Computability in ergodic theory

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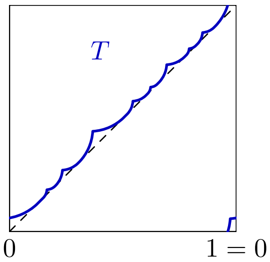
Given a computable dynamical system,

- is it possible to compute its invariant measures? the ergodic ones?
- is it possible to compute the speed of convergence of Birkhoff averages?
- is it possible to compute the ergodic decomposition of invariant measures?

Computability of invariant measures

Proposition (Galatolo, H. & Rojas, 2009)

There exists a computable dynamical system $T : \mathscr{S}^1 \to \mathscr{S}^1$ *with no computable invariant measure.*



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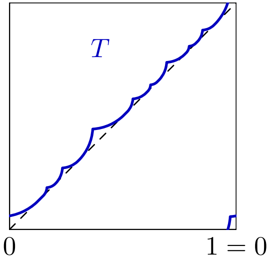

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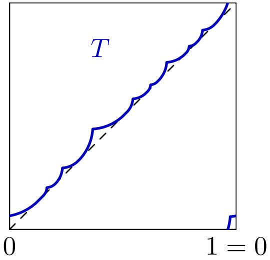

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Open question

What about the finitely ergodic case?

Birkhoff ergodic theorem

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map and μ a computable σ-invariant measure.

$$f^{(n)} = \frac{f + f \circ \sigma + \ldots + f \circ \sigma^{n-1}}{n} \underset{n\to\infty}{\longrightarrow} f^* \quad (L^1(\mu) \text{ and a.s.})$$

Theorem (V'yugin, 1997)

Let $f(x) = x_0$. There exists a computable shift-invariant measure μ such that the speed of convergence of $f^{(n)}$ to f^* is not computable.

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Theorem (Avigad, Gerhardy & Towsner, 2010)

The speed of convergence of $f^{(n)}$ to f^* is always computable from f and $\|f^*\|_2$.
In particular, if μ is ergodic then the speed is computable from f, as $\|f^*\|_2 = \|f\|_1$.

Computable probability measure

Definition

A **probability measure** μ **is computable** if the following equivalent conditions hold:

- there is an algorithm $A : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ such that

$$|A(w,n) - \mu[w]| < 2^{-n},$$

- there is a randomized algorithm computing a.s. a sequence $x \in \{0,1\}^{\mathbb{N}}$, whose distribution is μ, i.e.

$$\mathbb{P}(x \in [w]) = \mu[w].$$

Ergodic decomposition

- Let μ be a computable σ-invariant measure.
- By definition of *computable*, there is a randomized algorithm computing sequences with distribution μ.

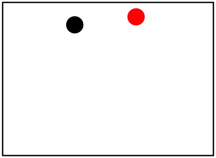
Definition

The **ergodic decomposition of μ is computable** if there is a randomized algorithm with *two* random oracles ω_1, ω_2 computing a.s. a sequence x, such that

- the distribution of x is μ,
- for a.e. fixed ω_1, the distribution of x is an ergodic measure.

Ergodic decomposition

The Pólya urn



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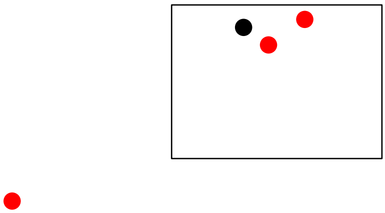

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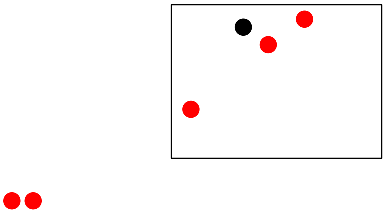

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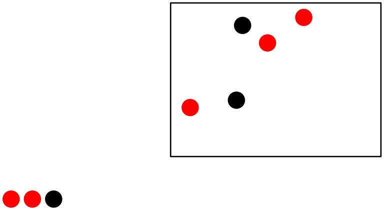

