

The Undecidability of Boolean BI through Phase Semantics

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Abstract

We solve the open problem of the decidability of Boolean BI logic (BBI), which can be considered as the core of separation and spatial logics. For this, we define a complete phase semantics for BBI and characterize it as trivial phase semantics. We deduce an embedding between trivial phase semantics for intuitionistic linear logic (ILL) and Kripke semantics for BBI. We single out a fragment of ILL which is both undecidable and complete for trivial phase semantics. Thus, we obtain the undecidability of BBI.

1. Introduction

The logic BI of bunched implications of Pym and O’Hearn [18] is a well-known sub-structural logic which freely combines additive connectives \wedge , \vee , \rightarrow and multiplicative connectives $*$, \multimap . The additives of BI behave either intuitionistically or classically giving rise to intuitionistic BI or Boolean BI (denoted BBI). The language of BI, and in particular its composition operators $*$ and \multimap , is at the heart of separation and spatial logics frameworks (see [15] for a discussion on these aspects).

It is striking that the proof-theoretical developments on BI have so far focused mainly on (intuitionistic) BI, especially since the numerous programs verification applications of BI – notably separation logic [9] or spatial logic [3] – are mainly based on its Boolean variant. Intuitionistic BI has been given a well behaved proof theory [20] composed of a bunched sequent calculus enjoying cut-elimination since its inception. Later, Galmiche *et al.* [6] gave BI a labelled tableaux system from which decidability was derived. On the contrary, the proof theory of BBI was reduced to the addition of a double negation principle to that of (intuitionistic) BI, as Pym did in [20]. For long, the main proof-theoretical result was the completeness of the corresponding Hilbert style system [5] and not much more. It was even unknown whether the relational Kripke semantics (corresponding to the Hilbert system) and

the partial monoidal Kripke semantics (corresponding to the labelled tableaux system) define the same notion of validity.

This situation evolved recently with two main families of results. On the one hand, in the spirit of his work with Calcagno on Classical BI [2], Brotherston [1] provided a Display Logic style proof system for relational BBI and derived a cut-elimination result from this Display framework. He then tried to obtain decidability with syntactic techniques similar to those Restall successfully used in relevant logics [21]. But for some fundamental logical reasons explained in this paper, his attempt was bound to fail. On the other hand, the authors recently obtained a sound and faithful embedding of BI into BBI (both defined with their partial deterministic Kripke semantics), illustrating the counter-intuitive fact that Boolean BI is surprisingly more expressive than intuitionistic BI [15]. The result is based on the study of the specific properties of the counter-models generated by proof-search in labelled tableaux systems.

Many questions remained open in relation to the proof theory and semantics of BBI:

1. Do the relational and the partial deterministic Kripke semantics define the same set of valid formulae?
2. Is validity decidable in either of these semantics?

In this paper we solve both questions and give them a negative answer. Indeed, we first show that the notion of invertibility, definable by the BBI formula $\mathcal{I} = \neg(\top \multimap \neg \bot)$, is not stable by composition in relational Kripke semantics whereas it is in partial deterministic Kripke semantics. Hence the formula $(\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$ distinguishes these two semantics. Then, we prove the main contribution of this paper which is *the undecidability of (universal) validity in BBI, be it relational BBI, partial deterministic BBI or even total deterministic BBI*.

We begin by exploring the relation between phase semantics and Kripke semantics in the context of BBI. Compared to the phase semantics of ILL, we characterize the phase semantics of BBI as trivial because it corresponds to the choice of the least stable closure: the identity closure.

We point out the direct correspondence between Kripke semantics and trivial phase semantics. From this correspondence, we derive a map from ILL sequents to BBI formulae that is a sound and faithful embedding, as soon as validity in ILL is defined by trivial phase semantics.

On the one hand, it could appear at first that we have only displaced the problem from the language of BBI to the language of ILL. On the other hand, the undecidability of various fragments of ILL is already known, but of course, not with validity defined by trivial phase semantics. It turns out we have changed the question: instead of searching for an undecidable fragment, we have to identify, among the known existing undecidable fragments of ILL, one which is at the same time complete for trivial phase semantics.

This fragment of ILL must include the bang ! connective because IMALL is decidable [13, 16]. As in phase semantics the definition of $\&$ and \multimap does not rely on the closure (as opposed to \oplus and \otimes), a naive idea would be to keep only those connectives in the desired fragment. If we consider the first fragments of linear logic that were proved undecidable like full propositional linear logic [16] or the $(!, \oplus)$ -Horn fragment [11], we observe that, unfortunately, they include both \otimes and \oplus . Recently, a fragment of IMELL denoted $\mathbf{s-IMELL}_0^\circ$ has been studied and characterized as equi-decidable to IMELL [4]. This fragment is important to us because it contains neither \otimes nor \oplus and it is very simple. However, decidability for IMELL is still an open question.

It turns out that $\mathbf{s-IMELL}_0^\circ$ is indeed complete for trivial phase semantics. We extend its goal-directed proof system [4] with the addition of the $\&$ connective, in the spirit of Kanovich's [11] and Lafont's [12] ideas for encoding Minsky machines in linear logic. The fragment obtained, denoted $\text{ILL}_{\multimap, \&, !}^0$, is sufficient to encode Minsky machines computations, because the $\&$ connective can be used to simulate forking. We show that, as with $\mathbf{s-IMELL}_0^\circ$, $\text{ILL}_{\multimap, \&, !}^0$ is still complete for trivial phase semantics.

2. Non-Deterministic Monoids, ILL and BBI

In this section, we define BBI and its non-deterministic (or relational) Kripke semantics, ILL and its non-deterministic phase semantics and establish a semantic link between those two logics: trivial phase semantics.

2.1. Non-Deterministic Monoids

Let us consider a set M . We denote by $\mathcal{P}(M)$ the powerset of M , i.e. its set of subsets. A binary function $\circ : M \times M \longrightarrow \mathcal{P}(M)$ is naturally extended to a binary operator on $\mathcal{P}(M)$ by $X \circ Y = \bigcup \{x \circ y \mid x \in X, y \in Y\}$ for any subsets X, Y of M . Using this extension, we can view an element m of M as the singleton set $\{m\}$ and derive the equations $m \circ X = \{m\} \circ X$ and $a \circ b = \{a\} \circ \{b\}$.

Definition 2.1. A non-deterministic (or relational) monoid is a triple (M, \circ, ϵ) where $\epsilon \in M$ and $\circ : M \times M \longrightarrow \mathcal{P}(M)$ for which the following conditions hold:

1. $\forall a \in M, \epsilon \circ a = \{a\}$ (neutrality)
2. $\forall a, b \in M, a \circ b = b \circ a$ (commutativity)
3. $\forall a, b, c \in M, a \circ (b \circ c) = (a \circ b) \circ c$ (associativity)¹

The extension of \circ to $\mathcal{P}(M)$ thus induces a commutative monoidal structure with unit element $\{\epsilon\}$ on $\mathcal{P}(M)$.

The term *non-deterministic* was introduced in [5] in order to emphasize the fact that the composition $a \circ b$ may yield not only one but an arbitrary number of results including the possible incompatibility of a and b in which case $a \circ b = \emptyset$. If (M, \bullet, \mathbf{e}) is a (usual) commutative monoid then, defining $a \circ b = \{a \bullet b\}$ and $\epsilon = \mathbf{e}$ induces a non-deterministic monoid (M, \circ, ϵ) . Using the bijection $x \mapsto \{x\}$ mapping elements of M to singletons in $\mathcal{P}(M)$, we can view (usual) commutative monoids as a particular case of non-deterministic monoids (later called total deterministic monoids). Partial monoids can also be represented using the empty set \emptyset as the result of undefined compositions.

The term *relational* is sometimes used because the map $\circ : M \times M \longrightarrow \mathcal{P}(M)$ can equivalently be understood as a ternary relation $- \circ - \ni - : M \times M \times M \longrightarrow \{0, 1\}$ through the Curry-Howard isomorphism and the axioms correspond to those of an internal monoid in the category of relations [7]. The two presentations are equivalent but we rather use the monoidal presentation in this paper because of the context of Kripke and phase semantics.

Definition 2.2. Let (M, \circ, ϵ) be a non-deterministic monoid. It is a partial deterministic monoid if for any $x, y \in M$, the composition $x \circ y$ is either empty or a singleton. (M, \circ, ϵ) is a total deterministic monoid if for any $x, y \in M$, the composition $x \circ y$ is a singleton. The class of non-deterministic (resp. partial deterministic, resp. total deterministic) monoids is denoted ND (resp. D, resp. T).

The reader may have noticed that total deterministic monoids (of class T) exactly correspond to those non-deterministic monoids derived from usual commutative monoid because the composition \circ is a functional relation (exactly one image for each couple of parameters). Another remark is that there is an obvious inclusion between those classes of monoids. The proofs of Theorem 2.6 and Proposition 2.7 contain witnesses of the strictness of the inclusions.

Proposition 2.3. $T \subsetneq D \subsetneq ND$

We now exploit the notion of non-deterministic monoid to establish links between the semantics of BBI and ILL.

¹ Associativity should be understood using the extension of \circ to $\mathcal{P}(M)$.

