On the theoretical complexity of the silhouette of a polyhedron

Mémoire soutenu le

vendredi 5 septembre 2003

par

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pour l'obtention du diplôme d'études approfondies

sous la direction de

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Abstract

We study conditions under which the silhouette of a polyhedron is guaranted to be sublinear, either on average or in the worst case, and give some counter-examples when not.

Contents

1 Introduction

Given a viewpoint, the apparent boundary of a polyhedron (in 3D), or *silhouette*, is the set of edges incident to a visible face and an invisible one, and in the neighbourhood of which one can see infinity; a face whose supporting plane contains the viewpoint is considered invisible. The worst-case upper bound on the complexity of a silhouette is $O(n)$. Notice that with this definition, the silhouette of a polytope (i.e. a convex bounded polyhedron) is a simple closed curve on its surface that separates visible and invisible faces.

Silhouettes arise in various problems in computer graphics, such as hidden surface removal [5] or shadow computations [2, 3]. The most common geometric primitives are polyhedra, so a better understanding of the size of their silhouette yields direct improvement in the theoretical complexity of algorithms in computer graphics.

Practical observations, supported by an experimental study by Kettner and Welzl [6], suggest that the number of silhouette edges of a polyhedron is often much smaller than the total number of edges. In the same paper, they proved that a polyhedral approximation of a sphere with Hausdorff distance ε and $\Theta(\frac{1}{\varepsilon})$ edges has a random orthographic silhouette of size $\Theta(\frac{1}{\sqrt{\varepsilon}})$.

In this paper, we investigate classes of polyhedra for which we can ensure that the silhouette has a sublinear complexity. We restrict ourselves to the case of orthographic perspective (viewpoint at infinity) and discuss extensions at the end. We first study the case of polytopes. Our approach is to consider the orthogonal projection of the polytope on a plane, since the boundary of the projected polygon is the projection of the silhouette. We measure the length of the boundary of the orthogonal projection of a silhouette, which we call its *apparent length*. We then try and relate it to the complexity of the silhouette. In a second part, we study polyhedra that are constructed as approximations of a given surface, and the way the complexity of their silhouette evolves when the approximation is refined.

2 Definitions and general remarks

2.1 View, silhouette

Several notions around the silhouette are interesting and deserve to be studied:

- complete view, transparent silhouette: the set of edges that are adjacent to a visible face and an invisible one, the visibility of a face being determined only by the direction of its tangent. It corresponds to the lines one can see if the object is transparent.
- view, complete silhouette: the edges of the complete view that are visible even when the faces of the polyhedron are not transparent.
- silhouette, shadow, apparent boundary: the edges of the view in the neighbourhood of which one can see at infinity.

In every case, a face that contains the viewpoint is considered invisible. Notice that for a polytope, all these definitions are equivalent.

2.2 Projective invariance

The easiest definition of the silhouette uses the normal to the surface. However, the silhouette is a projective invariant. Indeed, it can be defined as the set of lines going through the viewpoint and the object such that there are lines arbitrarily close to them going through the viewpoint that do not intersect the object. There may be some complications if the projective transformation takes an interior point of the polyhedron to infinity and the definitions need to be generalized, but this is of no interest here. The same result holds for the (complete) view.

2.3 kD-fatness

In this paper, we study polyhedra that satisfy nice regularity properties. One such property is fatness. In $\mathbb{R}^d$, an object is said to be fat if it has a ball of radius r in its interior and is contained in a ball of radius R such that R/r is smaller than some fixed constant.

A generalization of this is the notion of kD-fatness. An object in $\mathbb{R}^d$ is kD-fat if there exists a k-dimensional subspace such that the orthogonal projection of the object on this subspace is (in this subspace) a fat object with the same diameter as the original object. For instance, a plate or a coin is 2D-fat but not fat. Every fat object is in particular kD-fat for $k \leq d$ (see Lemma 1). In $\mathbb{R}^3$, an object is 2D-fat if and only if all its orthographic projections have the same diameter (up to some bounded factor).

