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The Semantics of BI and Resource Tableaux

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The logic of bunched implications, **BI**, provides a logical analysis of a basic notion of resource rich enough, for example, to form the logical basis for "pointer logic" and "separation logic" semantics for programs which manipulate mutable data structures. We develop a theory of semantic tableaux for **BI**, so providing an elegant basis for efficient theorem proving tools for **BI**. It is based on the use of an algebra of labels for **BI**'s tableaux to solve the resource-distribution problem, the labels being the elements of resource models. For **BI** with inconsistency, $\bot$, the challenge consists in dealing with **BI**'s Grothendieck topological models within such a proof-search method, based on labels. We prove soundness and completeness theorems for a resource tableaux method **TBI** with respect to this semantics and provide a way to build countermodels from so-called dependency graphs. Then, from these results on resource tableaux, we can define a new resource semantics of **BI**, based on partially defined monoids, and prove that this semantics is complete. We also give such a semantics for Affine **BI** (with weakening for the multiplicative conjunction). Such a semantics, based on partiality, is closely related to the semantics of **BI**'s (intuitionistic) pointer logic and of separation logics. Returning to the tableaux calculus, we propose a new version with liberalized rules for which the countermodels are closely related to the topological Kripke semantics of **BI**. As consequences of the relationships between semantics of **BI** and resource tableaux, we prove two strong new results for propositional **BI**: its decidability and the finite model property with respect to topological semantics.

Keywords: **BI**; resources; semantics; tableaux; decidability; finite model property.

1. Introduction

The notion of *resource* is a basic one in many fields, including economics, engineering and psychology, but it is perhaps most clearly illuminated in computer science. The location, ownership, access to and, indeed, consumption of, resources are central concerns in the design of systems (such as networks, within which processors must access devices such as file servers, disks and printers) and in the design programs, which access memory and manipulate data structures (such as pointers).

The development of a mathematical theory of resource is one of the objectives of the pro-

gramme of study of **BI**, the logic of bunched implications, introduced by O'Hearn and Pym (O'Hearn and Pym 1999; Pym 1999; Pym 2002; Pym 2004). The basic idea is to model directly the observed properties of resources and then to give a logical axiomatization. Initially, we require the following properties of resource, beginning with the simple assumption of a set R of elements of a resource: a *combination*, $\bullet$, of resources, together with a *zero* resource, e; a *comparison*, $\sqsubseteq$, of resources. Mathematically, we model this set-up with a (for now, commutative) preordered [†] monoid $\mathcal{R} = (R, \bullet, e, \sqsubseteq)$, in which $\bullet$, with unit e, has the property $m \sqsubseteq n$ and $m' \sqsubseteq n'$ imply $m \bullet m' \sqsubseteq n \bullet n'$, for any m, n, m', n'. Taking such a structure as an algebra of *worlds*, we obtain a forcing semantics for (propositional) **BI** which freely combines multiplicative (intuitionistic linear $\otimes$ and $-\!\circ$) and additive (intuitionistic $\wedge$, $\rightarrow$ and $\vee$) structure. A significant variation takes classical additives instead. **BI** is described in necessary detail in § 2. For now, the key property of the semantics is the *sharing interpretation* (O'Hearn and Pym 1999; O'Hearn 1999).

The (elementary) semantics of the multiplicative conjunction, $m \models \phi_1 * \phi_2$ iff there are n_1 and n_2 such that $n_1 \bullet n_2 \sqsubseteq m$, $n_1 \models \phi_1$ and $n_2 \models \phi_2$, is interpreted as follows: the resource m is sufficient to support $\phi_1 * \phi_2$ just in case it can be divided into resources n_1 and n_2 such that n_1 is sufficient to support ϕ_1 and n_2 is sufficient to support ϕ_2. The assertions ϕ_1 and ϕ_2 — think of them as expressing properties of programs — *do not share* resources. In contrast, in the semantics of the additive conjunction, $m \models \phi_1 \wedge \phi_2$ iff $m \models \phi_1$ and $m \models \phi_2$, the assertions ϕ_1 and ϕ_2 *share* the resource m. Similarly, the semantics of the multiplicative implication, $m \models \phi \mathrel{-\!\!*} \psi$ iff for all n such that $n \models \phi$, $m \bullet n \models \psi$, is interpreted as follows: the resource m is sufficient to support $\phi \mathrel{-\!\!*} \psi$ — think of the proposition as (the type of) a function — just in case for any resource n which is sufficient to support ϕ — think of it as the argument to the function — the combination $m \bullet n$ is sufficient to support ψ. The function and its argument *do not share* resources. In contrast, in the semantics of additive implication, $m \models \phi \rightarrow \psi$ iff for all $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$, the function and its argument *share* the resource n. For a simple example of resource as *cost*, let the monoid be given by the natural numbers with addition and unit zero, ordered by less than or equals. A more substantial example, "pointer logic", **PL**, and its spatial semantics, has been provided by Ishtiaq and O'Hearn (Ishtiaq and O'Hearn 2001). In fact, the semantics of pointer logic is based on *partial monoids*, in which the operation $\bullet$ is partially defined.

An elementary Kripke resource semantics, formulated in categories of *presheaves* on preordered monoids, has been defined for **BI** (O'Hearn and Pym 1999; Pym 1999; Pym 2002; Pym 2004) but it is sound and complete only for **BI** without inconsistency, $\bot$, the unit of the additive disjunction. This elementary forcing semantics handles inconsistency only by denying the existence of a world at which $\bot$ is forced. The completeness of **BI** *with* $\bot$ for a monoid-based forcing semantics is achieved, first, in categories of *sheaves* on *open topological monoids* and, second, in the more abstract topological setting of *Grothendieck sheaves* on preordered monoids (Pym 2002; Pym et al. 2004; Pym 2004). These different semantics of **BI** are sketched in § 2. In each of these cases, inconsistency is internalized in the semantics. The semantics of pointer logic can be incorporated into the Kripke semantics based on Grothendieck sheaves (Pym 2002;

[†] The preorder $\sqsubseteq$ is here the inverse of the original one proposed by O'Hearn and Pym. It corresponds to the one usually used in labelled deductive systems and thus allows us to directly relate the resources with labels in a traditional way.

Pym et al. 2004; Pym 2004). But it suggests partial monoids as a basis for a "Kripke resource semantics".

BI provides a logical analysis of a basic notion of resource (Pym 2002; Pym 2004), quite different from linear logic's "number-of-uses" reading (Girard 1987), which has proved rich enough to provide intuitionistic (*i.e.*, the additives) "pointer logic" semantics for programs which manipulate mutable data structures (Ishtiaq and O'Hearn 2001; O'Hearn et al. 2001; Pym et al. 2004; Pym 2004). In this context, efficient and useful proof-search methods are necessary. For many logics, semantic tableaux have provided elegant and efficient bases for tools based on both proof-search and countermodel generation (Fitting 1990). We should like to have bases for such tools for **BI** and **PL**. The main difficulty to be overcome in giving such a system for **BI** is the presence of multiplicatives. We need a mechanism for calculating the distribution of "resources" with multiplicative rules which, in **BI**'s sequent calculus, given in § 2, is handled via side-formulae. A solution is a specific use of *labels* that allow the capture of the semantic relationships between connectives during proof-search or proof-analysis (Balat and Galmiche 2000; Gabbay 1996; Harland and Pym 2003).

Recent work has proposed a tableaux calculus, with labels, for **BI** without $\bot$, which captures the elementary Kripke resource semantics (Galmiche and Méry 2001) but an open question until now has been whether a similar approach or calculus can be extended to full **BI**, including $\bot$, and its Grothendieck topological semantics. Such a calculus and its related tableaux method would provide a decision procedure for **BI** (decidability of **BI** has been conjectured, via a different method, in (Pym 2002; Pym 2004) but not explicitly proved). A real difficulty lies in the treatment of a monoid-based forcing semantics, like Grothendieck topological semantics (Pym 2002; Pym 2004), with such a labelled calculus.

In this paper, we are concerned mainly with the relationships and connections between semantics of **BI** and so-called "resource tableaux" which lead to new results from the perspectiveof of both proof-search and semantics. In § 2, we review briefly the **BI** logic and its sequent calculus and introduce different semantics of **BI**, namely the Kripke resource semantics and the Grothendieck topological semantics. We also explain the problems for completeness arising from the presence of inconsistency $\bot$.

In § 3, we define a system of labelled semantic tableaux, **TBI**, in which the labels are drawn from **BI**'s algebra of worlds and which use **BI**'s forcing semantics, based on Grothendieck sheaves. The rules are similar to the ones of (Galmiche and Méry 2001) with introduction of label constraints (called assertions and requirements) but the specific way to deal with $\bot$ topologically involves delicate new closure and provability conditions. Moreover we introduce a specific graph called dependency graph (or Kripke resource graph) that is built in parallel with the tableau expansion and reflects the information that can be derived from a given set of assertions. Two examples illustrate how resource tableaux deal with $\bot$ and how provability in **BI** can be analyzed. We study, in § 4, the soundness of **TBI** that can only be proved for so-called basic Grothendieck resource models. We need new results on semantics, developed in the next section, to be able to directly prove the soundness of **TBI**. Moreover, we show the completeness of **TBI** with respect to the Grothendieck topological semantics. Moreover, we use our completeness proof to show that in the case of a *failed* tableau, *i.e.*, non-provability, we can build a *a countermodel* from a *dependency graph*. Moreover, observing that a dependency graph only deals with the relevant resources needed to decide provability, it seems possible to propose a new resource

semantics for **BI**, called PDM semantics, that corresponds to an alternative way of dealing with $\bot$ by considering partially defined monoids. In § 5, we define such a semantics that was previously expected but not developed in (Pym 2002; Pym et al. 2004; Pym 2004). For that, we start by defining a new relational semantics of **BI**, based on a ternary-relation $Rxyz$ with particular properties, and then prove its soundness and completeness with respect to **BI**. This semantics is such that the class of PDM models is included in the class of the relational models and therefore, the PDM semantics corresponding to this relational semantics with a specific relation defined by $Rxyz \equiv x \bullet y \sqsubseteq z$. Thus, we can solve the problem of soundness of **TBI** by showing that **TBI** is sound with respect to the relational semantics.

Returning to the PDM semantics, considered as a specialization of the relational semantics, we have now a new resource semantics that is naturally related to our study of resource tableaux, through labels and constraints, and that is proved complete with respect to **BI**. Then, we propose a similar semantics for a variant of **BI**, called Affine **BI**, in which the multiplicative conjunction satisfies the structural rule of weakening. It illustrates the power of the partiality in this context, knowing that such fragments of **BI** are the logical bases of pointer logic (Ishtiaq and O'Hearn 2001) and separation logic (Reynolds 2000).

In § 6, we study how, by a special treatment of the additive disjunction, we can propose liberalized rules for **TBI**, that is an improvement of the initial version of the calculus. This new version is related to the topological Kripke semantics of **BI** (Pym et al. 2004; Pym 2004) which allows $\bot$ to be taken into account together with a non-indecomposable treatment of the disjunction. In fact, the topological semantics considers open sets, while the canonical interpretation of **BI** we define, considers sets that are closed under deduction. We show that a new semantic clause for the additive disjunction is required to achieve a suitable canonical forcing relation. However, the semantic changes made to $\vee$ have a syntactic counterpart and the corresponding initial expansion rules have to be accordingly modified. In § 7, we give new expansion rules for **TBI** and thus a resulting tableau system called **TBI'**. We prove its soundness and completeness but the construction of countermodels is less direct than in **TBI** because of the extension of the label algebra. Moreover, those countermodels are related to **BI**-algebras which are themselves closely related to the topological Kripke semantics, from which they can be viewed as an algebraic counterpart. Therefore **TBI'** appears as the syntactic reflection of the forcing semantics in the category of sheaves over a topological monoid and dependency graphs can be viewed as (partial) topological Kripke models.

In § 8, we prove two *new* results for propositional **BI**, namely, the *finite model property* with respect to Grothendieck topological semantics, and the *decidability* of propositional **BI**, conjectured but not proved in (Pym 2002; Pym 2004). The relationships identified between resources, labels, dependency graphs, proof-search and resource semantics are central in this study. Moreover, dependency graphs can be seen directly as countermodels in this new semantics. We conclude, in § 9, with a summary of our contribution and a brief discussion of future directions for this research.

2. The Semantics and Proof Theory of BI

We review briefly the semantics and proof theory of **BI** (with $\bot$). **BI** freely combines linear conjunction, $*$ with unit I, and linear implication, $-\!\!*$, with intuitionistic conjunction, $\wedge$ with

unit $\top$, disjunction, $\vee$ with unit $\bot$, and implication, $\to$. There is an elementary Kripke resource semantics which, because of the interaction between $-\!\!*$ and $\bot$ (Pym 2002; Pym et al. 2004; Pym 2004), is complete only for **BI** without $\bot$. In order to have completeness with $\bot$, it is necessary to use the topological setting introduced in (O'Hearn and Pym 1999; Pym 2002; Pym et al. 2004; Pym 2004) and described below, which is a significant step over the elementary case.

Definition 2.1 (propositions). The propositional language of **BI** consists of: a multiplicative unit I, the multiplicative connectives $*$, $-\!\!*$, the additive units $\top$, $\bot$, the additive connectives $\wedge$, $\to$, $\vee$, a countable set $L = \mathrm{p}, \mathrm{q}, \ldots$ of propositional letters. $\mathcal{P}(L)$, the collection of **BI** propositions over L, is given by the following inductive definition:

$$\phi ::= p \mid I \mid \phi * \phi \mid \phi -\!\!* \phi \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \vee \phi.$$

The additive connectives correspond to those of intuitionistic logic (**IL**) whereas the multiplicative connectives correspond to those of multiplicative intuitionistic linear logic (**MILL**). The antecedents of logical consequences are structured as bunches, in which there are two ways to combine operations that respectively display additive and multiplicative behaviour.

Definition 2.2. Bunches are given by the following grammar:

$$\Gamma ::= \phi \mid \emptyset_a \mid \Gamma\,;\,\Gamma \mid \emptyset_m \mid \Gamma\,,\,\Gamma.$$

Equivalence, $\equiv$, is given by commutative monoid equations for "," and ";", whose units are $\emptyset_m$ and $\emptyset_a$ respectively, together with the evident substitution congruence for sub-bunches — we write $\Gamma(\Delta)$ to denote a sub-bunch Δ of Γ — determined by the grammar.

Judgements are expressions of the form $\Gamma \vdash \phi$, where Γ is a "bunch" and ϕ is a proposition. The **LBI** sequent calculus is given in Figure 1.[‡] The Cut-elimination theorem holds (see (Pym 2002; Pym 2004) for a proof):

Theorem 2.1 (Cut-elimination). If $\Gamma \vdash \phi$ is provable in **LBI** including Cut, then it is provable in **LBI** without Cut.

A proposition ϕ is a theorem if $\emptyset_a \vdash \phi$ or $\emptyset_m \vdash \phi$ is provable in **LBI**, but the following theorem provides a more simple definition (Pym 1999; Pym 2002):

Theorem 2.2. $\emptyset_a \vdash \phi$ (resp. $\emptyset_m \vdash \phi$) is provable in **LBI** iff $\top \vdash \phi$ (resp. $I \vdash \phi$) is provable in **LBI**.

Corollary 2.1. A proposition ϕ is a theorem iff $\emptyset_m \vdash \phi$ is provable in **LBI**.

[‡] In the definition of **LBI** given in (Pym 2002), the $\vee L$ rule is mis-stated: it is corrected in (Pym 2004) and the corrected version is as given in Figure 1. This error was known prior to the publication of (Pym 2002) but persisted because of an editing error by the author. The error was also propagated to Gal02d.

$$\frac{}{\phi \vdash \phi}\ Axiom \qquad \frac{\Gamma \vdash \phi \quad \Delta(\phi) \vdash \psi}{\Delta(\Gamma) \vdash \psi}\ Cut \qquad \frac{\Gamma(\Delta) \vdash \phi}{\Gamma(\Delta;\Delta') \vdash \phi}\ W \qquad \frac{\Gamma(\Delta;\Delta) \vdash \phi}{\Gamma(\Delta) \vdash \phi}\ C$$

$$\frac{}{\Gamma(\bot) \vdash \phi}\ \bot L \quad \frac{\Gamma \vdash \phi}{\Delta \vdash \phi}\ (\Gamma \equiv \Delta)\ E \quad \frac{\Gamma(\emptyset_m) \vdash \phi}{\Gamma(I) \vdash \phi}\ I\,L \qquad \frac{}{\emptyset_m \vdash I}\ I\,R$$

$$\frac{\Gamma(\emptyset_a) \vdash \phi}{\Gamma(\top) \vdash \phi}\ \top L \qquad \frac{}{\emptyset_a \vdash \top}\ \top R \quad \frac{\Delta \vdash \phi \quad \Gamma(\psi,\Delta') \vdash \chi}{\Gamma(\Delta, \phi \mathrel{-\!\!*} \psi, \Delta') \vdash \chi}\ \mathrel{-\!\!*} L \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi}\ \mathrel{-\!\!*} R$$

$$\frac{\Gamma(\phi,\psi) \vdash \chi}{\Gamma(\phi * \psi) \vdash \chi}\ * L \quad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma,\Delta \vdash \phi * \psi}\ * R \quad \frac{\Delta \vdash \phi \quad \Gamma(\psi\,;\Delta') \vdash \chi}{\Gamma(\Delta\,;\phi \to \psi\,;\Delta') \vdash \chi}\ \to L \quad \frac{\Gamma\,;\phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ \to R$$

$$\frac{\Gamma(\phi_1;\phi_2) \vdash \psi}{\Gamma(\phi_1 \wedge \phi_2) \vdash \psi}\ \wedge L \quad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma\,;\Delta \vdash \phi \wedge \psi}\ \wedge R \quad \frac{\Gamma(\phi) \vdash \chi \quad \Gamma(\psi) \vdash \chi}{\Gamma(\phi \vee \psi) \vdash \chi}\ \vee L \quad \frac{\Gamma \vdash \phi_{i\ (i=1,2)}}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee R$$

Figure 1. The **LBI** sequent calculus

2.1. **Kripke Resource Models**

As explained in the introduction, **BI** has a simple and natural truth-functional semantics, presented as Kripke-like forcing relation relative to a preordered commutative monoid or worlds, which may be seen as freely combining Kripke's semantics for intuitionsitic logic (Kripke 1965) with Urquhart's semantics for the relevant connectives (Urquhart 1972) which comprise the multiplicative fragment of intuitionistic linear logic (**MILL**) (O'Hearn and Pym 1999; Pym 2002; Pym et al. 2004; Pym 2004). As we have seen, the meaning of the formal semantics may be explained in terms of *resource*, an appropriate generalization of Urquhart's notion of *pieces of information*.

So our basic semantic structure is a preordered commutative monoid upon which we impose a bifunctoriality condition. This condition, though credible from the point of view of resource semantics, is motivated mathematically. Other properties which are well-motivated by resource semantics, such as "aggregation":

(A): for all m and n, $m \sqsubseteq m \bullet n$ and $n \sqsubseteq m \bullet n$,

are not required for our mathematical development and are not adopted here.

Definition 2.3. A *Kripke resource monoid* (*KRM*) $\mathcal{M} = \langle M, \bullet, e, \sqsubseteq \rangle$ is a preordered commutative monoid in which $\bullet$ is bifunctorial w.r.t. $\sqsubseteq$:

(P): if $m \sqsubseteq n$ and $m' \sqsubseteq n'$, then $m \bullet m' \sqsubseteq n \bullet n'$.

Having established the basic structure, we only require a notion of interpretation in order to be able to define a class of elementary models. At this stage, we require a condition (K) which will ensure the hereditary property of Kripke models holds in our setting.

Definition 2.4. Let $\mathcal{M}$ be a *KRM* and $\mathcal{P}(L)$ be the collection of **BI** propositions over a language

L of propositional letters, then an *elementary Kripke resource interpretation* (EKRI) is a function $[\![-]\!] : L \to \mathcal{P}(M)$ that satisfies:

(K): for all $m, n \in M$ such that $m \sqsubseteq n$, if $m \in [\![\mathrm{p}]\!]$, then $n \in [\![\mathrm{p}]\!]$.

All that remains for an elementary semantics is to give a forcing relation. The clauses for the additives ($\top$, $\wedge$, $\bot$, $\vee$, $\to$) exploit just the ordering on worlds, $\sqsubseteq$, and exactly those for intuitionistic Kripke models (Kripke 1965). Note, in particular, the clause for $\bot$: the model has no internal representative for inconsistency. The clauses for the multiplicatives (I, $\otimes$, $-\!\!*$) require the combination of worlds, $\bullet$, and follow Urquhart's semantics (Urquhart 1972).

Definition 2.5. An *elementary Kripke resource model* (EKRM), is a triple $\mathcal{K} = \langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$ in which $\mathcal{M}$ is a ***KRM***, $[\![-]\!]$ is an EKRI, and $\models$ is a forcing relation on $M \times \mathcal{P}(L)$ satisfying the following conditions:

- $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
- $m \models \top$ iff always
- $m \models \bot$ iff never
- $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
- $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
- $m \models \phi \to \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, the $n \models \psi$
- $m \models I$ iff $e \sqsubseteq m$
- $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$, $n \models \phi$ and $n' \models \psi$
- $m \models \phi -\!\!* \psi$ iff for all $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

The semantics of propositions given by the relation $\models$ is parametrized by the interpretation $[\![-]\!]$ for which the property (K) of definition 2.4 holds for atomic propositions. It is easy to prove, by structural induction on propositions, that if (K) holds for atomic propositions, then it also holds for any proposition.

Let $\mathcal{K}$ be a EKRM and Φ_Γ be the formula obtained from a bunch Γ by replacing each ";" by $\wedge$, each "," by $*$ Γ and also $\emptyset_a$ (resp. $\emptyset_m$) by $\top$ (resp. I), with association respecting the tree structure of Γ, the sequent $\Gamma \vdash \phi$ is said to be *valid in* $\mathcal{K}$ (notation: $\Gamma \models_{\mathcal{K}} \phi$), if and only if, for any world $m \in M$, $m \models \Phi_\Gamma$ implies $m \models \phi$. The sequent $\Gamma \vdash \phi$ is said to be *valid* (notation: $\Gamma \models \phi$), if and only if, for any EKRM $\mathcal{K}$, $\Gamma \models_{\mathcal{K}} \phi$.

Theorem 2.3 (soundness). If $\Gamma \vdash \phi$ is provable in **LBI**, then $\Gamma \models \phi$.

The unit $\bot$ of the $\vee$ connective internalizes inconsistency in **BI** but the elementary Kripke resource semantics does not account for inconsistency ($\bot$ is nowhere forced). Accordingly, $\bot$ must be excluded to obtain the completeness result w.r.t. this semantics and the completeness result is only proved for **BI** without $\bot$ (Pym 2002; Pym 2004). In fact, the incompleteness arises from the interaction between multiplicative implication ($-\!\!*$) and the unit $\bot$.

To understand this incompleteness, it is necessary to take a slightly more abstract point of view. The elementary semantics can be formulated quite conveniently in presheaf categories $[\mathcal{M}^{op}, Set]$ (Lambek and Scott 1986). Here $\mathcal{M}$ is a KRM, considered as a category (of worlds). All of the connectives, except $\bot$, can be defined in this setting by exploiting the Cartesian closed (Lambek and Scott 1986) structure, carried by any functor category (we neglect concerns about

size here), and the monoidal closed structure on $[\mathcal{M}^{op}, Set]$ induced by $\mathcal{M}$ via Day's constructions (Day 1970; O'Hearn and Pym 1999; Pym et al. 2004; Pym 2004). To see that the completeness argument fails in this setting, we must the consider the interaction between $\bot$ and $-\!\!*$. It may readily be checked that the sequent

$$\phi\,,\ \phi -\!\!* \bot \vdash \bot$$

is provable in **LBI**. But, in the term model which must be constructed to establish completeness (Pym 2002; Pym 2004), the bunch $\phi, \phi -\!\!* \bot$ is equivalent to $\bot$. It then follows that we must have a *world* which represents, and so forces, $\bot$. Such a world is not present in the elementary semantics.