2.2. Kripke Semantics for BBI

The syntax of BBI is exactly the syntax of BI augmented with negation, although negation could be defined by $\neg A = A \rightarrow \perp$ like in classical logic. Thus, the formulae of BBI are defined as following: starting from a set Var , they are freely build using the *logical variables* in Var , the *logical constants* in $\{\top, \perp\}$, the unary connective \neg or the binary connectives in $\{*, \multimap, \wedge, \vee, \rightarrow\}$. Formally, the set of formulae is described by the following grammar: $A ::= v \mid c \mid \neg A \mid A \boxtimes A$ with $v \in \text{Var}$, $c \in \{\top, \perp\}$ and $\boxtimes \in \{*, \multimap, \wedge, \vee, \rightarrow\}$.

Validity in BBI has not always been unequivocally defined. Indeed, the initial proposition of Pym [20] was simply to add a double negation principle to the cut-free bunched proof system of BI. But of course, this does not lead to a proof-theoretically well behaved proof-system for BBI: it does not enjoy cut-elimination, sub-formula property, etc. Then, the syntax of BBI has been used as a foundation for various forms of separation logic with the common property that the additive operator \rightarrow is interpreted classically where it is interpreted intuitionistically in BI. The removal of the pre-order in the Kripke semantics is moreover necessary for the interpretation of classical negation \neg .

In this paper, we choose to present BBI as a family of logics defined by their Kripke semantics rather than proof-systems. Given a non-deterministic monoid (M, \circ, ϵ) and an interpretation $\delta : \text{Var} \rightarrow \mathcal{P}(M)$ of propositional variables, we define the Kripke forcing relation by induction on the structure of formulae:

$$\begin{aligned} m \Vdash v & \text{ iff } m \in \delta(v) \\ m \Vdash \perp & \text{ iff never} & m \Vdash \neg A & \text{ iff } m \not\Vdash A \\ m \Vdash \top & \text{ iff always} & m \Vdash \mathbf{I} & \text{ iff } m = \epsilon \\ m \Vdash A \wedge B & \text{ iff } m \Vdash A \text{ and } m \Vdash B \\ m \Vdash A \vee B & \text{ iff } m \Vdash A \text{ or } m \Vdash B \\ m \Vdash A \rightarrow B & \text{ iff } m \not\Vdash A \text{ or } m \Vdash B \\ m \Vdash A * B & \text{ iff } \exists a, b, m \in a \circ b \text{ and } a \Vdash A \text{ and } b \Vdash B \\ m \Vdash A \multimap B & \text{ iff } \forall a, b (b \in m \circ a \text{ and } a \Vdash A) \Rightarrow b \Vdash B \end{aligned}$$

A formulae F is *valid* in a non-deterministic monoid (M, \circ, ϵ) if for any interpretation $\delta : \text{Var} \rightarrow \mathcal{P}(M)$ of propositional variables, the relation $m \Vdash F$ holds for any $m \in M$. A *counter-model* of the formula F is given by a non-deterministic monoid (M, \circ, ϵ) , an interpretation $\delta : \text{Var} \rightarrow \mathcal{P}(M)$ and an element $m \in M$ such that $m \not\Vdash F$.

In some papers, you might find BBI defined by non-deterministic monoidal Kripke semantics [1, 5], in other papers it is defined by partial but deterministic monoidal Kripke semantics and generally separation logic models are particular instances of partial (deterministic) monoids. See [15] for a general discussion about these issues.

Definition 2.4. We denote by BBI_{ND} (resp. BBI_{D} , resp.

BBI_{T}) the set of formulae of BBI which are valid in every monoid of the class ND (resp. D, resp. T).

On the proof-theoretic side, we briefly recall that BBI_{ND} has been proved sound and complete w.r.t. a Hilbert proof-system [5] and also, more recently w.r.t. a Display Logic based proof-system [1] enjoying cut-elimination. BBI_{D} can be proved sound and complete w.r.t. the semantic constraints based tableaux proof-system presented in [15] (although only the soundness proof is presented in that particular paper) and the adaptation of this tableaux system to BBI_{T} should be straightforward (contrary to BBI_{ND}).

As it turns out, the three different classes of models ND, D and T define three different logics, i.e. universally valid formulae differ from one class of models to another. The relation of *strict inclusion* between BBI_{ND} and BBI_{D} was, to our knowledge, an undecided proposition.

Theorem 2.5. $\text{BBI}_{\text{ND}} \subsetneq \text{BBI}_{\text{D}} \subsetneq \text{BBI}_{\text{T}}$

Proof. The following inclusion relations $\text{T} \subseteq \text{D} \subseteq \text{ND}$ hold between the classes of models which respectively define those three logics. Hence, only the strictness of the inclusion of validities is not obvious. This strictness is established by upcoming Theorem 2.6 and Proposition 2.7. \square

Consider the formula $\mathcal{I} = \neg(\top \multimap \neg \mathbf{I})$ and a non-deterministic monoid (M, \circ, ϵ) . Since \mathcal{I} does not contain any variable, its Kripke interpretation does not depend on the choice of δ . One can check that for any $x \in M$, $x \Vdash \mathcal{I}$ iff there exists $x' \in M$ s.t. $\epsilon \in x \circ x'$. So \mathcal{I} expresses “invertibility” in Kripke semantics. The formula $(\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$ expresses stability of invertibility by monoidal composition.

Theorem 2.6. The formula $(\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$ is valid in every partial deterministic monoid, with $\mathcal{I} = \neg(\top \multimap \neg \mathbf{I})$. There exists a non-deterministic monoid which is a counter-model to $(\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$.

Proof. First the counter-model. Consider the following non-deterministic monoid $(\{\epsilon, x, y\}, \circ, \epsilon)$ where $x \circ x = \{\epsilon, y\}$ and $y \circ \alpha = \{y\}$ for any $\alpha \in \{\epsilon, x, y\}$. We let the reader check that the axioms of non-deterministic monoids hold for this structure. Then $x \Vdash \mathcal{I}$ because there exists α ($\alpha = x$) such that $\epsilon \in x \circ \alpha$. On the other hand, $y \not\Vdash \mathcal{I}$ because there is no α such that $\epsilon \in y \circ \alpha$ holds. So, as $y \in x \circ x$, we have $y \Vdash \mathcal{I} * \mathcal{I}$. Thus $y \not\Vdash (\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$.

Now let us prove that $(\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$ is valid in every partial deterministic monoid. Let (M, \circ, ϵ) be a partial deterministic monoid. Let us choose $a \in M$ and let us prove that $a \Vdash (\mathcal{I} * \mathcal{I}) \rightarrow \mathcal{I}$. So we suppose $a \Vdash \mathcal{I} * \mathcal{I}$ holds and we have to prove $a \Vdash \mathcal{I}$. As $a \Vdash \mathcal{I} * \mathcal{I}$, there exist $b, c \in M$ such that $a \in b \circ c$, $b \Vdash \mathcal{I}$ and $c \Vdash \mathcal{I}$. Thus there exist $b', c' \in M$ such that $\epsilon \in b \circ b'$ and $\epsilon \in c \circ c'$. As M is (partial) deterministic, we have $b \circ b' = \{\epsilon\}$, $c \circ c' = \{\epsilon\}$ and $b \circ c = \{a\}$. Thus we have $(b \circ b') \circ (c \circ c') = \{\epsilon\} \circ \{\epsilon\} = \{\epsilon\}$.

If $b' \circ c' = \emptyset$ then we would have $(b \circ c) \circ (b' \circ c') = \{a\} \circ \emptyset = \emptyset$ but also $(b \circ b') \circ (c \circ c') = \{\epsilon\}$ and thus $\emptyset = \{\epsilon\}$ by associativity/commutativity, which is absurd. Thus $b' \circ c' = \{a'\}$ and we obtain $(b \circ c) \circ (b' \circ c') = \{a\} \circ \{a'\} = a \circ a'$ and then $a \circ a' = \{\epsilon\}$ by associativity/commutativity. Hence, $\epsilon \in a \circ a'$ and $a \Vdash \mathcal{I}$. \square

The formula $(\neg \mid \multimap \perp) \rightarrow \mid$ is inspired from the example given to establish the incompleteness of (total) monoidal Kripke semantics w.r.t. (intuitionistic) BI (see [20] page 63).

Proposition 2.7. *The formula $(\neg \mid \multimap \perp) \rightarrow \mid$ is valid in every total deterministic monoid. There exists a partial deterministic monoid which is a counter-model to $(\neg \mid \multimap \perp) \rightarrow \mid$.*

The proof can be found in Appendix A. Having defined the Kripke semantics of BBI within the framework of non-deterministic monoids, let us consider non-deterministic phase semantics for ILL.

2.3. Non-Deterministic Phase Spaces for ILL

The formulae of ILL are defined by the following grammar: $A ::= v \mid c \mid !A \mid A \boxtimes A$ with $v \in \text{Var}$, $c \in \{1, \top, \perp\}$ and $\boxtimes \in \{\otimes, \multimap, \&, \oplus\}$. A *sequent* is a pair denoted $\Gamma \vdash A$ where Γ is a *multiset of formulae* and A is a *single formula*. The *sequent calculus Sill* (see Figure 1) is provided for ILL and the *set of derivable sequents* is the least set closed under its rules. Notice that Γ, Δ denote multisets of formulae and A, B, C denote formulae. In rule $\langle !R \rangle$, $! \Gamma$ denotes the multiset $! \Gamma = !A_1, \dots, !A_k$ if $\Gamma = A_1, \dots, A_k$.

