c.e.,
- The function $S \mapsto d(S, X)$ is right-c.e.,
- The function $S \mapsto d(X, S)$ is left-c.e.

E.2 Proof

Let $D_n[s]$ be a decreasing computable sequence of closed convex cones whose intersection is D_n. Let $C_n = D_n^*$ and $C_n[s] = D_n[s]^*$. One has $C_{n+1} \subseteq C_n$ and $C_n = \bigcup_s C_n[s]$.

We build a computable rational sequence x_s converging to some x. Let $(U_n)_{n\in\mathbb{N}}$ be an effective enumeration of the effective open sets. We will satisfy the following requirements by an application of the finite injury method:

$$R_n\colon\ x \in U_n \text{ or there exists } \epsilon > 0 \text{ such that } B(x,\epsilon) \cap U_n \cap (x + C_n) = \emptyset.$$

Construction. We start with some x_0 and let $B_n[0] = B(x_0, 2^{-n})$.

At stage s, if there exists $n \leq s$ such that $B_n[s] \cap U_n[s] \cap (x_s + C_n[s]) \neq \emptyset$ and R_n is not declared satisfied, then we choose the minimal such n and act by taking x_{s+1} in the intersection and $\overline{B}_n[s+1]$ centered at x_{s+1} and contained in $B_n[s] \cap U_n[s]$. For $m \geq n$, let $\overline{B}_{m+1}[s+1] \subseteq B_m[s+1]$ be a ball centered at x_{s+1} with radius decreasing to 0. We declare R_n satisfied and all requirements R_m with $m > n$ unsatisfied.

If there is no such $n \leq s$ then simply take $x_{s+1} = x_s$ and $B_n[s+1] = B_n[s]$ for all n.

Verification. For each n, $B_n[s]$ eventually settles to a ball B_n. One has $\overline{B}_{n+1} \subseteq B_n$, the radius of B_n converges to 0 so their intersection contains a point x. The sequence $(x_s)_{s\in\mathbb{N}}$ converges to x, as it eventually enters and stay in each ball B_n.

Each requirement R_n is satified: if $x \notin U_n$ and $B_n \cap U_n \cap (x + C_n) \neq \emptyset$ then for sufficiently large s, $R_0, \dots, R_n$ do not act from stage s, $B_n[s] = B_n$ and $B_n \cap U_n[s] \cap (x + C_n) \neq \emptyset$. As a result, for sufficiently large s one has $B_n \cap U_n[s] \cap (x_s + C_n[s]) \neq \emptyset$, so R_n should act and force x to be in U_n, contradicting $x \notin U_n$.

We now show that for each n, x is D_n-c.e. Indeed, let s be a stage after which no requirement $R_0, \dots, R_{n-1}$ acts. For $t \geq s$, one has $x_{t+1} - x_t \in C_m[t] \subseteq C_m \subseteq C_n$ for some $m \geq n$, so for any $d \in D_n$, the sequence $\langle d, x_t \rangle$ is non-decreasing after stage s, hence its limit $\langle d, x \rangle$ is left-c.e., uniformly in $d \in D_n$. Observe that this may not be uniform in n because we do not know at which stage the n first requirements stop acting.

Let now K be a co-c.e. closed convex cone that is not contained in any D_n. Assume that x is K-c.e. and let $U = \{y : \min_{d\in K_1} \langle d, y - x \rangle < 0\}$. As K_1 is effectively compact, U is effectively open so $U = U_n$ for some n. As K is not contained in D_n, let $d \in K \setminus D_n$ and let $c \in C_n$ be such that $\langle d, c \rangle < 0$. One has $\langle d, x + \epsilon c \rangle < \langle d, x \rangle$, so $x + \epsilon c \in U_n \cap (x + C_n)$ for all $\epsilon > 0$. As R_n is satified, we conclude that $x \in U_n$ which is impossible. The contradiction implies that x is not K-c.e.

F Proof of Theorem 4.8

Let $C = D^*$ (which implies that $D = C^*$). We build x that is D-c.e. and generic inside $x + C$.

Let $(d_n)_{n\in\mathbb{N}}$ be a computable enumeration of the rational vectors of our space. For rational vectors d and $s \in \mathbb{N}$, let $R(d, s)$ be a decidable predicate such that $d \in C^*$ iff $R(d, s)$ holds for infinitely many s, and if $s < n$ then $R(d_n, s)$ is false. Say that d is *active* at stage s if $R(d, s)$ holds. At any stage, there are finitely many active vectors, among $d_0, \dots, d_n$. At stage s, we say that $r < s$ is the *previous activation stage* of d if d is active at stage r but not at stages $r + 1, \dots, s - 1$.

▷ Claim F.1. We can assume w.l.o.g. that if C^* is contained in an open half-space, then for sufficiently large s, the only active vectors belong to that half-space.

