Labelled Calculi for Łukasiewicz Logics

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Abstract. In this paper, we define new decision procedures for Łukasiewicz logics. They are based on particular integer-labelled hypersequents and of logical proof rules for such hypersequents. These rules being proved strongly invertible our procedures naturally allow one to generate countermodels. From these results we define a "merge"-free calculus for the infinite version of Łukasiewicz logic and prove that it satisfies the sub-formula property. Finally we also propose for this logic a new terminating calculus by using a focusing technique.

1 Introduction

Łukasiewicz logics, including finite and infinite versions, are among the most studied many-valued logics [10] and the infinite version Ł is, like Gödel-Dummett logic (LC) and Product logic (Π), one of the fundamental *t-norm based* fuzzy logics [8]. There exist various calculi and methods dedicated to proof-search in these logics that are based on sequents [1,12], hypersequents [4,12] or relational hypersequents [3] and on tableaux [13] or goal-directed approach [11]. In this paper, we consider proof-search in propositional Łukasiewicz logics through a particular approach that consists firstly in reducing (by a proof-search process) a hypersequent into a set of so-called irreducible hypersequents and then secondly in deciding these specific hypersequents by a particular procedure. Such an approach has been studied for Gödel-Dummett logic [2] and also the infinite version Ł of Łukasiewicz logics [3] but not for the finite versions. In this context we are interested in deciding irreducible hypersequents through a countermodel search process and thus in providing new decision procedures that generate countermodels.

Therefore we define labelled hypersequents, called $\mathbb{Z}$-hypersequents, in which components are labelled with integers, such labels introducing semantic information in the search process. Then we define proof rules that deal with labels by using the addition and subtraction and then prove that they are strongly invertible. It is important to notice that we define a same set of simple proof rules for both finite and infinite versions of Łukasiewicz logic. By application of these rules we show how we can reduce the decision problem of every $\mathbb{Z}$-hypersequent to the decision problem of a set of so-called atomic $\mathbb{Z}$-hypersequents that only contain atomic formulae. To solve the later problem we associate a set of particular inequalities to these hypersequents and then strongly relate the existence of a countermodel to the existence of a solution for this set of inequalities. Thus, by using results from linear and integer programming [16], we can decide any atomic $\mathbb{Z}$-hypersequent and also generate a countermodel in case of non-validity. Thus, from the same set of rules, we provide a new decision procedure for the infinite version but also one for the finite versions of Łukasiewicz logic, both including countermodel generation. After this first contribution we focus, in the rest of the paper, on the infinite version denoted Ł. The next contribution is the definition of a new calculus for this logic that is characterized by a single form of axioms and the absence of the "merge" rule that is not appropriate for proof-search. In addition our labelling of components by integers can be seen as a kind of merge-elimination technique that could be applied to hypersequent calculi given in [4,12]. From a refinement of the notion $\mathbb{Z}$-hypersequent, by using a focusing technique defined in [12], the last contribution is a terminating calculus for Ł, that is proved sound and complete, in which only one rule is not (strongly) invertible. We complete these results by showing, in the appendix, how to obtain a labelled calculus for Bounded Łukasiewicz logics $\mathsf{ŁB_n}$ with $n \geqslant 2$ [4].

2 Łukasiewicz Logics

We consider here the family of Łukasiewicz logics denoted $Ł_n$ with $n \in \mathbb{N}^1 = \{2, \ldots\} \cup \{\infty\}$, set of natural numbers with its natural order $\leqslant$, augmented with a greatest element ∞. In the case $n = \infty$, $Ł_\infty$, also denoted by Ł, is one of the most interesting multi-valued logics and one of the fundamental t-norms based fuzzy logics (see [8] for more details). In the case $n \neq \infty$, $Ł_n$ denotes the finite versions of Łukasiewicz logics.
The set of propositional formulae, denoted Form, is inductively defined from a set of propositional variables with a bottom constant $\perp$ (absurdity) by using the connectives $\wedge, \vee$, $\odot$ (strong conjunction) and $\oplus$ (strong disjunction). All the connectives can be expressed by using the $\supset$ connective: $\neg A =_{def} A \supset \perp$, $A \oplus B =_{def} \neg A \supset B$, $A \odot B =_{def} \neg(A \supset \neg B)$, $A \vee B =_{def} (A \supset B) \supset B$ and $A \wedge B =_{def} \neg(\neg A \vee \neg B)$.
In the case of Ł, the logic has a following Hilbert axiomatic system:

Ł1 $A \supset (B \supset A)$
Ł2 $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
Ł3 $((A \supset B) \supset B) \supset ((B \supset A) \supset A)$
Ł4 $((A \supset \perp) \supset (B \supset \perp)) \supset (B \supset A)$

with the rule $\dfrac{A \supset B \qquad A}{B}\ [mp]$

Another Hilbert axiomatic system can be obtained by adding axioms Ł1 and Ł3 to any axiomatization of the multiplicative additive fragment of Linear Logic [14].
For the finite versions $\mathsf{Ł_n}$ with $n \neq \infty$, a Hilbert axiomatic system is obtained by adding to the previous axioms of Ł the following axioms : $(n-1)A \supset nA \odot nA \supset (n-1)A$ and $(pA^{p-1})^n \supset mA^p \odot mA^p \supset (pA^{p-1})^n$ for every integer $p = 2, \ldots, n-2$ that does not divide $n-1$, with $kA = A \oplus \ldots \oplus A$ (k times) and $A^k = A \odot \ldots \odot A$ (k times).
A valuation for $\mathsf{Ł_n}$ is a function $[\![\cdot]\!]$ from the set of propositional variables Var to $[0,1]$ if $n = \infty$ and to $[0, 1/(n-1), \ldots, (n-2)/(n-1), 1]$ if $n \neq \infty$. It is inductively extended to formulae as follows:

$[\![A \supset B]\!] = min(1, 1 - [\![A]\!] + [\![B]\!])$
$[\![\perp]\!] = 0$
$[\![\neg A]\!] = 1 - [\![A]\!]$
$[\![A \odot B]\!] = max(0, [\![A]\!] + [\![B]\!] - 1)$

$[\![A \oplus B]\!] = min(1, [\![A]\!] + [\![B]\!])$
$[\![A \wedge B]\!] = min([\![A]\!], [\![B]\!])$
$[\![A \vee B]\!] = max([\![A]\!], [\![B]\!])$

A formula A is valid in $\mathsf{Ł_n}$, written $\models_{\mathsf{Ł_n}} A$, iff $[\![A]\!] = 1$ for all valuations $[\![\cdot]\!]$ for $\mathsf{Ł_n}$.

In this paper we study proof-search in the finite and infinite versions of Łukasiewicz logics. Our approach based on labelled calculi is an alternative to existing works based on sequents [1,12], on multisets of sequents, called hypersequents [4,12] and relational hypersequents [3] but also on tableaux [13] or goal-directed approach [11]. It consists first in reducing (by a proof-search process) a hypersequent into a set of so-called irreducible hypersequents and then in deciding these hypersequents. It has been studied for LC [2] and also the infinite version Ł [3] but not for the finite versions. Like for recent works in Gödel-Dummett Logics [7,9] we aim at deciding irreducible hypersequents through a countermodel search process and then at providing new calculi and decision procedures that allow us to generate countermodels.