The notion of sequent calculus proof is defined as usual: an ordered tree where each node together with its sons correspond to an instance of one of the rules of Sill. Hence, a sequent is derivable if and only if there exists a proof of it in Sill. By historical definition of ILL [8], the sequents which are provable in Sill are exactly the *valid sequents* of ILL, and a formula A of ILL is valid if $\vdash A$ is a valid sequent.

We extend the notion of intuitionistic phase space [8] to non-deterministic monoids and show that this semantic interpretation is sound and complete w.r.t. Sill, and thus equivalent to the original notion (see Corollary 2.12).

Definition 2.8. *A non-deterministic (intuitionistic) phase space is given by a non-deterministic monoid $\mathcal{M} = (M, \circ, \epsilon)$ together with a stable closure operator $(\cdot)^\diamond : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ and a sub-monoid K included in $J = \{x \in M \mid x \in \{\epsilon\}^\diamond \cap (x \circ x)^\diamond\}$.*

The *closure property* corresponds to the condition $X \subseteq Y^\diamond$ iff $X^\diamond \subseteq Y^\diamond$ for any $X, Y \in \mathcal{P}(M)$. We recall that the monoidal composition \circ is naturally extended to $\mathcal{P}(M)$ by $X \circ Y = \bigcup \{x \circ y \mid x \in X, y \in Y\}$ providing a (commutative) monoidal structure on $\mathcal{P}(M)$ with unit $\{\epsilon\}$.

The *stability property*² corresponds to the condition $X^\diamond \circ Y^\diamond \subseteq (X \circ Y)^\diamond$ for any $X, Y \in \mathcal{P}(M)$. A subset X of M is $(\cdot)^\diamond$ -closed (or simply closed when the closure is obvious from the context) if $X^\diamond = X$ or equivalently $X^\diamond \subseteq X$. The set of closed subsets is denoted $\mathcal{M}^\diamond = \{X \in \mathcal{P}(M) \mid X^\diamond = X\}$, not to be confused with M^\diamond where M is viewed as the (total) subset of M (and in this case, $M^\diamond = M$). Any intersection of closed subsets is a closed subset and thus \mathcal{M}^\diamond is invariant under arbitrary intersections, inducing a complete lattice structure on $(\mathcal{M}^\diamond, \subseteq)$.

The set K is just a *sub-monoid* of \mathcal{M} included in J , i.e. K verifies $\epsilon \in K \subseteq J$ and $K \circ K \subseteq K$. Let \multimap be the binary operator on $\mathcal{P}(M)$ defined by $X \multimap Y = \{k \in M \mid k \circ X \subseteq Y\}$. In the lattice $(\mathcal{P}(M), \subseteq)$, the operator \multimap is contra-variant in its first parameter and co-variant in its second and $K \subseteq X \multimap Y$ iff $K \circ X \subseteq Y$ for any $K, X, Y \in \mathcal{P}(M)$. By stability of the closure operator $(\cdot)^\diamond$, the subset $X \multimap Y$ is closed as soon as Y is closed and $X \multimap Y^\diamond = X^\diamond \multimap Y^\diamond$ for any $X, Y \in \mathcal{P}(M)$.

We see that we have a (quite direct) generalization of the usual notion of phase space in the case where the monoid is neither supposed to be total nor deterministic. In the particular case of total deterministic monoids, we recover the usual notion of phase space.

The interpretation of ILL connectives is done in the following way. Given an *interpretation* $\llbracket \cdot \rrbracket : \text{Var} \rightarrow \mathcal{M}^\diamond$ of logical variables into closed subsets, this interpretation is extended to all the formulae of ILL by structural induction as following:

$$\begin{aligned} \llbracket \perp \rrbracket &= \emptyset^\diamond & \llbracket A \oplus B \rrbracket &= (\llbracket A \rrbracket \cup \llbracket B \rrbracket)^\diamond \\ \llbracket \top \rrbracket &= M & \llbracket A \& B \rrbracket &= \llbracket A \rrbracket \cap \llbracket B \rrbracket \\ \llbracket 1 \rrbracket &= \epsilon^\diamond & \llbracket A \otimes B \rrbracket &= (\llbracket A \rrbracket \circ \llbracket B \rrbracket)^\diamond \\ \llbracket !A \rrbracket &= (K \cap \llbracket A \rrbracket)^\diamond & \llbracket A \multimap B \rrbracket &= \llbracket A \rrbracket \multimap \llbracket B \rrbracket \end{aligned}$$

Again, when the interpretation is done in a total deterministic monoid, we obtain exactly the same value for $\llbracket A \rrbracket$ as in the usual phase semantics interpretation. A sequent $A_1, \dots, A_k \vdash B$ of ILL is *valid in the interpretation* $\llbracket \cdot \rrbracket$ if $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \subseteq \llbracket B \rrbracket$. We recall the soundness theorem which states that provability in Sill entails semantic validity in non-deterministic intuitionistic phase semantics.

Theorem 2.9. *If the sequent $A_1, \dots, A_k \vdash B$ has a proof in Sill, then the inclusion $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \subseteq \llbracket B \rrbracket$ holds.*

Proof. The proof of this theorem can be done directly by generalizing the soundness proof of usual phase semantics [8], or else, as done in Appendix C, by using the algebraic semantic characterization of ILL of [22]. \square

Definition 2.10. *We denote by ILL_p the set of sequents which have a proof in Sill. We denote by ILL_{ND} (resp. ILL_{D} ,*

²We call it “stability” but it seems that there is no well established terminology for this property throughout the literature.

$$\begin{array}{c}
\frac{}{A \vdash A} \langle \text{id} \rangle \quad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \langle \text{cut} \rangle \quad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \langle w \rangle \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \langle c \rangle \quad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \langle !L \rangle \quad \frac{! \Gamma \vdash B}{! \Gamma \vdash !B} \langle !R \rangle \\
\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \langle \otimes_L \rangle \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C} \langle \multimap_L \rangle \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \langle \otimes_R \rangle \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \langle \multimap_R \rangle \\
\frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \langle \&_L^1 \rangle \quad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C} \langle \&_L^2 \rangle \quad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C} \langle \oplus_L \rangle \quad \frac{}{\Gamma, \perp \vdash A} \langle \perp_L \rangle \quad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \langle 1_L \rangle \\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \langle \&_R \rangle \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \langle \oplus_R^1 \rangle \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \langle \oplus_R^2 \rangle \quad \frac{}{\Gamma \vdash \top} \langle \top_R \rangle \quad \frac{}{\vdash 1} \langle 1_R \rangle
\end{array}$$

Figure 1. Sequent calculus Sill for ILL

resp. ILL_T) the set of sequents which are valid in every non-deterministic phase semantic interpretation where the base monoid is of the class ND (resp. D, resp. T).

Let us consider the following inclusion sequence:

$$\text{ILL}_p \subseteq \text{ILL}_{\text{ND}} \subseteq \text{ILL}_D \subseteq \text{ILL}_T \subseteq \text{ILL}_p \quad (1)$$

The first inclusion $\text{ILL}_p \subseteq \text{ILL}_{\text{ND}}$ is given by Theorem 2.9. The two following inclusions $\text{ILL}_{\text{ND}} \subseteq \text{ILL}_D \subseteq \text{ILL}_T$ are obvious consequences of the inclusions $T \subseteq D \subseteq \text{ND}$ between classes of non-deterministic monoids. The last inclusion $\text{ILL}_T \subseteq \text{ILL}_p$ is just a reformulation of the completeness of the (usual) phase semantics w.r.t. Sill:

Proposition 2.11. *If $\Gamma \vdash A$ is valid in every non-deterministic phase semantic interpretation $(M, \circ, \epsilon, (\cdot)^\circ, K, \llbracket \cdot \rrbracket)$ with (M, \circ, ϵ) of the class T, then $\Gamma \vdash A$ has a proof in Sill.*

Proof. Total deterministic monoids (of the class T) are in one to one correspondence with (usual) commutative monoids and this correspondence trivially extends to phase semantics. The result is simply a reformulation of the completeness of usual phase semantics [8, 22] for ILL. \square

Corollary 2.12. $\text{ILL}_p = \text{ILL}_{\text{ND}} = \text{ILL}_D = \text{ILL}_T$ and non-deterministic intuitionistic phase semantics is both sound and complete w.r.t. Sill.

Proof. With Proposition 2.11, we have closed the circular inclusion sequence (1). In particular $\text{ILL}_p = \text{ILL}_{\text{ND}}$. \square

2.4. Trivial Phase vs. Kripke Semantics

In this section, we define trivial phase semantics which is a particular case of phase semantics that can be viewed as Kripke semantics put in a particular form.