2.4 Duality

There is an interesting duality for polytopes. The polar set of a set K is $K^* = \{x \in \mathbb{R}^3$ such that $x \cdot y \leq 1$ for all $y \in K\}$. Notice that K^* is always convex. If $conv(K)$ the convex hull of K contains the origin then $(conv(K))^* = K^*$. Moreover if K is convex, then $K^{**} = K$. Here is a way to compute a silhouette. First add the viewpoint to the polytope, and compute the convex hull. The edges passing through the viewpoint correspond to the silhouette vertices (actually not exactly, because we considered the faces containing the viewpoint as invisible, but it remains the same problem). . In the dual, adding a new point means intersecting the polyhedron by a half space, and looking at the edges of this new face. The problem of the silhouette is then dual to that of the intersection by a plane, for a polytope. We did not find anything interesting through this dual approach, but we can notice that the dual of a fat polytope is fat, the dual of a triangulated polytope is a polytope whose vertices have degree 3, and for fat polytopes the length of an edge corresponds to the exterieur angle between two faces with à $\Theta(1)$ factor.

3 Polytopes

3.1 First examples

This section examines three examples of ill-shaped polytopes with silhouette of linear complexity, and identifies the reasons for this behavior.

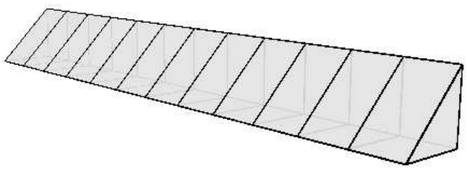

Figure 1: A triangulated polytope with bounded-length edges but not 2D-fat.

The example of Fig. 1 is characteristic of polytopes that are much longer along one dimension than along the others. This kind of behavior can be ruled out by considering *2D-fat* polytopes.

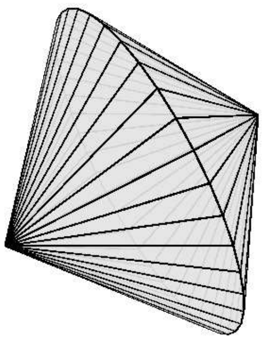

Figure 2: A fat triangulated polytope with uneven edges.

Our second example (see Fig. 2) illustrates the impact of the length of the edges over the silhouette. The ratio of the length of the longest edge to the length of the smallest is $\Omega(n)$, where n is the total number of points. To avoid such behaviors, we require that our polytopes have *bounded-length edges*, i.e. that the ratio between the lengths of the longest and the smallest edges is $O(1)$. Usually we will assume edges to be of length $\Theta(1)$.

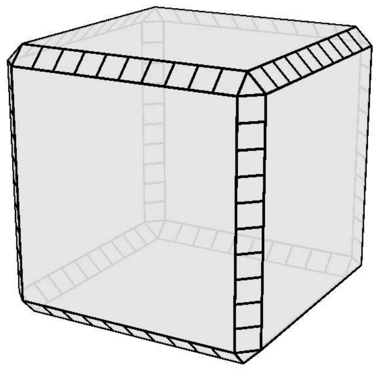

Figure 3: A fat polytope with bounded-length edges but with a face of large complexity.

Our last example, in Fig. 3, exhibits a linear-size silhouette due to faces with order n edges. We therefore consider polytopes with faces of bounded complexity. Without loss of generality, we simply assume that our polytopes are triangulated.

This set of conditions is minimal in the sense that each of the previous examples satisfies all but one condition.

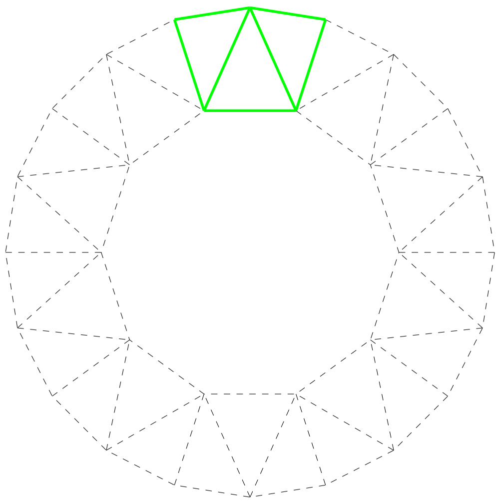

Figure 4: A triangulation scheme for polygons.