3 Labelled Proof Rules for $Ł_n$

In this section, we present for $Ł_n$ the definition of integer-labelled hypersequents, labels introducing semantic information in the search process, and of labelled proof rules that are strongly invertible in order to generate countermodels. Let us remind that the hypersequent structure $\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_k \vdash \Delta_k$ has been introduced as a natural generalization of Gentzen's sequents [2]. It is a multiset of sequents, called components, with "|" denoting a disjunction at the meta-level.

Definition 1. *A* $\mathbb{Z}$-hypersequent *is a hypersequent of the form:* $\Gamma_1 \vdash_{n_1} \Delta_1 \mid \ldots \mid \Gamma_k \vdash_{n_k} \Delta_k$ *where for* $i = 1, \ldots, k$, $n_i \in \mathbb{Z}$, *and* Γ_i *and* Δ_i *are multisets of formulae.*

Definition 2. *A* $\mathbb{Z}$*-hypersequent* $G = \Gamma_1 \vdash_{n_1} \Delta_1 \mid \ldots \mid \Gamma_k \vdash_{n_k} \Delta_k$ *is* valid *in* $Ł_\mathsf{n}$ *iff for any valution* $[\![\cdot]\!]$ *for* $Ł_\mathsf{n}$, *there exists* $i \in \{1, \ldots, n\}$ *such that:* $\llfloor \Gamma_i \rrfloor \leqslant \llceil \Delta_i \rrceil - n_i$ *where* $\llfloor \emptyset \rrfloor = 1$, $\llceil \emptyset \rrceil = 0$, $\llfloor \Gamma_i \rrfloor = 1 + \sum_{A \in \Gamma_i} ([\![A]\!] - 1)$ *and* $\llceil \Delta_i \rrceil = \sum_{B \in \Delta_i} [\![B]\!]$.

A formula A is valid in $Ł_\mathsf{n}$ if and only if the $\mathbb{Z}$-hypersequent $\vdash_0 A$ is valid in $Ł_\mathsf{n}$. Moreover the $\mathbb{Z}$-hypersequent $A^1_1, \ldots, A^1_{l_1} \vdash_0 B^1_1, \ldots, B^1_{m_1} \mid \ldots \mid A^k_1, \ldots, A^k_{l_1} \vdash_0 B^k_1, \ldots, B^k_{m_k}$ is valid in $Ł_\mathsf{n}$ if and only if $(A^1_1 \odot \ldots \odot A^1_{l_1}) \supset (B^1_1 \oplus \ldots \oplus B^1_{m_1}) \vee \ldots \vee (A^k_1 \odot \ldots \odot A^k_{l_1}) \supset (B^k_1 \oplus \ldots \oplus B^k_{m_k})$ is valid in $Ł_\mathsf{n}$.

In comparison with hypersequents in [4,12] where the interpretation of components is such that one has disjunctions ($\oplus$) on the both sides, our aim here is to recover the standard interpretation with conjunctions ($\odot$) on the left-hand side and disjunctions ($\oplus$) on the right-hand side.

Now we define a set of proof rules, presented in Figure 1, dealing with these structures. They mainly decompose the principal formula and simply modify the labels by addition or substraction of 1.

Considering a proof rule as composed of premises H_i with a conclusion C, it is *sound* if, for any instance of the rule, the validity of the premises H_i entails the validity of C. It is *strongly sound* if, for any instance of the rule and any valuation $[\![\cdot]\!]$, if $[\![\cdot]\!]$ is a model of all the H_i then it is a model of C. Moreover a proof rule is *invertible* if, for any instance of the rule, the non-validity of at least one H_i entails the non-validity of C. It is *strongly invertible* if, for any instance of the rule and any valuation $[\![\cdot]\!]$, if $[\![\cdot]\!]$ is a countermodel of at least one H_i then it is a countermodel of C. We can observe that strong invertibility (resp. soundness) implies invertibility (resp. soundness).

$$\frac{G \mid \Gamma, A, B \vdash_n \Delta \qquad G \mid \Gamma \vdash_{n-1} \Delta}{G \mid \Gamma, A \odot B \vdash_n \Delta} \; [\odot_L] \qquad \frac{G \mid \Gamma \vdash_n \Delta \mid \Gamma \vdash_{n+1} A, B, \Delta}{G \mid \Gamma \vdash_n A \odot B, \Delta} \; [\odot_R]$$

$$\frac{G \mid \Gamma \vdash_n \Delta \mid \Gamma, A, B \vdash_{n+1} \Delta}{G \mid \Gamma, A \oplus B \vdash_n \Delta} \; [\oplus_L] \qquad \frac{G \mid \Gamma \vdash_n A, B, \Delta \qquad G \mid \Gamma \vdash_{n-1} \Delta}{G \mid \Gamma \vdash_n A \oplus B, \Delta} \; [\oplus_R]$$

$$\frac{G \mid \Gamma \vdash_n \Delta \mid \Gamma, B \vdash_{n+1} A, \Delta}{G \mid \Gamma, A \supset B \vdash_n \Delta} \; [\supset_L] \qquad \frac{G \mid \Gamma, A \vdash_n B, \Delta \qquad G \mid \Gamma \vdash_{n-1} \Delta}{G \mid \Gamma \vdash_n A \supset B, \Delta} \; [\supset_R]$$

$$\frac{G \mid \Gamma, A \vdash_n \Delta \mid \Gamma, B \vdash_n \Delta}{G \mid \Gamma, (A \wedge B) \vdash_n \Delta} \; [\wedge_L] \qquad \frac{G \mid \Gamma \vdash_n A, \Delta \qquad G \mid \Gamma \vdash_n B, \Delta}{G \mid \Gamma \vdash_n A \wedge B, \Delta} \; [\wedge_R]$$

$$\frac{G \mid \Gamma, A \vdash_n \Delta \qquad G \mid \Gamma, B \vdash_n \Delta}{G \mid \Gamma, A \vee B \vdash_n \Delta} \; [\vee_L] \qquad \frac{G \mid \Gamma \vdash_n A, \Delta \mid \Gamma \vdash_n B, \Delta}{G \mid \Gamma \vdash_n A \vee B, \Delta} \; [\vee_R]$$

Fig. 1. Proof rules for $\mathbb{Z}$-hypersequents in Ł$_\mathsf{n}$

Theorem 1 (Soundness). *The rules of Figure 1 are strongly sound for Ł$_\mathsf{n}$.*

Proof. We only develop the cases of $[\supset_L]$ and $[\oplus_R]$ rules, the other cases being similar. Case $[\supset_L]$. Let $[\![\cdot]\!]$ be a model of $G \mid \Gamma \vdash_k \Delta \mid \Gamma, B \vdash_{k+1} A, \Delta$ in Ł$_\mathsf{n}$. Then we have $[\![\cdot]\!]$ is a model of G, $\llbracket \Gamma \rrbracket \leqslant \llbracket \Delta \rrbracket - k$ or $\llbracket \Gamma \rrbracket + ([\![B]\!] - 1) \leqslant \llbracket \Delta \rrbracket + [\![A]\!] - (k+1)$. Thus, we obtain $[\![\cdot]\!]$ is a model of G, $\llbracket \Gamma \rrbracket + (1-1) \leqslant \llbracket \Delta \rrbracket - k$ or $\llbracket \Gamma \rrbracket + ((1 - [\![A]\!] + [\![B]\!]) - 1) \leqslant \llbracket \Delta \rrbracket - k$. We deduce that $[\![\cdot]\!]$ is a model of G or $\llbracket \Gamma \rrbracket + (min(1, 1 - [\![A]\!] + [\![B]\!]) - 1) \leqslant \llbracket \Delta \rrbracket - k$. Therefore $[\![\cdot]\!]$ is a model of $G \mid \Gamma, A \supset B \vdash_n \Delta$.
Case $[\oplus_R]$. Let $[\![\cdot]\!]$ be a model of $G \mid \Gamma \vdash_k A, B, \Delta$ and of $G \mid \Gamma \vdash_{k-1} \Delta$ in Ł$_\mathsf{n}$. Thus, $[\![\cdot]\!]$ is a model of G, or $\llbracket \Gamma \rrbracket \leqslant \llbracket \Delta \rrbracket + [\![A]\!] + [\![B]\!] - k$ and $\llbracket \Gamma \rrbracket \leqslant \llbracket \Delta \rrbracket - (k-1)$ hold. Then $[\![\cdot]\!]$ is a model of G or the inequality $\llbracket \Gamma \rrbracket \leqslant \llbracket \Delta \rrbracket + min(1, [\![A]\!] + [\![B]\!]) - k$ holds. Thus, $[\![\cdot]\!]$ is a model of $G \mid \Gamma \vdash_n A \oplus B, \Delta$.