Proof. The set of rational open half-spaces H_i containing C^* is $\emptyset'$-c.e., so there is a decidable predicate $Q(i, s)$ such that H_i contains C^* if and only if $Q(i, s)$ holds for almost all s, and $Q(i, s)$ does not hold if $i > s$. We replace $R(d, s)$ by $R'(d, s)$, which is true iff $R(d, s)$ is true and for all $i \leq s$, either $d \in H_i$ or there exists $t > s$ such that $Q(i, t)$ is false. If $d \in C^*$ and $R(d, s)$ holds then necessarily $R'(d, s)$ holds. If C^* is contained in H_i then for sufficiently large s, $Q(i, s)$ holds so if $R'(d, s)$ holds then $d \in H_i$. ◀

We build a computable rational sequence x_s converging to some x. Let $(U_n)_{n\in\mathbb{N}}$ be an effective enumeration of the effective open sets. The proof is similar to the proof of Theorem 4.6, with some changes in the details. We will satisfy the following requirements by an application of the finite injury method:

R_n: $x \in U_n$ or there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap U_n \cap (x + C) = \emptyset$.

Construction. We start with some x_0 and let $B_n[0] = B(x_0, 2^{-n})$.

At stage s, let T_s be the set of points y such that for every d that is active at stage s, and every $r < s$ where d is active, $\langle d, y\rangle \geq \langle d, x_r\rangle$.

If there exists $n \leq s$ such that $B_n[s] \cap U_n[s] \cap T_s \neq \emptyset$ and R_n is not declared satisfied, then we choose the minimal such n and act by taking x_{s+1} in the intersection and $\overline{B}_n[s+1]$ centered at x_{s+1} and contained in $B_n[s] \cap U_n[s]$. For $m \geq n$, let $\overline{B}_{m+1}[s+1] \subseteq B_m[s+1]$ be a ball centered at x_{s+1} with radius decreasing to 0. We declare R_n satisfied and all requirements R_m with $m > n$ unsatisfied.

If there is no such $n \leq s$ then simply take $x_{s+1} = x_s$ and $B_n[s+1] = B_n[s]$ for all n.

Verification. For each n, $B_n[s]$ eventually settles to a ball B_n. One has $\overline{B}_{n+1} \subseteq B_n$, the radius of B_n converges to 0 so their intersection contains a point x. The sequence $(x_s)_{s\in\mathbb{N}}$ converges to x, as it eventually enters and stay in each ball B_n.

Each requirement R_n is satified: if $x \notin U_n$ and $B_n \cap U_n \cap (x + C_n) \neq \emptyset$ then for sufficiently large s, $R_0, \ldots, R_n$ do not act from stage s, $B_n[s] = B_n$ and $B_n \cap U_n[s] \cap (x + C_n) \neq \emptyset$. As a result, for sufficiently large s one has $B_n \cap U_n[s] \cap (x_s + C_n[s]) \neq \emptyset$, so R_n should act and force x to be in U_n, contradicting $x \notin U_n$.

We now show that x is C^*-c.e. If $d \in C^*$ then d is active for infinitely many s. By definition of T_s, the subsequence of $\langle d, x_s\rangle_{s\in\mathbb{N}}$ at these stages is non-decreasing. As a result, $\langle d, x\rangle$ is the limit of $\langle d, x_s\rangle$ at these stages, so it is left-c.e. (uniformly in d).

We show that x is generic inside $x + C$.

▶ **Lemma F.2.** $\liminf_s T_s$ *is dense in* $x + C$.

Proof. As C^* is salient, C is full-dimensional so its interior is dense in C.

Let u belong to the interior of C and $y = x+u$. There exists $\delta > 0$ such that $\langle u, d\rangle > \delta$ for all $d \in C^*$, which means that C^* is contained in some open half-space, hence for sufficiently large s the same inequality holds for every active d at stage s.

Let s_0 be such that $\|x - x_s\| < \delta$ for all $s \geq s_0$. There exists s_1 such that if d is active at a stage $s \geq s_1$ then either it is active for the first time or its previous activation stage is at least s_0.

Let $s \geq s_1$, d be active at stage s and $r \geq s_0$ be its previous activation stage. One has $\langle y, d\rangle > \langle x, d\rangle + \delta > \langle x_r, d\rangle$ so $y \in T_s$.

As a result, $y \in \liminf_s T_s$. ◀

If $B_n \cap U_n \cap (x + C) \neq \emptyset$ then for sufficiently large s, $B_n[s] \cap U_n[s] \cap T_s \neq \emptyset$ so R_n must act, so $x \in U_n$. As a result, x is generic inside $x + C$.

G Proof of Theorem 5.2

We show that the pairs $(f, g) \in \mathcal{F}$ are exactly the functions that can be read on convex cones in $\mathbb{R}^3$ in some way.

▶ **Lemma G.1.** *Let* $C \subseteq \mathbb{R}^3$ *be a convex cone containing* $(0, 0, 1)$ *but not* $(0, 0, -1)$.

Let $f_C(X) = \inf\{z : (-X, -1, z) \in C\}$ *and* $g_C(X) = -\inf\{z : (X, 1, z) \in C\}$. *The pair* (f, g) *belongs to* $\mathcal{F}$.