Notice that the bounded-length condition cannot be replaced by an assumption on the fatness of the faces. Indeed, using recursively the technique of Fig 4, one can build a regular polygon of arbitrarily large size triangulated with a number of fat triangles linear with respect to the complexity of the polygon. Glueing two such polygons along their boundary and making the whole thing convex gives an example of a polytope with a linear-size silhouette.

In summary, in the next section we will consider *triangulated 2D-fat* polytopes with *bounded-length edges*.

3.2 Apparent length of a polytope

Recall that the apparent length of a silhouette is defined as the length of the orthogonal projection of the silhouette on a plane. In this section, we give bounds on the apparent length of the silhouette of a polytope.

We first recall a classical result on measures of convex sets. A proof can be found in [7][1].

Lemma 1 *Let $\mathcal{O}$ and $\mathcal{O}'$ be two convex objects in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) such that $\mathcal{O}$ contains $\mathcal{O}'$. Then the length (resp. area) of $\partial\mathcal{O}$ is larger than that of $\partial\mathcal{O}'$.*

For a polytope $\mathcal{P}$, let $\mathcal{A}(\mathcal{P})$ denote its surface area, and $\mathcal{L}(\mathcal{P})$ be the maximum apparent length of its silhouettes. The following lemma relates those two quantities.

Lemma 2 *If $\mathcal{P}$ is a 2D-fat polytope, then*

$$\mathcal{L}(\mathcal{P}) = \Theta(\sqrt{\mathcal{A}(\mathcal{P})}).$$

Proof. Let d be the diameter of $\mathcal{P}$. Since $\mathcal{P}$ is contained in a ball of radius $O(d)$, Lemma 1 shows that $\mathcal{A}(\mathcal{P}) = O(d^2)$. For any orthographic projection of $\mathcal{P}$, the projected object has a diameter at most d, and since it is convex, Lemma 1 proves that $\mathcal{L}(\mathcal{P}) = O(d)$. Now consider a projection of $\mathcal{P}$ that is fat and has diameter d. Obviously $\mathcal{L}(\mathcal{P}) = \Omega(d)$ (use Lemma 1 with a segment for example). Also, the area of this surface is $\Omega(d^2)$ (it contains a disk of radius $\Omega(d)$). Projection decreases the surface, so $\mathcal{A}(\mathcal{P}) = \Omega(d^2)$. All in all, $\mathcal{L}(\mathcal{P}) = \Theta(\sqrt{\mathcal{A}(\mathcal{P})})$. ◻

Notice that the reciprocal of this lemma also holds.

The next lemma bounds the surface area of a polytope with bounded-length edges.

Lemma 3 *If $\mathcal{P}$ is a triangulated polytope with edges of length $O(1)$, then $\mathcal{A}(\mathcal{P}) = O(n)$.*

Proof. Since the polytope has edges of length $O(1)$, the area of any of its triangles is $O(1)$. By Euler's formula, a triangulated polytope with n edges has $O(n)$ triangles, and the result follows. ◻

We can conclude with the following corollary, directly deduced from Lemmas 2 and 3.

Corollary 4 *If $\mathcal{P}$ is a triangulated 2D-fat polytope with n bounded-length edges, then $\mathcal{L}(\mathcal{P}) = O(\sqrt{n})$.*

To exploit the upper bound on the apparent length of the silhouette, we simply bound from below the contribution of silhouette edges to the apparent length. However, the contribution of an edge can be arbitrarily small, as it can be parallel to the direction of projection, and a triangulated fat polytope with bounded-length edges can have a linear number of such silhouette edges.