Theorem 2. *The rules of Figure 1 are strongly invertible for Ł$_\mathsf{n}$.*

Proof. We only develop the cases of $[\supset_L]$ and $[\oplus_R]$ rules, the other cases being similar. Case $[\supset_L]$. Let $[\![\cdot]\!]$ be a countermodel of $G \mid \Gamma \vdash_k \Delta \mid \Gamma, B \vdash_{k+1} A, \Delta$ in Ł$_\mathsf{n}$. Then $[\![\cdot]\!]$ is a countermodel of G and the inequalities $\llbracket \Gamma \rrbracket > \llbracket \Delta \rrbracket - k$ or $\llbracket \Gamma \rrbracket + ([\![B]\!] - 1) > \llbracket \Delta \rrbracket + [\![A]\!] - (k+1)$ hold. Therefore, $[\![\cdot]\!]$ is a countermodel of G and the inequality $\llbracket \Gamma \rrbracket + (min(1, 1 - [\![A]\!] + [\![B]\!]) - 1) > \llbracket \Delta \rrbracket - k$ holds. We deduce that $[\![\cdot]\!]$ is a countermodel of $G \mid \Gamma, A \supset B \vdash_n \Delta$.
Case $[\oplus_R]$. Let $[\![\cdot]\!]$ be a countermodel of $G \mid \Gamma \vdash_k A, B, \Delta$ in Ł$_\mathsf{n}$. Then we have $[\![\cdot]\!]$ is a countermodel of G and $\llbracket \Gamma \rrbracket > \llbracket \Delta \rrbracket + [\![A]\!] + [\![B]\!] - k$. Thus, the inequality $\llbracket \Gamma \rrbracket > \llbracket \Delta \rrbracket + min(1, [\![A]\!] + [\![B]\!]) - k$ holds. Therefore, $[\![\cdot]\!]$ is a countermodel of $G \mid \Gamma \vdash_n A \oplus B, \Delta$. Let $[\![\cdot]\!]$ be a countermodel of $G \mid \Gamma \vdash_{k-1} \Delta$. Then $[\![\cdot]\!]$ is a countermodel of G and $\llbracket \Gamma \rrbracket > \llbracket \Delta \rrbracket - (k-1)$ holds. Thus, $[\![\cdot]\!]$ is a countermodel of G and $\llbracket \Gamma \rrbracket > \llbracket \Delta \rrbracket + min(1, [\![A]\!] + [\![B]\!]) - k$ holds. Then we deduce that $[\![\cdot]\!]$ is a countermodel of $G \mid \Gamma \vdash_n A \oplus B, \Delta$.

Having proved these properties we now define what an atomic $\mathbb{Z}$-hypersequent is and show that we can reduce any $\mathbb{Z}$-hypersequent $\mathcal{H}$ into a set $\mathcal{S}$ of atomic $\mathbb{Z}$-hypersequents, such that $\mathcal{H}$ is valid iff the elements of $\mathcal{S}$ are valid.

Definition 3. *An* atomic $\mathbb{Z}$-hypersequent *is a $\mathbb{Z}$-hypersequent which only contains atomic formulae.*

Theorem 3. *The application of the rules of Figure 1 to a given $\mathbb{Z}$-hypersequent terminates with atomic $\mathbb{Z}$-hypersequents.*

Proof. To prove the termination, we show that for every rule, its conclusion is more complex than its premises by using a measures of complexity over the formulae [6]. This measure, called α, is defined by: $\alpha(A) = 1$ where $(A \in \mathsf{Var} \cup \{\top, \bot\})$; $\alpha(A \circledast B) = \alpha(A) + \alpha(B) + 1$ where $\circledast \in \{\wedge, \vee, \supset, \oplus, \odot\}$; and $\alpha(\neg A) = \alpha(A) + 1$. We can see that the order relation $<$ on formulae, defined by $A < B$ iff $\alpha(A) < \alpha(B)$, is well-founded. Let Γ_1 and Γ_2 two multisets of formulae, we have $\Gamma_1 >_m \Gamma_2$ iff Γ_2 is obtained form Γ_1 by replacing a formula by a finite number of formulae, each in which is of lower measure than the replaced formula. Since the relation order on pure formulae and sentences is well-founded, the order relation $>_m$ is well-founded, for more details [5]. Similarly, we define a well-founded relation $>>_m$ on $\mathbb{Z}$-hypersequents, induced by the order relation $>_m$, by: $G_1 >>_m G_2$ iff G_2 is obtained form G_1 by replacing a component of $G1$ by a smaller finite set of components, where a component $\Gamma_2 \vdash_{n_2} \Delta_2$ is smaller than $\Gamma_1 \vdash_{n_1} \Delta_1$ iff $\Gamma_1 \cup \Delta_1 >_m \Gamma_2 \cup \Delta_2$. By using this order relation, it is easy to prove for every rule, its premises are smaller than its conclusion. Finally, there is always a rule for any sequent which is not atomic. Therefore, we deduce that the application of our rules to a given $\mathbb{Z}$-hypersequent terminates with atomic $\mathbb{Z}$-hypersequents.

4 New Decision Procedures for Ł$_\mathsf{n}$

By using Theorem 3 we can generate, from a given $\mathbb{Z}$-hypersequent, to a set of atomic $\mathbb{Z}$-hypersequents by application of our logical rules. After this step of bottom-up proof-search we now consider the resulting set of atomic $\mathbb{Z}$-hypersequents in the perspective of countermodel generation. For respectively Ł and Ł$_n$ with $n \neq \infty$, we associate to each atomic $\mathbb{Z}$-hypersequent a set of particular inequalities and then relate the existence of a countermodel to the existence of solution for this set.