Completeness in the presence of $\bot$ can be recovered be adopting a semantics which has an internal representation of $\bot$. One such semantics is provided by moving from the presheaf-theoretic setting of the elementary semantics, to the topological setting of sheaves. By replacing the category of worlds with a kind of topological monoid, considered as category: we then obtain models in the category of sheaves over a topological space of worlds in which the empty set, which is an open set, provides a representative for $\bot$ (Pym 2002; Pym 2004). Whilst technically possible, and of some conceptual value (see § 6), a simpler construction is possible in the setting of Grothendieck sheaves (Mac Lane and Moerdijk 1992).

2.2. **Grothendieck Sheaf-theoretic Models**

BI's Kripke semantics may be adapted to take account for $\bot$ by moving from presheaves to sheaves on a topological monoid (Pym 2002; Pym et al. 2004; Pym 2004). Such a semantics considers an inconsistant world, at which $\bot$ is forced, together with the so-called non-indecomposable treatment of $\vee$: $U \models \phi \vee \psi$ iff, for some open sets V, V' such that $U = V \cup V'$, $V \models \phi$ and $V' \models \psi$ (see § 6 for some consequences of this analysis of $\vee$). This semantics, not given in detail here, is complete. However, we give here an algebraic generalization of it. This setting recovers the simplicity of the previous elementary preordered monoid semantics and also the topological treatment of inconsistency.

The basic idea in Grothendieck sheaves is to represent the essential topological structure in terms of the underlying preordered commutative monoid using a map J (the "Grothendieck topology") to associate sets of sets of worlds with each world. Most of the necessary conditions, given in Definition 2.6, are quite standard (Mac Lane and Moerdijk 1992), and are required to handle the additive (intuitionistic) part of **BI**. We require on additional condition — continutity — to handle the monoid operation, $\bullet$, and so provide the additional structure that is necessary to interpret the multiplicatives. This condition amounts to a requirement that J respects the composition induced by $\bullet$.

Definition 2.6 (GTM). A *Grothendieck topological monoid* (GTM) is given by a quintuple $\mathcal{M} = \langle M, \bullet, e, \sqsubseteq, J \rangle$, where $\langle M, \bullet, e, \sqsubseteq \rangle$ is a preordered commutative monoid, in which $\bullet$ is bifunctorial w.r.t. $\sqsubseteq$, and J is a map $J : M \to \wp(\wp(M))$ satisfying the following:

1 *Sieve:* for any $m \in M$, $S \in J(m)$ and $m' \in S$, $m \sqsubseteq m'$;
2 *Maximality:* for any n' such that $n' = n$, $\{n'\}$ is in $J(n)$;

3 *Stability:* for any m, $n \in M$ and $S \in J(m)$ such that $m \sqsubseteq n$, there exists $S' \in J(n)$ such that for any $n' \in S'$, there exists $m' \in S$ such that $m' \sqsubseteq n'$;
4 *Transitivity:* for any $m \in M$, $S \in J(m)$ and $\{S_{m'} \in J(m')\}_{m' \in S}$, $\bigcup_{m' \in S} S_{m'} \in J(m)$;
5 *Continuity:* for any m, $n \in M$ and $S \in J(m)$, $\{m' \bullet n \mid m' \in S\} \in J(m \bullet n)$.

Such a J is usually called a *Grothendieck topology*.

Definition 2.7 (GTI). Let $\mathcal{M}$ be a *GTM* and $\mathcal{P}(L)$ be the collection of **BI** propositions over a language L of propositional letters, a *Grothendieck Topological Interpretation* is a function $[\![-]\!] : L \to \wp(M)$ satisfying:

6 (K): for any $m, n \in M$ such that $n \sqsubseteq m$, $n \in [\![\mathrm{p}]\!]$ implies $m \in [\![\mathrm{p}]\!]$;
7 (Sh): for any $m \in M$ and $S \in J(m)$, if, for all $m' \in S$, $m' \in [\![\mathrm{p}]\!]$, then $m \in [\![\mathrm{p}]\!]$.

It is shown in (Pym 2002; Pym et al. 2004; Pym 2004) that given an interpretation which makes (K) and (Sh) hold for atomic propositions, (K) and (Sh) also hold for any proposition of **BI** in that interpretation.

Definition 2.8 (GRM). A *Grothendieck resource model* (GRM) is a triple $\mathcal{G} = \langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$ in which $\mathcal{M} = \langle\, M, \bullet, e, \sqsubseteq, J \,\rangle$ is a *GTM*, $[\![-]\!]$ is a *GTI* and $\models$ is a forcing relation on $M \times \mathcal{P}(L)$ satisfying the following conditions:

- $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
- $m \models \top$ iff always
- $m \models \bot$ iff $\emptyset \in J(m)$
- $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
- $m \models \phi \vee \psi$ iff there exists $S \in J(m)$ such that for any $m' \in S$, $m' \models \phi$ or $m' \models \psi$
- $m \models \phi \to \psi$ iff for any $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
- $m \models I$ iff there exists $S \in J(m)$ such that for any $m' \in S$, $e \sqsubseteq m'$
- $m \models \phi * \psi$ iff there exists $S \in J(m)$ such that for any $m' \in S$, there exist n,$n' \in M$ such that $n \bullet n' \sqsubseteq m'$, $n \models \phi$ and $n' \models \psi$
- $m \models \phi \mathrel{-\!\!*} \psi$ iff for any $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

Let $\mathcal{G}$ be a *GRM* and ϕ_Γ be the formula obtained from a bunch Γ by replacing each ";" by $\wedge$ and each "," by $*$ with association respecting the tree structure of Γ. A sequent $\Gamma \vdash \phi$ is said to be *valid in* $\mathcal{G}$, written $\Gamma \models_{\mathcal{G}} \phi$, if and only if, for any world $m \in M$, $m \models \phi_\Gamma$ implies $m \models \phi$. A sequent $\Gamma \vdash \phi$ is *valid*, written $\Gamma \models \phi$, iff, for any *GRM* $\mathcal{G}$, it is valid in $\mathcal{G}$.

Theorem 2.4 (soundness and completeness). $\Gamma \vdash \phi$ is provable in **LBI** iff $\Gamma \models \phi$.

Proof. Proof based on a term model construction for **BI**, with respect to GRMs. (Pym et al. 2004; Pym 2002; Pym 2004). □

As a corollary, we obtain *validity*, *i.e.*, a proposition ϕ is valid iff for any *GRM* $\mathcal{G}$, $e \models_{\mathcal{G}} \phi$.

Having in mind these completeness results, for **BI** with or without $\bot$, we aim to study the proof-theoretic foundations of **BI** and to propose proof-search methods that build proofs or countermodels. For that, the challenge is to capture the semantics of the logic, *i.e.*, Grothendieck models, through labels in the spirit of labelled deductive systems (Gabbay 1996). In the case of **BI** without $\bot$, we have already provided a labelling algebra which syntactically reflects the Kripke

resource semantics (Galmiche and Méry 2001) and use it to define a proof-search procedure. For **BI** and its Grothendieck topological semantics, the analysis is much more delicate. A key step in this semantic analysis is the use of *dependency graphs*, explained in § 3.3.

3. Resource Tableaux for BI

We set up the theory of labelled semantic tableaux for **BI**. We assume a basic knowledge of tableaux systems (Fitting 1990). We begin with algebras of labels, which provide the connection between the underlying syntactic tableaux and the semantics of the connectives used to regulate the multiplicative structure.

3.1. A Labelling Algebra

We define a set of labels and constraints and a corresponding labelling algebra, *i.e.*, a preordered monoid whose elements are denoted by labels.

Definition 3.1. A *labelling language* consists of the following symbols: a unit symbol 1, a binary function symbol $\circ$, a binary relation symbol $\leq$, a countable set of constants $c_1, c_2, \ldots$. *Labels* are inductively defined from the unit 1 and the constants as expressions of the form $x \circ y$ in which x and y are labels. *Atomic labels* are labels which do not contain any $\circ$, while *compound labels* contain at least one $\circ$. *Label constraints* are expressions of the form $x \leq y$, where x and y are labels.

Definition 3.2. Labels and constraints are interpreted in an order-preserving preordered commutative monoid of labels, or *labelling algebra* $\mathcal{L} = \langle L, \circ, 1, \leq \rangle$, more precisely: 1. L is a set of labels; 2. $\leq$ is a preorder; 3. Equality on labels is defined by : $x = y$ iff $x \leq y$ and $y \leq x$; 4. $\circ$ is a binary operation on L such that: associativity: $(x \circ y) \circ z = x \circ (y \circ z)$, commutativity: $x \circ y = y \circ x$, identity: $x \circ 1 = 1 \circ x = x$, compatibility: $x \circ z \leq y \circ z$ if $x \leq y$. We say that x is a *sublabel* of y (notation: $x \preceq y$), if there exists a label z such that $y = x \circ z$. We say $x \prec y$ if $x \preceq y$ and $x \neq y$. $\wp(x)$ denotes the set of the sublabels of x.

For notational simplicity, we can omit the binary symbol $\circ$ when writing labels and then xy represents $x \circ y$. We deal with *partially defined* labelling algebras, obtained from sets of constraints by means of a *closure operator*.

Definition 3.3. The *domain* of a set K of label constraints is the set of all sublabels occurring in some constraints of K, *i.e.*, $\mathcal{D}(K) = \bigcup_{x \leq y \in K} (\wp(x) \cup \wp(y))$. The *closure* $\overline{K}$ of K is defined as the smallest set satisfying

- $K \subseteq \overline{K}$;
- Reflexivity: if $x \in \mathcal{D}(\overline{K})$, then $x \leq x \in \overline{K}$;
- Transitivity: if $x \leq y \in \overline{K}$ and $y \leq z \in \overline{K}$, then $x \leq z \in \overline{K}$;
- Compatibility: if $y \circ z \in \mathcal{D}(\overline{K})$, then, $x \leq y \in \overline{K}$ implies $x \circ z \leq y \circ z \in \overline{K}$.

We do not distinguish between the closure of a set of label constraints and the (partially defined) labelling algebra it generates.

3.2. **Expansion Rules**

We can now define the expansion rules of **TBI**.

Definition 3.4. A *signed formula* is a triple $\langle\, Sg, \phi, l \,\rangle$, denoted $\mathrm{Sg}\ \phi : l$, Sg ($\in \{F, T\}$) being the sign of the formula ϕ ($\in \mathcal{P}(L)$) and l ($\in \mathcal{L}$) its label.

Definition 3.5 (TBI). A **TBI** *tableau* t is a rooted tree whose nodes are labelled with a signed formula and built according to the expansion rules of Figure 2

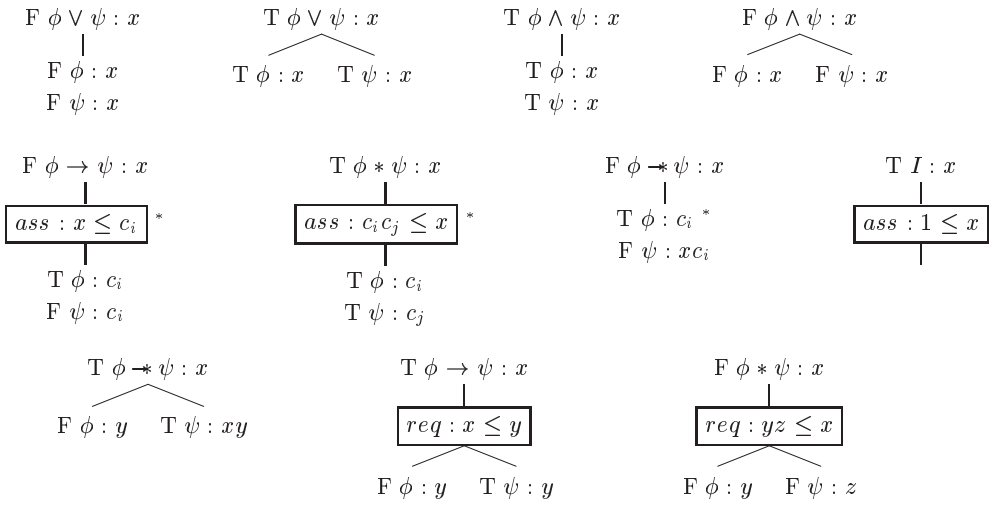


Figure 2. The **TBI** expansion rules

Given a tableau branch $\mathcal{B}$, $\mathcal{F}(\mathcal{B})$ denotes the set of all its signed formulae. Moreover, $\mathcal{B}$ is associated two particular sets of label constraints, $Ass(\mathcal{B})$, with elements "ass", and $Req(\mathcal{B})$, with elements "req", that are, respectively, the set of its *assertions* and of its *requirements*, or *obligations*. In fact, the assertions behave as known facts (or hypothesis) while the obligations express goals that must be satisfied (using assertions if necessary).

The *domain* $\mathcal{D}(\mathcal{B})$ of a branch $\mathcal{B}$ is the set of all sublabels occurring in its assertions, *i.e.*, $\mathcal{D}(\mathcal{B})$ = $\mathcal{D}(Ass(\mathcal{B}))$. $C(\mathcal{B})$ is the subset of all constants of $\mathcal{D}(\mathcal{B})$. The notation for branches extends to tableaux as follows: $f(t) = \bigcup_{\mathcal{B}\in t} f(\mathcal{B})$, where f is one of $\mathcal{F}$, Ass, Req, $\mathcal{D}$ or C.

The rules for $\wedge$ and $\vee$ are the usual α,β ones. Those introducing *assertions* including $F{-\!\!*}$ for which the assertion $c_i \leq c_i$ is implicitly assumed are called $\pi\alpha$ and those introducing *requirements* are called $\pi\beta$. Notice also that $\pi\alpha$ rules create new (atomic) labels while $\pi\beta$ reuse existing ones. Whilst the additive units are handled implicitly by the calculus (as in intuitionistic logic) a specific rule for the multiplicative unit I is required. It introduces an assertion of the form $1 \leq x$. To have a better intuition about labels and constraints, one can notice for each connective the relationship between the expansion rules and the clauses of the elementary Kripke semantics

(see Definition 2.5), having in mind that labels represent worlds, $\pi\alpha$ rules (with introduction of new labels) correspond to existential quantification on worlds and $\pi\beta$ rules (with variables that must be instanciated by known labels) correspond to universal quantification on worlds. This understanding includes the specific rule for I.

Definition 3.6. Let ϕ be a **BI** proposition. A *tableau sequence for* ϕ is a sequence of tableaux $t_1, t_2, \ldots$ for which t_1 is the one-node tree defined by $\mathcal{F}(t_1) = \{\mathrm{F}\ \phi : 1\}$, $Ass(t_1) = \{1 \leq 1\}$, $Req(t_1) = \emptyset$ and t_{i+1} is obtained from t_i by applying, on a branch of t_i, an expansion rule of Definition 3.5.

Building a labelled tableau for an initial formula ϕ, following such expansion rules, the key problem is to define so-called closure conditions such that either the tableau is closed and then ϕ is valid or there exists an open branch and then ϕ is not valid (Fitting 1990). Moreover, in the later case, we aim to build, from an open branch a countermodel of ϕ. Tableaux methods have been studied for various logics (classical, intuitionistic, linear, modal, *etc.*) and in each case, the particular definitions of *complementary formulae* and *closure conditions* allow to capture the semantics of the logic in order to analyze the provability. Other problems, like termination of tableau construction (or loops detection), are also studied and solved in different ways depending of the logic.

We illustrate the key notions and related problems in the case of **BI** with the example of Figure 3. Here we start with $\mathrm{F}\ (\mathrm{p} * (\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r}))) \mathrel{-\!\!*} (\mathrm{q} \to (I \mathrel{-\!\!*} \mathrm{r})) : 1$ and apply the expansion rules. A first key point is that we expand first the $\pi\alpha$ formulae that introduce new constants c_i and c_j and the related assertions. Step 1 introduces a new constant c_1. Steps 2 and 3 introduce new constants c_2, c_3, c_4 and the assertions $c_2c_3 \leq c_1$ and $c_1 \leq y$. Step 4 introduces $\mathrm{T}\ I : c_6$ with an implicit assertion $1 \leq c_6$. Note that each time we add a new assertion we consider the compatible closure of the assertion set. Thus, here we add the assertion $c_4 \leq c_4c_6$. We observe that we build a specific graph, called a dependency graph or Kripke resource graph, in parallel with expansions of $\pi\alpha$ formulae. It reflects the reflexive, transitive and compatible closure of the assertions. Now we can expand the $\pi\beta$ formulae that introduce requirements that must be satisfied from the dependency graph,*i.e.*, by using the existing constants introduced by the $\pi\alpha$ rules. Note that the left branch is closed by a standard complementarity between $\mathrm{T}\ p : c_2$ and $\mathrm{F}\ p : c_2$. Step 6 corresponds to the expansion of $\mathrm{T}\ q \to r : c_3c_2$ with y such that $c_3c_2 \leq y$. We consider $y = c_2c_3$ knowing that we consider label composition modulo commutativity. Then we can close a branch with $\mathrm{T}\ r : c_2c_3$ and $\mathrm{F}\ r : c_4c_6$ because $c_3c_2 \leq c_4c_6$ holds (see the dependency graph). Step 7 corresponds to a new expansion of $\mathrm{T}\ q \to r : c_3c_2$ with $y = c_4$, the requirement $c_3c_2 \leq c_4$ being satisfied (see the dependency graph). Consequently, the third branch is closed because we have $\mathrm{T}\ r : c_4$ and $\mathrm{F}\ r : c_4c_6$ and $c_4 \leq c_4c_6$ holds (see the dependency graph).

It is important to note that we can expand a $\pi\beta$ formula several times in order to close a branch with other instantiations of variables.

Now, we must formally define the different notions introduced through this example. Then, we introduce first the notions of *complementary formulae* and of *dependency graph*.

Definition 3.7. Two signed formulae $\mathrm{T}\ \phi : x$, $\mathrm{F}\ \phi : y$ are *complementary* in a branch $\mathcal{B}$ if and only if $x \leq y \in \overline{Ass}(\mathcal{B})$, *i.e.*, iff the constraint $x \leq y$ belongs to the reflexive, transitive and compatible closure of the assertions of $\mathcal{B}$.

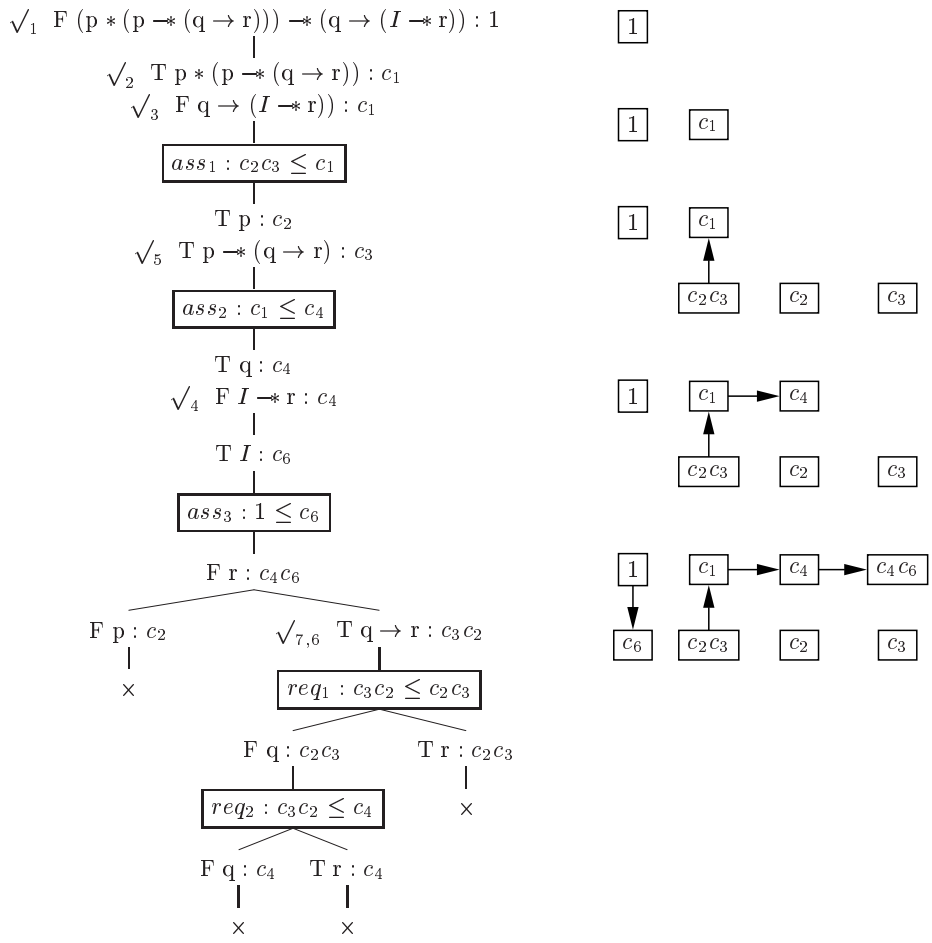


Figure 3. Tableau for $(\mathrm{p} * (\mathrm{p} -\!\!* (\mathrm{q} \to \mathrm{r}))) -\!\!* (\mathrm{q} \to (I -\!\!* \mathrm{r}))$

A specific graph, called the dependency graph or Kripke resource graph, is built in parallel with the tableau expansion. It reflects the information that can be derived from a given set of assertions. More formally, the dependency graph $DG(\mathcal{B}) = [N(\mathcal{B}), A(\mathcal{B})]$ associated to a given tableau branch $\mathcal{B}$ is defined as the following directed graph: the set of nodes $N(\mathcal{B})$ is the set of labels $\mathcal{D}(\mathcal{B})$ and the set of arrows $A(\mathcal{B})$ is built from the set of assertions $\overline{Ass}(\mathcal{B})$ such that there is an arrow $x \to y$ in $A(\mathcal{B})$ iff there is an assertion $x \leq y$ in $\overline{Ass}(\mathcal{B})$.

We can formally define a procedure that builds, in parallel with tableau expansions, the dependency graph $DG(\mathcal{B})$ of a branch $\mathcal{B}$ and so, the closure $\overline{Ass}(\mathcal{B})$. The expansion rules of a dependency graph are such that the given graph is only expanded by the $\pi\alpha$ rules, all the other rules, introducing neither new constants, nor new assertions, simply leave it unchanged. On a dependency graph $DG(\mathcal{B})$, the fact that a requirement $x \leq y$ holds with respect to $\overline{Ass}(\mathcal{B})$ corresponds to the existence of a path from the node x to the node y.

