On the worst-case complexity of the silhouette of a polytope

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Abstract

We give conditions under which the worst-case size of the silhouette of a polytope is sub-linear.

1 Introduction

Given a viewpoint, the apparent boundary of a polyhedron, or *silhouette*, is the set of edges incident to a visible face and an invisible one; a face whose supporting plane contains the viewpoint is considered invisible. The worst-case upper bound on the complexity of a silhouette is $O(n)$. With this definition, the silhouette of a polytope (i.e., a convex bounded polyhedron) is a simple closed curve on its surface that separates visible and invisible faces.

Silhouettes arise in various problems in computer graphics, such as hidden surface removal [4] or shadow computations [1, 2], so a better understanding of the size of the silhouette of polyhedra directly improves the theoretical complexity of algorithms in computer graphics.

Practical observations, supported by an experimental study by Kettner and Welzl [5], suggest that the number of silhouette edges of a polyhedron is often much smaller than the total number of edges. In the same paper, they proved that a polyhedral approximation of a sphere with Hausdorff distance ε has $\Theta(1/\varepsilon)$ edges, and a random orthographic silhouette of such a polyhedron has size $\Theta(1/\sqrt{\varepsilon})$.

In this paper, we investigate the worst-case size of the silhouette of a polytope observed under orthographic projection. We prove that some classes of polytopes have orthographic silhouettes with sub-linear complexity in the worst-case. We also give examples with linear-size silhouette when some of our conditions are not satisfied.

Our approach is to consider the orthogonal projection of the polytope on a plane, since the boundary of the projected polygon is the projection of the silhouette. We measure the length of the boundary of this polygon, which we call the *apparent length* of the polytope. First we show that all silhouettes of a triangulated fat object with n edges of length $\Theta(1)$ have apparent length $O(\sqrt{n})$. Secondly we derive bounds on the number of silhouette edges, using an additional condition on the repartition of the directions of the edges.

This paper is organized as follows. In Section 2, we review some examples of ill-shaped polytopes with silhouettes of linear complexity. Next, Section 3 studies the apparent length of the silhouette, and Section 4 relates it to the number of silhouette edges. Finally, Section 5 discusses extensions and applications of our results.

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2 Examples

The goal of this paper is to find conditions under which polytopes have sub-linear sized silhouettes in the worst-case. This section examines three examples of ill-shaped polytopes with silhouette of linear complexity, and identifies the reasons for this behavior.

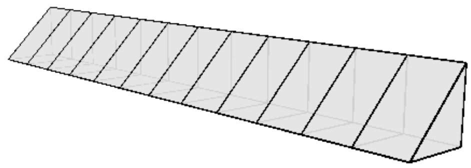

Figure 1: A non-fat triangulated polytope with bounded-length edges.

The example of Fig. 1 is characteristic of polytopes that are much longer along one dimension than along the others. This kind of behavior can be ruled out by considering *fat* polytopes, i.e., polytopes such that the ratio of the radius of the smallest enclosing to the largest enclosed sphere is $O(1)$.

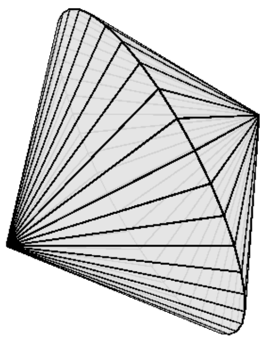

Figure 2: A fat triangulated polytope with uneven edges.

Our second example (see Fig. 2) illustrates the impact of the length of the edges on the silhouette. The ratio of the length of the longest edge to the length of the smallest is $\Omega(n)$, where n is the total number of edges. To avoid such behavior, we require that our polytopes have *bounded-length edges*, i.e., that all edges are of length $\Theta(1)$.

Our last example, in Fig. 3, exhibits a linear-size silhouette due to faces with order n edges. We therefore consider polytopes with faces of bounded complexity. Without loss of generality, we assume that our polytopes are triangulated.