Definition 4 **($SI_{\mathcal{H}}$).** *Let $\mathcal{H} = \Gamma_1 \vdash_{n_1} \Delta_1 \mid \ldots \mid \Gamma_k \vdash_{n_k} \Delta_k$ be an atomic $\mathbb{Z}$-hypersequent and x_p be a real variable associated to every propositional variable p. We define the set of inequalities $SI_{\mathcal{H}}$ associated to $\mathcal{H}$ by: $SI_{\mathcal{H}} = \{(\bigodot \Gamma_1) > (\bigoplus \Delta_1) - n_1, \ldots, (\bigodot \Gamma_k) > (\bigoplus \Delta_k) - n_k\}$ where $\bigodot \emptyset = 1$, $\bigoplus \emptyset = 0$, $\bigodot(\Gamma_i) = 1 + \sum_{A \in \Gamma_i} (x_A - 1)$ and $\bigoplus(\Delta_i) = \sum_{A \in \Delta_i} x_A$ with $x_\bot = 0$.*

Theorem 4. *An atomic $\mathbb{Z}$-hypersequent $\mathcal{H}$ has a countermodel in Ł iff $SI_{\mathcal{H}}$ has a solution over $[0,1]$.*

Definition 5 **($SI^n_{\mathcal{H}}$).** *Let $\mathcal{H} = \Gamma_1 \vdash_{m_1} \Delta_1 \mid \ldots \mid \Gamma_k \vdash_{m_k} \Delta_k$ be an atomic $\mathbb{Z}$-hypersequent and x_p be a real variable associated to every propositional variable p. We define the set of inequalities $SI^n_{\mathcal{H}}$ associated to $\mathcal{H}$ by: $SI^n_{\mathcal{H}} = \{(\bigodot_n \Gamma_1) - 1 \geq (\bigoplus_n \Delta_1) - ((n-1) * m_1), \ldots, (\bigodot_n \Gamma_k) - 1 \geq (\bigoplus_n \Delta_k) - ((n-1) * m_k)\}$, where $\bigodot_n \emptyset = n-1$, $\bigoplus_n \emptyset = 0$, $\bigodot_n(\Gamma_i) = (n-1) + \sum_{A \in \Gamma_i} (x_A - (n-1))$ and $\bigoplus_n(\Delta_i) = \sum_{A \in \Delta_i} x_A$ where $x_\bot = 0$.*

Theorem 5. *An atomic $\mathbb{Z}$-hypersequent H has a countermodel in $Ł_n$ with $n \neq \infty$ iff $SI^n_{\mathcal{H}}$ has a solution over the set of integers $\{0,\ldots,n-1\}$.*

The proofs of the above theorems are given in appendix B.

By using linear and integer programming [16], we can decide a $Ł_n$ atomic $\mathbb{Z}$-hypersequent in polynomial time. If $(x_{A_1} = r_1,\ldots,x_{A_k} = r_k)$ is a solution of the set $SI^n_{\mathcal{H}}$ (resp. $SI_{\mathcal{H}}$), where $\{x_{A_1},\ldots,x_{A_k}\}$ is the set of all its variables, then the valuation $[\![\cdot]\!]$ such that $\forall i \in \{1,\ldots,k\}$, $[\![A_i]\!] = r_i/(n-1)$ (resp. $[\![A_i]\!] = r_i$) is a countermodel of $\mathcal{H}$ in $Ł_n$ (resp. in Ł). For a given $\mathbb{Z}$-hypersequent, by Theorem 3 we can generate a set of atomic $\mathbb{Z}$-hypersequents by application of rules of Figure 1. Then we can build the set $SI_{\mathcal{H}}$ (resp. $SI^n_{\mathcal{H}}$) associated to each atomic $\mathbb{Z}$-hypersequent H and decide by using linear (resp. integer) programming if it has a countermodel or not and thus decide its validity in Ł (resp. $Ł_n$ with $n \neq \infty$).
These two main steps, namely proof search followed by countermodel search (based on the above theorems) provide new decision procedures for Łukasiewicz logics. A key point here is the generation of countermodels because of the strong invertibility of rules: any countermodel of an atomic $\mathbb{Z}$-hypersequent on the leaf of the derivation tree is a countermodel of the initial $\mathbb{Z}$-hypersequent on the root of this tree.
We illustrate our new procedure through examples. If we consider $\mathcal{H}_1 = \vdash_0 A \supset (B \supset A)$ and $\mathcal{H}_2 = \vdash_0 A \vee (A \supset \bot)$, by application of proof rules we obtain the derivations:

$$\dfrac{\dfrac{A,B \vdash_0 A \quad A \vdash_{-1}}{A \vdash_0 B \supset A}[\supset_R] \qquad \vdash_{-1}}{\vdash_0 A \supset (B \supset A)}[\supset_R] \qquad\qquad \dfrac{\dfrac{\vdash_0 A \mid A \vdash_0 \bot \quad \vdash_0 A \mid \vdash_{-1}}{\vdash_0 A \mid \vdash_0 A \supset \bot}[\supset_R]}{\vdash_0 A \vee (A \supset \bot)}[\vee_R]$$

Thus, $\mathcal{H}_1$ has a countermodel in Ł if one of the inequalities $1 > 1$ ($\vdash_{-1}$), $x_A > 1$ ($A \vdash_{-1}$) and $x_B > 1$ ($A,B \vdash_0 A$) has a solution over $[0,1]$. Since $1 > 1$, $x_A > 1$ and $x_B > 1$ have no solution over $[0,1]$, we deduce that $\mathcal{H}_1$ is valid in Ł. For $\mathcal{H}_2$, since $x_A = 1$ is an integer solution of the system $\{2 > x_A, x_A > 0\}$, the valuation $[\![\cdot]\!]$ defined by $[\![A]\!] = \frac{1}{2}$ is a countermodel of $\vdash_0 A \mid A \vdash_0 \bot$ in $Ł_3$. Then it is a countermodel of $\mathcal{H}_2$ in $Ł_3$.

5 The $\mathbb{Z}$Ł Calculus

In this section we propose a new calculus for Ł called $\mathbb{Z}$Ł, that is defined by the rules in Figure 1 and the following axiom, special rules and structural rules:

$$\dfrac{}{\vdash_n}[Ax](n<0) \qquad \dfrac{G \mid \Gamma \vdash_{n-1} \Delta}{G \mid \Gamma, \bot \vdash_n \Delta}[\bot_L] \qquad \dfrac{G \mid \Gamma \vdash_{n-1} \Delta}{G \mid \Gamma, A \vdash_n \Delta, A}[SR]$$

$$\dfrac{G}{G \mid \Gamma \vdash_n \Delta}[EW] \qquad \dfrac{G \mid \Gamma \vdash_n \Delta}{G \mid \Gamma, A \vdash_n \Delta}[IW_L] \qquad \dfrac{G \mid \Gamma \vdash_n \Delta}{G \mid \Gamma \vdash_n \Delta, A}[IW_R]$$

$$\dfrac{G \mid \Gamma \vdash_n \Delta \mid \Gamma \vdash_n \Delta}{G \mid \Gamma \vdash_n \Delta}[EC] \qquad \dfrac{G \mid \Gamma_1, \Gamma_2 \vdash_{n_1+n_2+1} \Delta_1, \Delta_2}{G \mid \Gamma_1 \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2}[SP]$$

Theorem 6 (Soundness). *The rules of the $\mathbb{Z}$Ł calculus are sound.*

Proof. From Theorem 1 the logical rules of $\mathbb{Z}$Ł are sound. Similar arguments are used for the other rules.

Theorem 7 (Completeness). *If a $\mathbb{Z}$-hypersequent is valid in Ł then it is derivable in the $\mathbb{Z}$Ł calculus.*

Proof. See appendix B.