Definition 2.13. *Given a non-deterministic monoid $\mathcal{M} = (M, \circ, \epsilon)$, the trivial phase space is defined by taking the identity map on $\mathcal{P}(M)$ as closure operator (i.e. for all $X \in \mathcal{P}(M)$, $X^\circ = X$) and by taking $K = \{\epsilon\}$.*

It is clear that the identity on $\mathcal{P}(M)$ is both a closure and stable. Obviously also, $K = \{\epsilon\}$ verifies the conditions $\epsilon \in K \subseteq J$ and $K \circ K \subseteq K$.³ In a trivial phase space, every subset of M is closed and thus $\mathcal{M}^\circ = \mathcal{P}(M)$. The interpretation of ILL connectives becomes:

$$\begin{array}{ll}
\llbracket \perp \rrbracket = \emptyset & \llbracket A \oplus B \rrbracket = \llbracket A \rrbracket \cup \llbracket B \rrbracket \\
\llbracket \top \rrbracket = M & \llbracket A \& B \rrbracket = \llbracket A \rrbracket \cap \llbracket B \rrbracket \\
\llbracket 1 \rrbracket = \{\epsilon\} & \llbracket A \otimes B \rrbracket = \llbracket A \rrbracket \circ \llbracket B \rrbracket \\
\llbracket !A \rrbracket = \{\epsilon\} \cap \llbracket A \rrbracket & \llbracket A \multimap B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket
\end{array}$$

Replacing $1/\perp$, \oplus/\vee , $\&/\wedge$, $\otimes/*$ and \multimap/\rightarrow in the previous equations and defining $\llbracket A \rightarrow B \rrbracket = M \setminus \llbracket A \rrbracket \cup \llbracket B \rrbracket$ and $\llbracket \neg A \rrbracket = M \setminus \llbracket A \rrbracket$ provides a complete (non-deterministic) trivial phase semantics to BBI, in direct correspondance to its Kripke semantics. Thus, there is an embedding of the connectives of ILL into BBI, which can be formalized with the following inductively defined map $(\cdot)^* : \text{ILL} \rightarrow \text{BBI}$:

$$\begin{array}{ll}
v^* = v & \text{for } v \in \text{Var} \\
\perp^* = \perp & (A \oplus B)^* = A^* \vee B^* \\
\top^* = \top & (A \& B)^* = A^* \wedge B^* \\
1^* = 1 & (A \otimes B)^* = A^* * B^* \\
(!A)^* = 1 \wedge A^* & (A \multimap B)^* = A^* \rightarrow B^*
\end{array}$$

Lemma 2.14. *In trivial phase semantics, if the phase interpretation $\llbracket \cdot \rrbracket : \text{Var} \rightarrow \mathcal{M}^\circ$ and the Kripke interpretation $\delta : \text{Var} \rightarrow \mathcal{P}(M)$ are identical maps then the phase semantics and the Kripke semantics are in the following relation:*

$$\forall F \in \text{ILL}, \forall m \in M, m \in \llbracket F \rrbracket \text{ iff } m \Vdash F^* \quad (2)$$

Proof. By induction on the structure of F . We consider the case $F = A \otimes B$ as an example. Let $m \in M$. We have $m \in \llbracket A \otimes B \rrbracket$ iff $m \in \llbracket A \rrbracket \circ \llbracket B \rrbracket$ iff $\exists a \in \llbracket A \rrbracket, \exists b \in \llbracket B \rrbracket, m \in a \circ b$ iff $\exists a, b, (a \Vdash A^* \text{ and } b \Vdash B^* \text{ and } m \in a \circ b)$ iff $m \Vdash A^* * B^*$ iff $m \Vdash (A \otimes B)^*$. \square

So if the interpretation of logical variables coincide, trivial phase semantics and Kripke semantics correspond

³In fact, there is no other choice for K because $J = \{x \in M \mid x \in \{\epsilon\}^\circ \cap (x \circ x)^\circ\} = \{\epsilon\}$ when $(\cdot)^\circ$ is the identity map on $\mathcal{P}(M)$.

to each other through the map $(\cdot)^{\circledast}$. Given a *sequence* $S = A_1, \dots, A_k$ of formulae of ILL, we define S^{\circledast} by structural induction on S :

$$(\cdot)^{\circledast} = \text{I} \quad (A_1, \dots, A_{k+1})^{\circledast} = A_1^{\circledast} * (A_2, \dots, A_{k+1})^{\circledast}$$

When $\llbracket \cdot \rrbracket$ and δ are identical maps on propositional variables, it is then straightforward to prove the equivalence by induction on k :

$$m \in \llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \quad \text{iff} \quad m \Vdash (A_1, \dots, A_k)^{\circledast} \quad (3)$$

Lemma 2.15. *If the sequent $A_1, \dots, A_n \vdash B$ has a proof in Sill then the formula $(A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$ is valid in BBI_{ND} .*

Proof. Let us suppose that the sequent $A_1, \dots, A_k \vdash B$ is provable in Sill. We show that $(A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$ is valid in BBI_{ND} . Let us consider a non-deterministic monoid (M, \circ, ϵ) and an interpretation $\delta : \text{Var} \rightarrow \mathcal{P}(M)$. For the non-deterministic trivial phase space associated to (M, \circ, ϵ) , we choose the phase interpretation $\llbracket v \rrbracket = \delta(v)$ for any variable $v \in \text{Var}$. By soundness of non-deterministic phase semantics (see Theorem 2.9), we obtain the inclusion $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \subseteq \llbracket B \rrbracket$. Then, by (2) and (3), for any $m \in M$ we have $m \Vdash (A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$. As this holds for any non-deterministic Kripke interpretation, we deduce that $(A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$ is valid in BBI_{ND} . \square

Lemma 2.16. *If the sequent $A_1, \dots, A_k \vdash B$ has a counter-model in total deterministic trivial phase semantics then the formula $(A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$ is invalid in BBI_{T} .*

Proof. Let us suppose that there exists a total deterministic monoid (M, \circ, ϵ) (of class T) and a trivial phase semantics interpretation $\llbracket \cdot \rrbracket : \text{Var} \rightarrow \mathcal{P}(M)$ such that $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \not\subseteq \llbracket B \rrbracket$. Considering the Kripke interpretation defined by $\delta(v) = \llbracket v \rrbracket$ for any $v \in \text{Var}$, by equivalences (2) and (3), there exists $m \in M$ such that $m \not\Vdash (A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$. So $(A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$ has a Kripke counter-model in the class T. Hence, this formula is invalid in BBI_{T} . \square

3. The Undecidability of Boolean BI

From the preceding results, we establish the undecidability of BBI. We define a reverse map from multisets of formulae of ILL into lists of formulae by choosing an arbitrary *computable total order* between the formulae of ILL (e.g. lexicographic ordering). Then, for any multiset Γ of formulae of ILL, there exists a unique and computable ordered sequence of formulae A_1, \dots, A_k s.t. $\Gamma = \{A_1, \dots, A_k\}$ and we define $\Gamma^{\circledast} = (A_1, \dots, A_k)^{\circledast}$. The map $(\cdot)^{\circledast} : \text{ILL} \rightarrow \text{BBI}$ defined by $(\Gamma \vdash B) \mapsto (\Gamma^{\circledast} \rightarrow B^{\circledast})$ is thus a computable map from sequents of ILL into formulae of BBI.

We introduce the key result that links undecidability in ILL and in BBI. The fragment $\text{ILL}_{\rightarrow, \&, !}^0$ is a recursive subset of the set of sequents of ILL (see Definition 3.4).

Theorem 3.1. *In $\text{ILL}_{\rightarrow, \&, !}^0$, validity is both undecidable and complete for total deterministic trivial phase semantics.*

The proof of this theorem spans over the remaining sections of the paper (see Theorems 3.7 and 3.14). There is no ambiguity: “recursive” applies to the membership predicate whereas “undecidable” applies to the validity predicate.

Theorem 3.2. *Let $x \in \{\text{ND}, \text{D}, \text{T}\}$. The restricted map $(\cdot)^{\circledast} : \text{ILL}_{\rightarrow, \&, !}^0 \rightarrow \text{BBI}_x$ is a sound and faithful embedding.*

Proof. Let us consider a given sequent $\Gamma \vdash B$ of $\text{ILL}_{\rightarrow, \&, !}^0$. Let $\Gamma = \{A_1, \dots, A_k\}$ where the sequence A_1, \dots, A_k is sorted according to the previously chosen total order. Then $(\Gamma \vdash B) = (A_1, \dots, A_k \vdash B)$ and $\Gamma^{\circledast} \rightarrow B^{\circledast} = (A_1, \dots, A_k)^{\circledast} \rightarrow B^{\circledast}$. On the one hand, if $\Gamma \vdash B$ is valid in ILL, then it has a proof in Sill and thus, according to Lemma 2.15, the formula $\Gamma^{\circledast} \rightarrow B^{\circledast}$ is valid in BBI_{ND} . It is thus valid in BBI_x because $\text{BBI}_{\text{ND}} \subseteq \text{BBI}_x$ holds (Theorem 2.5). On the other hand, if $\Gamma \vdash B$ is invalid in ILL, then as it belongs to the fragment $\text{ILL}_{\rightarrow, \&, !}^0$ which is complete w.r.t. total deterministic trivial phase semantics, it has a counter-model in total deterministic trivial phase semantics. By Lemma 2.16, the formula $\Gamma^{\circledast} \rightarrow B^{\circledast}$ is invalid in BBI_{T} . It is thus invalid in BBI_x because $\text{BBI}_x \subseteq \text{BBI}_{\text{T}}$ holds (Theorem 2.5). \square

Theorem 3.3. *(Universal) validity in the logic BBI_{ND} (resp. BBI_{D} , resp. BBI_{T}) is undecidable.*

Proof. Indeed, for any $x \in \{\text{ND}, \text{D}, \text{T}\}$, by Theorem 3.2, a decision procedure for BBI_x would lead to a decision procedure for $\text{ILL}_{\rightarrow, \&, !}^0$ by composition with the obviously computable map $(\cdot)^{\circledast}$, which contradicts Theorem 3.1. \square

Before we explore the description of $\text{ILL}_{\rightarrow, \&, !}^0$ and the proof of Theorem 3.1, we point out the inclusion sequence $\text{ILL}_{\text{ND}}^t \subseteq \text{ILL}_{\text{D}}^t \subseteq \text{ILL}_{\text{T}}^t$ as a remaining open question where ILL_x^t is defined by trivial phase semantics with the monoid belonging to the class $x \in \{\text{ND}, \text{D}, \text{T}\}$. The question is of course: *are these two inclusions strict?* We remark that the counter-examples of Theorem 2.6 and Proposition 2.7 cannot be used as is because both formulae contain a negation.