3.3 Worst-case complexity for polytopes

3.3.1 The cylinder example

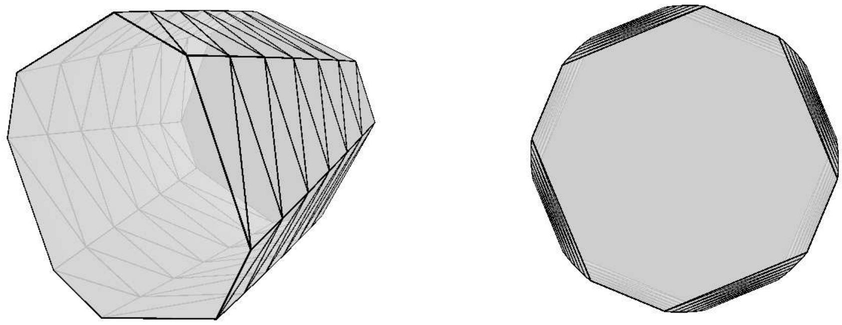

Figure 5: A triangulated fat polytope with bounded-length edges and a linear-size silhouette (the front and back faces, of complexity $\sqrt{n}$, were not triangulated for clarity).

Here is a description of the example of Figure 5.

[1]In fact, the proof in [7] is much more general than our statement, and applies to any Minkowski measure, in any dimension.

First, start with two almost regular polygons of diameter $\Theta(\sqrt{n})$ facing each other at distance $\Theta(\sqrt{n})$. These polygons both have $\Theta(\sqrt{n})$ edges of alternating length $1 \pm \varepsilon$. Note that corresponding edges on the two polygons have different length. Next, connect each pair of corresponding edges by a strip of length $\Theta(\sqrt{n})$ and width $\Theta(1)$. Thus, each strip is an almost rectangular trapezoïd.

We then triangulate the extremal polygons and the strips, inserting points on the edges and inside the faces, so that the triangles have edges of length $\Theta(1)$. A triangulation of the $\Theta(\sqrt{n})$-gons can be made with $\Theta(n)$ triangles. There are $\Theta(\sqrt{n})$ strips, each triangulated with $\Theta(\sqrt{n})$ triangles. So the total size of the polyhedron is $\Theta(n)$.

Now, when looking along the axis of this cylinder-like polytope, the silhouette is made of $\Theta(\sqrt{n})$ polygon edges and all the sides of the strips, that is $\Theta(\sqrt{n})$ collections of $\Theta(\sqrt{n})$ edges. This is thus an example of a polytope with a linear-size silhouette. Yet, this polytope is fat with an aspect ratio close to $\sqrt{2}$, with triangular faces, and bounded-length edges.

This example shows that we need to bound from above the number of silhouette edges that can be close to the direction of projection. We give two distinct additional conditions that ensure a sub-linear size for the silhouette. The first one is a local condition.

3.3.2 Local theorem

Lemma 5 *Let ε be a positive real number and $\mathcal{P}$ be a polytope with n bounded-length edges such that any two incident edges make an angle in the interval $[\varepsilon, \pi - \varepsilon]$. Then, any silhouette of $\mathcal{P}$ has $O(\mathcal{L}(\mathcal{P}))$ edges.*

Proof. Let $\vec{\delta}$ be a viewing direction . As any two incident edges make an angle in the interval $[\varepsilon, \pi - \varepsilon]$, two consecutive silhouette edges contribute $\Omega(\varepsilon)$ to the apparent length of the silhouette. It follows that the number of silhouette edges is $O(\mathcal{L}(\mathcal{P}))$. Note that the constant in the O depends on ε. □

Combining Corollary 4 with Lemma 5 yields:

Theorem 6 *Let ε be a positive real number and $\mathcal{P}$ be a triangulated fat polytope with n bounded-length edges such that any two incident edges make an angle in the interval $[\varepsilon, \pi - \varepsilon]$. Then, any silhouette of $\mathcal{P}$ has $O(\sqrt{n})$ edges.*

3.3.3 Global theorem

The second condition corresponds to a regular repartition of the directions of the edges of the polytope and is thus global. The idea is that if the directions of the edges do not accumulate along a few directions, the number of edges almost collinear with any direction is bounded, and so is the complexity of the silhouette. The meaning of this accumulation hypothesis is explained in the next Lemma.