We illustrate our calculus by considering our example $\mathcal{H}_1 = \vdash_0 A \supset (B \supset A)$. By application of proof rules we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{-1}}{B \vdash_{-1}}[IW_L]}{A, B \vdash_0 A}[SR] \quad \dfrac{\vdash_{-1}}{A \vdash_{-1}}[IW_L]}{A \vdash_0 B \supset A}[\supset_R] \qquad \vdash_{-1}}{\vdash_0 A \supset (B \supset A)}[\supset_R]$$

From the $\mathbb{Z}$Ł calculus we can show that the weakening rules ($[EW]$, $[IW_L]$ and $[IW_R]$) can be *"absorbed"* in the axiom by using an approach similar to the one of [17]. Thus we obtain a new simplified calculus $\mathbb{Z}Ł'$ without these rules and with the following axiom: $\dfrac{}{G \mid \Gamma \vdash_n \Delta}\ [Ax](n < 0)$.

Proposition 1. *The $\mathbb{Z}$Ł calculus satisfies the subformula property, namely any formula appearing in a proof of $\mathcal{H}$ in $\mathbb{Z}$Ł is a subformula of a formula in $\mathcal{H}$.*

An important point of these calculi is that they are "merge"-free. It means that the following rule, called *merge*, is not needed.

$$\dfrac{G \mid \Gamma_1 \vdash \Delta_1 \qquad G \mid \Gamma_2 \vdash \Delta_2}{G \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [M]$$

In hypersequent calculi for Ł in [4,3] a challenge, in the perspective of proof-search, consists in eliminating this rule that is not appropriate because it is not invertible and context splittings on the left and right sides could be very expensive. The rule has been eliminated in [15] by replacing the existing axioms by the following axiom $G \mid \Gamma, \overbrace{\bot, \ldots, \bot}^{n} \vdash A_1, \ldots, A_n, \Delta$, where $n \geqslant 0$. But our approach based on the labelling of components by integers allows us to eliminate the merge rule without having to complicate the form of axioms.

6 A Terminating Calculus for Ł

Now, we consider an approach based on a focusing technique in [12] in order to provide a terminating calculus for Ł. Thus we consider now so-called *focused hypersequents*.

Definition 6. *A* focused $\mathbb{Z}$-hypersequent *is a structure of the form $[p]\mathcal{H}$ where $\mathcal{H}$ is a $\mathbb{Z}$-hypersequent, p a propositional variable, and $[p]\mathcal{H}$ is valid in Ł iff $\mathcal{H}$ is valid in Ł.*

Let $\mathcal{H} = \Gamma_1 \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2 \mid \dots \mid \Gamma_k \vdash_{n_k} \Delta_k$ be a $\mathbb{Z}$-hypersequent. We denote by $left(\mathcal{H})$ the multiset $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ and by $right(\mathcal{H})$ the multiset $\Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_k$. We define a new calculus, called $\mathbb{Z}$Ł**T**, that consists of the logical rules in Figure 1 with the same focus for premises and conclusion, and of these following rules:

$$\frac{}{[p]G \mid \Gamma \vdash_n \Delta}\ [Ax](n<0) \qquad \frac{[p]G \mid \Gamma \vdash_{n-1} \Delta}{[p]G \mid \Gamma, \bot \vdash_n \Delta}\ [\bot_L] \qquad \frac{[p]G \mid \Gamma \vdash_{n-1} \Delta}{[p]G \mid \Gamma, A \vdash_n \Delta, A}\ [SR]$$

$$\frac{[q]\mathcal{H}}{[p]\mathcal{H}}\ [F] \quad \text{where } q \in left(\mathcal{H}) \cap right(\mathcal{H}) \text{ and } p \notin left(G) \cap right(G)$$

$$\frac{[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid S}{[p]G \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p}\ [R]$$

where $G, \Gamma_1, \Gamma_2, \Delta_1$ and Δ_2 are atomic and $k_1 > 0, k_2 > 0, p \notin \Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2$.
S is $\Gamma_1, k_1 p \vdash_{n_1} \Delta_1$ or $\Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ and $n' = k_2 * n_1 + k_1 * n_2 + k_1 + k_2 - (k_1 * k_2 + 1)$.

Theorem 8. *All the rules of $\mathbb{Z}$Ł**T** except $[R]$ are strongly invertible.*

Proof. From Theorem 2, the logical rules of $\mathbb{Z}$Ł**T** are strongly invertible. For the other rules we use similar arguments.

Definition 7. *An* irreducible focused $\mathbb{Z}$-hypersequent $[p]\mathcal{H}$ *is an atomic focused $\mathbb{Z}$-hypersequent where $left(\mathcal{H}) \cap right(\mathcal{H}) = \emptyset$, $\bot \notin left(\mathcal{H})$ and for every component $\Gamma \vdash_n \Delta$ of $\mathcal{H}$, we have $n \geqslant 0$.*

Definition 8. *An* inv-irreducible focused $\mathbb{Z}$-hypersequent $[p]\mathcal{H}$ *is an atomic $\mathbb{Z}$-hypersequent where $p \in left(\mathcal{H}) \cap right(\mathcal{H})$, and for every component $\Gamma \vdash_n \Delta$ of $\mathcal{H}$, we have $\bot \notin \Gamma$, $\Gamma \cap \Delta = \emptyset$ and $n \geqslant 0$.*

Proposition 2. *Any irreducible focused $\mathbb{Z}$-hypersequent has a countermodel.*

Proof. Let $[p]\mathcal{H}$ be an irreducible focused $\mathbb{Z}$-hypersequent. Let $[\![\cdot]\!]$ a valuation defined by: for every $A \in left(\mathcal{H})$ we have $[\![A]\!] = 1$, and for every $B \in right(\mathcal{H})$ we have $[\![B]\!] = 0$. It is easy to prove that $[\![\cdot]\!]$ is a countermodel of $[p]\mathcal{H}$.

Theorem 9. *The application of $\mathbb{Z}$Ł**T** calculus to every focused $\mathbb{Z}$-hypersequent terminates with axioms or irreducible focused $\mathbb{Z}$-hypersequents.*

Proof. From Theorem 3, we see that the application of the logical rules of $\mathbb{Z}$Ł**T** to a given focused $\mathbb{Z}$-hypersequent terminates with atomic focused $\mathbb{Z}$-hypersequents. By using the order $>>_m$ defined in the proof of Theorem 3, $G \mid \Gamma, \bot \vdash_n \Delta >>_m G \mid \Gamma \vdash_{n-1} \Delta$ and $G \mid \Gamma, A \vdash_n \Delta, A >>_m G \mid \Gamma \vdash_{n-1} \Delta$ hold. Now considering the rule $[R]$, we can see that its application with the focus p decreases strictly the number of p's. Therefore, in any derivation in $\mathbb{Z}$Ł**T**, the number of applications of the rules $[R]$ and $[F]$ is finite. Thus, the application of $\mathbb{Z}$Ł**T** calculus to every focused $\mathbb{Z}$-hypersequent terminates.
Since there is always a rule for any $\mathbb{Z}$-hypersequent which is not an axiom or an irreducible $\mathbb{Z}$-hypersequent, we deduce that The application of $\mathbb{Z}$Ł**T** calculus to every focused $\mathbb{Z}$-hypersequent terminates with axioms or irreducible focused $\mathbb{Z}$-hypersequents.

Theorem 10 (Soundness). *The rules of $\mathbb{Z}\text{Ł}\boldsymbol{T}$ are sound.*

Proof. The soundness of the logical rules and the rules $[Ax]$, $[\perp_L]$ and $[SR]$ comes from Theorem 6. The soundness of $[F]$ is trivial. For the rule $[R]$ we consider arguments similar to those of proof of Theorem 1.