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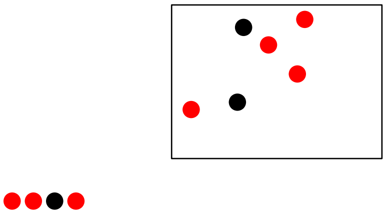

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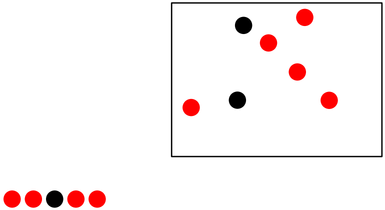

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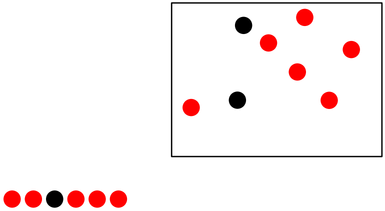

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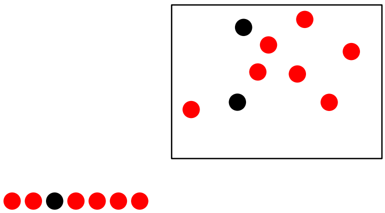

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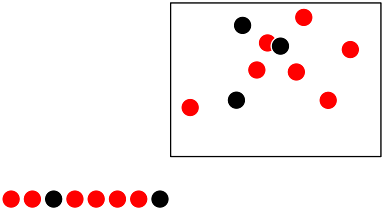

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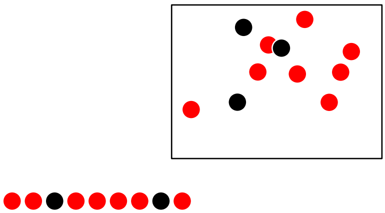

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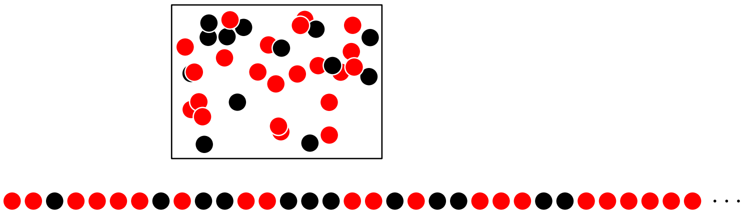

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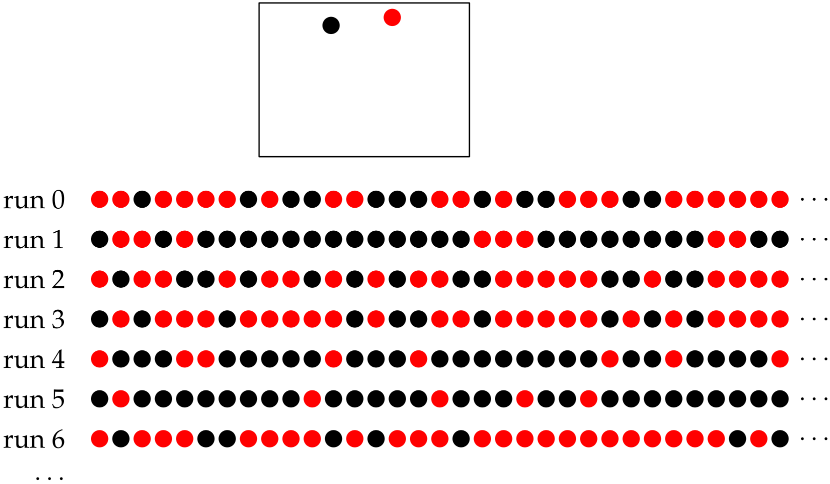

Ergodic decomposition

The Pólya urn

$$\mu[w] = \frac{|w|_0! \times |w|_1!}{(|w|+1)!}.$$

- μ is σ-invariant
- μ is the uniform average of the Bernoulli measures μ_p, $0 \leq p \leq 1$:

$$\mu[w] = \int_0^1 \mu_p[w]\,\mathrm{d}p.$$

- its ergodic decomposition is computable: for each oracle ω_1, the algorithm $A(\omega_2)$ simulates μ_p where $p = 0.\omega_1$.