This set of conditions is minimal in the sense that each of the previous examples satisfies all but one condition.

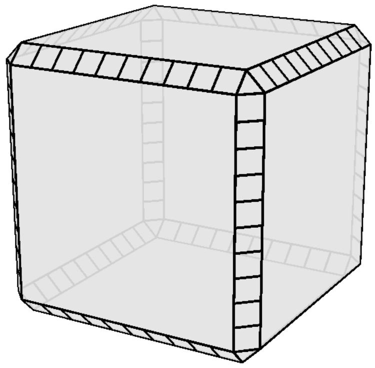

Figure 3: A fat polytope with bounded-length edges but with a face of large complexity.

In summary, in the rest of this paper we consider *triangulated fat* polytopes with *bounded-length edges*.

3 Apparent length

Recall that the apparent length of a silhouette is defined as the length of the orthogonal projection of the silhouette on a plane. In this section, we give bounds on the apparent length of the silhouette of a polytope.

We first recall a classical result on measures of convex sets. A proof can be found in [6][1].

Lemma 1 *Let $\mathcal{O}$ and $\mathcal{O}'$ be two convex objects in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) such that $\mathcal{O}$ contains $\mathcal{O}'$. Then the length (resp. area) of $\partial\mathcal{O}$ is larger than that of $\partial\mathcal{O}'$.*

For a polytope $\mathcal{P}$, let $\mathcal{A}(\mathcal{P})$ denote its surface area, and $\mathcal{L}(\mathcal{P})$ be the maximum apparent length of its silhouettes. The following lemma relates those two quantities.

Lemma 2 *If $\mathcal{P}$ is a fat polytope, then $\mathcal{L}(\mathcal{P}) = \Theta(\sqrt{\mathcal{A}(\mathcal{P})})$.*

Proof. Let r be the radius of the largest enclosed sphere of $\mathcal{P}$, and λr be the radius of the smallest enclosing sphere. Since $\mathcal{P}$ is fat, λ is $\Theta(1)$.

First, we apply Lemma 1 to $\mathcal{P}$ and its biggest enclosed sphere, and to $\mathcal{P}$ and its smallest enclosing sphere. This yields that $\mathcal{A}(\mathcal{P}) = \Theta(r^2)$. Next, consider an orthogonal projection of $\mathcal{P}$. Each of the two spheres projects onto a circle whose radius is the same as the radius of the corresponding sphere. Since the projection of $\mathcal{P}$ is convex, we can apply Lemma 1 to these circles and the boundary of that projection, and obtain that the length of that boundary is $\Theta(r)$. Taking the maximum over all possible orthogonal projections, we obtain that $\mathcal{L}(\mathcal{P}) = \Theta(r)$. It follows that $\mathcal{L} = \Theta(\sqrt{\mathcal{A}(\mathcal{P})})$. ◻

The next lemma bounds the area of a polytope with bounded-length edges.

Lemma 3 *If $\mathcal{P}$ is a triangulated polytope with bounded-length edges, then $\mathcal{A}(\mathcal{P}) = O(n)$.*

Proof. Since the polytope has bounded-length edges, the area of any of its triangles is $O(1)$. By Euler's formula, a triangulated polytope with n edges has $O(n)$ triangles, and the result follows. ◻

We can conclude with the following corollary, directly deduced from Lemmas 2 and 3.

Corollary 4 *If $\mathcal{P}$ is a triangulated fat polytope with n bounded-length edges, then the apparent length of any of its silhouettes is $O(\sqrt{n})$.*

4 Complexity of the silhouette

This section uses Corollary 4 to measure the complexity of the silhouette. To exploit the upper bound on the apparent length of the silhouette, we bound from below the contribution of silhouette edges to the apparent length. However, the contribution of an edge can be arbitrarily small, as it can be parallel to the direction of projection, and a triangulated fat polytope with bounded-length edges can have a linear number of such silhouette edges, as shown[2] in Fig. 4. Thus, we need to bound from above the number of silhouette edges that can be close to the direction of projection.