Lemma 7 *Let $\mathcal{P}$ be a polytope with n bounded-length edges and apparent length $O(\sqrt{n})$ such that for any direction $\vec{\delta}$, the number of edges of $\mathcal{P}$ making an angle smaller than $\Theta(n^{-1/6})$ with $\vec{\delta}$ is $O(n^{2/3})$. Then any silhouette of $\mathcal{P}$ has $O(n^{2/3})$ edges.*

Proof. Let us fix a direction $\vec{\delta}$, and let $\alpha < 1$ be a real number. We count separately the silhouette edges that make an angle greater than α with $\vec{\delta}$, and the others, and find the value of α yielding the best trade-off.

If we represent the set of directions by a unit sphere, the directions that make an angle smaller than α with $\vec{\delta}$ form a spherical cap of area $\Theta(\alpha^2)$. The sphere can be covered by $\Theta(1/\alpha^2)$ such spherical caps and the directions of the n edges are distributed over the sphere, so one of the caps has to contain $\Omega(\alpha^2 n)$ edge directions. This means that, for some viewing direction, there are $\Omega(\alpha^2 n)$ edges that make an angle less than α. Thus, the best we can ask is that the number of silhouette edges having a negligible contribution to the apparent length is $O(\alpha^2 n)$.

Let k denote the number of silhouette edges that make an angle greater than α with $\vec{\delta}$. The contribution of these k edges to the apparent length of the silhouette is $\Omega(k\alpha)$. Thus, $k = O(\mathcal{L}/\alpha) = O(\sqrt{n}/\alpha)$.

If we ask that at most $O(\alpha^2 n)$ edges of the polytope make an angle less than α with any given direction, then the complexity of the silhouette is bounded from above by

$$O\left(\sqrt{n}/\alpha + \alpha^2 n\right).$$

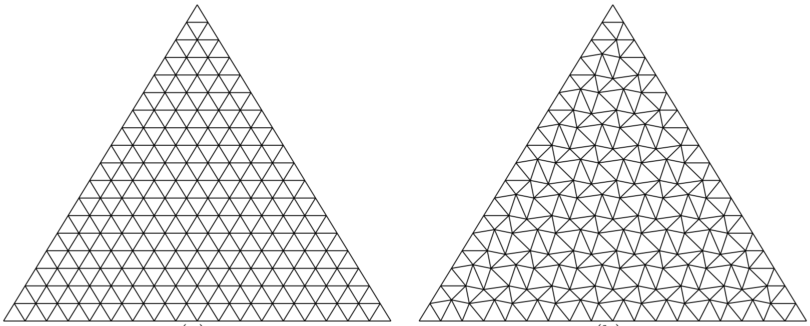

Figure 6: A face of a polytope for Theorem 6.

The best trade-off one can achieve is to choose $\sqrt{n}/\alpha = \Theta(\alpha^2 n)$, which means $\alpha = \Theta(n^{-1/6})$. In that case, the number of silhouette edges is $O(n^{2/3})$, and the regular distribution assumption is the one mentioned in the statement of the lemma. $\square$

Note that the proof of Lemma 7 establishes a more general result: a weaker condition on the repartition of the directions of the edges still yields a sub-linear bound on the complexity of the silhouette, which is in between $O(n^{2/3})$ and $O(n)$. Besides, if the repartition condition is satisfied for a given direction $\vec{\delta}$, then the orthographic silhouette along this direction has a complexity $O(n^{2/3})$.

Combining Corollary 4 with Lemma 7 yields:

Theorem 8 *Let $\mathcal{P}$ be a triangulated fat polytope with n bounded-length edges such that for any direction $\vec{\delta}$, the number of edges of $\mathcal{P}$ making an angle smaller than $\Theta(n^{-1/6})$ with $\vec{\delta}$ is $O(n^{2/3})$. Then any silhouette of $\mathcal{P}$ has $O(n^{2/3})$ edges.*

3.3.4 Existence of such polytopes

The requirements on the polytopes in Theorems 5 and 7 are very strong, and it might be that they describe empty classes of polytopes.

However, notice that not all edges have to satisfy all the requirements for Corollary 6. Indeed, if $O(\sqrt{n})$ edges are of length $o(1)$ or make a flat angle with a neighbouring edge, the result still holds. This allows us to give an example of a polytope in this slightly larger class.