Proof. If $f(X) < g(X)$ then $(0, 0, -1) \in C$. Indeed, let z_1, z_2 satisfy $z_1 < -z_2$, $(-X, -1, z_1) \in C$ and $(X, 1, z_2) \in C$. Summing these two vectors we see that $(0, 0, -1) \in C$.

The functions f and $-g$ are convex, because their epigraphs is the intersection of C with planes, so they are convex.

We show that the line segment joining $(X, f(X))$ and $(Z, g(Z))$ is above the graph of g.

Let $Y = \lambda X + (1-\lambda)Z$. Let $\alpha = \frac{\lambda}{1-\lambda} \geq 0$. As $A := (-X, -1, f(X))$ and $B := (Y, 1, -g(Y))$ belong to C, their combination $P := \alpha A + (\alpha+1)B$ must belong to C. Let $Z = -\alpha X + (\alpha+1)Y$. One has $P = (Z, 1, \alpha f(X) - (\alpha+1)g(Y))$. As $P \in C$, we have $\alpha f(X) - (\alpha+1)g(Y) \geq -g(Z)$. In other words, $g(Y) \leq \lambda f(X) + (1-\lambda)g(Z)$. As this is true for every $\lambda \in [0,1]$, the line segment between $(X, f(X))$ and $(Z, g(Z))$ is above the graph of g.

Showing that the line segment is below the graph of f is similar, by using the fact that the point $(\alpha+1)A + \alpha B$ belongs to C. ◀

We now show that evey pair in (f, g) can be obtained from some convex cone.

▶ **Lemma G.2.** *Let $(f, g) \in \mathcal{F}$. Let C be the convex cone induced by the points $(0, 0, 1)$ and the points $(-X, -1, f(X))$ and $(X, 1, -g(X))$ where X ranges over $\mathbb{R}$. One has $(f_C, g_C) = (f, g)$.*

Proof. Let $F = \{(-X, -1, z) : X \in \mathbb{R}, z \geq f(X)\}$ and $G = \{(X, 1, z) : X \in \mathbb{R}, z \geq -g(X)\}$. As f and $-g$ are convex and $(0, 0, 1) \in C$, $F \cup G \subseteq C$. It is sufficient to prove that if $A = (-X, -1, f(X))$, and $B = (Y, 1, -g(Y))$ and $P := \alpha A + \beta B$ (with $\alpha, \beta \geq 0$) belongs to the planes $y = \pm 1$, then $P \in F \cup G$.

In that case, one has $\beta - \alpha = \pm 1$. Assume that $\beta = \alpha + 1$, the other case is similar. Let $Z = (\alpha+1)Y - \alpha X$. One has $P = (Z, 1, -(\alpha+1)g(Y) + \alpha f(X))$ so we have to show that $P \in G$, i.e. that $-(\alpha+1)g(Y) + \alpha f(X) \geq -g(Z)$ or equivalently, $g(Y) \leq \lambda f(X) + (1-\lambda)g(Z)$ where $\lambda = \frac{\alpha}{\alpha+1} \in [0,1]$. As $Y = \lambda X + (1-\lambda)Z$, this condition exactly means that the line segment joining $(X, f(X))$ and $(Z, g(Z))$ is above the graph of g.

The case $\alpha = \beta + 1$ gives similarly that the line segment is below the graph of f. ◀

These results can now be applied to our problem.

▶ **Lemma G.3.** *Let $P = (a, b, c)$ where c is left-c.e. not computable. One has $(\overline{S}(aX + b, c), \underline{S}(aX + b, c)) = (f_C, g_C)$, where $C = \text{range}(P)$.*

Proof. For computable X, one has

$$\begin{aligned} \overline{S}(aX+b, c) &= \inf\{q \in \mathbb{Q} : qc - (aX+b) \text{ is left-c.e.}\} \\ &= \inf\{q \in \mathbb{Q} : (-X, -1, q) \in \text{range}(P)\} \\ &= f_{\text{range}(P)}(X), \end{aligned}$$

and similarly for $\underline{S}(aX + b, c)$. ◀

Proof of Theorem 5.2. By Lemma G.3, the pair $(\overline{S}(aX + b, c), \underline{S}(aX + b, c))$ is (f_C, g_C) where $C = \text{range}(P)$, so it belongs to $\mathcal{F}$ by Lemma G.1 (observe that C contains $(0, 0, 1)$ but not $(0, 0, -1)$ as c is left-c.e. but not right-c.e.).

Conversely, let $(f, g) \in \mathcal{F}$.

If f is $\emptyset'$-left-c.e. and g is $\emptyset'$-right-c.e., then the induced cone C defined in G.2 is $\emptyset'$-co-c.e. and $(f_C, g_C) = (f, g)$. By Theorem 4.8 there exists $P = (a, b, c) \in \mathbb{R}^3$ such that $\text{range}(P) = C$. As a result, $(\overline{S}(aX + b, c), \underline{S}(aX + b, c)) = (f_C, g_C) = (f, g)$ by Lemma G.3.

If f is $\emptyset'$-right-c.e. and g is $\emptyset'$-left-c.e., then the interior of C is effectively open relative to $\emptyset'$, so by Theorem 4.6 it is the range of some point and we conclude in the same manner. ◀