Proposition 3. *If the atomic $\mathbb{Z}$-hypersequent $G \mid \Gamma_1 \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2$ is valid in Ł then either $G \mid \Gamma_1, \Gamma_2 \vdash_{n_1+n_2+1} \Delta_1, \Delta_2 \mid \Gamma_1 \vdash_{n_1} \Delta_1$ is valid in Ł or $G \mid \Gamma_1, \Gamma_2 \vdash_{n_1+n_2+1} \Delta_1, \Delta_2 \mid \Gamma_2 \vdash_{n_2} \Delta_2$ is valid in Ł.*

Proposition 4. *Let $[p]G \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ be an atomic focused $\mathbb{Z}$-hypersequent. If it is valid in Ł then either $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1$ is valid in Ł or $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ is valid in Ł, with $k_1 > 0, k_2 > 0, p \notin \Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2$ and $n' = k_2 * n_1 + k_1 * n_2 + k_1 + k_2 - (k_1 * k_2 + 1)$.*

Proofs of these propositions are given in appendix A.

Definition 9 (Proof-refutation tree). *A* proof-refutation tree *is a tree where the nodes are labelled by a focused $\mathbb{Z}$-hypersequents and satisfying the following properties:*

- *Every internal node n labelled by $\mathcal{H}$ which is not an inv-irreducible $\mathbb{Z}$-hypersequent has a maximum of two children: if n has two children (resp. a single child) labelled by $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}'$) then $\dfrac{\mathcal{H}_1 \quad \mathcal{H}_2}{\mathcal{H}}\ [r]$ (resp. $\dfrac{\mathcal{H}'}{\mathcal{H}}\ [r]$) is an instance of a strongly invertible rule.*
- *Every internal node n labelled by $\mathcal{H}$ which is an inv-irreducible $\mathbb{Z}$-hypersequent, namely $[p]G \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$, has two children labelled by $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1$ and by $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ where $n' = k_2 * n_1 + k_1 * n_2 + k_1 + k_2 - (k_1 * k_2 + 1)$.*

From Theorem 9, we can see that a proof-refutation tree is finite and its leaf nodes are indexed by axioms and irreducible $\mathbb{Z}$-hypersequents.

Theorem 11 (Completeness). *If $[p]\mathcal{H}$ is valid in Ł then $[p]\mathcal{H}$ is provable in $\mathbb{Z}\text{Ł}\boldsymbol{T}$.*

Proof. Let $[p]\mathcal{H}$ be a focus $\mathbb{Z}$-hypersequent and $\mathcal{P}$ its proof-refutation tree. We show how to decide if an index of a given node in $\mathcal{P}$ is valid or not. We start by the leaf nodes. From Theorem 9, we know that such leaf nodes are labelled by axioms or irreducible focused $\mathbb{Z}$-hypersequents. Thus, by using Proposition 2, we can decide all the leaf nodes. Now we see how, from the children of a given internal node, we can propagate validity or invalidity. Let $\mathcal{H}$ be an index of internal node. If $\mathcal{H}$ is not an inv-irreducible focused $\mathbb{Z}$-hypersequent then, from Definition 9, this node has a maximum of two children where if these children are labelled by $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}'$) then $\dfrac{\mathcal{H}_1 \quad \mathcal{H}_2}{\mathcal{H}}\ [r]$ (resp. $\dfrac{\mathcal{H}'}{\mathcal{H}}\ [r]$) is an instance of a strongly invertible rule. Thus, if $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}'$) are valid then $\mathcal{H}$ is valid because $[r]$ is sound. Else, from the strong invertibility of $[r]$, $\mathcal{H}$ has the same countermodels of its non-valid premises.

We now deal with the nodes labelled by inv-irreducible focused $\mathbb{Z}$-hypersequent. Let n be an internal node labelled by an inv-irreducible $\mathbb{Z}$-hypersequent $\mathcal{H}$. Thus $\mathcal{H}$ is of the form $[p]G \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$. and the children of n are labelled by $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1$ and $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ where $n' = k_2 * n_1 + k_1 * n_2 + k_1 + k_2 - (k_1 * k_2 + 1)$. By using Proposition 4, if one of the indexes of the children of n is valid then $\mathcal{H}$ is valid else $\mathcal{H}$ is not valid. Therefore, if a focused $\mathbb{Z}$-hypersequent is valid then it is derivable in $\mathbb{Z}$Ł**T**.

In the completeness proof (Theorem 11) we give a decision procedure for Ł based on the concept of proof-refutation tree. Let $\mathcal{H} = \vdash_0 A \supset B \vee B \supset A$. A proof-refutation tree of $\mathcal{H}$ is given by:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]\vdash_{-1} \mid A\vdash_0 B}{[A]B\vdash_0 B \mid B\vdash_0 A}[SR] \quad \dfrac{[A]\vdash_{-1} \mid A\vdash_0 B}{[A]B\vdash_0 B \mid A\vdash_0 B}[SR]}{[A]A\vdash_0 B \mid B\vdash_0 A}[R] \qquad [A]\vdash_{-1} \mid B\vdash_0 A}{[A]\vdash_0 A\supset B \mid B\vdash_0 A}[\supset_R] \qquad [A]\vdash_0 A\supset B \mid \vdash_{-1}}{\dfrac{[A]\vdash_0 A\supset B \mid \vdash_0 B\supset A}{[A]\vdash_0 A\supset B \vee B\supset A}[\vee_R]}[\supset_R]$$

From this proof refutation tree, we then deduce that $\mathcal{H}$ is valid.

Our method based on proof-refutation trees cannot be applied to the terminating calculus in [12] because the merge and weakening rules are not invertible. Our terminating calculus that does not contain these rules is then more efficient because all its rules except one are (strongly) invertible: the conclusion of an invertible rule is valid iff its premises are valid.

7 Conclusion and Perspectives

In this work, we provide new decision procedures with countermodel generation for Łukasiewicz logics, using the approach proposed in [2]. A key point is the use of strongly invertible rules and consequently the ability to generate countermodels. An important contribution is the definition of a new terminating calculi for the infinite version Ł. In comparison with the calculi based on hypersequents [3,4] our calculus improves proof-search because it has a single form of axiom and moreover does not contain the merge rule. In further works we will define such labelled terminating calculi for the finite versions of Łukasiewicz logics and also for Bounded Łukasiewicz logics (see preliminary results in appendix C) for which cut-elimination will be studied. We will also study the possible design of labelled systems for other fuzzy logics.