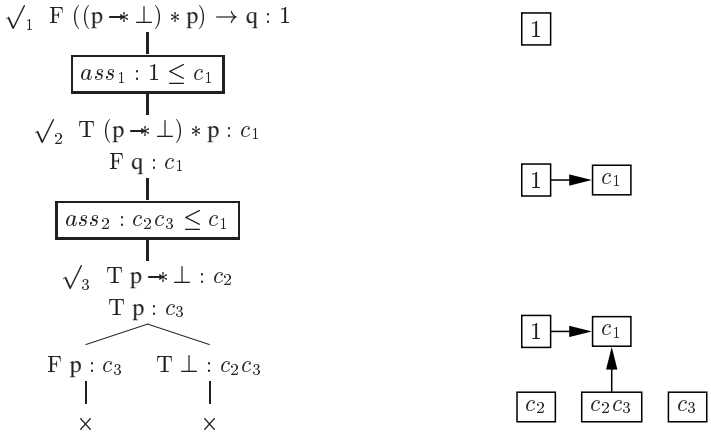


Figure 4. Tableau and Dependency Graph for $((\mathrm{p} \mathrel{-\!\!*} \bot) * \mathrm{p}) \to \mathrm{q}$

3.3. **Resource Tableaux for BI**

Before to define the closure conditions for the **TBI** calculus, let us emphasize that the labelling algebra is defined in order to capture the semantics inside the tableaux calculus. If we consider **BI** without $\bot$ the labels and constraints of **TBI** clearly reflect the elementary Kripke semantics at the syntactic level and thus provide resource tableaux with soundness and completeness properties (Galmiche and Méry 2001). But our aim is to consider **BI** and its complete Grothendieck topological semantics. Thus, we have to give an appropriate definition of closed tableau which takes the specificity of $\bot$ into account.

If we consider the elementary Kripke semantics, which is not complete for **BI** a branch is closed when it contains a signed formula $\mathrm{T}\ \bot : x$. Indeed, such a branch cannot have a model in the elementary semantics and thus it would be in contradiction with the fact that $\bot$ should never be forced by any world. In Grothendieck topological semantics, however, we can have worlds at which $\bot$ is forced. We will denote such worlds inconsistent worlds. Then, a branch cannot be considered as closed simply because it contains a signed formula $\mathrm{T}\ \bot : x$. Additional conditions have to be defined in order to allow a branch to contain such a signed formula while still being realizable in some Grothendieck resource model.

We introduce first the notion of inconsistent label in a tableau branch.

Definition 3.8. Let $\mathcal{B}$ be a branch, a label x is *inconsistent in* $\mathcal{B}$ if there exists a label y such that $y \leq x \in \overline{Ass}(\mathcal{B})$ and a label z in $\wp(y)$ (set of sub-labels of y) such that $\mathrm{T}\ \bot : z$ occurs in $\mathcal{B}$. A label x is *consistent* in $\mathcal{B}$ if it is not inconsistent.

Recall that if a world m is inconsistent, then it forces any proposition ϕ because **LBI** is sound and complete with respect to Grothendieck resource models and $\bot \vdash \phi$ is an axiom. Moreover, because of the continuity axiom, we notice that if $m \models \bot$ then, for all world n, we have $m \bullet n \models \bot$. In other words, each world that can be obtained by a composition involving an inconsistent world is itself inconsistent.

Definition 3.9. A tableau t is *closed* iff any branch $\mathcal{B}$ in t satisfies the following conditions:
(i) (1) there are two formulae $\mathrm{T}\ \phi : x$, and $\mathrm{F}\ \phi : y$ that are complementary in $\mathcal{B}$, or (2) there is $\mathrm{F}\ \top : x$ in $\mathcal{B}$, or (3) there is $\mathrm{F}\ I : x$ in $\mathcal{B}$ with $1 \leq x \in \overline{Ass}(\mathcal{B})$, or (4) there is $\mathrm{T}\ I : x$ in $\mathcal{B}$ with $1 \leq x \notin \overline{Ass}(\mathcal{B})$, or (5) there is $\mathrm{F}\ \phi : x$ in $\mathcal{B}$ with x inconsistent in $\mathcal{B}$;
(ii) $\forall\, x \leq y \in Req(\mathcal{B}), x \leq y \in \overline{Ass}(\mathcal{B})$.

A tableaux sequence $t_1, t_2, \ldots$ is *closed* if it contains a closed tableau.

If we suppress, in the above definition, the condition (5) in (i) then we have the closure conditions that fit well with **BI** without $\bot$ and its elementary semantics. The challenge here is to show that with this additional condition we can cope with **BI** and its Grothendieck topological semantics.

Before we study the properties of the **TBI** calculus, we illustrate the particular treatment of $\bot$ with two examples, one for a provable formula and another one for an unprovable formula.

First, we consider the provable formula $((\mathrm{p} \mathrel{-\!\!*} \bot) * \mathrm{p}) \to \mathrm{q}$. Its closed tableau is given in Figure 4. The first two steps generate two assertions and the associated dependency graph. After Step 3, we have a tableau with two branches. The first branch is closed since it contains complementary formulae, namely, $\langle\, \mathrm{T}\ \mathrm{p} : c_3, \mathrm{F}\ \mathrm{p} : c_3 \,\rangle$. The second, however, contains no complementary formulae. We notice that the branch contains the formula $\mathrm{T}\ \bot : c_2c_3$. Thus, c_2c_3 is what we have called an inconsistent label and, by assertion $ass_2 : c_2c_3 \leq c_1$, c_1 is also inconsistent. Therefore, the branch is closed because it contains the formula $\mathrm{F}\ \mathrm{q} : c_1$ with label c_1 being inconsistent. Then we can deduce that the formula is provable.

Consider another example with the formula $((\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot) \mathrel{-\!\!*} (((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot)$ that leads to an unclosed tableau (see Figure 5). The first steps are similar to the other examples and after Step 6, the tableau has only so-called completed branches (see Definition 4.5), meaning that all signed formula have been completely analyzed. The second branch is closed with $\langle\, \mathrm{T}\ \mathrm{p} : c_3, \mathrm{F}\ \mathrm{p} : c_3 \,\rangle$, the third one is closed with $\langle\, \mathrm{T}\ \bot : c_2c_3, \mathrm{F}\ \bot : c_2c_3 \,\rangle$ and the fourth one is closed with $\langle\, \mathrm{T}\ \bot : c_2, \mathrm{F}\ \bot : c_2 \,\rangle$. The first branch, on the contrary, remains open since the only way to close it would be to have $\langle\, \mathrm{T}\ \mathrm{p} : c_3, \mathrm{F}\ \mathrm{p} : 1 \,\rangle$, but $c_3 \leq 1$ cannot be deduced from the (closure of the) assertions of the branch. We will see in a next section how to build a countermodel from such an open branch.

Now we must show that this labelled calculus, whose restriction to **BI** without $\bot$ is sound and complete for the elementary semantics, is also sound and complete for **BI** with respect to the Grothendieck topological semantics.

4. Properties of the TBI Calculus

We aim to show the soundness and completeness of **TBI** with respect to GRMs but the soundness can only be proved with respect to so-called basic GRMs. This deductive framework allows not only a proof procedure but also, in the case of non-provability, the systematic generation of countermodels.

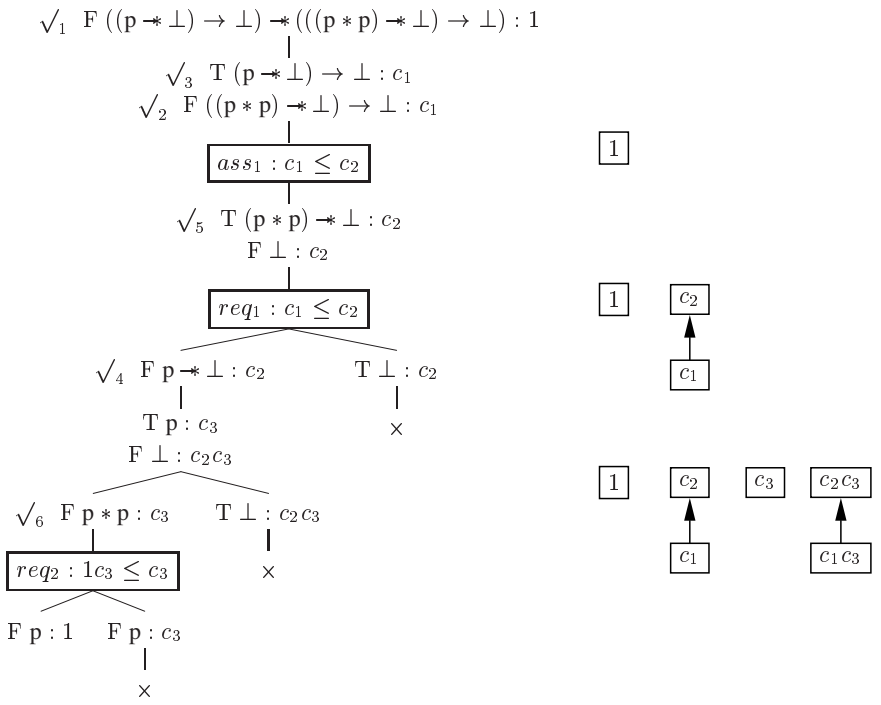


Figure 5. Tableau and Dependency Graph for $((\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot) \mathrel{-\!\!*} (((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot)$

4.1. **Soundness**

We prove here the soundness of **TBI** with respect to particular GRMs called basic GRMs, following a classical development, subject to the usual adaptations to **BI** from a notion of *realizability* that is preserved by the expansion rules (Galmiche and Méry 2001). But we cannot prove it with respect to GRMs and consequently with respect to the Grothendieck topological semantics.

Taking $\bot$ into account, the proof of soundness of **TBI** cannot be a simple extension of the one of (Galmiche and Méry 2003). It becomes more delicate because we have to deal with Grothendieck topological semantics. The best way to solve the problem consists first in restricting first the initial proof to so-called basic GRMs, and then in proving soundness of **TBI** with respect to a new relational semantics that is complete and closely related to the **TBI** calculus.

Definition 4.1. Let $\mathcal{G} = \langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$ be a *GRM* with $\mathcal{M} = (M, \bullet, e, \sqsubseteq, J)$ and $\mathcal{B}$ be a tableau branch, a *realization of* $\mathcal{B}$ *in* $\mathcal{G}$ is a mapping $\|-\| : \mathcal{D}(\mathcal{B}) \to M$, from the domain of $\mathcal{B}$ to the worlds of M, that satisfies

1 $\|1\| = e$,
2 $\|x \circ y\| = \|x\| \bullet \|y\|$,
3 for any $\mathrm{T}\ \phi : x$ in $\mathcal{B}$, $\|x\| \models \phi$,
4 for any $\mathrm{F}\ \phi : x$ in $\mathcal{B}$, $\|x\| \not\models \phi$,
5 for any $x \leq y$ in $Ass(\mathcal{B})$, $\|x\| \sqsubseteq \|y\|$.

Lemma 4.1. Let t be a tableau, $\mathcal{B}$ a branch of t and $\|-\|$ a realization of $\mathcal{B}$ in a *GRM* $\mathcal{G}$. Then, for any $x \leq y \in \overline{Ass}(\mathcal{B})$, $\|x\| \sqsubseteq \|y\|$ holds in $\mathcal{G}$.

Proof. By a straightforward induction. □

Definition 4.2. A tableau branch $\mathcal{B}$ is *realizable* if there exists a realization of $\mathcal{B}$ in some *GRM* $\mathcal{G}$. A tableau t is *realizable* if it contains a realizable branch.

Lemma 4.2. A closed tableau is not realizable.

Proof. Let t be a closed tableau that is also realizable. Then, t contains a branch $\mathcal{B}$ which is realizable in some *GRM* $\mathcal{G} = \langle \mathcal{M}, \models, [\![-]\!] \rangle$. If the branch is closed because of complementary formulae $\langle \mathrm{T}\ \phi : x, \mathrm{F}\ \phi : y \rangle$ then, by definition, we have $x \leq y \in \overline{Ass}(\mathcal{B})$ which, by Lemma 4.1, implies $\|x\| \sqsubseteq \|y\|$. But, since $\|-\|$ realizes $\mathcal{B}$, we also have $\|x\| \models \phi$ and $\|y\| \not\models \phi$. Therefore, we reach a contradiction because, by property (K), we should have $\|y\| \models \phi$. If the branch is closed because of a formula $\mathrm{F}\ \phi : x$, whose label x is inconsistent in $\mathcal{B}$, then, by definition, there exists a label y such that $y \leq x \in \overline{Ass}(\mathcal{B})$ and a label z in $\wp(y)$ such that $\mathrm{T}\ \bot : z \in \mathcal{B}$. Since $\|-\|$ realizes $\mathcal{B}$ we have $x \not\models \phi$ and $z \models \bot$. Since z is a sublabel of y, the continuity axiom of J implies that $y \models \bot$. Therefore, as Lemma 4.1 implies $\|y\| \sqsubseteq \|x\|$, (K) yields $x \models \bot$ and, once again, we reach a contradiction because, if $x \models \bot$ then, for any ϕ, we should have $x \models \phi$. Other cases are similar. □

Compared to the soundness proof for **BI** without $\bot$ and its elementary Kripke models (EKRMs) (Galmiche and Méry 2003), we must introduce a restriction on the GRMs in order to have a soundness result for the **TBI** calculus.

Definition 4.3 (basic GRM). Let $\mathcal{G} = \langle \mathcal{M}, \models, [\![-]\!] \rangle$ be a *GRM* with $\mathcal{M} = (M, \bullet, e, \sqsubseteq, J)$, $\mathcal{G}$ is *basic* iff M contains a greatest element π such that for any $m \in M$, $\pi \bullet m = \pi$ and J is basic, *i.e.*, is given by $J(m) = \{\{m\}\}$ if $m \neq \pi$ and $J(\pi) = \{\{\pi\}, \emptyset\}$.

All the previous lemmas that hold for GRMs also hold for basic GRMs. Now, let us consider a lemma that only holds for the restricted models. We say that a tableau branch $\mathcal{B}$ is *b-realizable* if there exists a realization of $\mathcal{B}$ in some basic *GRM* $\mathcal{G}$ and a tableau t is *b-realizable* if it contains a b-realizable branch.

Lemma 4.3. If t' is a tableau obtained from a tableau t by application of an expansion rule of **TBI**, then if t is b-realizable, t' is also b-realizable.

Proof. Since t is realizable, it contains a branch $\mathcal{B}$ which is realizable in some basic *GRM* for some realization $\|-\|$. If the signed formula $\mathrm{Sg}\ X : x$ that has been expanded to obtain t' does not belong to $\mathcal{B}$, then t' is realizable since it still contains $\mathcal{B}$. Otherwise, we show by case analysis on $\mathrm{Sg}\ X : x$ that the corresponding expansion rule preserves realizability.

- Case $\mathrm{T}\ \phi * \psi : x$.
  $\mathcal{B}$ is expanded into $\mathcal{B}'$ with $\mathcal{F}(\mathcal{B}') = \mathcal{F}(\mathcal{B}) \cup \{\mathrm{T}\ \phi : c_i, \mathrm{T}\ \psi : c_j\}$ and $Ass(\mathcal{B}') = Ass(\mathcal{B}) \cup \{c_i c_j \leq x\}$, c_i and c_j being new constants. Since $\|-\|$ realizes $\mathcal{B}$, we have $\|x\| \models \phi * \psi$. Therefore, there exists $S \in J(\|x\|)$ such that for any $m' \in S$, there exist $n_1, n_2 \in M$ such that $n_1 \bullet n_2 \sqsubseteq m'$, $n_1 \models \phi$ and $n_2 \models \psi$. As we consider a basic GRM (cf. Definition 4.3) we have $J(\|x\|) = \{\{\|x\|\}\}$. We simply extend $\|-\|$ to c_i, c_j by $\|c_i\| = n_1$ and $\|c_j\| = n_2$

and consider $m' = \|x\|$. Thus we directly deduce that $\|c_i\| \bullet \|c_j\| \sqsubseteq \|x\|$. Therefore, $\mathcal{B}'$ is realizable and, consequently, t' is realizable.

- Case $\mathrm{F}\ \phi * \psi : x$.
  $\mathcal{B}$ splits into $\mathcal{B}'$ and $\mathcal{B}''$ such that $\mathcal{F}(\mathcal{B}') = \mathcal{F}(\mathcal{B}) \cup \{\mathrm{F}\ \phi : y\}$, $\mathcal{F}(\mathcal{B}'') = \mathcal{F}(\mathcal{B}) \cup \{\mathrm{F}\ \psi : z\}$ and $Req(\mathcal{B}') = Req(\mathcal{B}'') = Req(\mathcal{B}) \cup \{yz \leq x\}$. A valid application of the $\mathrm{F} *$ rule requires that $yz \leq x$ should be in $\overline{Ass}(\mathcal{B})$. Thus, by Lemma 4.1, we have $\|y\| \bullet \|z\| \sqsubseteq \|x\|$. Since $\|-\|$ realizes $\mathcal{B}$, we have $\|x\| \not\models \phi * \psi$. Therefore, for any $m, n \in M$ such that $m \bullet n \sqsubseteq \|x\|$, either $m \not\models \phi$, or $n \not\models \psi$, which implies that either $\|y\| \not\models \phi$, or $\|z\| \not\models \psi$. Then, either $\mathcal{B}'$, or $\mathcal{B}''$ is realizable and, consequently, t' is realizable.
- Case $\mathrm{T}\ \phi \vee \psi : x$.
  $\mathcal{B}$ splits into $\mathcal{B}'$ with $\mathcal{F}(\mathcal{B}') = \mathcal{F}(\mathcal{B}) \cup \{\mathrm{T}\ \phi : x\}$ and $\mathcal{B}''$ with $\mathcal{F}(\mathcal{B}'') = \mathcal{F}(\mathcal{B}) \cup \{\mathrm{T}\ \psi : x\}$. Moreover, $Ass(\mathcal{B}') = Ass(\mathcal{B}'') = Ass(\mathcal{B})$. Since $\|-\|$ realizes $\mathcal{B}$, we have $\|x\| \models \phi \vee \psi$. Therefore, there exist $S \in J(\|x\|)$ such that for any $m' \in S$, either $m' \models \phi$ or $m' \models \psi$. As we consider a basic GRM we obtain either $\|x\| \models \phi$ or $\|x\| \models \psi$. Therefore, either $\mathcal{B}'$ or $\mathcal{B}''$ is realizable and, consequently, t' is realizable.
- Other cases are similar.

□

It is important to notice that we cannot prove this lemma without the restriction to the basic GRMs. If we consider the above $*$ and $\vee$ cases with general GRMs, it is clear that we cannot conclude.

Corollary 4.1. Let $t_1, t_2, \ldots$ be a tableaux sequence, if t_i is b-realizable, then, for $j > i$, t_j is b-realizable.

Proof. Directly from the previous lemma. □

Theorem 4.1 (soundness w.r.t. basic GRMs). Let ϕ be a proposition of **BI**. If there exists a closed tableaux sequence $\mathcal{T}$ for ϕ, then ϕ is valid in basic Grothendieck topological resource models.

Proof. Let $\mathcal{T} = t_1, t_2, \ldots, t_n$ be a closed tableaux sequence. Suppose that ϕ does not hold in basic Grothendieck resource semantics. Then, by validity result (§2) there exists a basic *GRM* $\mathcal{G}$ for which $e \not\models \phi$. Then, the initial tableau t_1 is trivially b-realizable and Lemma 4.3 implies that all t_i such that $i > 1$ are also b-realizable. It follows from Lemma 4.2 that none of the t_i can be closed and, consequently, $\mathcal{T}$ cannot be closed. □

We observe that we are not in position to prove the soundness of **TBI** but we show, in the next section, how to solve this problem by analyzing **BI**'s semantics and by defining a new relational semantics of **BI** that is naturally related to the **TBI** calculus and reflects in a better way the semantical interactions between connectives. First, we study the completeness of **TBI** that needs no such restrictions on models.

4.2. **Countermodel Construction**

We describe how to construct a countermodel of ϕ from an open branch in a tableau for ϕ. The proof of the finite model property, in a next section, relies critically on the introduction of a special element, here called π, used to collect the inessential (and possibly infinite) parts of the model.

Definition 4.4. Let $\mathcal{B}$ be a tableau branch. A signed formula $\mathrm{Sg}\ \chi : x$ is *fulfilled*, or *completely analyzed*, in $\mathcal{B}$, denoted $\mathcal{B} \Vdash \mathrm{Sg}\ \chi : x$, if it satisfies one of the following conditions:

1 $\mathcal{B} \Vdash \mathrm{F}\ \top : x$ iff always;
2 $\mathcal{B} \Vdash \mathrm{F}\ \bot : x$ iff always;
3 $\mathcal{B} \Vdash \mathrm{F}\ I : x$ iff $1 \leq x \notin \overline{Ass}(\mathcal{B})$;
4 $\mathcal{B} \Vdash \mathrm{F}\ \mathrm{p} : x$ iff there is $\mathrm{F}\ \mathrm{p} : y \in \mathcal{B}$ s.t. $x \leq y \in \overline{Ass}(\mathcal{B})$;
5 $\mathcal{B} \Vdash \mathrm{F}\ \phi \wedge \psi : x$ iff $\mathcal{B} \Vdash \mathrm{F}\ \phi : x$ or $\mathcal{B} \Vdash \mathrm{F}\ \psi : x$;
6 $\mathcal{B} \Vdash \mathrm{F}\ \phi \vee \psi : x$ iff $\mathcal{B} \Vdash \mathrm{F}\ \phi : x$ and $\mathcal{B} \Vdash \mathrm{F}\ \psi : x$;
7 $\mathcal{B} \Vdash \mathrm{F}\ \phi \to \psi : x$ iff there is $y \in \mathcal{D}(\mathcal{B})$ s.t. $x \leq y \in \overline{Ass}(\mathcal{B})$ and both $\mathcal{B} \Vdash \mathrm{T}\ \phi : y$ and $\mathcal{B} \Vdash \mathrm{F}\ \psi : y$;
8 $\mathcal{B} \Vdash \mathrm{F}\ \phi * \psi : x$ iff, for any $y, z \in \mathcal{D}(\mathcal{B})$ s.t. $yz \leq x \in \overline{Ass}(\mathcal{B})$, $\mathcal{B} \Vdash \mathrm{F}\ \phi : y$ or $\mathcal{B} \Vdash \mathrm{F}\ \psi : z$;
9 $\mathcal{B} \Vdash \mathrm{F}\ \phi \mathrel{-\!\!*} \psi : x$ iff there exist $y, xy \in \mathcal{D}(\mathcal{B})$ s.t. both $\mathcal{B} \Vdash \mathrm{T}\ \phi : y$ and $\mathcal{B} \Vdash \mathrm{F}\ \psi : xy$;
10 $\mathcal{B} \Vdash \mathrm{T}\ \top : x$ iff always;
11 $\mathcal{B} \Vdash \mathrm{T}\ \bot : x$ iff always;
12 $\mathcal{B} \Vdash \mathrm{T}\ I : x$ iff $1 \leq x \in \overline{Ass}(\mathcal{B})$;
13 $\mathcal{B} \Vdash \mathrm{T}\ \mathrm{p} : x$ iff there is $\mathrm{T}\ \mathrm{p} : y \in \mathcal{B}$ s.t. $y \leq x \in \overline{Ass}(\mathcal{B})$;
14 $\mathcal{B} \Vdash \mathrm{T}\ \phi \wedge \psi : x$ iff both $\mathcal{B} \Vdash \mathrm{T}\ \phi : x$ and $\mathcal{B} \Vdash \mathrm{T}\ \psi : x$;
15 $\mathcal{B} \Vdash \mathrm{T}\ \phi \vee \psi : x$ iff $\mathcal{B} \Vdash \mathrm{T}\ \phi : x$ or $\mathcal{B} \Vdash \mathrm{T}\ \psi : x$;
16 $\mathcal{B} \Vdash \mathrm{T}\ \phi \to \psi : x$ iff, for any $y \in \mathcal{D}(\mathcal{B})$ s.t. $x \leq y \in \overline{Ass}(\mathcal{B})$, $\mathcal{B} \Vdash \mathrm{F}\ \phi : y$ or $\mathcal{B} \Vdash \mathrm{T}\ \psi : y$;
17 $\mathcal{B} \Vdash \mathrm{T}\ \phi * \psi : x$ iff there are $y, z \in \mathcal{D}(\mathcal{B})$ s.t. $yz \leq x \in \overline{Ass}(\mathcal{B})$ and both $\mathcal{B} \Vdash \mathrm{T}\ \phi : y$ and $\mathcal{B} \Vdash \mathrm{T}\ \psi : z$;
18 $\mathcal{B} \Vdash \mathrm{T}\ \phi \mathrel{-\!\!*} \psi : x$ iff, for any y, $xy \in \mathcal{D}(\mathcal{B})$, $\mathcal{B} \Vdash \mathrm{F}\ \phi : y$ or $\mathcal{B} \Vdash \mathrm{T}\ \psi : xy$.