Computational consequences in terms of memory [Freer & Roy, 2009].

Ergodic decomposition

Let μ be a computable σ-invariant measure. The following are equivalent:

- the ergodic decomposition of μ is computable,
- there exists a probabilistic algorithm computing a.s. an ergodic measure ν, and such that

$$\mu[w] = \mathbb{E}(\nu[w]),$$

- the speed of convergence of $\mathbf{1}_{[w]}^{(n)}$ to $\mathbf{1}_{[w]}^{*}$ is computable (unif. in w),
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When $\mu = \alpha_1\mu_1 + \ldots + \alpha_n\mu_n$ ($0 < \alpha_i \leq 1$, $\sum \alpha_i = 1$, μ_i ergodic), the decomposition of μ is computable iff all α_i, μ_i are computable.

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$$\mu = \sum_i 2^{-i} \mu_{t(i)}$$
where $t(i) \in \mathbb{N} \cup \{\infty\}$ is the halting time of program number i.

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- μ is computable but its decomposition is not.

□

Proof on an example.

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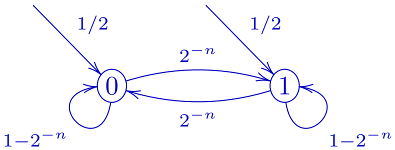
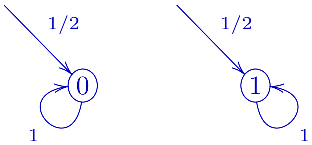
<tr><td>
</td><td>
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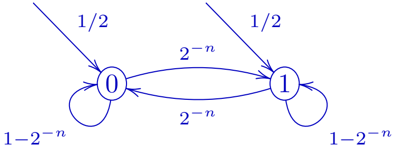
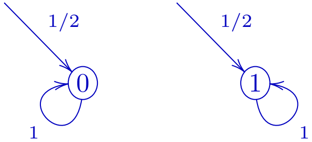
<tr><td>
</td><td>
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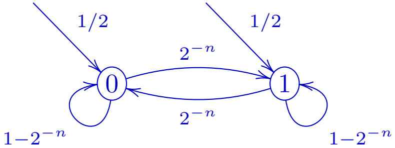
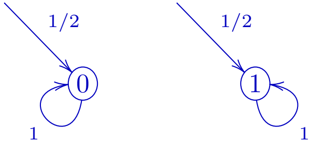
<tr><td>
</td><td>
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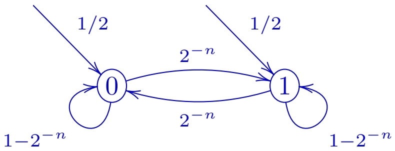	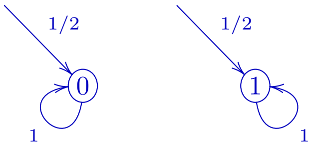
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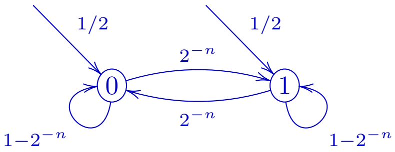	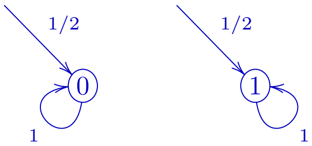
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- The three events are "isolated from each other", hence distinguishable: their probabilities are computable if the decomposition of μ is computable.
- But $\mathbb{P}(\nu[1] = \frac{1}{2}) = \sum_{i \in K} 2^{-i}$ is not computable! □

The technic is much more general

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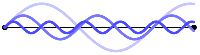

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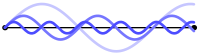

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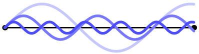