3.1. A Trivially Complete Fragment of ILL

We define and characterize an extension of the fragment s-IMELL_0° of ILL [4] which we denote $\text{ILL}_{\rightarrow, \&, !}^0$. We provide a simple goal-directed proof system, denoted Gill^0 , which is itself an extension of the goal-directed proof system of s-IMELL_0° , obtained by the addition of a new additive rule. Then we show that the proof system Gill^0 and

$$\begin{array}{c}
\frac{}{! \Sigma, a \vdash a} \langle \text{Ax} \rangle \quad \frac{! \Sigma, \Gamma \vdash a}{! \Sigma, \Gamma \vdash b} a \multimap b \in \Sigma \quad \frac{! \Sigma, \Gamma, a \vdash b}{! \Sigma, \Gamma \vdash c} (a \multimap b) \multimap c \in \Sigma \\
\\
\frac{! \Sigma, \Gamma \vdash a \quad ! \Sigma, \Delta \vdash b}{! \Sigma, \Gamma, \Delta \vdash c} a \multimap (b \multimap c) \in \Sigma \quad \frac{! \Sigma, \Gamma \vdash a \quad ! \Sigma, \Gamma \vdash b}{! \Sigma, \Gamma \vdash c} (a \& b) \multimap c \in \Sigma
\end{array}$$

Figure 2. Gill⁰: a goal-directed sequent calculus for ILL⁰_{→, &, !}

trivial phase semantics are both sound and complete w.r.t. the fragment ILL⁰_{→, &, !}.

Definition 3.4. A formula of ILL is (→, &)-elementary if it is of the form $u \multimap v$, $(u \multimap v) \multimap w$, $u \multimap (v \multimap w)$ or $(u \& v) \multimap w$ where u , v and w are logical variables. The sequents of the fragment ILL⁰_{→, &, !} are those of the form $! \Sigma, \Gamma \vdash c$ where Γ is a multiset of variables, c is a variable and Σ is a multiset of (→, &)-elementary formulae.

From this definition, it is obvious that membership in the fragment ILL⁰_{→, &, !} is a recursive property. Compared to s-IMELL⁰_→, the only new form is $(u \& v) \multimap w$. The validity of sequents in ILL⁰_{→, &, !} can be established using the proof system Sill but we rather provide an alternative goal-directed proof system called Gill⁰ in Figure 2. Apart for the axiom rule ⟨Ax⟩, each other rule ⟨→⟩, ⟨(→)→⟩, ⟨→(→)⟩ or ⟨(&)→⟩ is named according to the form of its side condition. Compared to s-IMELL⁰_→, the only new rule is ⟨(&)→⟩ (see [4]). In this paper, the authors did not provide a proof of soundness/completeness of the system s-IMELL⁰_→, letting it to the reader. Here we present a full proof of soundness/completeness for our extension Gill⁰ not only to please the reader but also to derive completeness of the fragment w.r.t. trivial phase semantics.

Hence, even though validity in ILL⁰_{→, &, !} is the same as in the whole ILL (established for instance by a proof in Sill), here we show that in this specific fragment, validity is also sound and complete both w.r.t. the system Gill⁰ and w.r.t. total deterministic trivial phase semantics.

Lemma 3.5. Every proof of a sequent in Gill⁰ can be transformed into a proof (of the same sequent) which uses only rules ⟨id⟩, ⟨w⟩, ⟨c⟩, ⟨→_L⟩, ⟨→_R⟩, ⟨!_L⟩ and ⟨&_R⟩ of Sill.

Lemma 3.6. If the sequent $! \Sigma, \Gamma \vdash c$ of ILL⁰_{→, &, !} is valid in every total deterministic trivial phase semantic interpretation then it has a proof in Gill⁰.

Proof. The proof of Lemma 3.5 can be found in Appendix B. For the proof of Lemma 3.6, we apply a technique similar to the one of Okada [19] for obtaining strong completeness through phase semantics. Let us consider a fixed multiset Σ of (→, &)-elementary formulae. We consider the free commutative monoid over the set of logical variables, i.e. the set of finite multisets of variables endowed

with multiset addition (denoted by the comma) as monoidal composition and with the empty multiset (denoted $\epsilon = [\emptyset]$) as neutral element. We write $[a, a, b]$ for the multiset composed of two occurrences of a and one of b . Let us define the total deterministic monoid $(M, \circ, \epsilon = [\emptyset])$ where M is the set of finite multisets of variables and $\circ : M \times M \rightarrow \mathcal{P}(M)$ is defined by $[\Gamma] \circ [\Delta] = \{[\Gamma, \Delta]\}$.

We define the following semantic interpretation in the trivial phase space based on (M, \circ, ϵ) :

$$[[c]] = \{[\Gamma] \mid ! \Sigma, \Gamma \vdash c \text{ has a proof in Gill}^0\} \text{ for } c \in \text{Var}$$

Let us now show that $\epsilon \in [[\sigma]]$ holds for any $\sigma \in \Sigma$. We pick one $\sigma \in \Sigma$ and proceed by case analysis.

If $\sigma = u \multimap v$. Then $\epsilon \in [[u \multimap v]]$ iff $\epsilon \circ [[u]] \subseteq [[v]]$ iff $[[u]] \subseteq [[v]]$. So let us consider one $[\Gamma]$ such that $[\Gamma] \in [[u]]$. Let us prove that $[\Gamma] \in [[v]]$. By definition of $[[u]]$, the sequent $! \Sigma, \Gamma \vdash u$ has a proof in Gill⁰. Then, by rule ⟨→⟩, the sequent $! \Sigma, \Gamma \vdash v$ has a proof in Gill⁰. So we deduce $[\Gamma] \in [[v]]$. Hence $[[u]] \subseteq [[v]]$ and we obtain $\epsilon \in [[\sigma]]$.

If $\sigma = (u \multimap v) \multimap w$. We have $\epsilon \in [(u \multimap v) \multimap w]$ iff $[[u]] \multimap [[v]] \subseteq [[w]]$. Let us choose $[\Gamma] \in [[u]] \multimap [[v]]$. Then $\{[\Gamma]\} \circ [[u]] \subseteq [[v]]$. By rule ⟨Ax⟩, $! \Sigma, u \vdash u$ has a proof in Gill⁰ and thus $[u] \in [[u]]$. Thus $\{[\Gamma, u]\} = [\Gamma] \circ [u] \subseteq [[v]]$. Thus $! \Sigma, \Gamma, u \vdash v$ has a proof in Gill⁰. By rule ⟨(→)→⟩, $! \Sigma, \Gamma \vdash w$ has a proof in Gill⁰. We conclude $[\Gamma] \in [[w]]$. Thus $[[u]] \multimap [[v]] \subseteq [[w]]$ holds, hence $\epsilon \in [[\sigma]]$.

If $\sigma = u \multimap (v \multimap w)$. We have $\epsilon \in [[u \multimap (v \multimap w)]]$ iff $[[u]] \circ [[v]] \subseteq [[w]]$. Let us choose $[\Gamma] \in [[u]]$ and $[\Delta] \in [[v]]$ and let us prove $[\Gamma] \circ [\Delta] \subseteq [[w]]$. Both $! \Sigma, \Gamma \vdash u$ and $! \Sigma, \Delta \vdash v$ have a proof in Gill⁰. By rule ⟨→(→)⟩, the sequent $! \Sigma, \Gamma, \Delta \vdash w$ has a proof in Gill⁰. Thus $[\Gamma] \circ [\Delta] = \{[\Gamma, \Delta]\} \subseteq [[w]]$. We deduce $[[u]] \circ [[v]] \subseteq [[w]]$ and thus conclude $\epsilon \in [[\sigma]]$.