Start with a regular tetrahedron, and triangulate each of its faces regularly as shown in Figure 6(a). Then, for each face, perturb the interior points as shown in Figure 6(b). The vertices on every second horizontal line are moved alternatively upward and to the left, and the remaining vertices are moved in the direction opposite to that of their top-left neighbour. The scale of the perturbation is chosen to be proportional to the size of the triangles, so that the angles between edges do not depend on the size of the triangulation. Notice that this perturbation is 2-periodic along each of the 3 main directions of the triangulation

Now, we slightly inflate the faces of the perturbed polytope so that no two triangles are coplanar. The resulting polytope is fat, triangulated, has bounded-length edges and only the $O(\sqrt{n})$ edges included in the edges of the initial tetrahedron are aligned with some of their neighbours.

Notice that a similar perturbation scheme applied to the triangulation of the lateral surface of the cylinder in Figure 5, brings the size of its silhouette from $\Theta(n)$ down to $\Theta(\sqrt{n})$.

3.4 Average complexity for polytopes

First, notice that in Fig. 5, the silhouette is not linear for every direction. Actually, the set of directions that give a linear silhouette is quite negligible. We shall see here that the average complexity of the silhouette is $\sqrt{n}$ even in some cases were there exists a direction that gives a linear silhouette. The average is computed among all viewpoints at infinity with the usual measure on the sphere of directions. The intuitive idea is that the average apparent length of an edge is a constant times its length, so Corollary 4 is sufficient, but we write the proof a slightly different way in order to introduce an interesting formula.

Instead of looking at the silhouette globally, it is easier to study each edge individually. The set of directions such that this edge is on the silhouette is two symmetric quarters of orange, and their angle is the exterior angle of the two faces incident to this edge (which we call α_i for an edge e_i of length l_i). The average complexity of the silhouette is:

$$\frac{1}{\pi}\sum_{e_i}\alpha_i$$

. For the same reason, it is easy to compute the average of the apparent length of the polytope (that is the length of the surface of the polytope):

$$\frac{1}{2\pi}\sum_{e_i}\alpha_i l_i$$

.

Theorem 9 *Let $\mathcal{P}$ be a polytope with n edges of length $\Omega(1)$ and with average apparent length $O(\sqrt{n})$. Then the average complexity of the silhouette of $\mathcal{P}$ is $O(\sqrt{n})$.*

Proof. The average complexity is:

$$\frac{1}{\pi}\sum_{e_i}\alpha_i = O(1)\cdot\frac{1}{2\pi}\sum_{e_i}\alpha_i l_i = O(\sqrt{n})$$

. □

Corollary 10 *Let $\mathcal{P}$ be a triangulated 2D-fat polytope with bounded-length edges. Then the average complexity of the silhouette of $\mathcal{P}$ is $O(\sqrt{n})$.*

3.5 Lower bound for polytopes

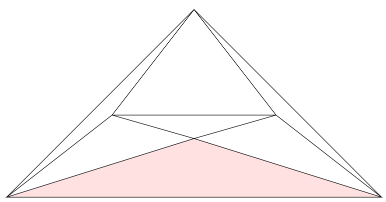

Figure 7: A triangulation with long edges

It is also possible to give a lower bound on the complexity of the silhouette of a polytope. However, the assumptions are not exactly the same. Indeed, recursively applying the triangulation of Fig. 7, it is possible to triangulate an equilateral triangle with only edges of length at least one fifth of the size of the initial triangle. Applying this to a tetrahedron gives an exemple of a triangulated fat polytope with bounded-length edges and a constant size silhouette.