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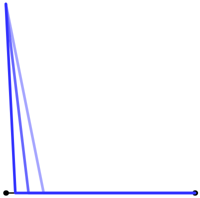

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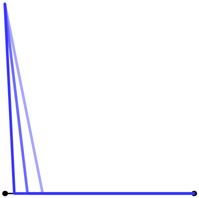

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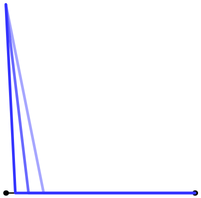

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More generally

Theorem (Pour-El & Richards, 1989)

Let X and Y be effective Banach spaces and $T : X \to Y$ a linear operator with c.e. closed graph. If T is unbounded then there exists a computable point x such that $T(x)$ is not computable.

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Examples

The following operators are unbounded

- id : $L^1[0,1] \to L^2[0,1]$,
- id : $L^1[0,1] \to \mathscr{C}[0,1]$,
- $\frac{d}{dx} : \mathscr{C}[0,1] \to \mathscr{C}[0,1]$,
- solution operator of the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Main question

- V'yugin's example is an infinite combination of ergodic measures.
- What about the finite case?
- If $\mu = \frac{\mu_1 + \mu_2}{2}$ (with μ_1, μ_2 ergodic) is computable, are μ_1 and μ_2 computable as well?

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Theorem (H., 2011)

There exist ergodic measures μ_1, μ_2 that are not computable relative to $\frac{\mu_1+\mu_2}{2}$.

Main question

- V'yugin's example is an infinite combination of ergodic measures.
- What about the finite case?
- If $\mu = \frac{\mu_1+\mu_2}{2}$ (with μ_1, μ_2 ergodic) is computable, are μ_1 and μ_2 computable as well?

The preceding technic does not apply. Still,

Theorem (H., 2011)

There exist ergodic measures μ_1, μ_2 that are not computable relative to $\frac{\mu_1+\mu_2}{2}$.

Theorem (H., 2012)

There exist ergodic measures μ_1, μ_2 that are not computable while $\frac{\mu_1+\mu_2}{2}$ is computable.

First result

Theorem (H., 2011)

There exist ergodic measures μ_1, μ_2 *that are not computable relative to* $\frac{\mu_1+\mu_2}{2}$.

The set of such pairs is even *co-meager*!

First result

$\mathcal{M}_\sigma = \{\sigma$-invariant measures$\}$.

Lemma

Let $C \subseteq \mathcal{M}_\sigma \times \mathcal{M}_\sigma$ be such that the function $(\mu_1, \mu_2) \mapsto \frac{\mu_1+\mu_2}{2}$ restricted to C is one-to-one and has a continuous inverse. C is nowhere dense.

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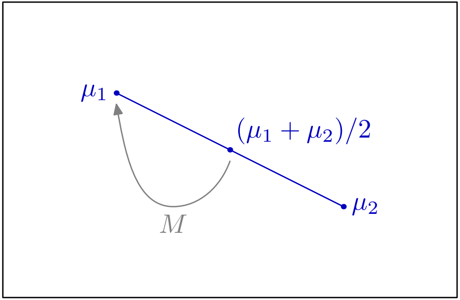
Proof.

- Let $(\mu_1, \mu_2) \in C$



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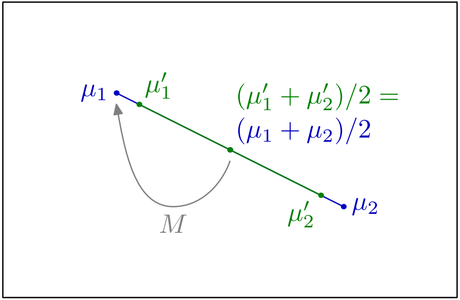
Proof.