If $\sigma = (u \& v) \multimap w$. We have $\epsilon \in [(u \& v) \multimap w]$ iff $[[u]] \cap [[v]] \subseteq [[w]]$. If $[\Gamma] \in [[u]] \cap [[v]]$ then both $! \Sigma, \Gamma \vdash u$ and $! \Sigma, \Gamma \vdash v$ have a proof in Gill⁰. By rule ⟨(&)→⟩, the sequent $! \Sigma, \Gamma \vdash w$ has a proof in Gill⁰. Thus $[\Gamma] \in [[w]]$. We have proved that $[[u]] \cap [[v]] \subseteq [[w]]$ and we conclude $\epsilon \in [[\sigma]]$.

Now let $\Sigma = \{\sigma_1, \dots, \sigma_k\}$. Then for any $i \in [1, k]$ we have $[[! \sigma_i]] = \{\epsilon\} \cap [[\sigma_i]] = \{\epsilon\}$. Consider $\Gamma = [a_1, \dots, a_p]$ and let us suppose that the sequent $! \Sigma, \Gamma \vdash c$ of ILL⁰_{→, &, !} is valid in every total deterministic trivial phase semantics interpretation. As a particular case, it is valid in the inter-

pretation $(M, \circ, \epsilon, \llbracket \cdot \rrbracket)$ and thus the inclusion

$$\llbracket !\sigma_1 \rrbracket \circ \dots \circ \llbracket !\sigma_k \rrbracket \circ \llbracket a_1 \rrbracket \circ \dots \circ \llbracket a_p \rrbracket \subseteq \llbracket c \rrbracket$$

holds. By rule $\langle \text{Ax} \rangle$, for any $i \in [1, p]$ the sequent $! \Sigma, a_i \vdash a_i$ has a proof in Gill^0 and thus we have $\llbracket a_i \rrbracket \in \llbracket a_i \rrbracket$. Also remember that for any $i \in [1, k]$, we have $\epsilon \in \llbracket !\sigma_i \rrbracket$. So

$$\llbracket \Gamma \rrbracket \in \{ \llbracket a_1, \dots, a_p \rrbracket \} = \epsilon \circ \dots \circ \epsilon \circ \llbracket a_1 \rrbracket \circ \dots \circ \llbracket a_p \rrbracket \subseteq \llbracket c \rrbracket$$

holds and we conclude that $! \Sigma, \Gamma \vdash c$ has a proof in Gill^0 . \square

Theorem 3.7. *The system Gill^0 and total deterministic trivial phase semantics are both sound and complete for the fragment $\text{ILL}_{\neg, \&, !}^0$.*

Proof. For a given sequent $! \Sigma, \Gamma \vdash c$ of $\text{ILL}_{\neg, \&, !}^0$, if it has a proof in Gill^0 then, by Lemma 3.5, it has a proof in Sill . If $! \Sigma, \Gamma \vdash c$ is provable in Sill then, as a particular case of Theorem 2.9, it is valid in every total deterministic trivial phase semantics interpretation. Finally, if $! \Sigma, \Gamma \vdash c$ is valid in every total deterministic trivial phase semantics interpretation then, by Lemma 3.6, it is provable in Gill^0 . \square

3.2. Encoding Minsky machines in $\text{ILL}_{\neg, \&, !}^0$

We propose an encoding of two counter Minsky machines in the fragment $\text{ILL}_{\neg, \&, !}^0$ of ILL . Kanovich [10, 11] already proved that Minsky machines can be encoded into the $(!, \oplus)$ -Horn fragment of ILL . In his encoding, the recovery of computations from proofs is done through some form of proof normalization and the \oplus connective is used to simulate forking. Lafont later showed that the use of proof normalization can be avoided and replaced by a phase semantics argument [12, 14]. In our encoding of Minsky machines in $\text{ILL}_{\neg, \&, !}^0$, the $\&$ connective is used to simulate forking and we will show that a trivial phase semantics argument is sufficient to recover computability from provability.

Let a and b be two distinct counter symbols. A (deterministic) two counter Minsky machine is a pair (l, ψ) where $l > 0$ is a strictly positive natural number of instructions and

$$\psi : [1, l] \longrightarrow \uplus \begin{cases} \{+\} \times \{a, b\} \times [0, l] \\ \{-\} \times \{a, b\} \times [0, l] \times [0, l] \end{cases}$$

is a total map representing the list of instructions (here, \uplus represents disjoint set union). Minsky machines instructions (incrementation, zero test/decrementation) are encoded as illustrated in the two following examples:

$$\psi(1) = (+, a, 3) \rightsquigarrow 1: a := a + 1; \text{ goto } 3$$

$$\psi(2) = (-, b, 4, 5) \rightsquigarrow 2: \begin{cases} \text{if } b = 0 \text{ then goto } 4 \\ \text{else } b := b - 1; \text{ goto } 5 \end{cases}$$

Given a two counter Minsky machine $\mathcal{M} = (l, \psi)$, we define the set $\mathcal{S}(\mathcal{M})$ of states of the machine by $\mathcal{S}(\mathcal{M}) =$

$[0, l] \times \mathbb{N} \times \mathbb{N}$ and a (binary) transition relation between states $\rightarrow_{\mathcal{M}} \subseteq \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M})$. We define the following notations: $\bar{a} = (1, 0)$, $\bar{b} = (0, 1)$, $(m, n)_a = m$ and $(m, n)_b = n$. Then for any two states (i, m, n) and (i', m', n') , we define the relation $(i, m, n) \rightarrow_{\mathcal{M}} (i', m', n')$ by:

$$\begin{aligned} & \psi(i) = (+, x, i') \text{ and } (m', n') = (m, n) + \bar{x} \\ \text{or } & \psi(i) = (-, x, i', k), (m, n)_x = 0 \text{ and } (m', n') = (m, n) \\ \text{or } & \psi(i) = (-, x, j, i'), (m, n)_x \neq 0 \\ & \text{and } (m', n') + \bar{x} = (m, n) \end{aligned}$$

holds for some $x \in \{a, b\}$ and $j, k \in [0, l]$. Remark that $(i, m, n) \rightarrow_{\mathcal{M}} (i', m', n')$ does not hold if $i = 0$ because $\psi(0)$ is not defined. Let $\rightarrow_{\mathcal{M}}^*$ be the reflexive and transitive closure of the relation $\rightarrow_{\mathcal{M}}$. We say that the machine \mathcal{M} accepts the input (m, n) if starting from the state $(1, m, n)$, there exists a sequence of transitions leading to the state $(0, 0, 0)$ and we define the set $\mathcal{A}(\mathcal{M})$ of accepted inputs:

$$\mathcal{A}(\mathcal{M}) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid (1, m, n) \rightarrow_{\mathcal{M}}^* (0, 0, 0)\}$$

Theorem 3.8. *There exists a two counter Minsky machine \mathcal{M} for which the set $\mathcal{A}(\mathcal{M})$ is not recursive [17].*

Let us consider the two counter symbols a and b as two (different) logical variables and let us choose two new variables u and v so that the set $\{a, b, u, v\}$ has cardinal four. Let us choose an infinite set⁴ of new logical variables $\{q_i \mid i \in \mathbb{N}\}$ such that $q_i \neq q_j$ unless $i = j$ and $\{a, b, u, v\} \cap \{q_i \mid i \in \mathbb{N}\} = \emptyset$.

Let Σ_0 be the following multiset composed of five $(\neg, \&)$ -elementary formulae:

$$\Sigma_0 = \left\{ \begin{aligned} & a \neg (u \neg u), b \neg (v \neg v), \\ & (a \neg a) \neg u, (a \neg a) \neg v, (a \neg a) \neg q_0 \end{aligned} \right\}$$

We define the two abbreviations $\underline{a} = v$ and $\underline{b} = u$. Given a Minsky machine $\mathcal{M} = (l, \psi)$, for $i \in [1, l]$, we define the multisets $\Sigma_1, \dots, \Sigma_l$ of $(\neg, \&)$ -elementary formulae by:

$$\begin{aligned} \Sigma_i &= \{(x \neg q_j) \neg q_i\} \quad \text{when } \psi(i) = (+, x, j) \\ \text{and } \Sigma_i &= \{(\underline{x} \& q_j) \neg q_i, x \neg (q_k \neg q_i)\} \\ & \quad \text{when } \psi(i) = (-, x, j, k) \end{aligned}$$

Let $\Sigma_{\mathcal{M}}$ be the multiset $\Sigma_{\mathcal{M}} = \Sigma_0, \Sigma_1, \dots, \Sigma_l$. Given a natural number n and a logical variable $x \in \{a, b\}$, we define $x^n = x, x, \dots, x$ as the multiset composed of n occurrences of the variable x . Then, it is trivial to verify that for any natural numbers m, n and any $i \in [0, l]$, the sequent $! \Sigma_{\mathcal{M}}, a^m, b^n \vdash q_i$ belongs to the fragment $\text{ILL}_{\neg, \&, !}^0$.

Theorem 3.9. *For any two counter Minsky machine \mathcal{M} and for any pair $m, n \in \mathbb{N}$, we have $(m, n) \in \mathcal{A}(\mathcal{M})$ if and only if the sequent $! \Sigma_{\mathcal{M}}, a^m, b^n \vdash q_1$ is provable in Gill^0 .*

⁴For our particular purpose, we only need as many q_i 's as there are instructions in the Minsky machine obtained from Theorem 3.8.