Lemma 4.4. Let $\mathcal{B}$ be a tableau branch. The property of being fulfilled, given in Definition 4.4, satisfies Kripke monotonicity, *i.e.*, (i) $\mathcal{B} \Vdash \mathrm{F}\ \phi : x$ and $y \leq x \in \overline{Ass}(\mathcal{B})$ imply $\mathcal{B} \Vdash \mathrm{F}\ \phi : y$, and (ii) $\mathcal{B} \Vdash \mathrm{T}\ \phi : x$ and $x \leq y \in \overline{Ass}(\mathcal{B})$ imply $\mathcal{B} \Vdash \mathrm{T}\ \phi : y$.

Proof. By structural induction on $\mathrm{Sg}\ \phi : x$. □

Definition 4.5. A tableau branch $\mathcal{B}$ is *completed* if any signed formula $\mathrm{Sg}\ \phi : x$ in $\mathcal{B}$ is fulfilled. A tableau is *completed* if it has a branch that is completed. A tableau branch $\mathcal{B}$ is an *H-branch* if it is open and completed.

Lemma 4.5. If $\mathcal{B}$ is an H-branch then, for any proposition ϕ, we have not both $\mathcal{B} \Vdash \mathrm{T}\ \phi : x$ and $\mathcal{B} \Vdash \mathrm{F}\ \phi : x$.

Proof. Suppose that both $\mathcal{B} \Vdash \mathrm{T}\ \mathrm{p} : x$ and $\mathcal{B} \Vdash \mathrm{F}\ \mathrm{p} : x$. Then, $\mathrm{T}\ \mathrm{p} : y \in \mathcal{B}$ for some y such that $y \leq x \in \overline{Ass}(\mathcal{B})$ and $\mathrm{F}\ \mathrm{p} : z \in \mathcal{B}$ for some z such that $x \leq z \in \overline{Ass}(\mathcal{B})$. Therefore, by transitivity, we have $y \leq z \in \overline{Ass}(\mathcal{B})$, which implies that $\mathcal{B}$ is closed. It is in contradiction with

the definition of a completed branch. The property can be extended by structural induction to non-atomic propositions. □

The dependency graph related to a formula ϕ during the resource tableau construction represents the closure of the assertions in the sense of Definition 3.3 and so captures the computational content of ϕ. Therefore, if a formula ϕ happens to be unprovable, we should have enough information in its dependency graph to extract a countermodel for ϕ. For that, we must provide a preordered commutative monoid together with a Grothendieck topology and a forcing relation which falsifies ϕ in some world. The idea behind the countermodel construction is to regard the dependency graph itself as the desired countermodel, thereby considering it as a central semantic structure. For that, we take the nodes (labels) of the graph as the elements of a monoid whose multiplication is given by the composition of the labels. The preordering relation is then given by the arrows and the forcing relation simply reflects the property of being fulfilled.

The key problem is that, since the closure operator induces a *partially defined* labelling algebra, the dependency graph only deals with those pieces of information (resources) that are relevant for deciding provability. Therefore, the monoidal product should be completed with suitable values for those compositions which are undefined. The problem of undefinedness is solved in Definition 4.6 by the introduction of a particular element, denoted π, to which all undefined compositions are mapped and for which the equation $(\forall x)(x \bullet \pi = \pi \bullet x = \pi)$, meaning that any composition with something undefined is itself undefined, is assumed.

We must, however, be careful because introducing a new element may affect the property of a formula $\phi \multimap\!\!* \psi$ of being realized in a world x although the signed formula $\mathrm{T}\ \phi \multimap\!\!* \psi : x$ was fulfilled in the dependency graph. Indeed, if π forces ϕ then, since $x \bullet \pi = \pi$, we also need π to force ψ. But, if π forces any formula ψ, then everything works as it should. On the other hand, we know that an inconsistent world necessarily forces any formula ψ because $\bot \vdash \psi$ is an axiom. Therefore, making π an inconsistent world by setting $\emptyset \in J(\pi)$ just solves the problem.

Definition 4.6 (M-structure, π). Let $\mathcal{B}$ be an H-branch. The M-structure

$$\mathcal{M}(\mathcal{B}) = \langle\, M, \bullet, 1, \sqsubseteq, J \,\rangle$$

is defined as follows: (i) M is the subset of labels of $\mathcal{D}(\mathcal{B})$ consistent in $\mathcal{B}$, extended with a particular element π; (ii) the multiplication $\bullet$ is defined by

$$x \bullet 1 = 1 \bullet x = x \qquad x \bullet y = y \bullet x = \begin{cases} xy & \textit{if } xy \in M \\ \pi & \textit{otherwise;} \end{cases}$$

(iii) the relation $\sqsubseteq$ between elements of M is defined by

$$x \sqsubseteq y \quad \text{iff} \quad y = \pi \text{ or } x \leq y \in \overline{Ass}(\mathcal{B});$$

and (iv) the map $J : M \to \wp(\wp(M))$, called the J-map of $\mathcal{B}$, is defined by

$$J(\pi) = \{\{\pi\}, \emptyset\} \quad J(x) = \{\{y\} \mid y = x\}.$$

Lemma 4.6. Let $\mathcal{B}$ be an H-branch, the M-structure $\mathcal{M}(\mathcal{B}) = \langle\, M, \bullet, 1, \sqsubseteq, J \,\rangle$ is a *GTM*, *i.e.*, (i) $\langle\, M, \bullet, 1, \sqsubseteq \,\rangle$ is a preordered commutative monoid, and (ii) J is a Grothendieck topology.

Proof. A routine calculation shows that $\langle M, \bullet, 1, \sqsubseteq \rangle$ is a preordered monoid. The commutativity of $\bullet$ is by definition and the associativity and compatibility properties come from those of $\circ$. Therefore, we only show that J is a Grothendieck topology, *i.e.*, that J satisfies all of the axioms of Definition 2.6.

1. Sieve. We show $(\forall m \in M)(\forall S \in J(m))\ (\forall m' \in S)(m \sqsubseteq m')$:

— if $m \equiv \pi$ then,
  - if $S = \emptyset$ then, the condition is trivial since there is no $m' \in S$;
  - if $S = \{\pi\}$ then, obviously, we have $\pi \sqsubseteq \pi$;

— if $m \equiv x$ then, $S = \{y\}$ and, by definition of J, $y = x$.

2. Maximality. We show $(\forall m, n \in M)(m = n \ \Rightarrow\ \{n\} \in J(m))$:

— if $m \equiv \pi$ then, by definition of J, $\{\pi\} \in J(\pi)$ and π is the only element such that $\pi = \pi$;

— if $m \equiv x$ then, by definition of J, $y = x$ implies $\{y\} \in J(x)$.

3. Stability. We show $(\forall m, n \in M)(\forall S \in J(m))(m \sqsubseteq n \Rightarrow (\exists S' \in J(n))(\forall n' \in S')(\exists m' \in S)(m' \sqsubseteq n'))$:

— if $m \equiv \pi$ then, necessarily, $n \equiv \pi$ and we simply choose $S' = S$;

— if $m \equiv x$ then, let $\{y\} \in J(x)$ and suppose $x' \in M$ such that $x \sqsubseteq x'$. By definition of J, we have $y = x$, which implies $y \sqsubseteq x'$. Since $\{x'\} \in J(x')$, we choose $S' = \{x'\}$.

4. Transitivity. We show $(\forall m \in M)(\forall S \in J(m))(\forall \{S_{m'} \in J(m')\}_{m' \in S})(\bigcup_{m' \in S} S_{m'} \in J(m))$:

— if $m \equiv \pi$ then,
  - if $S = \emptyset$ then, $\bigcup_{m' \in S} S_{m'} = \emptyset$ and $\emptyset \in J(\pi)$;
  - if $S = \{\pi\}$ then, the condition reduces to $(\forall S_\pi \in J(\pi))(S_\pi \in J(\pi))$, which is trivial;

— if $m \equiv x$ then, the condition reduces to $(\forall \{y\} \in J(x))(\forall \{z\} \in J(y))(\{z\} \in J(x))$, which is straightforward since, by definition of J, we have $y = x$ and $z = y$, which implies $z = x$, *i.e.*, $\{z\} \in J(x)$;

5. Continuity. We show $(\forall m, n \in M)(\forall S \in J(m))(\{m' \bullet n \mid m' \in S\} \in J(m \bullet n))$.

— if $m \equiv \pi$ then,
  - if $S \equiv \emptyset$ then, $S \bullet n = \emptyset$ and, by definition of J, we have $\emptyset \in J(\pi)$;
  - if $S \equiv \{\pi\}$ then, $S \bullet n = \{\pi\}$ and, by definition of J, we have $\{\pi\} \in J(\pi)$;

— if $m \equiv x$ then, $S = \{y\}$ such that $y = x$. Therefore, by compatibility of $\bullet$, we have $y \bullet n = x \bullet n$, and so, by definition of J, we have $\{y \bullet n\} \in J(x \bullet n)$.

□

Definition 4.7. Let $\mathcal{M}(\mathcal{B}) = \langle M, \bullet, 1, \sqsubseteq, J \rangle$ be the M-structure of an H-branch $\mathcal{B}$ and $\mathcal{P}(L)$ denote the collection of **BI** propositions over a language L of propositional letters. The interpretation $[\![-]\!]_{\mathcal{B}} : L \to \wp(M)$ is, for any atomic proposition p, $[\![\mathrm{p}]\!]_{\mathcal{B}} = \{\pi\} \cup \{x \mid \mathcal{B} \Vdash \mathrm{T\ p} : x\}$.

Lemma 4.7. $[\![-]\!]_{\mathcal{B}}$ is a *GTI*, *i.e.*, it satisfies properties (K) and (Sh) of Definition 2.7.

Proof.

(K): We prove $(\forall m, n \in M)((n \sqsubseteq m \text{ and } n \in [\![\mathrm{p}]\!]_{\mathcal{B}}) \Rightarrow m \in [\![\mathrm{p}]\!]_{\mathcal{B}})$:

— if $m \equiv \pi$ then, $\pi \in [\![\mathrm{p}]\!]_{\mathcal{B}}$ by definition;

— if $m \equiv x$ then, $n \sqsubseteq m$ implies that $n \equiv y$ such that $y \leq x \in \overline{Ass}(\mathcal{B})$. Moreover, if $n \in [\![\mathrm{p}]\!]_{\mathcal{B}}$ then, by definition of $[\![\mathrm{p}]\!]_{\mathcal{B}}$, we have $\mathcal{B} \Vdash \mathrm{T}\ \mathrm{p} : y$ and Lemma 4.4 yields $\mathcal{B} \Vdash \mathrm{T}\ \mathrm{p} : x$, *i.e.*, $m \in [\![\mathrm{p}]\!]_{\mathcal{B}}$.

(Sh): We prove $(\forall m \in M)(\forall S \in J(m))((\forall m' \in S)(m' \in [\![\mathrm{p}]\!]_{\mathcal{B}}) \Rightarrow m \in [\![\mathrm{p}]\!]_{\mathcal{B}})$

— if $m \equiv \pi$ then, $\pi \in [\![\mathrm{p}]\!]_{\mathcal{B}}$ by definition;

— if $m \equiv x$ then, $S = \{y\}$ such that $y = x$. Therefore, if $y \in [\![\mathrm{p}]\!]_{\mathcal{B}}$, property (K) entails $x \in [\![\mathrm{p}]\!]_{\mathcal{B}}$.

□

Theorem 4.2. Let $\mathcal{B}$ be an H-branch. Then $\langle\, \mathcal{M}(\mathcal{B}), \models, [\![-]\!]_{\mathcal{B}} \,\rangle$ is a Grothendieck resource model of $\mathcal{B}$, *i.e.*, for any proposition ϕ, we have: (i) $\pi \models \phi$; (ii) $\mathcal{B} \Vdash \mathrm{T}\ \phi : x$ implies $x \models \phi$; (iii) $\mathcal{B} \Vdash \mathrm{F}\ \phi : x$ implies $x \not\models \phi$.

Proof. By extension of the proof for **BI** without $\bot$ to the Grothendieck topological semantics. □

We remark that if we aim to deal with partial orders instead of preorders, the definition of the J-map could be simplified to $J(x) = \{\{x\}\}$. It is standard to embed a preorder into a partial order by taking the quotient of $\mathcal{D}(\mathcal{B})$ under the equivalence induced by the equality on labels defined in 3.2.

Returning to the example of Figure 5, we show how to build a countermodel from the open branch. As the reader might check, all formulae in the open branch are fulfilled and $\mathcal{B}$ is therefore what we have called an H-branch. First, following the steps of Definition 4.6, we build from $\mathcal{B}$ a GTM $\mathcal{M}(\mathcal{B}) = \langle\, M, \bullet, 1, \sqsubseteq, J \,\rangle$.

(i) M is the subset of labels of $\mathcal{D}(\mathcal{B})$ that are consistent, to which we add the element π, *i.e.*,

$$M = \{1, c_1, c_2, c_3, c_1c_3, c_2c_3, \pi\}.$$

Notice that, because of the presence in $\mathcal{B}$ of both the assertion $ass_1 : c_1 \leq c_2$ and of the label c_2c_3, the label c_1c_3, although not initially present in $\mathcal{B}$, is added by the closure operation in order to respect the compatibility requirement.

(ii) the multiplication $\bullet$ is

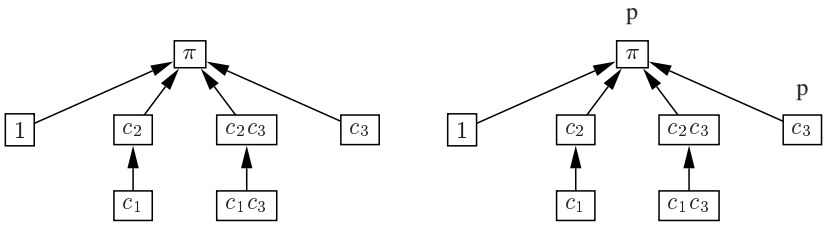


Figure 6. Countermodel for $((\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot) \mathrel{-\!\!*} (((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot)$

$\bullet$	1	c_1	c_2	c_3	c_1c_3	c_2c_3	π
1	1	c_1	c_2	c_3	c_1c_3	c_2c_3	π
c_1	c_1	π	π	c_1c_3	π	π	π
c_2	c_2	π	π	c_2c_3	π	π	π
c_3	c_3	c_1c_3	c_2c_3	π	π	π	π
c_1c_3	c_1c_3	π	π	π	π	π	π
c_2c_3	c_2c_3	π	π	π	π	π	π
π	π	π	π	π	π	π	π

(iii)The preordering relation $\sqsubseteq$ reflects the structure of the assertions $\overline{Ass}(\mathcal{B})$. If we omit reflexive and π relations, we have two non-trivial relations, namely, $c_1 \sqsubseteq c_2$ and $c_1c_3 \sqsubseteq c_2c_3$. The corresponding diagram is depicted on the left-hand side of Figure 6.

(iv)The Grothendieck topology J is given by the following table:

x	1	c_1	c_2	c_3	c_1c_3	c_2c_3	π
$J(x)$	$\{\{1\}\}$	$\{\{c_1\}\}$	$\{\{c_2\}\}$	$\{\{c_3\}\}$	$\{\{c_1c_3\}\}$	$\{\{c_2c_3\}\}$	$\{\{\pi\},\emptyset\}$

Second, we apply Definition 4.7 to the only atomic proposition p occurring in the branch $\mathcal{B}$, which leads to the GTI $[\![\mathrm{p}]\!]_{\mathcal{B}} = \{\pi, c_3\}$. This, in turn, finally gives rise to the GRM $\mathcal{G} = \langle \mathcal{M}(\mathcal{B}), \models, [\![-]\!]_{\mathcal{B}} \rangle$, the desired countermodel depicted on the right-hand side of Figure 6.

Now, we check that (i) $c_1 \models (\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot$ and (ii) $c_1 \not\models ((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot$. For (i), we have $c_3 \models \mathrm{p}$ because $c_3 \in [\![\mathrm{p}]\!]_{\mathcal{B}}$ and $c_2c_3 \not\models \bot$ because $\emptyset \notin J(c_2c_3)$. Thus, $c_2 \not\models \mathrm{p} \mathrel{-\!\!*} \bot$ and, since $c_1 \sqsubseteq c_2$ we obtain, by (K), $c_1 \not\models \mathrm{p} \mathrel{-\!\!*} \bot$. Therefore, we have $c_1 \models (\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot$.

For (ii), we notice that π is the only world that forces $\mathrm{p} * \mathrm{p}$. Thus, we have $c_2 \models (\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot$ only if $c_2 \bullet \pi \models \bot$, which is the case because $c_2 \bullet \pi = \pi$ and $\pi \models \bot$. Note that it would not be

the case in the elementary semantics for which no world can force $\bot$. On the other hand, $c_2 \not\models \bot$ because $\emptyset \notin J(c_2)$. Therefore, $c_1 \not\models ((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot$. Then the initial formula, although valid in the elementary semantics, is not provable in **BI**.

4.3. **Tableau Construction and Completeness**

In the previous section, we have explained how to build a model from an H-branch. To show the completeness theorem, we now need a tableau construction procedure which, given a formula ϕ, builds a tableaux sequence $t_1, t_2, \ldots$ until there exists a tableau t_i that is either closed or that contains a (possibly infinite) H-branch.

BI has such a procedure, with $\mathrm{F}\ \phi : 1$ as initial formula. Until $\mathcal{T}$ is closed or completed, choose an open branch $\mathcal{B}$; if there is an unfulfilled α or $\pi\alpha$ formula ($\mathrm{Sg}\ \phi : x$) in $\mathcal{B}$, then apply the related expansion rule; else if there is an unfulfilled β or $\pi\beta$ formula ($\mathrm{Sg}\ \phi : x$) in $\mathcal{B}$, then apply the corresponding expansion rule, with all labels for which the formula is not fulfilled.

We remark, in the case of $\mathrm{T}\ I : x$, that although there is no explicit expansion rule in **TBI**, the fulfilled condition requires the addition of the constraint $1 \leq x$ to the set of assertions $Ass(\mathcal{B})$.

Theorem 4.3 (completeness). If $I \models \phi$, then there is a closed tableau sequence for ϕ.

Proof. Suppose there is no closed tableau sequence for ϕ. Then, the above tableau construction procedure yields a tableau in which there is a completed branch $\mathcal{B}$. Since $\mathcal{B}$ contains the initial formula $\mathrm{F}\ \phi : 1$, Theorem 4.2 implies that we can build a Grothendieck resource model $\langle \mathcal{M}(\mathcal{B}), \models, [\![-]\!]_{\mathcal{B}} \rangle$ of $\mathcal{B}$, such that $1 \not\models \phi$, which means that ϕ is not valid in the Grothendieck resource semantics. □

5. BI's Semantics Revisited

As discussed in the introduction, the initial semantics of **BI**, based on pre-ordered commutative monoids, may be motivated by modelling units of resource as entities which may be zero, combined, and compared. In (Pym 2002; Pym et al. 2004; Pym 2004), and in the preceeding sections, it has been shown that a great deal of logical theory may be developed quite naturally and that this simple model of resource quite naturally encompasses a wide range of examples of resource, including ambients, Petri nets, memory allocation and deallocation, logic programming, and money (Ishtiaq and O'Hearn 2001). However, it may readily be seen that models based not on monoids with total combinations but rather on monoids with partial combination operations, $\bullet : M \times M \rightharpoonup M$ would not only more naturally encompass these examples but also may be motivated abstractly by a desire to capture the notion of *separation* (Reynolds 2000; Ishtiaq and O'Hearn 2001). The key idea here is that two units of resource may be combined only if they are disjoint, or separated, or non-interfering. An excellent example arises quite simply in the "pointer logic" model of **BI** given in (Reynolds 2000; Ishtiaq and O'Hearn 2001; O'Hearn et al. 2001), in which we may illustrate the composition by taking "resource" to mean "portion of computer memory". The pointer logic has, in addition to $*$, as a form of assertion, the "points-to" relation, $\mapsto$, which is used to make statements about the contents of heap cells. For example, $(x \mapsto 3, y) * (y \mapsto 4, x)$ says that x and y denote distinct binary cells in memory, where the

second part of x is a pointer to y, the second part of y is a pointer to x, and where the first parts contain 3 and 4.

In this context, an open question arises: is it possible to propose a semantics of **BI** that is based on monoids like for instance the pointer logic model of **BI**? In this section, we provide, in the intuitionistic setting, a positive answer through the definition of a new semantics for **BI**, based on partially defined pre-ordered commutative monoids (PDM semantics), that is intermediate between the elementary and the Grothendieck topological semantics. This semantics arises from our study of resource tableaux and their specific relationships with **BI**'s semantics.

5.1. **A New Relational Semantics for BI**

We first introduce a relational semantics of **BI**, based on specific ternary relations, such that the PDM semantics will be a particular case (or instantiation) of this relational semantics that we prove sound and complete for **BI**.

Definition 5.1 (BI frame). A **BI** *frame* is a structure $\mathcal{F} = \langle M, e, R, \sqsubseteq \rangle$, in which M is a set of resources with two distinguished elements, e and π, and R is a ternary relation on $M \times M \times M$ that satisfies the following conditions:

- $\forall x.\ Rexx$ (reflexivity),
- $\forall x \forall y \forall z.\ (Rxyz \rightarrow Ryxz)$ (commutativity),
- $\forall x \forall y \forall z \forall t.\ \exists u(Rxyu \wedge Ruzv) \leftrightarrow \exists t(Ryzt \wedge Rxtv)$ (associativity),
- $\forall x \forall y \forall z \forall x'.\ (Rxyz \wedge x \sqsubseteq x') \rightarrow Rx'yz$ (compatibility),
- $\forall x \forall y \forall z \forall z'.\ (Rxyz \wedge z \sqsubseteq z') \rightarrow Rxyz'$ (transitivity),
- $\forall x \forall y. Rxy\pi$ (π-max),
- $\forall x \forall y. (R\pi xy \rightarrow\ y \sqsubseteq \pi)$ (π-abs),

where $x \sqsubseteq y$ is defined as $x \sqsubseteq y \equiv Rexy$.

We observe that e is neutral for the ternary relation R. Moreover the π-max condition entails that $\forall x. x \sqsubseteq \pi$, *i.e.*, π is the greatest element for the preorder induced by the R relation.

Definition 5.2 (relational interpretation). Let M be is a set of resources with a greatest element π (w.r.t. a preorder $\sqsubseteq$) and $\mathcal{P}(L)$ be the collection of **BI** propositions over a language L of propositional letters, then an *relational interpretation* (*RI*) is a function $[\![-]\!] : L \rightarrow \mathcal{P}(M)$ that satisfies:

(K) for any $m, n \in M$ such that $m \sqsubseteq n$, $m \in [\![\mathrm{p}]\!]$ implies $n \in [\![\mathrm{p}]\!]$;
(B) for any m such that $\pi \sqsubseteq m$, we have $m \in [\![\mathrm{p}]\!]$.