- Let $(\mu_1, \mu_2) \in C$
- $(\mu_1', \mu_2') \notin C$



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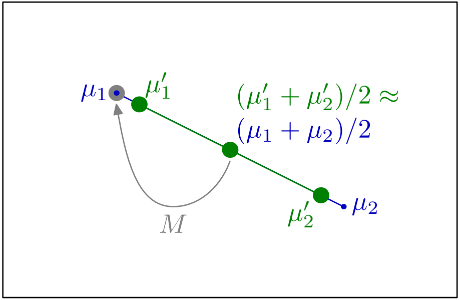
Proof.

- Let $(\mu_1, \mu_2) \in C$
- $(\mu_1', \mu_2') \notin C$ ...
- ... for (μ_1', μ_2') in an open set.



First result

Hence

- For each oracle Turing machine M, the set

$$C_M := \{(\mu_1, \mu_2) \in \mathcal{M}_\sigma \times \mathcal{M}_\sigma : M^{\frac{\mu_1+\mu_2}{2}} \text{ computes } \mu_1\}$$

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  is co-meager.
- As $\mathcal{M}_\sigma \times \mathcal{M}_\sigma$ is a Baire space, the set is non-empty.

Second result

Theorem (H., 2012)
There exist ergodic measures μ_1, μ_2 that are not computable while $\frac{\mu_1+\mu_2}{2}$ is computable.

Second result

- The construction is a game between a player and a countably infinite number of opponents (the programs).
- The player privately builds μ_1 and μ_2 and publicly describes $\frac{\mu_1+\mu_2}{2}$.
- Each opponent tries to guess μ_1, i.e. to publicly describe μ_1.
- The player wins if the opponent fails.
- Any public statement is irrevocable.

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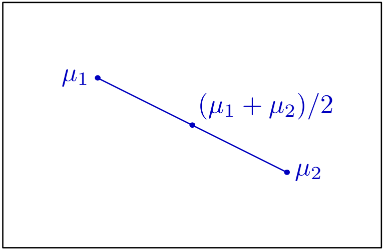
Theorem

The player has a computable winning strategy.

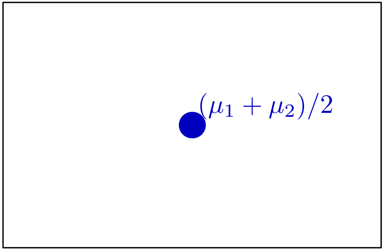
Second result

Against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



Knowledge of the player



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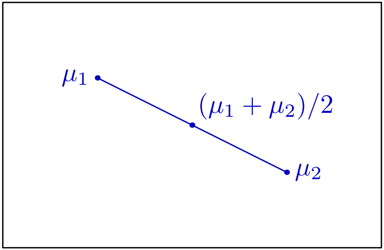

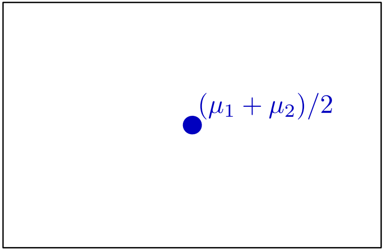

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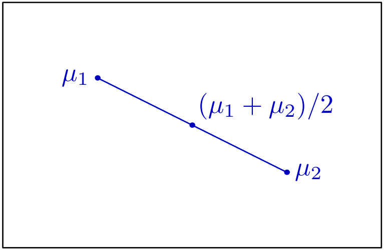

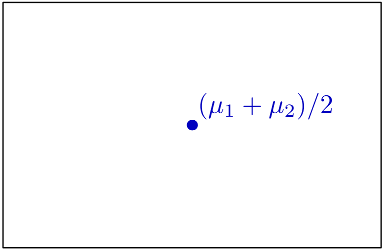

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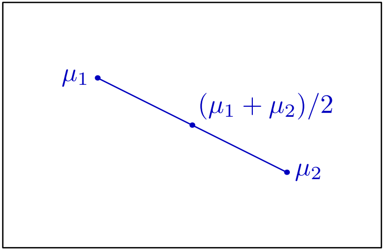

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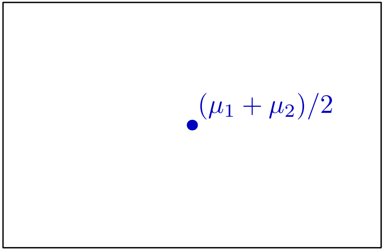

Knowledge of the opponent

Three cases:

1. the opponent remains silent forever: do nothing.