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A Proofs of Propositions 3 and 4

Proposition 3. *If the atomic $\mathbb{Z}$-hypersequent $G \mid \Gamma_1 \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2$ is valid in Ł then either $G \mid \Gamma_1, \Gamma_2 \vdash_{n_1+n_2+1} \Delta_1, \Delta_2 \mid \Gamma_1 \vdash_{n_1} \Delta_1$ is valid in Ł or $G \mid \Gamma_1, \Gamma_2 \vdash_{n_1+n_2+1} \Delta_1, \Delta_2 \mid \Gamma_2 \vdash_{n_2} \Delta_2$ is valid in Ł.*

Proof. Let $\mathcal{H} = G \mid \Gamma_1 \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2$ be an atomic $\mathbb{Z}$-hypersequent where G being $\Gamma'_1 \vdash_{n'_1} \Delta'_1 \mid \ldots \mid \Gamma'_k \vdash_{n'_k} \Delta'_k$. By using linear programming [16] $\mathcal{H}$ is valid iff there exist $\alpha'_1, \ldots, \alpha'_k, \alpha_1, \alpha_2 \in \mathbb{N}$ where $\alpha'_i > 0$ or $\alpha_j > 0$ for some $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant 2$, such that for every valuation $[\![\cdot]\!]$, $\sum_{i=1}^{k}(\alpha'_i * \llbracket\Gamma'_i\rrbracket) + \sum_{i=1}^{2}(\alpha_i * \llbracket\Gamma_i\rrbracket) \leqslant \sum_{i=1}^{k}(\alpha'_i * \llbracket\Delta'_i\rrbracket) + \sum_{i=1}^{2}(\alpha_i * \llbracket\Delta_i\rrbracket) - (\sum_{i=1}^{k}(\alpha'_i * n'_i) + \sum_{i=1}^{2}\alpha_i * n_i)$. We suppose that $\alpha_1 \geqslant \alpha_2$. Then for every valuation $[\![\cdot]\!]$ we have $\sum_{i=1}^{k}(\alpha'_i * \llbracket\Gamma'_i\rrbracket) + (\alpha_1 - \alpha_2) * \llbracket\Gamma_1\rrbracket + \alpha_2 * (\llbracket\Gamma_1 + \Gamma_2\rrbracket) \leqslant \sum_{i=1}^{k}(\alpha'_i * \llbracket\Delta'_i\rrbracket) + (\alpha_1 - \alpha_2) * \llbracket\Delta_1\rrbracket + \alpha_2 * (\llbracket\Gamma_1 + \Gamma_2\rrbracket) - (\sum_{i=1}^{k}(\alpha'_i * n'_i) + (\alpha_1 - \alpha_2) * n_1 + \alpha_2 * (n_1 + n_2 + 1))$. Then $G \mid \Gamma_1, \Gamma_2 \vdash_{n_1+n_2+1} \Delta_1, \Delta_2 \mid \Gamma_1 \vdash_{n_1} \Delta_1$ is valid in Ł. The case of $\alpha_2 \geqslant \alpha_1$ is symmetrical.

Proposition 4. *Let $[p]G \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ be an atomic focused $\mathbb{Z}$-hypersequent. If it is valid then one of the following focused $\mathbb{Z}$-hypersequents is valid:*

- $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1$
- $[p]G \mid k_2\Gamma_1, k_1\Gamma_2 \vdash_{n'} k_2\Delta_1, k_1\Delta_2 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$

where $k_1 > 0, k_2 > 0, p \notin \Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2$ and $n' = k_2 * n_1 + k_1 * n_2 + k_1 + k_2 - (k_1 * k_2 + 1)$.

Proof. We first prove by induction on k that $[p]G \mid \Gamma \vdash_m \Delta$ is valid iff $G \mid k\Gamma \vdash_n k\Delta$ where $n = k * m + (k-1)$. Then, by Proposition 3, if $[p]G \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1 \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$ is valid then one of the following focused $\mathbb{Z}$-hypersequents is valid:

- $[p]G \mid k_2\Gamma_1, k_1\Gamma_2, (k_1 * k_2)p \vdash_{n''} k_2\Delta_1, k_1\Delta_2, (k_1 * k_2)p \mid \Gamma_1, k_1 p \vdash_{n_1} \Delta_1$
- $[p]G \mid k_2\Gamma_1, k_1\Gamma_2, (k_1 * k_2)p \vdash_{n''} k_2\Delta_1, k_1\Delta_2, (k_1 * k_2)p \mid \Gamma_2 \vdash_{n_2} \Delta_2, k_2 p$

where $n' = k_2 * n_1 + k_1 * n_2 + k_1 + k_2 - 1$. Finally we prove the following result by induction: $[p]G \mid \Gamma, kp \vdash_n \Delta, kp$ is valid in Ł iff $[p]G \mid \Gamma \vdash_{n-k} \Delta$ is valid in Ł. Therefore we deduce the result.

B Proofs of Theorems 4, 5 and 7

Theorem 4. *An atomic $\mathbb{Z}$-hypersequent $\mathcal{H}$ has a countermodel in Ł iff $SI_{\mathcal{H}}$ has a solution over $[0,1]$.*

Proof. Let $\mathcal{H} = \Gamma_1 \vdash_{n_1} \Delta_1 \mid \ldots \mid \Gamma_k \vdash_{n_k} \Delta_k$ be an atomic $\mathbb{Z}$-hypersequent. $[\![\cdot]\!]$ is a countermodel of $\mathcal{H}$ in Ł iff for all $i \in \{1, \ldots, k\}$, the inequality $\llbracket\Gamma_i\rrbracket > \llbracket\Delta_i\rrbracket - n_i$ holds. Thus, $[\![\cdot]\!]$ is a countermodel of $\mathcal{H}$ iff for all $i \in \{1, \ldots, k\}$, $(x_p = [\![p]\!] \mid p \in \Gamma_i \cup \Delta_i)$ is a solution of $\bigodot\Gamma_1 > \bigoplus\Delta_1 - n_1$. Therefore, $\mathcal{H}$ has a countermodel in Ł iff $SI_{\mathcal{H}}$ has a solution. This solution is over $[0,1]$ because the valuations in Ł are from Var to $[0,1]$.

Theorem 5. *An atomic $\mathbb{Z}$-hypersequent H has a countermodel in $Ł_n$ for $n \neq \infty$ iff $SI^n_{\mathcal{H}}$ has a solution over the set of integers $\{0, \ldots, n-1\}$.*

Proof. Let $\mathcal{H} = \Gamma_1 \vdash_{m_1} \Delta_1 \mid \ldots \mid \Gamma_k \vdash_{m_k} \Delta_k$ be an atomic $\mathbb{Z}$-hypersequent. By using arguments used in the proof of Theorem 4, we show that $\mathcal{H}$ has a countermodel in $Ł_n$

iff the inequality $1+\sum_{A\in\Gamma_i}(x_A-1)>\sum_{A\in\Delta_i}x_A$ has a solution over $[0,1/(n-1),\ldots,(n-2)/(n-1),1]$. Thus $\mathcal{H}$ has a countermodel in Ł_n iff $(n-1)+\sum_{A\in\Gamma_i}(x_A-(n-1))>\sum_{A\in\Delta_i}x_A$ has a solution over $\{0,\ldots,n-1\}$.

Theorem 7. *If a $\mathbb{Z}$-hypersequent is valid in Ł then it is derivable in $\mathbb{Z}$Ł.*