We give two distinct additional conditions that ensure a sub-linear size for the silhouette. The first one is local.

Lemma 5 *Let ε be a positive real number and $\mathcal{P}$ be a polytope with n bounded-length edges such that any two incident edges make an angle in the interval $[\varepsilon, \pi - \varepsilon]$. Then, any silhouette of $\mathcal{P}$ has $O(\mathcal{L}(\mathcal{P}))$ edges.*

Proof. Let $\vec{\delta}$ be a viewing direction . As any two incident edges make an angle in the interval $[\varepsilon, \pi-\varepsilon]$, two consecutive silhouette edges contribute $\Omega(\varepsilon)$ to the apparent length of the silhouette. It follows that the number of silhouette edges is $O(\mathcal{L}(\mathcal{P}))$. Note that the constant in the O depends on ε. ◻

Notice that if $O(\mathcal{L}(\mathcal{P}))$ edges do not satisfy the angle hypothesis, the same reasoning can be applied to the remaining edges on the silhouette, and the result of Lemma 5 still holds.

Combining Corollary 4 with Lemma 5 yields:

Theorem 6 *Let ε be a positive real number and $\mathcal{P}$ be a triangulated fat polytope with n bounded-length edges such that any two incident edges make an angle in the interval $[\varepsilon, \pi - \varepsilon]$. Then, any silhouette of $\mathcal{P}$ has $O(\sqrt{n})$ edges. This still holds if $O(\sqrt{n})$ edges do not satisfy the angle hypothesis.*

The second condition corresponds to a regular repartition of the directions of the edges of the polytope and is thus global. The idea is that if the directions of the edges do not accumulate along a few directions, the number of edges almost collinear with any direction is bounded, and so is the complexity of the silhouette. The meaning of this accumulation hypothesis is explained in the next Lemma.

Lemma 7 *Let $\mathcal{P}$ be a polytope with n bounded-length edges and apparent length $O(\sqrt{n})$ such that for any direction $\vec{\delta}$, the number of edges of $\mathcal{P}$ making an angle smaller*

[1] In fact, the proof in [6] is much more general than our statement, and applies to any Minkowski measure, in any dimension.

[2] See Appendix A for details.

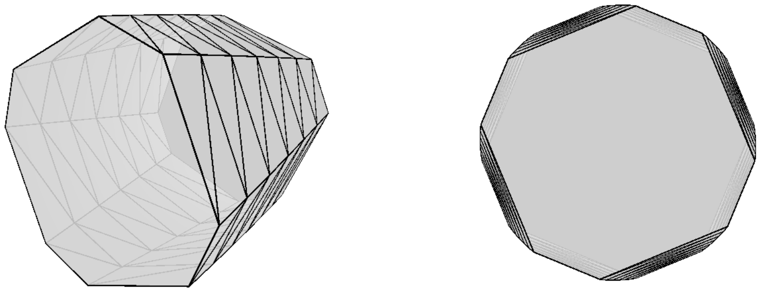

Figure 4: A triangulated fat polytope with bounded-length edges and a linear-size silhouette. The front and back faces, of complexity $O(\sqrt{n})$, were not triangulated for clarity.

than $\Theta(n^{-1/6})$ with $\vec{\delta}$ is $O(n^{2/3})$. Then any silhouette of $\mathcal{P}$ has $O(n^{2/3})$ edges.

Proof. Let us fix a direction $\vec{\delta}$, and let α be a real number. We count separately the silhouette edges that make an angle greater than α with $\vec{\delta}$, and the others, and find the value of α yielding the best trade-off.

If we represent the set of directions by a unit sphere, the directions that make an angle smaller than α with $\vec{\delta}$ form a spherical cap of area $\Theta(\alpha^2)$. The sphere can be covered by $\Theta(1/\alpha^2)$ such spherical caps and the directions of the n edges are distributed over the sphere, so one of the caps has to contain $\Omega(\alpha^2 n)$ edge directions. This means that, for some viewing direction, there are $\Omega(\alpha^2 n)$ edges that make an angle less than α. Thus, the best we can ask is that the number of silhouette edges having a negligible contribution to the apparent length is $O(\alpha^2 n)$.