Definition 5.3 (relational model). Let $\mathcal{P}(L)$ be the collection of **BI** propositions over a language L of propositional letters, a *Relational Model* (*RM*) is a structure $\mathcal{R} = \langle M, e, R, \sqsubseteq, [\![-]\!], \models \rangle$, in which $\langle M, e, R, \sqsubseteq \rangle$ is a *BI frame*, $[\![-]\!]$ is a relational interpretation, and $\models$ is a forcing relation on $M \times \mathcal{P}(L)$, satisfying the following conditions:

- $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
- $m \models \top$ iff always
- $m \models \bot$ iff $m = \pi$
- $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$

- $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
- $m \models \phi \rightarrow \psi$ iff, for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
- $m \models I$ iff $e \sqsubseteq m$
- $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $Rnn'm$, $n \models \phi$ and $n' \models \psi$
- $m \models \phi \mathrel{-\!\!*} \psi$ iff, for all $n, n' \in M$, $n \models \phi$ and $Rmnn'$ entails $n' \models \psi$.

Given a relational model $\mathcal{R}$, the validity is defined as follows:

$$\phi \models_{\mathcal{R}} \psi \text{ iff } \mathcal{R}, e \models \phi \mathrel{-\!\!*} \psi \text{ and } \phi \models \psi \text{ iff } \forall \mathcal{R}, \phi \models_{\mathcal{M}} \psi.$$

Theorem 5.1 (soundness). BI is sound with respect to the relational semantics.

Proof. We show, by case analysis, that every **LBI**-rule preserves validity.

- $(*_R)$ We assume $\Gamma \models \phi$ and $\Delta \models \psi$ and then show that $\Gamma, \Delta \models \phi * \psi$. Let $\mathcal{R}$ be a relational model and m be a world in $\mathcal{R}$ such that $m \models \Phi_{\Gamma,\Delta}$, we have to show that $m \models \phi * \psi$. Since $\Phi_{\Gamma,\Delta} = \Phi_\Gamma * \Phi_\Delta$, we have $m \models \Phi_\Gamma * \Phi_\Delta$. Therefore, there exist n and n' in $\mathcal{R}$ such that $Rnn'm$, $n \models \Phi_\Gamma$ and $n' \models \Phi_\Delta$. From $\Gamma \models \phi$ and $n \models \Phi_\Gamma$, we deduce $n \models \phi$. Similarly, $\Delta \models \psi$ and $n' \models \Phi_\Delta$ imply $n' \models \psi$. Consequently, we get $Rnn'm$, $n \models \phi$ and $n' \models \psi$. Hence, $m \models \phi * \psi$.
- $(*_L)$ Immediate since $\Phi_{\Gamma(\phi,\psi)} = \Phi_{\Gamma(\phi * \psi)}$.
- $(\mathrel{-\!\!*}_R)$ We assume $\Gamma, \phi \models \psi$ and then show that $\Gamma \models \phi \mathrel{-\!\!*} \psi$. Let $\mathcal{R}$ be a relational model and m be a world in $\mathcal{R}$ such that $m \models \Phi_\Gamma$, we have to show that $m \models \phi \mathrel{-\!\!*} \psi$. Suppose n and n' are worlds in $\mathcal{R}$ such that $Rmnn'$ and $n \models \phi$, then $n' \models \Phi_\Gamma * \phi$. Since $\Phi_\Gamma * \phi = \Phi_{\Gamma,\phi}$, we have $n' \models \Phi_{\Gamma,\phi}$ which, using the assumption $\Gamma, \phi \models \psi$, entails that $n' \models \psi$. Therefore, $m \models \phi \mathrel{-\!\!*} \psi$.
- $(\mathrel{-\!\!*}_L)$ We show that $\Delta \models \phi$ and $\Gamma(\psi, \Delta') \models C$ imply $\Gamma(\Delta, \phi \mathrel{-\!\!*} \psi, \Delta') \models C$ by induction on the structure of Γ.
  - Base case. Assuming $\Delta \models \phi$ and $\psi, \Delta' \models C$, we have to show that $\Delta, \phi \mathrel{-\!\!*} \psi, \Delta' \models C$. Let $\mathcal{R}$ be a relational model and m a world in $\mathcal{R}$ such that $m \models \Phi_{\Delta,\phi \mathrel{-\!\!*} \psi,\Delta'}$. Since $\Phi_{\Delta,\phi \mathrel{-\!\!*} \psi,\Delta'} = \Phi_\Delta * (\phi \mathrel{-\!\!*} \psi) * \Phi_{\Delta'}$, there exist two worlds n and n' in $\mathcal{R}$ such that $Rnn'm$, $n \models \Phi_\Delta$ and $n' \models (\phi \mathrel{-\!\!*} \psi) * \Phi_{\Delta'}$. Similarly, there are two worlds k and k' in $\mathcal{R}$ such that $Rkk'n'$, $k \models \phi \mathrel{-\!\!*} \psi$ and $k' \models \Phi_{\Delta'}$. Since we have a world n' such that $Rkk'n'$ and $Rnn'm$, by the associativity axiom (of Definition 5.1) for R, there exists a world t such that $Rnkt$ and $Rtk'm$. From $n \models \Phi_\Delta$ and $\Delta \models \phi$, we get $n \models \phi$. From $Rnkt$, $n \models \phi$ and $k \models \phi \mathrel{-\!\!*} \psi$, we get $t \models \psi$. Since we now have $Rtk'm$, $t \models \psi$ and $k' \models \Phi_{\Delta'}$, we can deduce $m \models \psi * \Phi_{\Delta'}$ which, using the assumption that $\psi, \Delta' \models C$, entails that $m \models C$.
  - Case $\Gamma = \Gamma'(\psi, \Delta'), \Gamma''$.
    Assuming $\Delta \models \phi$ and $\Gamma'(\psi, \Delta'), \Gamma'' \models C$, we get $\Gamma'(\psi, \Delta') \models \Phi_{\Gamma''} \mathrel{-\!\!*} C$. Then, by induction hypothesis, we obtain $\Gamma'(\Delta, \phi \mathrel{-\!\!*} \psi, \Delta') \models$ get $\Gamma'(\Delta, \phi \mathrel{-\!\!*} \psi, \Delta'), \Gamma'' \models C$.
  - Case $\Gamma = \Gamma'(\psi, \Delta'); \Gamma''$. Similar to the previous one.
- The other cases are similar.

□

We now consider the question of completeness. We aim to build a term model such that if $\Gamma \not\vdash \phi$ then there exists a world m in the model such that $m \models \Gamma$ and $m \not\models \phi$. For that, we consider the idea of *prime theory* in order to have the structure required by the semantic clauses for the connectives, in particular for $\vee$ and $*$ (Pym 2002; Pym 2004).

Here, we need some simple definitions. A bunch Γ is said to be *prime* if it verifies that $\Gamma \vdash \phi \vee \psi$ if $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$. A *prime extension* of Γ is a bunch Γ' such that Γ' is prime and $\Gamma' \models \Phi_\Gamma$. It can also be defined via extension, as $\Gamma' \equiv \Gamma; \Delta$, in the evident way. Moreover we extend the definition of $*$ on bunches in the following way: $\Delta * \Theta$ is defined as $\Phi_\Delta * \Phi_\Theta$. Similarly for $\Gamma \vdash \Delta$.

Definition 5.4 (term model). The term model is defined as $\mathcal{T} = \langle P, \emptyset_m, R, \sqsubseteq, [\![-]\!], \models \rangle$ in which

- P is the set $B/\dashv\vdash$ where B is the set of all the prime bunches and $\dashv\vdash$ is the equality generated by derivability,
- $\emptyset_m$ is the multiplicative unit of bunches (trivially prime),
- $R\Gamma\Delta\Theta$ is defined as $R\Gamma\Delta\Theta$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$,
- $\sqsubseteq$ is defined by $\Gamma \sqsubseteq \Delta$ iff $R\emptyset_m\Gamma\Delta$, [§]
- $[\![p]\!] = \{\Gamma \in P \mid \Gamma \vdash p\}$,
- $\models$ is defined by $\Gamma \models \phi$ iff $\Gamma \vdash \phi$ for any $\Gamma \in P$.

Let us first mention two results about the relation R of the term model $\mathcal{T}$.

Lemma 5.1. If we consider R the relation of the term model $\mathcal{T}$, $R\Gamma\Delta\Theta$ if and only if $\Theta \vdash \Gamma * \Delta$.

Proof. If $R\Gamma\Delta\Theta$ then $\forall\phi, \psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$. In particular, as $\Gamma \vdash \Gamma$ and $\Delta \vdash \Delta$ we deduce $\Theta \vdash \Gamma * \Delta$. If $\Theta \vdash \Gamma * \Delta$, then let us suppose $\Gamma \vdash \phi$ and $\Delta \vdash \psi$. By compatibility of $*$, we get $\Gamma * \Delta \vdash \phi * \psi$. Since $\Theta \vdash \Gamma * \Delta$, we finally have $\Theta \vdash \phi * \psi$. □

Corollary 5.1. If we consider $\sqsubseteq$, the preorder of the term model $\mathcal{T}$, $\Gamma \sqsubseteq \Delta$ if and only if $\Delta \vdash \Gamma$.

Proof. We have $\Gamma \sqsubseteq \Delta$ iff $R\emptyset_m\Gamma\Delta$ iff $\Delta \vdash \emptyset_m * \Gamma$ (by Lemma 5.1) iff $\Delta \vdash \Gamma$. □

Moreover, we can easily deduce that $\bot$ and $\top$ are respectively the greatest and the least elements of the term model w.r.t. the preorder, $\sqsubseteq$.

Having defined the term model, the next step consists in verifying that the relation R of Definition 5.4 satisfies the conditions of Definition 5.1 and that the forcing relation $\models$ satisfies the conditions of Definition 5.3. The proofs of these results rely on two fundamental following lemmas:

Lemma 5.2 (extension lemma). If $\Gamma \not\vdash \phi$ then there exists a prime extension Γ' of Γ such that $\Gamma' \not\vdash \phi$.

[§] In the corresponding constructions in (Pym 2002), this condition is mis-stated: it is corrected in (Pym 2004) and the corrected statement is as in Corollary 5.1, below. This error was known prior to the publication of (Pym 2002) but persisted because of an editing error by the author. Similarly, (Pym et al. 2004) requires the following: Erratum: p. 285, l. -12: ", for some $P', Q \equiv P; P'$ " should be "$P \vdash Q$".

Proof. Similar to the corresponding proof in (Pym 2002; Pym 2004). □

Lemma 5.3 (primeness lemma). If Γ is prime and $\Gamma \vdash \Delta * \Theta$ then there exists a prime extension Δ' of Δ such that $\Gamma \vdash \Delta' * \Theta$

Proof. Similar to the corresponding proofs in (Dunn 1986) or (Routley and Meyer 1972). □

Now we can show the completeness via the next two lemmas.

Lemma 5.4. In the term model, $\mathcal{T} = \langle P, \emptyset_m, R, \sqsubseteq, [\![-]\!], \models \rangle$, $\langle P, \emptyset_m, R, \sqsubseteq \rangle$ is a **BI** frame.

Proof. R is defined as $R\Gamma\Delta\Theta$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$. Now, let us verify each condition.

- Reflexivity: $\forall\phi, \forall\psi, \emptyset_m \vdash \phi$ and $\Delta \vdash \psi$ entails $\Delta \vdash \phi * \psi$. Then, we have $R\emptyset_m\Delta\Delta$.
- Commutativity: Trivial.
- Associativity: We show that $\exists\Theta(R\Gamma\Delta\Theta$ and $R\Theta\Psi\Sigma)$ iff $\exists\Phi(R\Delta\Psi\Phi$ and $R\Gamma\Phi\Sigma)$. We prove here the implication from left to right, the other being analogous. By hypothesis we have (i) $R\Gamma\Delta\Theta$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$ and (ii) $R\Delta\Psi\Sigma$ iff $\forall\phi, \forall\psi, \Delta \vdash \phi$ and $\Psi \vdash \psi$ entails $\Sigma \vdash \phi * \psi$. We consider $\Phi \equiv (\Delta * \Psi)'$ that is a prime bunch built by two applications of Lemma 5.3.
  - We have $R\Delta\Psi(\Delta * \Psi)'$ that is trivial.
  - We first show that we have $\Sigma \vdash (\Gamma * \Delta) * \Psi$. As $\Theta \vdash \Theta$ and $\Psi \vdash \Psi$, by (ii) we can deduce that $\Sigma \vdash \Theta * \Psi$. Moreover, as $\Gamma \vdash \Gamma$ and $\Delta \vdash \Delta$, by (i) we can deduce that $\Theta \vdash \Gamma * \Delta$ and therefore $\Theta * \Psi \vdash (\Gamma * \Delta) * \Psi$. Finally, we have $\Sigma \vdash (\Gamma * \Delta) * \Psi$. Then, we have $\Sigma \vdash (\Gamma * \Delta) * \Psi$ iff $\Sigma \vdash \Gamma * (\Delta * \Psi)$ iff $R\Gamma(\Delta * \Psi)\Sigma$.
- Transitivity: We show that $R\Gamma\Delta\Theta$ and $\Gamma' \sqsubseteq \Gamma$ entails $R\Gamma'\Delta\Theta'$. By hypothesis, we have (i) $R\Gamma\Delta\Theta$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$ and (ii) $\Gamma' \subseteq \Gamma$ iff $R\emptyset_m\Gamma'\Gamma'$ iff $\forall\phi, \forall\psi, \emptyset_m \vdash \phi$ and $\Gamma' \vdash \psi$ entails $\Gamma \vdash \phi * \psi$. If $\Gamma' \vdash \phi$ then since $\emptyset_m \vdash I$, by (ii) we have $\Gamma \vdash I * \phi$ and then $\Gamma \vdash \phi$. Moreover, if $\Delta \vdash \psi$ then, by (i) we get $\Theta \vdash \phi * \psi$ and therefore $R\Gamma'\Delta\Theta'$.
- (π-max) We have to show that $R\Gamma\Delta\bot$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\bot \vdash \phi * \psi$, that is trivial because it is an axiom.
- The other cases are similar.

□

Lemma 5.5. The term model $\mathcal{T} = \langle P, \emptyset_m, R, \sqsubseteq, [\![-]\!], \models \rangle$ is a relational model.

Proof. By induction on formulae ϕ.

- (monotonicity) Suppose that $\Gamma \models \phi$ and $\Gamma \sqsubseteq \Delta$ and prove that $\Delta \models \phi$. By definition we have $\Gamma \vdash \phi$. As $\Gamma \sqsubseteq \Delta$ iff $\Delta \vdash \Gamma$, by Corollary 5.1, we deduce $\Delta \vdash \phi$ by transitivity of $\vdash$.
- ($\wedge$) $\Gamma \models \phi \wedge \psi$ iff $\Gamma \vdash \phi \wedge \psi$ iff $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$.
- ($\vee$) $\Gamma \models \phi \vee \psi$ iff $\Gamma \vdash \phi \vee \psi$ iff $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$ (because Γ is prime).
- ($\rightarrow$) $\Gamma \models \phi \rightarrow \psi$ iff, for all Δ such that $\Gamma \sqsubseteq \Delta$, if $\Delta \models \phi$ then $\Delta \models \psi$. Suppose that $\Gamma \not\models \phi \rightarrow \psi$. Then $\Gamma; \phi \not\vdash \psi$ and then there exists, by Lemma 5.2, $(\Gamma; \phi)'$ such that $(\Gamma; \phi)' \not\vdash \psi$. Moreover, we have $(\Gamma; \phi)' \vdash (\Gamma; \phi)$ and thus $\Gamma \sqsubseteq (\Gamma; \phi)'$ and $(\Gamma; \phi)' \vdash \phi$. therefore, there exists $\Delta \equiv (\Gamma; \phi)$ such that $\Gamma \sqsubseteq \Delta$, $\Delta \models \phi$ and $\Delta \not\models \psi$. Suppose there exists Δ such that

$\Delta \models \phi$, $\Delta \not\models \psi$ and $\Gamma \sqsubseteq \Delta$. Thus, we have $\Delta \vdash \phi$ and $\Delta \nvdash \psi$ and also $\Delta \vdash \Gamma$ by Corollary 5.1. Then, we have $\Gamma \nvdash \phi \to \psi$ and then $\Gamma \not\models \phi \to \psi$.

- $(*)$ $\Gamma \models \phi * \psi$ iff $\exists \Delta, \Psi / R\Delta\Psi\Gamma$ and $\Delta \models \phi$ and $\Psi \models \psi$. Suppose that $\Gamma \models \phi * \psi$, then we have $\Gamma \vdash \phi * \psi$ and thus there exist α, β such that $\alpha \vdash \phi$ and $\beta \vdash \psi$ and $\Gamma \vdash \alpha * \beta$. We remark that α and β are not necessarily prime. By Lemma 5.3, applied twice, we can extend α (resp. β) into a prime bunch Δ (resp. Ψ) such that $\Gamma \vdash \Delta * \Psi$. Therefore, by Lemma 5.1, we have $R\Delta\Psi\Gamma$ and $\Delta \vdash \phi$ (resp. $\Psi \vdash \psi$) implies $\Delta \models \phi$ (resp. $\Psi \models \psi$). Suppose that $\exists \Delta, \Psi / R\Delta\Psi\Gamma$ and $\Delta \models \phi$ and $\Psi \models \psi$. Then, by Lemma 5.1, we have $\Gamma \vdash \Delta * \Psi$ and $\Delta \models \phi$ (resp. $\Psi \models \psi$) implies $\Delta \vdash \phi$ (resp. $\Psi \vdash \psi$). Then, we have $\Gamma \vdash \phi * \psi$ and thus $\Gamma \models \phi * \psi$.
- $(-\!\!*)$ $\Gamma \models \phi -\!\!* \psi$ iff $\forall \Delta, \Psi\ \Delta \models \phi$ and $R\Gamma\Delta\Psi$ entails $\Psi \models \psi$. Suppose that $\exists \Delta, \Psi / R\Gamma\Delta\Psi$ and $\Delta \models \phi$ entails $\Psi \models \psi$. $\Delta \models \phi$ entails $\Delta \vdash \phi$ and thus $\Gamma * \Delta \vdash \Gamma * \phi$. Moreover $R\Gamma\Delta\Psi$ entails $\Psi \vdash \Gamma * \Delta$ by Lemma 5.1. By transitivity of $\vdash$ we obtain $\Psi \vdash \Gamma * \phi$; if $\Gamma \vdash \phi -\!\!* \psi$ then $\Psi \vdash (\phi -\!\!* \psi) * \phi$. We can deduce $\Psi \vdash \psi$ that is a contradiction. Consequently, we have $\Gamma \nvdash \phi -\!\!* \psi$ which entails $\Gamma \not\models \phi -\!\!* \psi$. Suppose that $\Gamma \not\models \phi -\!\!* \psi$. Then $\Gamma \nvdash \phi -\!\!* \psi$ and thus $\Gamma, \phi \nvdash \psi$ that is equivalent to $\phi \nvdash \Gamma -\!\!* \psi$. As $\phi \nvdash \Gamma -\!\!* \psi$, by Lemma 5.2, there exists ϕ' such that $\phi' \nvdash \Gamma -\!\!* \psi$ that is equivalent to $\Gamma, \phi' \nvdash \psi$. If we consider $\Delta \equiv \phi'$ and $\Psi \equiv (\Gamma, \phi')'$ we get $\Gamma \vdash \phi$ and thus $\Delta \models \phi$. Moreover we have $R\Gamma\Delta\Psi$ and $\Psi \nvdash \psi$ entails $\Psi \not\models \psi$.
- $(\top)$ immediate since $\Gamma \vdash \top$ is an axiom.
- $(\bot)$ $\Gamma \models \bot$ iff $\Gamma = \bot$. $\Gamma \models \bot$ iff $\Gamma \vdash \bot$ iff $\Gamma \dashv\vdash \bot$ since $\bot \vdash \Gamma$ is an axiom. Then, we have $\Gamma \sqsubseteq \bot$ and $\bot \sqsubseteq \Gamma$.
- (I) $\Gamma \models I$ iff $I \sqsubseteq \Gamma$. $\Gamma \models I$ iff $\Gamma \vdash I$ iff $I \sqsubseteq \Gamma$ by Corollary 5.1.

□

Theorem 5.2 (completeness). **BI** is complete with respect to the relational semantics.

Proof. From Definition 5.4 (of the term model) and by Lemma 5.4 and Lemma 5.5. □

We have defined a new semantics for **BI** but the key points of this proposal are that we can relate it with a new semantics based-on partial monoids and also prove the soundness of the **TBI** calculus with respect to such a semantics (not achieved directly for the Grothendieck topological semantics).

5.2. **A Partially Defined Semantics for BI**

In § 4, we have studied how countermodels can be built from dependency graphs. We now observe that those models are very closely related to the ones recently proposed in the semantics of (intuitionistic) "pointer logic" (Ishtiaq and O'Hearn 2001; Pym et al. 2004; Pym 2004). Indeed, the Grothendieck topology used to characterize the pointer logic model exactly corresponds to our definition of the J-map for basic GRMs. Moreover, in our models, a special element called π is used to capture undefinedness as the image of all undefined compositions and is the only element to force $\bot$ (because $\emptyset$ only belongs to $J(\pi)$).

We now define what we call Kripke resource models and show that they correspond to a particular class of relational resource models, taking the relation $R_{\sqsubseteq}$ defined as

$$R_{\sqsubseteq} xyz \equiv x \bullet y \sqsubseteq z.$$

We now reconstruct the definitions of Kripke resource monoids, interpretations and Kripke resource models in the partially defined setting. Since no confusion is likely, we reuse their names.

Definition 5.5. A *Kripke resource monoid* (KRM) is a preordered commutative monoid $\mathcal{M} = \langle M, \bullet, e, \sqsubseteq \rangle$ which contains a greatest element, denoted π, such that $\pi \bullet m = \pi$ for any $m \in M$, and in which $\bullet$ is bifunctorial with respect to $\sqsubseteq$.

We consider such a collection of resources, a preordered commutative monoid, from the relational semantics perspective. In fact, we consider the relation $R_{\sqsubseteq}$ that naturally verifies the first three conditions of Definition 5.1. The conditions on π correspond to the satisfaction, for $R_{\sqsubseteq}$, of the (π-max) and (π-abs) conditions. The fact that $\bullet$ is bifunctorial with respect to $\sqsubseteq$, is captured by the $(compatibility)$ and $(transitivity)$ conditions for $R_{\sqsubseteq}$. It implies that this set of resources corresponds to a **BI** frame.

Definition 5.6. Let $\mathcal{M}$ be a *KRM* and $\mathcal{P}(L)$ be a language of **BI** propositions over a language L of propositional letters. Then, a *Kripke resource interpretation*, or KRI, is a function $[\![-]\!] : L \to \mathcal{P}(M)$ satisfying Kripke monotonicity (K) and such that for any p $\in L$, $\pi \in [\![\mathrm{p}]\!]$.

Again, such an interpretation can be seen, in a relational semantics perspective, as a relational interpretation (see Definition 5.2).

Definition 5.7. A *Kripke resource model* is a triple $\mathcal{K} = \langle \mathcal{M}, \models, [\![-]\!] \rangle$ in which $\mathcal{M}$ is a *KRM*, $[\![-]\!]$ is a KRI and $\models$ is a forcing relation on $M \times \mathcal{P}(L)$ satisfying the following conditions:

- $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
- $m \models \top$ iff always
- $m \models \bot$ iff $m = \pi$
- $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
- $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
- $m \models \phi \to \psi$ iff, for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
- $m \models I$ iff $e \sqsubseteq m$
- $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$, $n \models \phi$ and $n' \models \psi$
- $m \models \phi \mathrel{-\!\!*} \psi$ iff, for all $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

We observe that the above definition corresponds to a particular relational model (see Definition 5.3) in which we consider the relation $R_{\sqsubseteq}xyz \equiv x \bullet y \sqsubseteq z$. Therefore, it is clear that the class of Kripke resource models is included in the class of relational resource models.