If $\psi(i) = (-, x, j, k)$. Let us first prove that $(0, 0) \in \llbracket (x \& q_j) \multimap q_i \rrbracket$, i.e. $\llbracket x \rrbracket \cap \llbracket q_j \rrbracket \subseteq \llbracket q_i \rrbracket$. Let us consider $(m, n) \in \llbracket x \rrbracket \cap \llbracket q_j \rrbracket$. Then $(m, n)_x = 0$ and $(j, m, n) \rightarrow^* (0, 0, 0)$. Thus $(i, m, n) \rightarrow (j, m, n) \rightarrow^* (0, 0, 0)$ and $(m, n) \in \llbracket q_i \rrbracket$. Hence $\llbracket x \rrbracket \cap \llbracket q_j \rrbracket \subseteq \llbracket q_i \rrbracket$ holds. Let us finally prove that $(0, 0) \in \llbracket x \multimap (q_k \multimap q_i) \rrbracket$, i.e. $\llbracket x \rrbracket \circ \llbracket q_k \rrbracket \subseteq \llbracket q_i \rrbracket$. As $\llbracket x \rrbracket = \{\bar{x}\}$ for $x \in \{a, b\}$, let us choose an arbitrary pair $(m', n') \in \llbracket q_k \rrbracket$ and define $(m, n) = \bar{x} + (m', n')$. Then $(m, n)_x = 1 + (m', n')_x \neq 0$ and $(i, m, n) \rightarrow (k, m', n') \rightarrow^* (0, 0, 0)$. We obtain $(m, n) \in \llbracket q_i \rrbracket$ and thus conclude $\bar{x} + (m', n') \in \llbracket q_i \rrbracket$. Hence, for any $(m', n') \in \llbracket q_k \rrbracket$ we get $\llbracket x \rrbracket \circ (m', n') \subseteq \llbracket q_i \rrbracket$. Thus $\llbracket x \rrbracket \circ \llbracket q_k \rrbracket \subseteq \llbracket q_i \rrbracket$ holds.

As a consequence, for any $\sigma \in \Sigma$, we obtain $(0, 0) \in \llbracket \sigma \rrbracket$. The identity $\llbracket !\sigma \rrbracket = \{(0, 0)\}$ holds for any $\sigma \in \Sigma$. \square

Lemma 3.13. *For any $m, n \in \mathbb{N}$, if $!\Sigma, a^m, b^n \vdash q_1$ is provable in Gill^0 then $(m, n) \in \mathcal{A}(\mathcal{M})$ holds.*

Proof. Let $\Sigma = \{\sigma_1, \dots, \sigma_r\}$. We suppose that the sequent $!\Sigma, a^m, b^n \vdash q_1$ has a proof in Gill^0 . By the soundness part of Theorem 3.7, in our particular total deterministic trivial phase semantics interpretation, we have

$$\llbracket !\sigma_1 \rrbracket \circ \dots \circ \llbracket !\sigma_r \rrbracket \circ \llbracket a \rrbracket \circ \dots \circ \llbracket a \rrbracket \circ \llbracket b \rrbracket \circ \dots \circ \llbracket b \rrbracket \subseteq \llbracket q_1 \rrbracket$$

where a occurs m times and b occurs n times. By Proposition 3.12, we deduce $(m, n) = r \cdot (0, 0) + m \cdot (1, 0) + n \cdot (0, 1) \in \llbracket q_1 \rrbracket$ and thus $(1, m, n) \rightarrow^* (0, 0, 0)$ holds. \square

This concludes the proof of Theorem 3.9 as direct consequence of Lemma 3.11 and Lemma 3.13.

The reader may have noticed that more than the simple encoding of computability with provability, we can even show that computations and proofs match one to one. Even though this result is not necessary to our argumentation, this suggests that the system Gill^0 is a natural choice to illustrate the relations between Minsky machines and linear logic, may be more straightforward than the $(!, \oplus)$ -Horn fragment [11]. Whereas the decidability of s-IMELL_0° is still unclear (but nevertheless known to be equivalent to the decidability of MELL [4]), we have proved that the simple addition of the form $(\&) \multimap$ to s-IMELL_0° is sufficient to encode forking and thus, computations of Minsky machines.

Theorem 3.14. *Validity in $\text{ILL}_{\multimap, \&, !}^0$ is undecidable.*

Proof. By Theorem 3.8, let \mathcal{M} be a two counter Minsky machine s.t. $\mathcal{A}(\mathcal{M})$ is not recursive. Compute $\Sigma_{\mathcal{M}}$. If there is an algorithm that discriminates between provable and unprovable sequents of $\text{ILL}_{\multimap, \&, !}^0$, use it to decide $\mathcal{A}(\mathcal{M}) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid !\Sigma, a^m, b^n \vdash q_1 \text{ is provable in } \text{Gill}^0\}$. This identity is a direct consequence of Theorem 3.9. Thus $\mathcal{A}(\mathcal{M})$ would be recursive. We obtain a contradiction. \square

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A. A witness for $\mathbf{BBI}_T \not\subseteq \mathbf{BBI}_D$

Proposition A.1. *The formula $(\neg \vdash * \perp) \rightarrow \vdash$ is valid in every total deterministic monoid. There exists a partial deterministic monoid which is a counter-model to $(\neg \vdash * \perp) \rightarrow \vdash$.*

Proof. First the counter-model. Consider the following partial deterministic monoid $(\{\epsilon, x\}, \circ, \epsilon)$ where $x \circ x = \emptyset$. Then $x \neq \epsilon$ and thus $x \not\models \vdash$. Let us prove that $x \models \neg \vdash * \perp$. Let a, b such that $b \in x \circ a$ and $a \models \neg \vdash$. Then $a \neq \epsilon$ and thus $a = x$. Then $x \circ a = x \circ x = \emptyset$. We get a contradiction with $b \in x \circ a$. From this contradiction, we deduce $b \models \perp$. Hence, $x \models \neg \vdash * \perp$ and we conclude $x \not\models (\neg \vdash * \perp) \rightarrow \vdash$ and we have the counter-model.

Now let us prove that $(\neg \vdash * \perp) \rightarrow \vdash$ is valid in every total deterministic monoid. Let (M, \circ, ϵ) be a total deterministic monoid. Let us choose $a \in M$. There are two cases. Either $a = \epsilon$ or $a \neq \epsilon$. In the case $a = \epsilon$, we obviously have $a \models (\neg \vdash * \perp) \rightarrow \vdash$. In the case $a \neq \epsilon$, let us prove $a \models \neg \vdash * \perp$. Suppose $a \models \neg \vdash * \perp$. As $a \neq \epsilon$ we have $a \models \neg \vdash$. Also $a \circ a$ is not empty because \circ is total. Let $b \in a \circ a$. As $a \models \neg \vdash * \perp$, $b \in a \circ a$ and $a \models \neg \vdash$, we must have $b \models \perp$ which is impossible. Hence $a \not\models \neg \vdash * \perp$ and we conclude that $a \models (\neg \vdash * \perp) \rightarrow \vdash$ holds also in the case $a \neq \epsilon$. \square

B. The soundness of Gill⁰

Lemma B.1. *Every proof of a sequent in Gill⁰ can be transformed into a proof of the same sequent which uses only the rules $\langle \text{id} \rangle$, $\langle w \rangle$, $\langle c \rangle$, $\langle \neg \circ_L \rangle$, $\langle \neg \circ_R \rangle$, $\langle !_L \rangle$ and $\langle \&_R \rangle$ of Sill.*

We proceed by induction on the proofs in Gill⁰ and by case analysis, depending on the last rule applied. Let n be the cardinal of the multiset Σ . For each rule of Gill⁰, we propose the corresponding (open) proof tree in Sill:

Case of rule $\langle \text{Ax} \rangle$.

$$\frac{\frac{\frac{}{a \vdash a} \langle \text{id} \rangle}{a \vdash a} \langle w \rangle}{\vdots \text{ applied } n \text{ times}} \langle w \rangle$$

Case of rule $\langle \neg \circ \rangle$:

$$\frac{\frac{\frac{\frac{}{! \Sigma, \Gamma \vdash a} \langle \text{id} \rangle}{! \Sigma, \Gamma \vdash a} \langle \neg \circ_L \rangle}{! \Sigma, \Gamma, (a \neg \circ b) \vdash b} \langle !_L \rangle}{! \Sigma, \Gamma, !(a \neg \circ b) \vdash b} \langle c \rangle$$

Case of rule $\langle (\neg \circ) \neg \circ \rangle$:

$$\frac{\frac{\frac{\frac{}{! \Sigma, \Gamma, a \vdash b} \langle \text{id} \rangle}{! \Sigma, \Gamma \vdash a \neg \circ b} \langle \neg \circ_R \rangle}{c \vdash c} \langle \neg \circ_L \rangle}{! \Sigma, \Gamma, (a \neg \circ b) \neg \circ c \vdash c} \langle !_L \rangle$$