Theorem 11 *Let $\mathcal{P}$ be a triangulated polytope with bounded-length edges and fat faces. Then the average complexity of the silhouette is $\Omega(\sqrt{n})$. If besides the polytope is 2D-fat, then all orthographic silhouettes are of complexity $\Omega(\sqrt{n})$.*

Proof. Assume the length of the edges is $\Theta(1)$. Since all the faces are fat triangles, their area is $\Theta(1)$. The total surface of the polytope is then $\Theta(n)$; its diameter is $\Omega(\sqrt{n})$ (Lemma 1); its average apparent length is $\Omega(\sqrt{n})$. The same formula as in Section 3.4 says that the average complexity of the silhouette is $\Omega(\sqrt{n})$. If the polytope is 2D-fat, for any direction, the apparent length is $\Omega(\sqrt{n})$, and the complexity of the silhouette is $\Omega(\sqrt{n})$ since the length of the edges is $O(1)$. Notice that this proof still works if the faces are not triangles but polygons of width $\Omega(1)$. □

4 Approximation of a surface

In graphics, polyhedra usually model an object. We will then study the complexity of the silhouette of a polyhedron that approximates a given surface, and how it evolves when refining the approximation.

The basic idea of this paper is that a surface is a 2-dimensional object (hence the 2D-fatness assumptions in previous sections), whereas the silhouette is a one dimensional object. Intuition then says that if the first one is of size n, the second one is of size $\sqrt{n}$. The definitions of the silhouette can be extended from the polyhedron case to more general surfaces if instead of considering the edges between a visible and an invisible face we consider the points where the surface admits a normal orthogonal to the viewing direction, and a point is hidden only if a neighbourhood of this point is hidden. This way we may expect the silhouette of a polyhedral approximation of a surface to be close to the silhouette of the surface. Notice that the worst example we gave in Fig. 5 is a convex polyhedral approximation of a cylinder, and the transparent silhouette of a cylinder when looking along its axis is the whole lateral surface and not a one-dimensional object. Also notice that the definitions of the silhouettes of a polyhedron and a surface are different.

4.1 Kettner and Welzl

Their approach is to approximate a convex surface of bounded curvature by a convex polytope (they only mention the sphere, but their proof is a little more general). They assume that no point of the polytope has distance larger than $O(1/n)$ with the surface, and the reciprocal statement. These hypotheses give an upper bound on the exterior angle for every edge, and prove that the silhouette is of average complexity $O(\sqrt{n})$.

4.2 Average case

4.2.1 Smooth surface

Consider a family of polyhedra P_n of complexity $O(n)$. Assume:

1. Let $f_n : P_n \to S$ be the closest-point applications. f_n is an injective function,
2. the edges of P_n have length $O(1/\sqrt{n})$,
3. the faces of the polyhedra are fat triangles,
4. the vertices of P_n are on S,
5. S has bounded curvature.

Then the average complexity of the transparent silhouette of P_n is $O(\sqrt{n})$.

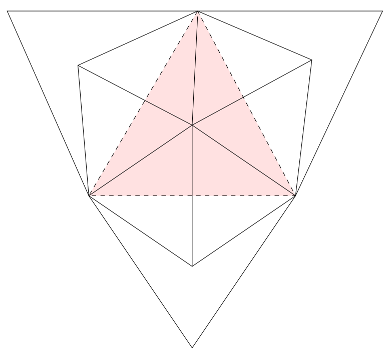

Figure 8: A way to fold a polytope. All the vertices are on a sphere.

The idea is that conditions 2 to 5 ensure that the geometric angle between any two adjacent faces is $O(1/\sqrt{n})$. However, without condition one, the polyhedron might be folded as in Figure 8. One way to avoid this is to assume that all exterior angles are smaller than $\pi/2$. Condition 1 is a funnier way to achieve the same goal. Notice that if the edges of P_n have length $\Theta(1/\sqrt{n})$, condition 4 may be weakened to: the vertices of P_n are at distance $O(1/n)$ from S.

4.2.2 Generalization

Consider a family of polyhedra P_n of complexity $O(n)$. Assume:

1. there exist continuous injective functions $f_n : P_n \to S$ such that $\max d(f_n(x), x) = O(1/n)$ (usually f_n can be the closest-point function),
2. the edges of P_n have length $\Theta(1/\sqrt{n})$,
3. the faces of the polyhedra are fat triangles,
4. $\int_S (|\lambda_1| + |\lambda_2|) dS < \infty$, where the λ_i are the principal curvatures of the surface.