Theorem 5.3 (soundness). **BI** is sound with respect to Kripke resource models.

Proof. Obvious from the proof of soundness with respect to the relational semantics (see Theorem 5.1) since relational models include Kripke resource models. □

We now return to the question of completeness. We observe that it is difficult to obtain a direct proof that **BI** is complete for Kripke resource models. The proof in (Pym 2002; Pym 2004) requires a delicate construction of sets of choices of "evaluated prime bunches" in order to ensure a consistent definition and, in the presence of $\bot$, the topological nature of the models

considered therein is, as we have seen, essential. In particular, it is necessary to have a world which forces $\bot$.

The idea with the partially defined semantics, based on monoids with the element π, is to have an elementary semantics, not requiring the topological structure. It seems that such a requirement is not compatible with obtaining a completeness theorem unless we move to the relational construction based on **BI** frames. Another advantage of this move is that we are able to work with our simpler notion of prime bunch, greatly reducing the technical and conceptual complexity of the completeness argument.

First, we relate a Kripke resource model with a basic GRM of Definition 4.3.

Lemma 5.6. The class of Kripke resource models coincides with the class of basic Grothendieck resource models.

Proof. Let $\mathcal{G}{=}\langle\, (M, \bullet, e, \sqsubseteq, J), \models_{\mathcal{G}}, [\![-]\!] \,\rangle$ be a basic GRM. Let us establish that it is a Kripke model. Since $\mathcal{G}$ is basic, we simply show that $\models_{\mathcal{G}}$ satisfies the conditions of Definition 2.5. In the case of $\bot$, since $\emptyset$ only belongs to $J(\pi)$, the condition $\emptyset \in J(m)$ is equivalent to $m = \pi$. Now, for any world $m \neq \pi$, we have $J(m) = \{\{m\}\}$. Thus, in the case of I, the condition $(\exists S \in J(m))$ $(\forall m' \in S)$ $(e \sqsubseteq m')$ simplifies to $(\forall m' \in \{m\})$ $(e \sqsubseteq m')$, which is equivalent to $e \sqsubseteq m$. The cases of $\vee$ and $*$ are similar. Conversely, endowing a Kripke model $\langle\, (M, \bullet, e, \sqsubseteq), \models_{\mathcal{K}}, [\![-]\!] \,\rangle$ with the basic topology turns it into a basic Grothendieck model We can easily show that, for such a J, that Kripke monotonicity for $[\![-]\!]$ implies (Sh). □

Now we return to the **TBI** calculus and its relationships with the Kripke resource semantics.

Theorem 5.4. **TBI** is sound w.r.t. Kripke resource models, *i.e.*,if there exists a closed tableaux sequence for a **BI** formula ϕ then ϕ is valid in Kripke resource models.

Proof. **TBI** is sound w.r.t. the basic GRMs. From Lemma 5.6 we deduce that **TBI** is sound w.r.t. Kripke resource models. □

Theorem 5.5. **TBI** is complete w.r.t. Kripke resource semantics, *i.e.*, if there is no closed tableau sequence for a **BI** formula ϕ then ϕ is not valid in the Kripke resource semantics.

Proof. **TBI** is complete w.r.t. the basic GRMs. From Lemma 5.6 we deduce that **TBI** is complete w.r.t. Kripke resource models. □

Finally, we are able to give a completeness theorem for **BI** w.r.t. the partially defined semantics, thereby establishing completeness for a class of models which includes pointer logic, and so demonstrating the strength of our resource semantics.

Theorem 5.6 (completeness). **BI** is complete with respect to Kripke resource models.

Proof. Suppose that $I \not\vdash \phi$ then, by Theorem 4.3, there exists a tableau containing a H-branch from which one can build, as explained in Definition 4.6, a basic *GRM* which is a countermodel of ϕ. From Lemma 5.6 it is also a Kripke resource model which is a countermodel of ϕ. Moreover, the Kripke resource semantics is a particular case of the relational semantics which has been proved complete (Theorem 5.2). □

Consequently, **BI** is proved sound and complete for this new semantics derived from the new relational semantics of **BI**. We now analyze more deeply this semantics in which the special element π is used to internalize undefinedness implicitly.

An alternative (and equivalent) way of dealing with $\bot$ is to handle undefinedness explicitly through a partially defined monoid (PDM), *i.e.*, a monoid in which the product $\bullet$ is a partial operation with no other requirement than $(x \bullet y) \bullet z$ being defined whenever $x \bullet (y \bullet z)$ is defined, and vice versa. This gives rise to the PDM semantics, obtained from the Kripke resource semantics by making the following changes to the forcing relation:

- $m \models \bot$ iff never
- $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \downarrow$, $n \bullet n' \sqsubseteq m$, $n \models \phi$ and $n' \models \psi$
- $m \models \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $n \models \phi$, $m \bullet n \downarrow$ implies $m \bullet n \models \psi$

where $\downarrow$ denotes definedness.

The PDM semantics is easily seen to be equivalent to the previous Kripke resource semantics. For example, in the case of $\bot$, given that, on one hand, π is the only element to force $\bot$ and, on the other hand, that π means undefinedness, then, no defined world should force $\bot$, *i.e.*, $\bot$ is nowhere forced. Similarly for $*$ and $\mathrel{-\!\!*}$. An immediate consequence of moving to the PDM semantics is that dependency graphs can straightforwardly be considered as countermodels in this semantics.

We conclude this section by showing the soundness of **TBI** that we were not in a position to prove in the previous section.

Theorem 5.7 (soundness of TBI). **TBI** is sound w.r.t. **BI**, *i.e.*, if there exists a closed tableaux sequence for a **BI** formula ϕ then ϕ is valid in **BI**.

Proof. **TBI** is sound w.r.t. Kripke resource models by Theorem 5.3. Moreover, the Kripke resource semantics is sound and complete w.r.t. BI by Theorem 5.4 and Theorem 5.5. Then we conclude about the soundness of **TBI**. □

This soundness theorem is the counterpart to the completeness theorem, Theorem 4.3.

5.3. **Partiality and Variations on BI**

A question naturally arises from these results: can we define such new semantics for some variants of **BI**, like for instance Affine **BI** or Boolean **BI**. In Affine **BI**, the comma, or $*$, admits weakening. this variant has been used as the basis for the program logics "pointer logic", introduced by Ishtiaq and O'Hearn (Ishtiaq and O'Hearn 2001) and "separation logic" (Reynolds 2000). It has also been used as basis of the type systems used to provide unified accounts (O'Hearn 1999; O'Hearn and Pym 1999; Pym 2002; Pym 2004) of Reynolds' Syntactic Control of Interference and Idealized Algol languages. In fact, both pointer logic and separation logic have both intuitionistic and classical versions, *i.e.*, the additives may be intuitionistic or classical. The system with classical additives is known as Boolean **BI**.

In this section, we focus on affine **BI**, proving soundness and completeness with respect to the resource semantics given by PDM models. We can derive a similar result directly for Boolean **BI** without the unit I but not for Boolean **BI**, seemingly because of particular interactions between

this unit and the additive conjunction.¶ A deeper analysis of this problem is out of the scope of this paper and will be provided in further work. We conclude this section with a brief discussion of how Reynolds' separation logics (Reynolds 2000), which may usefully be understood as specific models of **BI**, fit into our semantic framework.

5.3.1. **Affine BI**

An interesting variant of standard **BI** is the affine variant in which the multiplicative conjunction is supposed to satisfy the additional structural rule of weakening, *i.e.*, $\phi * \psi \vdash \psi$ for any ϕ and ψ. Compared to Definition 2.3, Kripke resource monoids of affine **BI** are characterized by monoidal products which satisfy the (weakening) condition that for any worlds m and n, $m \sqsubseteq m \bullet n$. Intuitively, the weakening condition demands that the composition of two resources should result in something bigger (w.r.t. the preordering) than the two components. Such a condition is met by many (if not most) natural resource compositions.

An immediate consequence is that e becomes the least element, and so, since any world m is now greater than e, the condition $m \models I$ iff $e \sqsubseteq m$ simplifies to $m \models I$ iff always, implying that $I = \top$. Conversely, if e is the least element, then, the functoriality of $\bullet$ implies the weakening condition. Therefore, affine Kripke resource monoids can equivalently be viewed as Kripke resource monoids having e as their least element. Another interesting consequence of the weakening for $\bullet$ is that the condition $n \bullet n' \sqsubseteq m$ in the forcing clause for $*$ can be simplified to an equality, namely, $n \bullet n' = m$, even in the presence of $\sqsubseteq$. Therefore, the modifications to Definition 2.5 in the case of affine **BI** are the following:

Definition 5.8. An *affine Kripke resource model* is a triple $\mathcal{K} = \langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$ in which $\mathcal{M}$ is a *KRM*, $[\![-]\!]$ is a KRI and $\models$ is a forcing relation on $M \times \mathcal{P}(L)$ satisfying:

- $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
- $m \models \top$ iff always
- $m \models \bot$ iff $m = \pi$
- $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
- $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
- $m \models \phi \to \psi$ iff, for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
- $m \models I$ iff always
- $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' = m$, $n \models \phi$ and $n' \models \psi$
- $m \models \phi \mathrel{-\!\!*} \psi$ iff, for all $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

Once again, using the arguments in section § 5.2 an affine PDM semantics can directly be derived from affine Kripke resource models, preserving all the results (soundness, completeness, finite model property):

- $m \models \bot$ iff never
- $m \models I$ iff always
- $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \downarrow$, $n \bullet n' = m$,$n \models \phi$ and $n' \models \psi$
- $m \models \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $n \models \phi$, $m \bullet n \downarrow$ implies $m \bullet n \models \psi$

¶ We are grateful to Hongseok Yang for his observations on this point.

With respect to **TBI**, handling the affine variant does not require any modification of the expansion rules as in the Boolean case, but the weakening condition has to be taken into account in the labelling algebra explicitly.

Theorem 5.8 (Soundness). Affine **BI** is sound w.r.t. affine Kripke resource models.

Proof. Similar to the proof of Theorem 5.3. □

Theorem 5.9 (Completeness). Affine **BI** is complete w.r.t. affine Kripke resource models.

Proof. Similar to the proof of Theorem 5.6. □

5.3.2. **Separation Logics**

In separation logic (Reynolds 2000), possible worlds are *stacks* and *heaps* arranged as a partially defined monoid, together with a forcing relation $s, h \models \phi$ meaning that the proposition ϕ is true of stack s and heap h. Recall that *heaps* are finite partial functions $Loc \rightharpoonup Val^+$ from a set of memory locations to non-empty sequences of values and that *stacks* are finite partial functions $Var \rightharpoonup Val$ from a set of variables to the set of values. The composition of two heaps $h_1.h_2$ is then defined as the union of functions h_1 and h_2 if h_1 and h_2 have disjoint domains and left undefined otherwise. Such a composition gives rise to a partially defined monoid of heaps on which assertions of the form $s, h \models \phi$ are interpreted.

Separation logic comes in two versions: one with intuitionistic additives and one with classical additives. The intuitionistic version presented in (Ishtiaq and O'Hearn 2001) is reminiscent of our affine PDM semantics, thereby emphasizing its utility in our resource semantics. The classical version, also presented in (Ishtiaq and O'Hearn 2001), requires forcing conditions that are related to our work on PDM semantics. Further work will be devoted to the design of resource tableaux for both versions of separation logic.

6. BI's Semantics and Liberalized Rules

In this section, we show how, by a special treatment of the additive disjunction, we can give liberalized rules for **TBI**, which can be seen as an improvement of the initial version of the calculus.

Liberalized rules have been proposed to improve free variable tableau methods dedicated to classical logic, for instance by giving new so-called δ rule (Hähnle and Schmitt 1994). In previous work (Galmiche and Méry 2003), we have given, starting from our calculus and its restrictions to **IL** and **MILL**, liberalized rules which improve the efficiency of the proof-search (by reducing the number of new constants to deal with) and which also provide easy arguments about termination.

As an illustration, let us consider for **MILL** the following liberalized rules,

$$\dfrac{\mathrm{F}\ \phi \multimap \psi : x}{\begin{array}{c}\mathrm{T}\ \phi : a \\ \mathrm{F}\ \psi : xa\end{array}} \qquad \dfrac{\mathrm{T}\ \phi * \psi : x\ {}^{1}}{\begin{array}{c}\boxed{ass : ab \leq x} \\ \mathrm{T}\ \phi : a \\ \mathrm{T}\ \psi : b\end{array}}$$

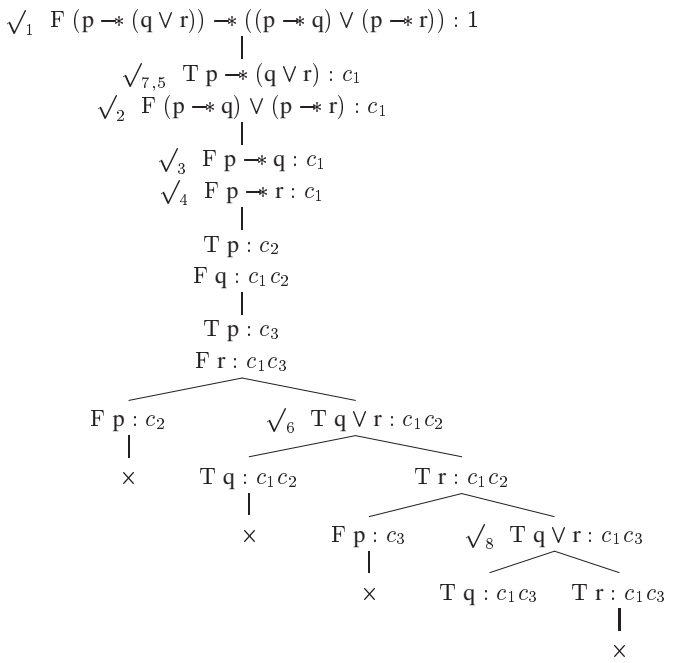


Figure 7. Tableau for $(\mathrm{p} -\!\!* (\mathrm{q} \vee \mathrm{r})) -\!\!* ((\mathrm{p} -\!\!* \mathrm{q}) \vee (\mathrm{p} -\!\!* \mathrm{r}))$

where a (resp. b) need not be new if $\mathrm{T}\ \phi : a$ (resp. $\mathrm{T}\ \psi : b$) has been already introduced in $\mathcal{B}$ by a previous $\pi\alpha$ tableau expansion.

These rules are proved admissible in the restriction of **TBI** to **MILL** (Galmiche and Méry 2003) and it would be rather interesting if they could be extended to **BI**.

Unfortunately, the previous liberalized rules are not sound for **TBI**, as seen with the formula $(\mathrm{p} -\!\!* (\mathrm{q} \vee \mathrm{r})) -\!\!* ((\mathrm{p} -\!\!* \mathrm{q}) \vee (\mathrm{p} -\!\!* \mathrm{r}))$, for which Figure 7 gives an open tableau. As the tableau contains an open branch which is also completed, we can, as previously explained, build a countermodel out of this H-branch. Therefore, the formula is not provable in **BI**.

However, if we use liberalized rules, the tableau of Figure 7 turns out to be closed and we eventually end up with a closed tableau for a non-provable formula. Indeed, using the liberalized version of F$-\!\!*$ we can reuse at Step 4 the constant c_2 introduced at Step 3, instead of creating the constant c_3. The corresponding tableau is obtained from Figure 7 by replacing each occurrence of c_3 by c_2 which leads to the closure of the previously open fourth branch due to $\langle\, \mathrm{T}\ \mathrm{q} : c_1c_2, \mathrm{F}\ \mathrm{q} : c_1c_2 \,\rangle$.

6.1. **The Canonical Interpretation**

To understand why the liberalized rules for **MILL** cannot be extended to **TBI**, we must recall their justification in the case of **MILL**, *i.e.*, the completeness of the logic with respect to *regular* Kripke resource models.

Definition 6.1. Let $\langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$ be a Kripke resource model. A world m is *ϕ-characteristic* for $\models$ in $\mathcal{M}$ if $m \models \phi$ and, for any world n such that $n \models \phi$, $m \sqsubseteq n$. The forcing relation $\models$ is *regular* on $\mathcal{M}$ if whenever there exists a world m such that $m \models \phi$, there also exists a ϕ-characteristic world. Finally, we say that $\langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$ is *regular* if its forcing relation $\models$ is regular on $\mathcal{M}$.

Theorem 6.1. **BI** is not complete with respect to regular Kripke resource models.

Proof. We prove that there exists no regular Kripke resource countermodel for the sequent $\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r}) \vdash (\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r})$. We suppose there is one such countermodel $\langle\, \mathcal{M}, \models, [\![-]\!] \,\rangle$, then there is a world m such that (1) $m \models \mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r})$ and (2) $m \not\models (\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r})$. As (2) implies $m \not\models \mathrm{p} \mathrel{-\!\!*} \mathrm{q}$, there is a world n such that $n \models \mathrm{p}$ and $m \bullet n \not\models \mathrm{q}$. Since $\mathcal{K}$ is regular, $n \models \mathrm{p}$ implies that there exists a ϕ-characteristic world c_ϕ. Then, $c_\phi \models \phi$ and $c_\phi \sqsubseteq n$ which, by order preservation, yields $m \bullet c_\phi \sqsubseteq m \bullet n$. It then follows from Kripke monotonicity and $m \bullet n \not\models \mathrm{q}$ that $m \bullet c_\phi \not\models \mathrm{q}$. As (2) also implies $m \not\models \mathrm{p} \mathrel{-\!\!*} \mathrm{r}$, we can similarly prove that $m \bullet c_\phi \not\models \mathrm{r}$. Therefore, $m \bullet c_\phi \not\models \mathrm{q} \vee \mathrm{r}$ and, since $c_\phi \models \phi$, we have shown $m \not\models \mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r})$ which contradicts 1). Thus, there exists no regular Kripke resource countermodel for the sequent $\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r}) \vdash (\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r})$. □

From the previous result, we observe that the presence of $\vee$ leads to the incompleteness of **BI** with respect to regular Kripke resource models. In (D'Agostino and Gabbay 1994), in which the multiplicative fragment ($\otimes$, $\multimap$, I) is considered, the regularity property arises from what is called the *canonical interpretation*. The canonical interpretation is a term model in which a world, also called an *information token*, is equivalent to the set of propositions it verifies. More precisely, information tokens are sets of propositions closed under deduction and partially ordered by set-inclusion. The forcing relation $x \models \phi$ between information tokens and propositions is given by $\phi \in x$. It immediately follows that $x \models \phi$ and $y \models \phi$ imply $x \cap y \models \phi$, in other words, satisfaction is preserved under arbitrary intersections. This, in turn, entails the existence of the *least* token that satisfies ϕ whenever there exists some token that satisfies ϕ. This least token intuitively corresponds to the computational content of ϕ, *i.e.*, the set $\{\psi \mid \phi \vdash \psi\}$.

Definition 6.2. Let $\mathcal{P}(L)$ denote the collection of **BI** propositions over a language L of propositional letters, the mapping $\|-\|$, from $\mathcal{P}(L)$ to sets of propositions, is defined as follows: $\|\varphi\| = \{\chi \mid \varphi \vdash_{\mathbf{LBI}} \chi\}$ for $\varphi \equiv \phi, I, \top, \bot$.

The *canonical interpretation* for **BI** is then given by the carrier set $H \equiv \{\|\phi\| \mid \phi \in \mathcal{P}(L)\}$, preordered by $\sqsubseteq$ seen as set-inclusion.

Whenever no confusion may arise, we shall write $\vdash$ instead of $\vdash_{\mathbf{LBI}}$.

Lemma 6.1. $\langle\, H, \sqsubseteq, \odot, \|\mathrm{I}\| \,\rangle$, where $\|\phi\| \odot \|\psi\|$ is defined by $\|\phi * \psi\|$, is a Kripke resource monoid.

Proof. We check that $\odot$ is a monoidal product on H. The identity w.r.t. $\|\mathrm{I}\|$, associativity and commutativity properties directly come from those of $*$. Moreover $\odot$ is order-preserving because, in **LBI**, one can derive $\phi * \psi \vdash \phi' * \psi'$ from the premises $\phi \vdash \phi'$ and $\psi \vdash \psi'$. □

Lemma 6.2. The canonical interpretation has the following properties:

(1) $\|\psi\| \sqsubseteq \|\phi\|$ iff $\phi \vdash \psi$;
(2) $\|\top\|$ is the least element;
(3) $\|\bot\|$ is the greatest element;
(4) $\|\phi \vee \psi\| = \|\phi\| \cap \|\psi\|$.

Proof.

(1) Suppose $\|\psi\| \sqsubseteq \|\phi\|$. Since $\psi \in \|\psi\|$, we have $\psi \in \|\phi\|$ and then, by definition, $\phi \vdash \psi$. We now suppose $\phi \vdash \psi$. Then, if $\chi \in \|\psi\|$, by definition, $\psi \vdash \chi$. Using cut with $\phi \vdash \psi$ yields $\phi \vdash \chi$, *i.e.*, $\chi \in \|\phi\|$.
(2) Immediate from (1) because, for any ϕ, $\phi \vdash \top$.
(3) Immediate from (1) because, for any ϕ, $\bot \vdash \phi$.
(4) We successively prove $\|\phi \vee \psi\| \sqsubseteq \|\phi\| \cap \|\psi\|$ and $\|\phi\| \cap \|\psi\| \sqsubseteq \|\phi \vee \psi\|$. First, since $\phi \vdash \phi \vee \psi$ and $\psi \vdash \phi \vee \psi$ hold in **LBI**, we have $\|\phi \vee \psi\| \sqsubseteq \|\phi\|$ and $\|\phi \vee \psi\| \sqsubseteq \|\psi\|$, which implies $\|\phi \vee \psi\| \sqsubseteq \|\phi\| \cap \|\psi\|$. Second, suppose $\chi \in \|\phi\| \cap \|\psi\|$. Then, by definition of $\|-\|$, we have $\phi \vdash \chi$ and $\psi \vdash \chi$, which, by $\vee_L$ of **LBI**, implies $\phi \vee \psi \vdash \chi$. Therefore, $\chi \in \|\phi \vee \psi\|$ and so $\|\phi\| \cap \|\psi\| \sqsubseteq \|\phi \vee \psi\|$.

□

Corollary 6.1. $\langle H, \sqsubseteq, \mathbin{⍉}, \|\bot\|, \|\top\| \rangle$, where $\|\phi\| \mathbin{⍉} \|\psi\|$ is defined by $\|\phi \vee \psi\|$, is a (complete) inf-semi-lattice with $\|\bot\|$ as the greatest element and $\|\top\|$ as the least element.

Proof. Immediate since property (4) of Lemma 6.2 implies that $\|\phi \vee \psi\|$ is the greatest lower bound of ϕ and ψ.