Let k denote the number of silhouette edges that make an angle greater than α with $\vec{\delta}$. The contribution of these k edges to the apparent length of the silhouette is $\Omega(k\alpha)$. Thus, $k = O(\mathcal{L}/\alpha) = O(\sqrt{n}/\alpha)$.

If we ask that at most $O(\alpha^2 n)$ edges of the polytope make an angle less than α with any given direction, then the complexity of the silhouette is bounded from above by

$$O\left(\sqrt{n}/\alpha + \alpha^2 n\right).$$

The best trade-off one can achieve is to choose $\sqrt{n}/\alpha = \Theta(\alpha^2 n)$, which means $\alpha = \Theta(n^{-1/6})$. In that case, the number of silhouette edges is $O(n^{2/3})$, and the regular distribution assumption is the one mentioned in the statement of the lemma. ◻

Note that the proof of Lemma 7 establishes a more general result: a weaker condition on the repartition of the directions of the edges still yields a sub-linear bound on the complexity of the silhouette, which is in between $O(n^{2/3})$ and $O(n)$. Besides, if the repartition condition is satisfied for a given direction $\vec{\delta}$, then the orthographic silhouette along this direction has a complexity $O(n^{2/3})$.

Combining Corollary 4 with Lemma 7 yields:

Theorem 8 *Let $\mathcal{P}$ be a triangulated fat polytope with n bounded-length edges such that for any direction $\vec{\delta}$, the number of edges of $\mathcal{P}$ making an angle smaller than $\Theta(n^{-1/6})$ with $\vec{\delta}$ is $O(n^{2/3})$. Then any silhouette of $\mathcal{P}$ has $O(n^{2/3})$ edges.*

The requirements on the polytopes in Lemmas 5 and 7 are strong, and may describe empty classes of polytopes. For the case of Theorem 6, Appendix B describes a class of polytopes that satisfy the requirements of the theorem when we allow $O(\sqrt{n})$ edges to miss the angular condition. However, to the best of our knowledge, whether there exists or not a polytope meeting the conditions of Lemma 7 is an open question.

5 Discussion

This section discusses our results, giving extensions as well as possible applications.

To begin with, notice that, in the results of Sections 3 and 4, the fatness assumption can be weakened: Lemma 2 holds for any polytope $\mathcal{P}$ with bounded-length edges that satisfies $d(\mathcal{P})^2 = O(\mathcal{A}(\mathcal{P}))$, where $d(\mathcal{P})$ is its diameter. This means having a fat orthogonal projection with the same diameter, i.e., to be fat along at least two dimensions.

Next, to extend our approach to the perspective case, the distance from the object to the viewpoint has to be taken into account. When the viewpoint is far from the polytope, the perspective case should behave as the orthographic case. But when the viewpoint is close to the polytope, the perspective projection introduces distortion: the length of the projection of a silhouette edge depends on its distance to the center of the view. Also, the global hypothesis on the distribution of the directions of the edges has to be adjusted accordingly.

The results of this paper are only a first step toward the understanding of the complexity of silhouettes, but they still have promising applications.

A first application is the computation of shadow boundaries. Drettakis and Duguet [1, 2] propose a solution based on a *visibility skeleton* restricted to the visual events generated by a punctual light source. In their detailed report [2], they show that their algorithm has complexity $O(ns_n)$, where n is the size of the polyhedron that casts a shadow, and s_n the size of its silhouette. Even the orthographic case is of interest, since it corresponds to a light source at infinity, a simple sun model for instance.

A second application is hidden surface removal, which has a long history as a problem difficult to address practically [3]. A solution proposed by Efrat et al. [4] is to render first the silhouettes of the objects, and then optimize the rendering in the single-object regions. They estimate the number of combinatorial changes to the rendered silhouettes of polytopes when the viewpoint moves along a line or an algebraic curve. Depending on the motion, this number depends either linearly or quadratically on the silhouette complexity, which they bound from above by the complexity of the polytope. Extension of our work to the perspective case would thus yield a direct improvement of their bounds.