The average complexity of the transparent silhouette of P_n is $O(\sqrt{n})$.

Here for each edge e_i of the polyhedron, we charge $\Omega(l_i \alpha_i)$ to a part of S in the integral above and there remains $O(1/\sqrt{n})$ as in the previous theorem.

4.2.3 Shadow

1. S has bounded curvature,
2. the vertices of P_n are at distance $O(1/n)$ from S,
3. if a point is not inside the volume defined by P_n then either it is not in the volume defined by S or it is at distance $O(1/n)$ from S,
4. the average length of the edges of P_n is $O(1/\sqrt{n})$.

When considering the shadow of a polyhedron, for an edge to be on the silhouette means for it to be visible "from both sides". Here we just give a bound by considering directions where it is obviously hidden on at least one side.

Then the average complexity of the shadow of P_n is $O(\sqrt{n})$.

4.3 Worst case

4.3.1 The sphere

1. The smallest eigenvalue of the curvature of S is bounded from below by a positive real constant,
2. P_n is convex,
3. the vertices of P_n are at distance $O(1/n)$ from S,
4. if a point is not inside the volume defined by P_n then either it is not in the volume defined by S or it is at distance $O(1/n)$ from S,
5. the edges of P_n have length $\Theta(1/\sqrt{n})$,
6. the faces of P_n are fat triangles.

Then the worst-case complexity of the silhouette of P_n is $O(\sqrt{n})$. This is close to Theorem 6.

4.3.2 Other surfaces

I lack some time to state an actual theorem here. An important formula should be the integral along the silhouette of the inverse of the square root of the curvature of the surface along the viewing direction.

There are actually two distinct problems depending on whether we fix a given direction and then refine the triangulation or refine the triangulation and then consider the worst direction for each P_n.

Another thing to think about is that I never really use the fact that points belong to the surface, only that they are close to it. But a construction like the cylinder (Fig. 5) cannot be done for a planar area if the vertices are required to be on the surface, it requires some curvature perpendicular to the viewing direction.

5 Conclusion

Our results only apply to the case where the viewpoint is at infinity. To extend our approach to the perspective case, one has to deal with three distinct issues. First, the distance from the object to the viewpoint has to be taken into account. When the viewpoint is far from the polytope, the perspective case should behave as the orthographic case. When the viewpoint is very close to the object, one can see only a constant number of faces. But in between, things get ugly. Then the apparent length is not well defined any more so our first results do not extend easily. Last, the notion of average complexity is much harder to define. In any case it is complicated.

The results of this paper are only a first step toward the understanding of the complexity of silhouettes, but they still have promising applications.

A first application is the computation of shadow boundaries. Drettakis and Duguet [2, 3] propose a solution based on a *visibility skeleton* restricted to the visual events generated by a punctual light source. In their detailed report [3], they show that their algorithm has complexity $O(ns_n)$, where n is the size of the polyhedron that casts a shadow, and s_n the size of its silhouette. Even the orthographic case is of interest, since it corresponds to a light source at infinity, a simple sun model for instance.

A second application is hidden surface removal, which has a long history as a problem difficult to address practically [4]. A solution proposed by Efrat et al. [5] is to render first the silhouettes of the objects, and then optimize the rendering in the single-object regions. They estimate the number of combinatorial changes to the rendered silhouettes of polytopes when the viewpoint moves along a line or an algebraic curve. Depending on the motion, this number depends either linearly or quadratically on the silhouette complexity, which they bound from above by the complexity of the polytope. Extension of our work to the perspective case would thus yield a direct improvement of their bounds.

Acknowledgements

I wish to thank Olivier Devillers for mentioning this problem during the Workshop on 3D Global Visibility in 2000 at Bonifacio, Corsica, Xavier Goaoc and Helmut Alt, coauthors of the CCCG version of this paper ([1]), Sylvain Lazard, my director, and Hazel Everett and Sylvain Petitjean for their helpful comments. Part of the research was done during the First McGill-INRIA Workshop on Geometry in Graphics at McGill Bellairs Research Institute, in 2002.

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