□

Although the canonical interpretation is closed under intersections, it is not closed under unions. Indeed, for any two atomic propositions p and q, we have $\mathrm{p}, \mathrm{q} \in \|\mathrm{p}\| \cup \|\mathrm{q}\|$. Since neither $\mathrm{p} \vdash \mathrm{p} \wedge \mathrm{q}$, nor $\mathrm{q} \vdash \mathrm{p} \wedge \mathrm{q}$ hold in **LBI**, it follows that $\mathrm{p} \wedge \mathrm{q} \notin \|\mathrm{p}\| \cup \|\mathrm{q}\|$. But suppose now that $\|\mathrm{p}\| \cup \|\mathrm{q}\| = \|\chi\|$, for some χ. Then $\chi \vdash \mathrm{p}$ and $\chi \vdash \mathrm{q}$ imply $\chi \vdash \mathrm{p} \wedge \mathrm{q}$, *i.e.*, $\mathrm{p} \wedge \mathrm{q} \in \|\mathrm{p}\| \cup \|\mathrm{q}\|$, that is a contradiction. Nevertheless, since the canonical interpretation is a complete inf-semi-lattice, it can still be embedded into a complete lattice by defining the least upper bound $\|\phi\| \mathbin{⍊} \|\psi\|$ of $\|\phi\|$ and $\|\psi\|$ as $\mathbin{⍉}\{\|\chi\| \mid \|\phi\| \sqsubseteq \|\chi\| \text{ and } \|\psi\| \sqsubseteq \|\chi\|\}$, which is easily seen to be equivalent to $\|\phi \wedge \psi\|$.

Theorem 6.2. Let $\mathcal{H}$ be the structure $\langle H, \sqsubseteq, \odot, \|\mathrm{I}\|, \mathbin{⍉}, \|\bot\|, \mathbin{⍊}, \|\top\| \rangle$ then, $\mathcal{H}$ is a **BI**-algebra, *i.e.*, a Heyting algebra equipped with an additional residuated commutative monoid structure.

Proof. From Lemma 6.2 we know that $\mathcal{H}$ is a complete lattice with least and greatest elements and, from Lemma 6.1, that it is also a Kripke resource monoid. Therefore, we only need to show that the lattice and the monoidal part of $\mathcal{H}$ are residuated. First, we prove $\|\phi \mathrel{-\!\!*} \psi\| = \mathbin{⍉}\{\|\chi\| \mid \|\psi\| \sqsubseteq \|\phi * \chi\|\}$. First, for any $\|\chi\|$, $\|\psi\| \sqsubseteq \|\phi * \chi\|$ iff $\phi * \chi \vdash \psi$ iff $\chi \vdash \phi \mathrel{-\!\!*} \psi$ iff $\|\phi \mathrel{-\!\!*} \psi\| \sqsubseteq \|\chi\|$. Second, $\|\phi \mathrel{-\!\!*} \psi\|$ belongs to the set because $\phi * (\phi \mathrel{-\!\!*} \psi) \vdash \psi$ entails $\|\psi\| \sqsubseteq \|\phi * (\phi \mathrel{-\!\!*} \psi)\|$. Therefore, $\|\phi \mathrel{-\!\!*} \psi\|$ is the least upper bound. Then, $\|\phi \to \psi\| = \mathbin{⍉}\{\|\chi\| \mid \|\psi\| \sqsubseteq \|\phi \wedge \chi\|\}$ can be proven similarly. □

Henceforth, we shall refer to the canonical interpretation as the structure $\mathcal{H}$ given in Theorem 6.2. Following the work in (D'Agostino and Gabbay 1994) for the fragment ($\otimes$, $\multimap$, I), we

equip the canonical interpretation with a satisfaction relation between information tokens and propositions as follows

Definition 6.3. The canonical forcing relation $\models$ is defined for an information token $\|\chi\|$ and a **BI** proposition ϕ by $\|\chi\| \models \phi$ iff $\phi \in \|\chi\|$, *i.e.*, iff $\chi \vdash \phi$.

Lemma 6.3. For any proposition ϕ, $\|\phi\|$ is ϕ-characteristic for $\models$ in $\mathcal{H}$.

Proof. First, we have $\|\phi\| \models \phi$ because $\phi \vdash \phi$ obviously holds in **LBI**. Then, if $\|\chi\|$ is such that $\|\chi\| \models \phi$ then, by definition, $\chi \vdash \phi$, which implies $\|\phi\| \sqsubseteq \|\chi\|$. □

Corollary 6.2. The canonical forcing relation $\models$ is regular on $\mathcal{H}$.

Proof. If any formula has a characteristic world in $\mathcal{H}$ then the regularity condition is satisfied. □

6.2. **A Special Treatment of the Disjunction**

In this subsection, we observe that the canonical forcing relation does not satisfy the usual clause for disjunction, *i.e.*, $\|\chi\| \models \phi \vee \psi$ iff $\|\chi\| \models \phi$ or $\|\chi\| \models \psi$. Indeed, this would require that $\chi \vdash \phi \vee \psi$ iff $\chi \vdash \phi$ or $\chi \vdash \psi$, which does not hold in general as p ∨ q ⊢ p ∨ q is provable and neither p ∨ q ⊢ p nor p ∨ q ⊢ q is provable in **LBI**. Nevertheless, we remark that the if-direction still holds since it simply corresponds to the pair of $\vee_R$-rules in **LBI**. So, what semantic clause for ∨ does the canonical forcing relation satisfy ?

Theorem 6.3. In the canonical interpretation $\mathcal{H}$, $\|\chi\| \models \phi \vee \psi$ if and only if there exist $\|\varphi\|$,$\|\varphi'\|$ such that $\|\varphi\| \circledcirc \|\varphi'\| \subseteq \|\chi\|$ and $\|\varphi\| \models \phi$ and $\|\varphi'\| \models \psi$.

Proof. For the if-direction, if $\|\chi\| \models \phi \vee \psi$ then, by definition, $\chi \vdash \phi \vee \psi$, which implies $\|\phi \vee \psi\| \subseteq \|\chi\|$. The result follows immediately since $\|\phi \vee \psi\| = \|\phi\| \circledcirc \|\psi\|$, $\|\phi\| \models \phi$ and $\|\psi\| \models \psi$.

For the only-if, we have, on the one hand, $\varphi \vdash \phi$ and $\varphi' \vdash \psi$ because, by hypothesis, $\|\varphi\| \models \phi$ and $\|\varphi'\| \models \psi$. On the other hand, since $\|\varphi \vee \varphi'\| = \|\varphi\| \circledcirc \|\varphi'\|$, $\|\varphi\| \circledcirc \|\varphi'\| \subseteq \|\chi\|$ yields $\chi \vdash \varphi \vee \varphi'$. The result is obtained from the following derivation in **LBI**:

$$\dfrac{\chi \vdash \varphi \vee \varphi' \qquad \dfrac{\dfrac{\varphi' \vdash \psi}{\varphi' \vdash \phi \vee \psi} \vee_R \quad \dfrac{\varphi \vdash \phi}{\varphi \vdash \phi \vee \psi} \vee_R}{\varphi \vee \varphi' \vdash \phi \vee \psi} \vee_L}{\chi \vdash \phi \vee \psi} Cut$$

□

As Theorem 6.3 shows, the semantics of disjunction in the canonical interpretation is unusual for a Kripke semantics. In fact, it is reminiscent of a more general topological semantics, as we will see in the next section.

6.3. **Sheaf-theoretic Kripke Models of BI**

As explained in the previous section, a special treatment of the disjunction, arising from considerations in **BI**-algebras, is needed to make the liberalized rules work. In (Pym 2002; Pym et al. 2004; Pym 2004), a topological Kripke semantics is introduced which, as for **BI**-algebras, allows $\bot$ to be taken into account together with a non-indecomposable treatment of the disjunction. This is done by moving from presheaves to sheaves over a topological space. We briefly review the topological semantics of **BI**, the details are in (Pym 2002; Pym et al. 2004; Pym 2004).

Definition 6.4 (TRM). A *topological resource monoid* (TRM) $\langle \mathcal{X}, \cdot, e \rangle$ is a commutative monoid in the category **Top** of topological spaces and continuous maps between them, *i.e.*, a topological space $|\mathcal{X}|$, with open sets $\mathcal{O}(\mathcal{X})$, on which a monoidal product $\cdot : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ of open sets is defined, together with its unit $e : 1 \to \mathcal{X}$, and such that $\cdot$ distributes over arbitrary unions of open sets:

$$V \cdot (\bigcup_i U_i) = \bigcup_i (V \cdot U_i)$$

The tensor product of two opens sets is not necessarily open, consequently, we must require that the monoidal structure be defined by *open maps*, *i.e.*,which map open sets to open sets. Thus, if $\langle |\mathcal{X}|, \mathcal{O}(\mathcal{X}) \rangle$ is a topological space and if $(\cdot, e)$ are defined as above, we speak of the *topological monoid* $\langle \mathcal{X}, \cdot, e \rangle$ on $\langle |\mathcal{X}|, \mathcal{O}(\mathcal{X}) \rangle$ and $(\cdot, e)$. If $(\cdot, e)$ are open, we speak of the *open topological monoid* $\langle \mathcal{X}, \cdot, e \rangle$ on $\langle |\mathcal{X}|, \mathcal{O}(\mathcal{X}) \rangle$ and $(\cdot, e)$.

The symmetric monoidal structure of a commutative topological monoid gives rise, via Day's construction of a tensor product (Day 1970), to a symmetric monoidal closed structure on the category $\mathrm{Sh}(\mathcal{X})$ of sheaves on $\mathcal{X}$.

Definition 6.5 (TKM). Let $\mathcal{X} = \langle \mathcal{X}, \cdot, e \rangle$ be an commutative open topological monoid. A *topological kripke model* (TKM) is a triple $\mathcal{T} = \langle \mathrm{Sh}(\mathcal{X}), \models, [\![-]\!] \rangle$, where $[\![-]\!] : \mathcal{P}(L) \rightharpoonup \mathrm{Sh}(\mathcal{X})$ is a partial function from the **BI**-propositions over a language L of propositional letters to the objects of $\mathrm{Sh}(\mathcal{X})$ such that:

Kripke monotonicity: if $V \subseteq U$ then $\forall\, \phi \in \mathcal{P}(L), U \models \phi$ implies $V \models \psi$

and $\models \subseteq \mathcal{O}(\mathcal{X}) \times \mathcal{P}(L)$ satisfies:

- $U \models \mathrm{p}$ iff $[\![\mathrm{p}]\!](U) \neq \emptyset$, for $\mathrm{p} \in L$
- $U \models \top$ for all $U \in \mathcal{O}(\mathcal{X})$
- $U \models \bot$ iff $U = \emptyset$
- $U \models I$ iff $U \subseteq I$
- $U \models \phi \wedge \psi$ iff $U \models \phi$ and $U \models \psi$
- $U \models \phi \vee \psi$ iff, for some $V, V' \in \mathcal{O}(\mathcal{X})$ such that $U = V \cup V'$, $V \models \phi$ and $V' \models \psi$
- $U \models \phi \to \psi$ iff, for all $V \subseteq U$, $V \models \phi$ implies $V \models \psi$
- $U \models \phi * \psi$ iff, for some $V, V' \in \mathcal{O}(\mathcal{X})$, $U \subseteq V \cdot V'$ and $V \models \phi$ and $V' \models \psi$
- $U \models \phi \mathrel{-\!\!*} \psi$ iff, for all $V \in \mathcal{O}(\mathcal{X})$, $V \models \phi$ implies $U \cdot V \models \psi$

As one can notice, the clause for $\vee$ is very similar to the one given in Theorem 6.3. Indeed, they are just dual since the topological semantics described above considers open sets, while the canonical interpretation considers sets that are closed under deduction. Therefore, the translation

from one to the other is simply obtained by taking the complement and **TBI'** appears as the syntactic reflection of the forcing semantics in the category of sheaves over a topological monoid. We shall see that the dependency graphs, which are defined in the case of liberalized rules, may be viewed as (partial) topological Kripke models.

7. Liberalized Resource Tableaux

In this section, we give new expansion rules for **TBI**. The resulting tableau system is called **TBI'**. In previous discussions, we have defined a canonical interpretation of **BI** and have also shown that a new semantic clause for the additive disjunction is required to achieve a suitable canonical forcing relation. However, the semantic changes made to $\vee$ have a syntactic counterpart and the corresponding initial expansion rules have to be modified accordingly. For that, we make several extensions to the initial framework.

7.1. An Extended Labelling Algebra

Definition 7.1. We enrich the labelling language given in Definition 3.1 with a new unit symbol t and a new binary function symbol $\sqcap$. Therefore, compound labels become expressions of the form $x \circ y$ or $x \sqcap y$ in which x and y are labels. We say that x is a *sublabel* of y (notation: $x \preceq y$), if there exists a label z such that $y = x \circ z$ or $y = x \sqcap z$. We note $x \prec y$ if $x \preceq y$ and $x \neq y$. $\mathcal{P}(x)$ denotes the set of the sublabels of x. All other definitions remain unchanged.

Definition 7.2. Labels and constraints are interpreted in a *labelling algebra* $\mathcal{L} = \langle L, \leq, \circ, 1, \sqcap, t \rangle$ in the following way:

- L is a set of labels;
- $\leq$ is a preordering;
- equality on labels is defined by : $x = y$ iff $x \leq y$ and $y \leq x$;
- $\langle L, \leq, \circ, 1 \rangle$ is an order preserving commutative monoid, *i.e.*, $\circ$ satisfies
  - associativity: $(x \circ y) \circ z = x \circ (y \circ z)$;
  - commutativity: $x \circ y = y \circ x$;
  - identity: $x \circ 1 = 1 \circ x = x$;
  - compatibility: $x \leq y$ implies $x \circ z \leq y \circ z$;
- $\langle L, \leq, \sqcap, t \rangle$ is a distributive complete semi-lattice, *i.e.*, $\sqcap$ satisfies;
  - associativity: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$;
  - commutativity: $x \sqcap y = y \sqcap x$;
  - identity: $x \sqcap t = t \sqcap x = x$;
  - compatibility: $x \leq y$ implies $x \sqcap z \leq y \sqcap z$;
  - contraction: $x \leq x \sqcap x$;
  - weakening: $x \sqcap y \leq x$;
  - distributivity: $(x \circ y) \sqcap (x \circ z) \leq x \circ (y \sqcap z)$.

Lemma 7.1. The labelling algebra $\mathcal{L} = \langle L, \leq, \circ, 1, \sqcap, t \rangle$ satisfies the following properties:

(1) $x \circ (y \sqcap z) \leq (x \circ y) \sqcap (x \circ z)$;
(2) $x \leq t$;
(3) $t \leq x \circ t$.

Proof.

(1) By the weakening axiom, we have both $y \sqcap z \leq y$ and $y \sqcap z \leq z$. Thus, by compatibility of $\circ$, we can derive $x \circ (y \sqcap z) \leq x \circ y$ and $x \circ (y \sqcap z) \leq x \circ z$. The compatibility of $\sqcap$ then entails $(x \circ (y \sqcap z)) \sqcap (x \circ (y \sqcap z)) \leq (x \circ y) \sqcap (x \circ z)$, from which the result immediately follows using the contraction axiom.
(2) The weakening axiom implies $x \sqcap t \leq t$ which, by identity, gives $x \leq t$.
(3) Property (2) yields $\sqcap\{y \mid t \leq x \circ y\} \leq t$. So, $x \circ \sqcap\{y \mid t \leq x \circ y\} \leq x \circ t$ follows by compatibility. Distributivity then gives $\sqcap\{x \circ y \mid t \leq x \circ y\} \leq x \circ t$. Since $\sqcap\{x \circ y \mid t \leq x \circ y\} = t$, it finally comes that $t \leq x \circ t$.

□

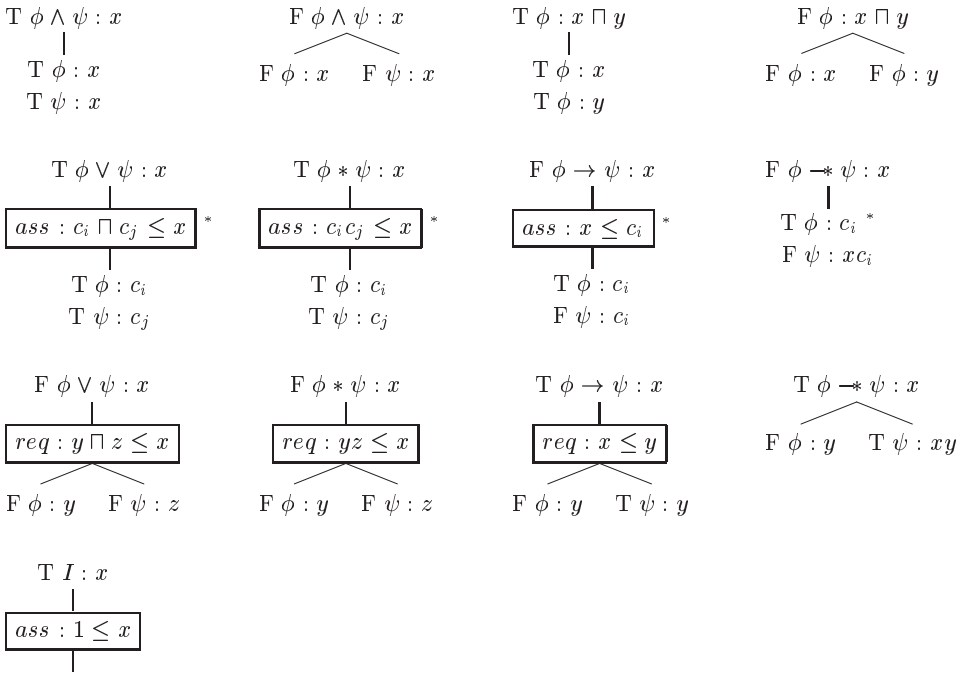
Notice that property (1), together with the distributivity axiom, imply that $\circ$ is "fully" distributive over $\sqcap$. Property (2) simply means that t is the greatest element in the labelling algebra and implies, with property (3), that t absorbs any other label in a multiplication, *i.e.*, $x \circ t = t$ for any x.

Definition 7.3. The *closure* $\overline{K}$ of a set K of label constraints is extended as follows:

- $K \subseteq \overline{K}$;
- if $x \in \mathcal{D}(\overline{K})$ then $x \leq x \in \overline{K}$ (reflexivity);
- if $x \leq y \in \overline{K}$ and $y \leq z \in \overline{K}$ then $x \leq z \in \overline{K}$ (transitivity);
- if $y \circ z \in \mathcal{D}(\overline{K})$ then $x \leq y \in \overline{K}$ implies $x \circ z \leq y \circ z \in \overline{K}$ (compatibility);
- if $y \sqcap z \in \mathcal{D}(\overline{K})$ then $x \leq y \in \overline{K}$ implies $x \sqcap z \leq y \sqcap z \in \overline{K}$ (compatibility);
- if $x \sqcap y \in \mathcal{D}(\overline{K})$ then $x \sqcap y \leq x$ and $x \sqcap y \leq y \in \overline{K}$ (weakening);
- if $(x \circ y) \sqcap (x \circ z)$ or $x \circ (y \sqcap z) \in \mathcal{D}(\overline{K})$ then $(x \circ y) \sqcap (x \circ z) = x \circ (y \sqcap z) \in \overline{K}$ (distributivity).

7.2. **Liberalized Rules**

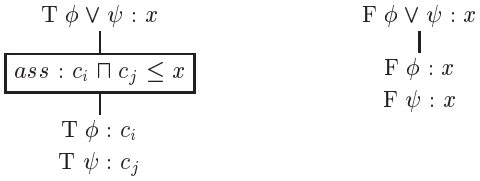
We now give, in Figure 8, the expansion rules of **TBI'**, which modifies **TBI** in order to reflect the semantic changes previously made. Compared to the initial labelled system, we notice the presence of two new rules, namely, $T\sqcap$ and $F\sqcap$. These rules are *structural* since they only operate on the label of their signed formula without decomposing the formula itself. Their computational contents simply reflect that $\sqcap$ corresponds to an intersection under the canonical interpretation. Then we notice that the $T\vee$ rule has been modified and is now a $\pi\alpha$ rule, introducing two new constants, c_i, c_j, and a new assertion $c_i \sqcap c_j \leq x$. The $F\vee$ rule, on the other hand, becomes a $\pi\beta$ rule re-using two labels, y, z, such that the requirement $y \sqcap z \leq x$ is satisfied by the closure of the assertions. The new rules for $\vee$ are justified by Theorem 6.3, of which they are the syntactic counterparts. The usual version of the $F\vee$ is admissible in **TBI'**, so one could also use



* c_i, c_j are new constants

Figure 8. **TBI'** Expansion Rules

the following pair of rules for the disjunction:



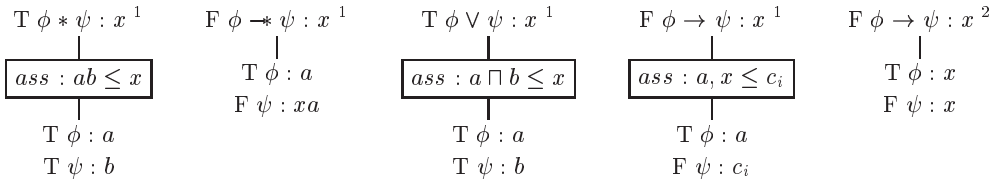
The initial pair of rules leads to a nicely symmetric treatment of the disjunction but has the drawback of solving requirements of the form $y \sqcap z \leq x$, which can be difficult. Moreover, it involves more splitting of the tableau branches, which can significantly increase the size of a tableau proof. In presence of explicit structural rules (Kripke monotonicity and $F\sqcap$), the two versions of the $F\vee$ are easily proven to be equivalent. Therefore, one can indifferently use one version or the other in a tableau proof.

Definition 7.4. A constant a is *ϕ-characteristic* in a tableau branch $\mathcal{B}$ if it appeared, for the first time, in a formula T $\phi : a$ which was introduced by a F$-\!\!*$, T$*$ or T$\vee$ expansion.

We remark that the constant c_i introduced by the F $\phi \to \psi : x$ rule is not ϕ-characteristic. This observation is semantically justified by the fact that, even if having c_i such that $x \sqsubseteq c_i$, $c_i \models \phi$

and $c_i \not\models \psi$ implies that there exists a ϕ-characteristic a, *i.e.*, an a such that $a \models \phi$ and $a \sqsubseteq c_i$, it does not necessarily imply that $x \sqsubseteq a$. However, the previous discussion shows that Definition 7.4 can still be extended to cover the case of the $F\,{\to}$ rule (and so, cover all the $\pi\alpha$ rules), by modifying it as prescribed by the fourth case of the following lemma.

Lemma 7.2. Let $\mathcal{B}$ be a tableau branch. The following liberalized rules



where a (b) need not be new, are admissible in **TBI'** provided

1 a (b) is ϕ-characteristic (ψ-characteristic) in $\mathcal{B}$, and
2 there exists T $\phi : y$ in $\mathcal{B}$ such that $y \leq x \in \overline{Ass}(\mathcal{B})$.

Proof. Let $\mathcal{K}$ be a regular Kripke resource model $\langle \mathcal{M}, \models, [\![-]\!] \rangle$. We only prove the result for $F\mathrel{-\!\!*}$, the other case being similar. The soundness of the liberalized $F\mathrel{-\!\!*}$ is justified by the following semantic equivalence : $x \not\models \phi \mathrel{-\!\!*} \psi$ iff there exists a ϕ-characteristic world a such that $x \bullet a \not\models \psi$. The if direction is obvious. For the only-if direction, suppose that $x \not\models \phi \mathrel{-\!\!*} \psi$. Then, there exists y such that $y \models \phi$ and $x \bullet y \not\models \psi$. Since $y \models \phi$, the regularity of $\models$ implies that there exists a ϕ-characteristic world a, which implies $a \sqsubseteq y$ and, by order preservation, $x \bullet a \sqsubseteq x \bullet y$. Kripke monotonicity finally entails $x \bullet a \not\models \psi$.