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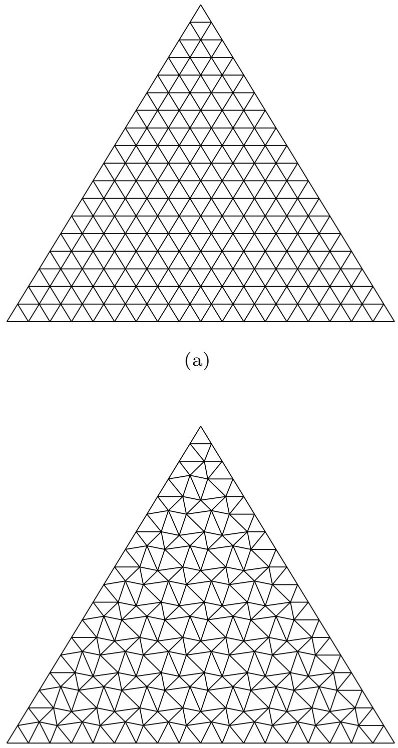
A The cylinder example

This appendix details the (not necessarily intuitive) construction of the polyhedron of Fig. 4.

First, start with two almost regular polygons of diameter $\Theta(\sqrt{n})$ facing each other at distance $\Theta(\sqrt{n})$. These polygons both have $\Theta(\sqrt{n})$ edges of alternating length $1 \pm \varepsilon$. Note that corresponding edges on the two polygons have different length. Next, connect each pair of corresponding edges by a strip of length $\Theta(\sqrt{n})$ and width $\Theta(1)$. Thus, each strip is an almost rectangular trapezoïd.

We then triangulate the extremal polygons and the strips, inserting points on the edges and inside the faces, so that the triangles have edges of length $\Theta(1)$. A triangulation of the $\Theta(\sqrt{n})$-gons can be made with $\Theta(n)$ triangles. There are $\Theta(\sqrt{n})$ strips, each triangulated with $\Theta(\sqrt{n})$ triangles. So the total size of the polyhedron is $\Theta(n)$.

Now, when looking along the axis of this cylinder-like polytope, the silhouette is made of $\Theta(\sqrt{n})$ polygon edges and all the sides of the strips, that is $\Theta(\sqrt{n})$ collections of $\Theta(\sqrt{n})$ edges. This is thus an example of a polytope with a linear-size silhouette. Yet, this polytope is fat with an aspect ratio close to $\sqrt{2}$, with triangular faces, and bounded-length edges.



(a)

(b)

Figure 5: a face of a polytope for Theorem 6.

B A polytope for Theorem 6

This appendix describes a class of polytopes that satisfy the requirements of Theorem 6 when we allow $O(\sqrt{n})$ edges to miss the angular condition.

Start with a regular tetrahedron, and triangulate each of its faces regularly as shown in Figure 5(a). Then, for each face, perturb the interior points as shown in Figure 5(b). The vertices on every second horizontal line are moved alternatively upward and to the left, and the remaining vertices are moved in the direction opposite to that of their top-left neighbour. The scale of the perturbation is chosen to be proportional to the size of the triangles, so that the angles between edges do not depend on the size of the triangulation. Notice that this perturbation is 2-periodic along each of the 3 main directions of the triangulation

Now, we slightly inflate the faces of the perturbed polytope so that no two triangles are coplanar. The resulting polytope is fat, triangulated, has bounded-length edges and only the $O(\sqrt{n})$ edges included in the edges of the initial tetrahedron are aligned with some of their neighbours.

Notice that a similar perturbation scheme applied to the triangulation of the lateral surface of the cylinder in Figure 4, brings the size of its silhouette from $\Theta(n)$ down to $\Theta(\sqrt{n})$.