Case of rule $\langle \neg \circ (\neg \circ) \rangle$:

$$\frac{\frac{\frac{\frac{\frac{}{! \Sigma, \Delta \vdash b} \langle \text{id} \rangle}{c \vdash c} \langle \neg \circ_L \rangle}{! \Sigma, \Gamma \vdash a} \langle \neg \circ_L \rangle}{! \Sigma, \Gamma, ! \Sigma, \Delta, a \neg \circ (b \neg \circ c) \vdash c} \langle !_L \rangle}{! \Sigma, \Gamma, ! \Sigma, \Delta, !(a \neg \circ (b \neg \circ c)) \vdash c} \langle c \rangle$$

Case of rule $\langle (\&) \neg \circ \rangle$:

$$\frac{\frac{\frac{\frac{}{! \Sigma, \Gamma \vdash a} \langle \text{id} \rangle}{! \Sigma, \Gamma \vdash b} \langle \&_R \rangle}{! \Sigma, \Gamma \vdash a \& b} \langle \neg \circ_L \rangle}{! \Sigma, \Gamma, (a \& b) \neg \circ c \vdash c} \langle !_L \rangle$$

C. Soundness of non-deterministic phase semantics for ILL

Theorem C.1. *Let $\mathcal{M} = (M, \circ, \epsilon, (\cdot)^\diamond, K)$ be a non-deterministic intuitionistic phase space and $\llbracket \cdot \rrbracket : \text{Var} \rightarrow \mathcal{M}^\diamond$ be an interpretation of logical variables. If the sequent $A_1, \dots, A_k \vdash B$ has a proof in Sill, then the inclusion $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \subseteq \llbracket B \rrbracket$ holds.*

The proof could be done by induction on ILL proofs but we rather use the algebraic semantic characterization of ILL of [22]. We prove that

$$(\mathcal{M}^\diamond, \cap, (\cdot \cup \cdot)^\diamond, \emptyset^\diamond, \neg, (\cdot \circ \cdot)^\diamond, \{\epsilon\}^\diamond, (K \cap \cdot)^\diamond)$$

is an IL-algebra with storage operator (where \neg is defined by $X \neg Y = \{k \in M \mid k \circ X \subseteq Y\}$).

First, it is obvious that $(\mathcal{M}^\diamond, \cap, (\cdot \cup \cdot)^\diamond, \emptyset^\diamond)$ is a complete lattice with bottom \emptyset^\diamond . This is the same proof as in the usual (monoidal) case because the (non-deterministic) monoidal structure does not play any role in this part of the proof. The principal argument is that $(\cdot)^\diamond$ is a closure operator in $\mathcal{P}(M)$.

Let us prove that $(\mathcal{M}^\diamond, (\cdot \circ \cdot)^\diamond, \{\epsilon\}^\diamond)$ is a commutative monoid. Obviously the set \mathcal{M}^\diamond is stable under the operator $(\cdot \circ \cdot)^\diamond$ which thus induces a binary operation on \mathcal{M}^\diamond . By stability, we obtain the inclusion $\{\epsilon\}^\diamond \circ X^\diamond \subseteq (\{\epsilon\} \circ X)^\diamond = X^\diamond$ and we deduce that for any closed subset X (i.e. $X = X^\diamond$), we have $(\epsilon^\diamond \circ X)^\diamond \subseteq X$. Also $X = \{\epsilon\} \circ X \subseteq \{\epsilon\}^\diamond \circ X \subseteq (\{\epsilon\}^\diamond \circ X)^\diamond$ by monotonicity of \circ and $(\cdot)^\diamond$. Thus $(\epsilon^\diamond \circ X)^\diamond = X$ for any closed subset $X \in \mathcal{M}^\diamond$ and thus $\{\epsilon\}^\diamond$ is a (left) unit for $(\cdot \circ \cdot)^\diamond$. Then, it is obvious that $(\cdot \circ \cdot)^\diamond$ is a commutative operation because \circ is itself commutative. We deduce that $\{\epsilon\}^\diamond$ is a unit for $(\cdot \circ \cdot)^\diamond$.

Let us prove that $(\cdot \circ \cdot)^\diamond$ is associative. Let $A, B, C \in \mathcal{M}^\diamond$. Then, by stability of $(\cdot)^\diamond$, we have $A \circ (B \circ C)^\diamond \subseteq A^\diamond \circ (B \circ C)^\diamond \subseteq (A \circ (B \circ C))^\diamond = (A \circ B \circ C)^\diamond$. Thus $(A \circ (B \circ C)^\diamond)^\diamond \subseteq (A \circ B \circ C)^\diamond$ holds. As $A \circ B \circ C = A \circ (B \circ C) \subseteq A \circ (B \circ C)^\diamond \subseteq (A \circ (B \circ C)^\diamond)^\diamond$, we deduce $(A \circ B \circ C)^\diamond \subseteq (A \circ (B \circ C)^\diamond)^\diamond$. By double inclusion, we conclude that $(A \circ B \circ C)^\diamond = (A \circ (B \circ C)^\diamond)^\diamond$. Associativity follows from this last identity and associativity/commutativity of \circ on $\mathcal{P}(M)$.

It is obvious that $(\cdot \circ \cdot)^\diamond$ is monotonic in both parameters because it is obtained by composition of two monotonic operators, namely \circ and $(\cdot)^\diamond$. Let us now prove that \multimap is a right-adjoint $(\cdot \circ \cdot)^\diamond$. First, $X \multimap Y$ is closed as soon as Y is closed and $X \multimap Y^\diamond = X^\diamond \multimap Y^\diamond$ holds for any $X, Y \in \mathcal{P}(M)$ just as in the usual (monoidal) case. Now let $A, B, C \in \mathcal{M}^\diamond$. We have $(A \circ B)^\diamond \subseteq C$ iff $A \circ B \subseteq C$ iff $A \subseteq B \multimap C$. Thus \multimap is indeed right-adjoint to $(\cdot \circ \cdot)^\diamond$. The fact that \multimap is contra-variant w.r.t. its first operand and co-variant w.r.t. its second operand is deducible from the monotonicity of \circ and the fact that \multimap is right adjoint to \circ .

We finish by proving that $X \mapsto (K \cap X)^\diamond$ is a *modality*. First, for any $X \in \mathcal{M}^\diamond$, as $K \cap X \subseteq X = X^\diamond$, we obtain $(K \cap X)^\diamond \subseteq X$. Then for $X, Y \in \mathcal{M}^\diamond$, if we suppose that $(K \cap Y)^\diamond \subseteq X$, then $K \cap Y \subseteq X$ and thus $K \cap Y \subseteq K \cap X$. Thus we obtain $(K \cap Y)^\diamond \subseteq (K \cap X)^\diamond$. Then, as $\epsilon \in K \subseteq \{\epsilon\}^\diamond$, we deduce $\{\epsilon\}^\diamond \subseteq K^\diamond = (K \cap M)^\diamond$.⁶ The last condition to check is $((K \cap X)^\diamond \circ (K \cap Y)^\diamond)^\diamond = (K \cap X \cap Y)^\diamond$ for any $X, Y \in \mathcal{M}^\diamond$. First we have $(K \cap X)^\diamond \circ (K \cap Y)^\diamond \subseteq ((K \cap X) \circ (K \cap Y))^\diamond$. As $K \subseteq \{\epsilon\}^\diamond$, we have $(K \cap X) \circ (K \cap Y) \subseteq \{\epsilon\}^\diamond \circ Y \subseteq Y^\diamond = Y$. We also have $(K \cap X) \circ (K \cap Y) \subseteq X$. As $K \circ K \subseteq K$ we have $(K \cap X) \circ (K \cap Y) \subseteq K$ and hence, we deduce $(K \cap X) \circ (K \cap Y) \subseteq K \cap X \cap Y$. Using stability, we compute $(K \cap X)^\diamond \circ (K \cap Y)^\diamond \subseteq ((K \cap X) \circ (K \cap Y))^\diamond \subseteq (K \cap X \cap Y)^\diamond$ and thus $((K \cap X)^\diamond \circ (K \cap Y)^\diamond)^\diamond \subseteq (K \cap X \cap Y)^\diamond$. Now let us prove the reverse inclusion. Let $z \in K \cap X \cap Y$. As $z \in K$ then $z \in J$ and we have $z \in (z \circ z)^\diamond \subseteq ((K \cap X) \circ (K \cap Y))^\diamond \subseteq ((K \cap X)^\diamond \circ (K \cap Y)^\diamond)^\diamond$. Hence, $K \cap X \cap Y \subseteq ((K \cap X)^\diamond \circ (K \cap Y)^\diamond)^\diamond$ and we deduce $(K \cap X \cap Y)^\diamond \subseteq ((K \cap X)^\diamond \circ (K \cap Y)^\diamond)^\diamond$.

We can then apply Theorem 8.21 (page 80) from [22]. If $A_1, \dots, A_k \vdash B$ has a proof in ILL, then the inclusion $\llbracket A_1, \dots, A_k \rrbracket \subseteq \llbracket B \rrbracket$ holds. It is obvious to prove that $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \subseteq \llbracket A_1, \dots, A_k \rrbracket$ by induction on k for example. So we deduce $\llbracket A_1 \rrbracket \circ \dots \circ \llbracket A_k \rrbracket \subseteq \llbracket B \rrbracket$.

⁶Recall the identity $\emptyset^\diamond \multimap \emptyset^\diamond = \emptyset \multimap \emptyset^\diamond = M$.