Now, suppose that we have a realization $\|-\|$ of $\mathcal{B}$ in $\mathcal{K}$ with T $\phi : a$ and F $\phi \mathrel{-\!\!*} \psi : x$ in $\mathcal{B}$. Then, $\|a\| \models \phi$ and $\|x\| \not\models \phi \mathrel{-\!\!*} \psi$. Since $\|a\|$ is assumed to be ϕ-characteristic, it follows from $\|x\| \not\models \phi \mathrel{-\!\!*} \psi$ and our previous discussion that $\|x\| \bullet \|a\| (= \|xa\|) \not\models \psi$, which means that the expansion of a liberalized $F\mathrel{-\!\!*}$ preserves realizability.

□

7.3. **Proof and Countermodel Construction**

We illustrate how **TBI'** works with some examples. The first example, *q.v.* Figure 9, shows two closed tableaux for $((\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r})) \mathrel{-\!\!*} (\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r}))$, which therefore holds in **BI**. The tableau on the left-hand side is obtained using the $\pi\beta$ version of the $F\vee$ rule, while the other is obtained with the usual α version. As one can see, for both tableaux, Step 2 deals with the new $T\vee$ rule and extends the branch with T $\mathrm{p} \mathrel{-\!\!*} \mathrm{q} : c_2$ and T $\mathrm{p} \mathrel{-\!\!*} \mathrm{r} : c_3$, introducing by the way two new constants c_2 and c_3 and a new assertion $c_2 \sqcap c_3 \leq c_1$. Step 4, which shows the correspondence between the two $F\vee$ rules, is more interesting. With the $\pi\beta$ version, we are required to find two labels x and y which verify the constraint $x \sqcap y \leq c_1 c_4$. Taking $x = c_2 c_4$ and $y = c_3 c_4$ is a suitable choice since, by compatibility on assertion $c_2 \sqcap c_3 \leq c_1$, we can deduce $(c_2 \sqcap c_3) c_4 \leq c_1 c_4$, which, by distributivity, gives $(c_2 c_4) \sqcap (c_3 c_4) \leq c_1 c_4$.

If we use the α version, Step 4 in the left tableau is simulated by Steps 4, 4' and 4" in the right one. Step 4 first expands F $\mathrm{q} \vee \mathrm{r} : c_1 c_4$ into F $\mathrm{q} \vee \mathrm{r} : (c_2 c_4) \sqcap (c_3 c_4)$ using Kripke monotonicity because, as previously explained, $(c_2 c_4) \sqcap (c_3 c_4) \leq c_1 c_4$ holds in the labelling algebra

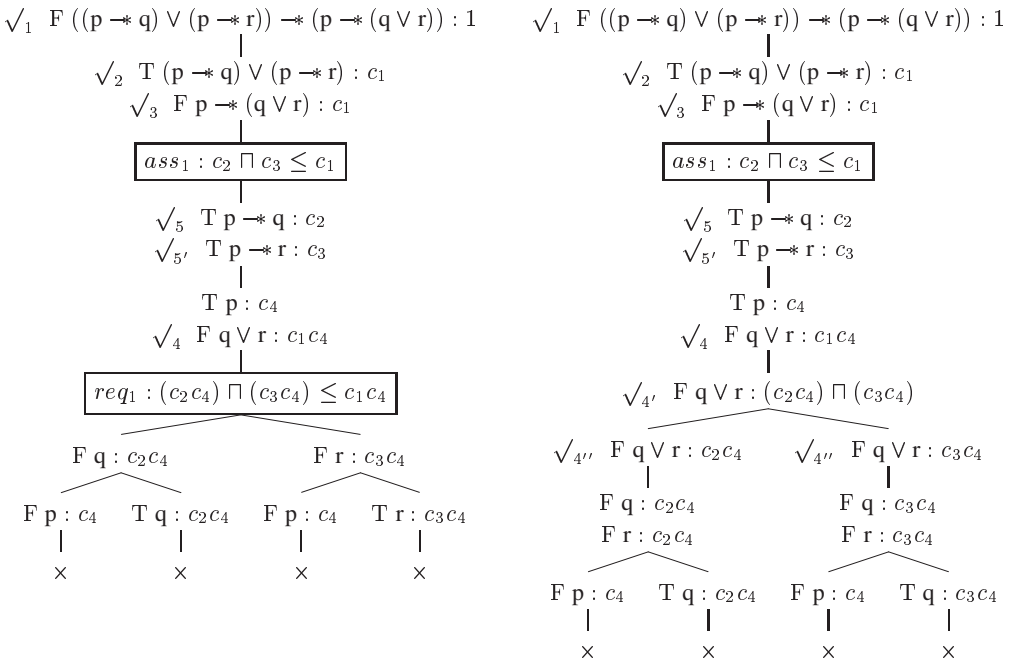


Figure 9. Two different tableaux for $((\mathrm{p} -\!\!* \mathrm{q}) \vee (\mathrm{p} -\!\!* \mathrm{r})) -\!\!* (\mathrm{p} -\!\!* (\mathrm{q} \vee \mathrm{r}))$.

with respect to the assertions. Step 4' then splits the tableau into two branches by decomposing $\mathrm{F\ q} \vee \mathrm{r} : (c_2c_4) \sqcap (c_3c_4)$ with the structural $F\sqcap$ rule. Finally, Step 4" reapplies the $F\vee$ rule on both $\mathrm{q} \vee \mathrm{r}$ resulting from the previous step. As the reader can notice, all other steps are exactly the same for both tableaux. Moreover, such a simulation of one $F\vee$ version into the other can always be performed, thus proving their equivalence.

For the second example, see Figure 10, the tableaux construction procedure ends up with an open branch. Therefore, the formula $(\mathrm{p} -\!\!* (\mathrm{q} \vee \mathrm{r})) -\!\!* ((\mathrm{p} -\!\!* \mathrm{q}) \vee (\mathrm{p} -\!\!* \mathrm{r}))$ does not hold in **BI**. Notice that, since the new version of the $T\vee$ rule introduces distinct constants for q and r when expanding $\mathrm{T\ q} \vee \mathrm{r} : c_1c_2$ instead of propagating the c_1c_2, the fact that we reuse c_2 in Step 4 does not lead to a closed tableau as it did for the introductory example of Figure 7.

It is routine to extend the notions of fulfilled formula, completed branch, H-branch and dependency graph to cover the introduction of $\sqcap$ in the labels. The model existence theorem for **TBI'** then follows immediately.

Theorem 7.1. An H-branch $\mathcal{B}$ has an (algebraic) Kripke resource model.

Proof. The problem is to embed the dependency graph of a given H-Branch into a **BI**-algebra. The main difficulty is that $\circ$ and $\sqcap$ are only partial operations on the dependency graph. So, we must complete $\circ$ and $\sqcap$ with suitable values to obtain satisfactory monoidal product and lattice meet operators. For the monoidal part, we follow the ideas of section § 4.2 and add a greatest element π, to which all undefined products are mapped. Similarly, for the lattice part, we add a least element ω to which all undefined meets are mapped (other completions, such

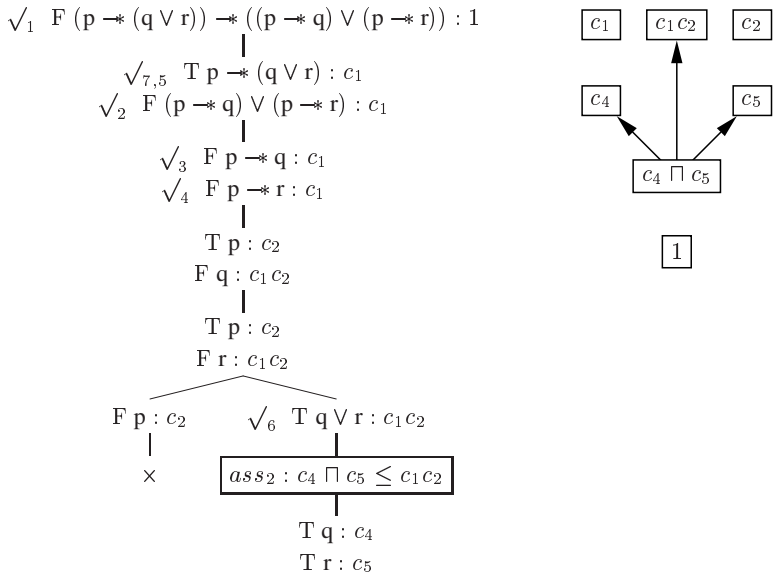


Figure 10. Tableau for $(\mathrm{p} -\!\!* (\mathrm{q} \vee \mathrm{r})) -\!\!* ((\mathrm{p} -\!\!* \mathrm{q}) \vee (\mathrm{p} -\!\!* \mathrm{r}))$

as the Mac Neille completion, could also be used). The result then becomes a straightforward adaptation of the proof of Theorem 4.2. □

7.4. **Properties of TBI'**

In this subsection we prove the soundness and completeness of **TBI'**. For the soundness we show that each rule of **TBI'** preserves realizability under the canonical interpretation.

Theorem 7.2 (soundness). Let ϕ be a **BI** proposition, if there exists a closed tableaux sequence for ϕ in **TBI'**, then ϕ is a theorem of **LBI**.

Proof. It is easy to prove that each rule of **TBI'** is sound under the canonical interpretation. We do only a few cases, knowing that the others are similar.

- Case $\mathrm{T}\ \phi \vee \psi : x$.
  suppose $||\chi|| \models \phi \vee \psi$ for some $||\chi||$. Then, $||\phi \vee \psi|| \sqsubseteq ||\chi||$. The result immediately follows since $||\phi \vee \psi|| = ||\phi|| \mathbin{\mathbb{O}} ||\psi||$, $||\phi|| \models \phi$ and $||\psi|| \models \psi$.
- Case $\mathrm{F}\ \phi \vee \psi : x$.
  suppose $||\chi|| \not\models \phi \vee \psi$ for some $||\chi||$. Then, $||\chi|| \not\models \phi$ for then, since $\phi \vdash \phi \vee \psi$, we should have $||\chi|| \models \phi \vee \psi$. Analogously, we prove $||\chi|| \not\models \psi$.

Now, to establish the soundness of **TBI'**, suppose that there exists a closed tableaux sequence for ϕ. Since all rules are sound under the canonical interpretation, the initial signed formula $\mathrm{F}\ \phi : 1$ is not realizable in $\mathcal{H}$. In other words, it is impossible that $||\mathrm{I}|| \not\models \phi$. Therefore, $||\mathrm{I}|| \models \phi$, which implies $I \vdash \phi$. □

Theorem 7.3 (completeness). Let ϕ be a proposition, if ϕ is provable in **LBI** then, there exists a closed tableaux sequence for ϕ in **TBI'**.

Proof. The completeness follows from Theorem 7.1 and the fact that the tableau construction procedure builds a tableau which is either closed or with a H-branch.

Another way to show completeness is to prove that **TBI'** is closed under each **LBI** rule, as explained in (D'Agostino and Gabbay 1994). For that, we define a transformation Γ^t on a bunch Γ by replacing each proposition ϕ by a ϕ-characteristic label c_ϕ, each "," by $\circ$ and sequences of the form $\phi_1; \phi_2; ...; \phi_n$ by a label $\sqcup(c_{\phi_1}, c_{\phi_2}, ..., c_{\phi_n})$. For example, $\phi, (\psi; \phi'; \psi' \rightarrow \phi)^t = c_\phi \circ (\sqcup(c_\psi, c_{\phi'}, c_{\psi' \rightarrow \phi}))$.The notation $\sqcup(c_\phi, c_\psi)$ is to mean that such a label is assumed to be the least upper bound of c_ϕ and c_ψ, and, therefore, the implicit assertion $c_\phi, c_\psi \leq \sqcup(c_\phi, c_\psi)$ is also assumed. Then, a sequent $\Gamma \vdash \psi$, where Γ is made upon a set of propositions ϕ_i, is provable in **TBI'** if the following tree is closed:

$$\begin{array}{c} \mathrm{T}\ \phi_i : c_{\phi_i} \\ \mathrm{F}\ \psi : \Gamma^t \\ | \\ \mathcal{T} \\ | \\ \times \end{array}$$

Now we prove the result for the $\rightarrow_R$ rule of **LBI**, the others being similar. For that, we show that if $\Gamma; \phi \vdash \psi$ is provable in **TBI'**, then, so is $\Gamma \vdash \phi \rightarrow \psi$. In other words, what we show, in the following figure, is that the tree corresponding to $\Gamma \vdash \phi \rightarrow \psi$ (on the right-hand side) can be closed if we assume that the one for $\Gamma; \phi \vdash \psi$ (on the left-hand side) is closed. The notation $\mathcal{T}[x/y]$ is to say that all occurrences of the label x in $\mathcal{T}$ are replaced by the label y.

$$\begin{array}{c} \mathrm{T}\ \phi_i : c_{\phi_i} \\ \mathrm{T}\ \phi : c_\phi \\ | \\ \boxed{ass : \Gamma^t, c_\phi \leq \sqcup(\Gamma^t, c_\phi)} \\ | \\ \mathrm{F}\ \psi : \sqcup(\Gamma^t, c_\phi) \\ | \\ \mathcal{T} \\ | \\ \times \end{array} \qquad \begin{array}{c} \mathrm{T}\ \phi_i : c_{\phi_i} \\ \surd_1\ \mathrm{F}\ \phi \rightarrow \psi : \Gamma^t \\ | \\ \boxed{ass : \Gamma^t, c_\phi \leq b} \\ | \\ \mathrm{T}\ \phi : c_\phi \\ \mathrm{F}\ \psi : b \\ | \\ \mathcal{T}[\sqcup(\Gamma^t, c_\phi)/b] \\ | \\ \times \end{array}$$

□

In this section, we have presented a liberalized tableau for **BI**. Liberalized rules are important in practice since they give a way to control the introduction of new constants and, thus, to limit the syntactic complexity of the labels which, in turn, leads to a more efficient proof-search (Hähnle and Schmitt 1994). However, having such liberalized rules requires a special treatment of the

additive disjunction which forces us to enrich the structure of the labels with a new symbol. For this reason, it becomes more difficult to build countermodels. Moreover, such countermodels are related to **BI**-algebras, which are themselves closely related to the topological Kripke semantics, of which they may be viewed as an algebraic counterpart.

8. Decidability and Finite Model Property for BI

In this section, we discuss the finite model property w.r.t. topological resource models and, as a consequence, the decidability of **BI**. For that, we investigate the situations in which a tableau may have infinite branches and thus use two central notions, namely *liberalized expansion rules* and *branch redundancy*, that have been introduced in the case of **BI** without $\bot$ (Galmiche and Méry 2003). The former provides a way to control the syntactic complexity of the labels by restricting the introduction of new constants during the tableau expansion. The latter characterizes the potential need of an infinite expansion process to achieve a completed branch and can roughly be viewed as a kind of loop-checking.

We summarize the situation. When $\pi\alpha$ formulae are in the scope of $\pi\beta$ formulae, the fulfillment of $\pi\alpha$ formulae requires the introduction of new constants which may destroy the fulfillment of $\pi\beta$ formulae. The first step towards termination of the H-branch construction process is to make use of the liberalized rules presented in Lemma 7.2. Doing so, only *finitely many distinct constants* can be introduced in a tableau branch $\mathcal{B}$, *i.e.*, only finitely many distinct atomic labels. But this not yet sufficient to prevent branches from growing infinitely, because, even with a finite number of atomic labels, one can still generate an infinite number of labels through composition. Anyway, since there are only *finitely many subformulae* of the initial formula to prove, after a given finite number of expansion steps, any newly introduced signed formula must have already been introduced, up to a fixed number of occurrences of the same constant. Such a situation happens when some formulae of the form $\mathrm{F}\ \phi \mathrel{-\!\!*} \psi\ :\ x$ occur in the scope of some formula $\mathrm{T}\ \phi' \mathrel{-\!\!*} \psi'\ :\ y$, with y being a sublabel of x. With such expansions, we can have sequences such as $\mathrm{F}\ \phi \mathrel{-\!\!*} \psi\ :\ x$, $\mathrm{F}\ \phi \mathrel{-\!\!*} \psi\ :\ xc$ (c being the constant introduced by the first expansion), $\mathrm{F}\ \phi \mathrel{-\!\!*} \psi\ :\ xcc, \ldots$ in a H-branch. Then, we have repetitions of the same signed formulae differing from each other only by one occurrence of the constant c, but without any additional computational content allowing to possibly close the branch. In order to solve this problem, we need the following notion of *branch redundancy* (Galmiche and Méry 2003).

Definition 8.1 (redundancy). A completed H-branch $\mathcal{B}$ is said to be *redundant for* the constant c if there exists $i \geq 1$ and $k > i$ such that for any $j \geq k$

1 $\mathrm{T}\ \phi : xc^j \in \mathcal{B}$ implies $\mathrm{T}\ \phi : xc^{j-i} \in \mathcal{B}$, and
2 $\mathrm{F}\ \phi : xc^j \in \mathcal{B}$ implies $\mathrm{F}\ \phi : xc^{j-i} \in \mathcal{B}$, and
3 $x \leq yc^j \in Ass(\mathcal{B})$ implies $x \leq yc^{j-i} \in Ass(\mathcal{B})$, and
4 $xc^j \leq y \in Ass(\mathcal{B})$ implies $xc^{j-i} \leq y \in Ass(\mathcal{B})$.

Moreover, as Definition 8.1 suggests, considering $\mathrm{T}\ \phi : xc^j$ ($\mathrm{F}\ \phi : xc^j$) and $\mathrm{T}\ \phi : xc^{j-i}$ ($\mathrm{F}\ \phi : xc^{j-i}$) as equivalent and since π captures the inessential parts of the model, the construction explained in Definition 4.6 always results in a finite countermodel.

Theorem 8.1. A completed branch $\mathcal{B}$ has a finite topological resource model.

Proof. Proof by induction on the number of constants for which $\mathcal{B}$ is redundant.
Base case. If there is no constant for which $\mathcal{B}$ is redundant then $\mathcal{B}$ is finite and by Theorem 7.1 topological resource model which is, by construction, obviously finite.
Inductive case. We assume that the proposition holds for any completed branch which is redundant for less than n constants. Suppose that $\mathcal{B}$ is redundant for n constants. We select one of those constants and denote it c. Then, we add the extra assertions $c^j = c^{j-i}$ for any $j \geq k$, where i, j and k refer to Definition 8.1. It is routine to show that the composition modulo rewriting $c^j = c^{j-i}$ preserves the properties of associativity, commutativity, and identity w.r.t. 1. Moreover, the properties 3 and 4 of Definition 8.1 ensure that the branch $\mathcal{B}$ remains non contradictory. Since now, we cannot have more than k occurrences of c in the labels, we have treated the redundancy of $\mathcal{B}$ w.r.t. the constant c and we apply the induction hypothesis to obtain the result. □

Theorem 8.2 (finite model property). If $I \nvdash \phi$ then there is a finite topological resource model such that $I \not\models \phi$.

Proof. If $I \nvdash \phi$ there is no closed tableau for ϕ. Thus, the tableau construction procedure yields a tableau with a completed branch $\mathcal{B}$. Then, by Theorem 8.1, we can build a finite topological resource model such that $I \not\models \phi$. □

Hence, we have the following result:

Theorem 8.3 (decidability). Propositional **BI** is decidable.

Proof. The tableau construction procedure, which is a semi-decision procedure, can be improved into a decision procedure by taking both the liberalized versions of the expansion rules and the notion of redundancy into account. Since, under the liberalized rules, an open branch cannot infinitely grow without becoming redundant, the termination of the procedure can be enforced by stopping the (potentially infinite) construction of a completed branch as soon as it has been recognized redundant. □

Note that full propositional linear logic, with exponentials, is undecidable even when restricted to the intuitionistic fragment, that the status of **MELL** is unknown, and that neither has the finite model property (Lafont 1997; Lincoln 1995).

By exploiting the capture of the semantics by labels, we have provided a decision procedure for **BI** which builds countermodels in Grothendieck topological semantics. Their study gives us a better understanding of the semantic information necessary to analyze provability and of the relationships between the elementary and topological settings.

9. Conclusion

Initially, resource tableaux were introduced for **BI** without $\bot$, with labels and constraints that directly capture the Kripke resource semantics that is complete for this logical fragment (Galmiche and Méry 2003). This paper presents new results for **BI** (with $\bot$), some of them being partially presented in (Galmiche et al. 2002). In this context, a first non-trivial problem was: is it possible to define resource tableaux for propositional **BI** (with $\bot$) and to capture the Grothendieck topological semantics that is complete for **BI**. In this paper, we provide a simple and nice solution

that is a new resource tableaux without introduction of new expansion rules but with a particular closure condition that deal with $\bot$ through labels.

We prove the soundness and completeness of the resource tableaux method with respect to this semantics and as consequences we deduce two strong new results for **BI** that are the decidability of propositional **BI** and the finite model property with respect to Grothendieck topological semantics. These results confirm that resource tableaux is the appropriate deductive framework for logics like **BI** in which different kinds of connectives cohabit and interact and thus from the capture of the semantics by labels, we provide a decision procedure for **BI** which builds countermodels in Grothendieck topological semantics. Their study gives us a better understanding of the semantic information necessary to analyze provability and of the relationships between the elementary and topological settings. From a proof-search perspective, we consider another non-trivial problem, namely, how to define a resource tableaux for **BI** with liberalized rules which improve efficiency in proof-search by reducing the number of new constants to deal with ? Therefore, we define such a liberalized resource tableaux that is based on a semantic analysis of the $\vee$ connective and a specific treatment that is needed to make liberalized rules work. The related extension of the label algebra involves a less direct construction of countermodels but surprisingly these countermodels are closely related to the topological Kripke semantics, that is complete for **BI** (Pym et al. 2004; Pym 2004), from which they can be viewed as an algebraic counterpart. These results emphasize the appropriateness of resource tableaux to deal with the different semantics which are available for **BI**.

Another important question arises from these relationships between semantics of **BI** and resource tableaux and mainly from the extraction of countermodels from the dependency graphs: it is possible to define a new semantics of **BI** such that a dependency graph can be directly considered as a countermodel ? We propose such a new Kripke semantics that can be seen as intermediate between the elementary and Grothendieck resource semantics. It emphasizes the central notion of dependency graph that captures the essential information necessary to analyze the provability in **BI** and leads to a simple semantics based on a partially defined monoid. The definition of this semantics is important by itself but the most interesting point is that it is strongly related to the semantics of (intuitionistic) "pointer logic" (Ishtiaq and O'Hearn 2001) and "separation logic" (Reynolds 2000), introduced in order to deal with mutable data structures.

Further work will be devoted to the study of pointer logic (Ishtiaq and O'Hearn 2001) and separation logic (Reynolds 2000), perhaps beginning with the specialization of resource tableaux to such logics. We will also study the relationships with other recent works on proof-search in **BI** based on free variable tableaux (Galmiche and Méry 2003) and on connection methods (Galmiche and Méry 2002). Moreover, as **BI** is conservative over intuitionistic logic (**IL**) and multiplicative intuitionistic linear logic (**MILL**) (O'Hearn and Pym 1999), these results on resource tableaux can be restricted to both logics in order to propose new proof-search methods based on labels and constraints. For instance, we will compare such a method for **IL** with existing methods from the efficiency and countermodels construction perspectives. In the case of **MILL** we will compare it with methods based on connections and proof nets construction (Galmiche 2000; Galmiche and Méry 2002). Moreover, from the semantic perspective, the impact of the results on **MILL** will be analyzed and compared with previous proposals about resource models (Galmiche and Larchey-Wendling 2000) and countermodels analysis. An important open prob-

lem is to provide a complete semantics for Boolean **BI** (*i.e.*, with classical additives) and so extend the analysis of this paper to the realm of classical pointer and separation logics.

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