Second result

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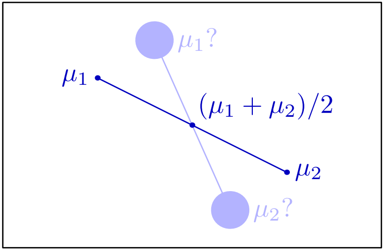

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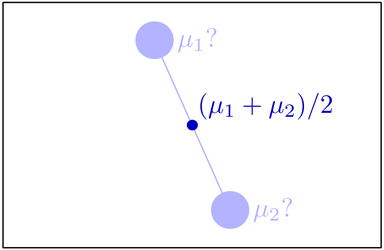


Knowledge of the opponent

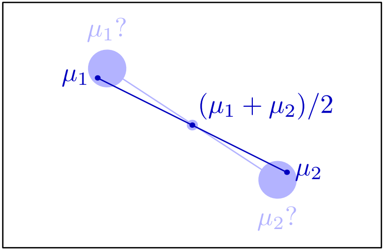
Three cases:

1. the opponent remains silent forever: do nothing.
2. the opponent eventually makes a wrong guess: do nothing.

Second result

Against one opponent

Start from any ergodic $\mu_1 \neq \mu_2$ and describe $\frac{\mu_1+\mu_2}{2}$.



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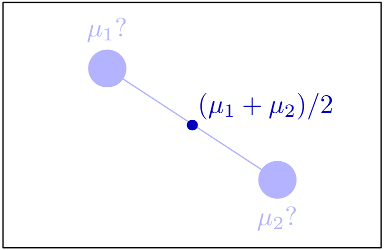

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Second result

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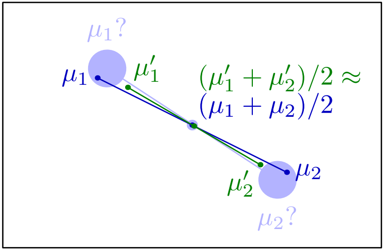

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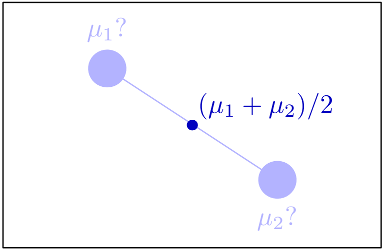

Knowledge of the opponent

Three cases:

1. the opponent remains silent forever: do nothing.
2. the opponent eventually makes a wrong guess: do nothing.
3. the opponent eventually makes a correct guess: move μ_1 and μ_2 much but $\mu_1 + \mu_2$ very little.

Second result

Against infinitely many opponents

Based on the "priority method with finite injury".

- Run the strategies S_i in parallel at different scales.

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- Run the strategies S_i in parallel at different scales.
- The strategies may interfer. Put a priority ordering: S_i has priority over S_j if $i < j$.

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- Every strategy eventually settles, so every strategy eventually acts without being injured any more ("finite injury").
- The limit measures μ_1 and μ_2 are not computed by any opponent. However the player computes $\frac{\mu_1+\mu_2}{2}$.

More generally

Let X be an effective Polish space and Y a second-countable topological space.

Definition

$f : X \to Y$ is **irreversible** if

- $\exists U \neq \emptyset$ open s.t. $\text{int}(f(U)) = \emptyset$ (inside $f(X)$),
- the same holds for the restriction $f_{|B}$ to any open B.

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Theorem (H., 2012)

If f is computable and computably irreversible then

- *the set $\{x \in X : x$ is not computable from $f(x)\}$ is co-meager,*
- *there exist a non-computable x such that $f(x)$ is computable.*