Proof. From Theorem 3, by applying the logical rules of $\mathbb{Z}$Ł to every $\mathbb{Z}$-hypersequent $\mathcal{H}$ we obtain a set $\mathcal{S}$ of atomic $\mathbb{Z}$-hypersequents such that $\mathcal{H}$ is valid iff all elements of $\mathcal{S}$ are valid. Let $\mathcal{H}=\Gamma_1\vdash_{m_1}\Delta_1\mid\ldots\mid\Gamma_k\vdash_{m_k}\Delta_k$ be an atomic $\mathbb{Z}$-hypersequent. We assume that $\mathcal{H}$ is valid. Hence, the set $SI_{\mathcal{H}}$ of inequalities is not feasible over $[0,1]$. Then, by using linear programming [16], there exists a positive nonnegative combination of the inequalities in $SI_{\mathcal{H}}$ inconsistent over $[0,1]$. Formally, $\exists\alpha_1,\ldots,\alpha_k\in\mathbb{N}$ such that for some $i\in 1,\ldots,K$ we have $\alpha_i>0$ and the inequality $\alpha_1*(\bigodot\Gamma_1)+\ldots+\alpha_k*(\bigodot\Gamma_k)>\alpha_1*(\bigoplus\Delta_1)-\alpha_1*m_1+\ldots+\alpha_k*(\bigoplus\Delta_k)-\alpha_k*m_k$ is inconsistent over $[0,1]$. We can easily show, by using Definition 4, that the last inequality is inconsistent over $[0,1]$ iff the $\mathbb{Z}$-hypersequent $\alpha_1\Gamma_1,\ldots,\alpha_k\Gamma_k\vdash_n\alpha_1\Delta_1,\ldots,\alpha_k\Delta_k$ is valid in Ł, where $n=\alpha_1*(m_1+1)+\ldots+\alpha_k*(m_k+1)-1$ and for all $i\in 1,\ldots,K$, $\alpha_i\Gamma_i$ (resp. $\alpha_1\Delta_i$) denotes the multiset obtained by the union of α_i copies of the multiset Γ_i (resp. Δ_i). This $\mathbb{Z}$-hypersequent can be obtained from $\mathcal{H}$ by using the external weakening ($[EW]$) and the external contraction rules ($[EC]$).
Let $\Gamma\vdash_n\Delta$ be an atomic $\mathbb{Z}$-hypersequent. We can easily prove that if there is a multiset of formulae Γ_1, subset of Γ and Δ, then $\Gamma\vdash_n\Delta$ is valid iff $\Gamma-\Gamma_1\vdash_{n-n'}\Delta-\Gamma_1$ is valid, where $n'=\mid\Gamma_1\mid$ such that $\mid S\mid$ denotes the number of elements in the multiset S. Moreover, if $l\bot\subseteq\Gamma$ such that $l\bot$ denotes the multiset containing l copies of $\bot$, then $\Gamma\vdash_n\Delta$ is valid iff $\Gamma-l\bot\vdash_{n-l}\Delta$ is valid. From these results we obtain $\Gamma\vdash_n\Delta$ is valid iff $\Gamma=\Gamma_1\cup\Gamma_2\cup l\bot$ such that $\bot\notin\Gamma_2$; $\Delta=\Delta_1\cup\Delta_2$; $\Gamma_1=\Delta_1$; $\Gamma_2\cap\Delta_2=\emptyset$; and $\mid\Gamma_2\mid\leqslant\mid\Gamma\mid-n-1$.Then, $\alpha_1\Gamma_1,\ldots,\alpha_k\Gamma_k\vdash_n\alpha_1\Delta_1,\ldots,\alpha_k\Delta_k$ such that $n=\alpha_1*(m_1+1)+\ldots+\alpha_k*(m_k+1)-1$ is derivable in $\mathbb{Z}$Ł by using $[Ax]$, $[SR]$, $[IW_L]$, $[IW_R]$ and $[\bot_L]$. If a $\mathbb{Z}$-hypersequent is valid in Ł then it is derivable in $\mathbb{Z}$Ł.

C Bounded Łukasiewicz Logic

Bounded Łukasiewicz logic ŁB_n for $n\geqslant 2$ is defined as the intersection of Ł_k for $k=2,\ldots,n$. A Hilbert axiomatic system for this logic consists of the same axioms and rules than Ł with $nA\supset(n-1)A$. Calculi for ŁB_n, called $G\text{ŁB}_n$ [4], are obtained by adding to the hypersequent calculus GŁ given in [12] the following rule:

$$\frac{G\mid\overbrace{\Gamma,\ldots\Gamma}^{n-1},\Gamma',\bot\vdash\overbrace{\Delta,\ldots,\Delta}^{n-1},\Delta'}{G\mid\Gamma\vdash\Delta\mid\Gamma'\vdash\Delta'}\ [nC]$$

It appears that this rule makes proof-search expensive because it duplicates the contexts Γ and Δ n-1 times. Here we introduce new calculi for Bounded Łukasiewicz logics that are simpler than $G\text{ŁB}_n$. We call $\mathbb{Z}\text{ŁB}_n$ the calculus obtained from $\mathbb{Z}\text{Ł}'$ by adding:

$$\frac{G \mid \Gamma_1 \vdash_{m_1} \Delta_1 \qquad G \mid \Gamma_2 \vdash_{m_1} \Delta_2}{G \mid \Gamma_1, \Gamma_2 \vdash_{m_1+m_2+1} \Delta_1, \Delta_2}\ [M] \qquad \frac{G \mid \Gamma, A \vdash_{m+1} \Delta, A}{G \mid \Gamma \vdash_m \Delta}\ [GCUT]$$

$$\frac{}{G \mid \Gamma \vdash_0 \Delta, \overbrace{A, \dots, A}^{n-1} \mid \Gamma', A \vdash_0 \Delta'}\ [Ax_n]$$

Theorem 12 (Soundness). *The rules of* $\mathbb{Z}\text{Ł}\mathsf{B}_\mathsf{n}$ *are sound.*

Proof. By Theorem 1 the logical rules are sound. The rules $[M]$, $[GCUT]$, $[SR]$, $[S]$ and $[\perp_L]$ are proved sound by similar arguments. Let us consider $[AX_n]$. We suppose that $\mathcal{H} = G \mid \Gamma \vdash_0 \Delta, \overbrace{A, \dots, A}^{n-1} \mid \Gamma', A \vdash_0 \Delta'$ has a countermodel. Thus, for $k \in \{2, \dots, n\}$ there is a valuation $[\![\cdot]\!]$ countermodel of $\mathcal{H}$ in Ł_k. Thus, there exists $i \in \{0, \dots, k-1\}$ such that $[\![A]\!] = \frac{i}{k-1}$. If $[\![A]\!] = 0$ then $\llfloor \Gamma', A \rrfloor \leqslant 0 \leqslant \llceil \Delta' \rrceil$ and we get a contradiction. Now, if $[\![A]\!] = \frac{i}{k-1}$ with $i \neq 0$ then $\llfloor \Gamma \rrfloor \leqslant 1 \leqslant (n-1) * \frac{i}{k-1} + \llceil \Delta \rrceil$ because $n \geqslant k$. This is a contradiction.

Theorem 13 (Completeness). *If A is valid in* $\text{Ł}\mathsf{B}_\mathsf{n}$ *then* $\vdash_0 A$ *is derivable in* $\mathbb{Z}\text{Ł}\mathsf{B}_\mathsf{n}$.

Proof. We have only to prove that (1) the axiom $nA \supset (n-1)A$ is derivable in $\mathbb{Z}\text{Ł}\mathsf{B}_\mathsf{n}$ and (2) the modus ponens rule is admissible in $\mathbb{Z}\text{Ł}\mathsf{B}_\mathsf{n}$. Then we have:

$$\cfrac{\cfrac{\vdash_0 (n-1)A \mid A \vdash_0}{\vdash_0 (n-1)A \mid (n-1)A, A \vdash_1 (n-1)A}\ [SR]}{A \oplus ((n-1)A) \vdash_0 (n-1)A}\ [\oplus_L]$$

and by using $[\oplus_R]$ n-1 times, we obtain the axiom $\vdash_0 \overbrace{A, \dots, A}^{n-1} \mid A \vdash_0$. A proof of (2) is given by the following derivation:

$$\cfrac{\cfrac{A \vdash_0 B \quad \vdash_0 A}{A \vdash_1 B, A}\ [M]}{\vdash_0 B}\ [GCUT]$$

In next works we will study the cut-elimination problem.