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A Connection-based Characterization of Bi-intuitionistic Validity

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Abstract We give a connection-based characterization of validity in propositional bi-intuitionistic logic in terms of specific directed graphs called R-graphs. Such a characterization is well-suited for deriving labelled proof-systems with counter-model construction facilities. We first define the notion of bi-intuitionistic R-graphs from which we then obtain a connection-based characterization of propositional bi-intuitionistic validity and derive a sound and complete cut-free free-variable labelled sequent calculus that enjoys variable splitting.

1 Introduction

Bi-intuitionistic logic BiInt is a conservative extension of intuitionistic logic that introduces a new connective $\prec$, called co-implication (also called exclusion or subtraction), which is dual to the implication connective $\rightarrow$. It was first studied by Rauszer who gives a Hilbert calculus with Kripke and algebraic semantics [14] and more recently by Crolard from the perspective of bicartesian closed categories with coexponents and the underlying type system with applications to type theory [3,4]. An interesting aspect of BiInt lies in the duality between implication and co-implication which motivates the definition of proof systems that work as programming languages in which values and continuations are handled in a symmetric way.

From a proof-theoretic point of view, a strong focus has been put on the achievement of cut-free proof-systems since cut-elimination in Gentzen-style (shallow) sequent calculi is particularly difficult to obtain. In this perspective some cut-free calculi for BiInt have been proposed from sequent structures like nested sequents [8] or display inference rules [13]. Another solution makes use of Negri's general methodology for designing labelled sequent calculi in modal logics [10] in order to provide a cut-free labelled sequent calculus where labels correspond to worlds in Kripke structures [12].

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In this paper, which is an extended version of [7], we give the first connection-based characterization of validity in bi-intuitionistic propositional logic in terms of bi-intuitionistic R-graphs. Let us note that similar structures have been defined in the case of BI or separation logics [5,6] in order to characterize validity. Our characterization is well-suited for deriving labelled proof-systems with counter-model construction facilities which, compared with the existing labelled proof-systems [12], easily integrate free-variables and variable splitting [1]. The main contributions of this work are: the definition of bi-intuitionistic R-graphs; a connection-based characterization of validity in propositional BiInt; a new sound and complete cut-free free-variable labelled sequent calculus that enjoys variable splitting; an algorithm for solving admissibility constraints and thus deriving a connection-based method.

In Section 2 we present the bi-intuitionistic logic BiInt, its syntax and semantics and recall the main existing works on calculi and proof-search in this logic. In Section 3 we introduce for BiInt the main notions that are common to most connection-based characterization of validity and in Section 4 we introduce the concept of bi-intuitionistic R-graphs (biRGs) that allows us to reformulate all the standard notions like paths, connections, spanning sets and admissible substitutions on the same graphical structure. In Section 5 we first introduce the notions of slice, concrete path and admissible R-graphs and then propose a connection-based characterization of propositional bi-intuitionistic validity. In Section 6 we present a new labelled sequent calculus, show its soundness and completeness w.r.t. bi-intuitionistic Kripke semantics and explain how to build counter-models from saturated biRGs. In Section 7 we propose an algorithm for solving constraints and in Section 8 we briefly discuss how the technique of variable splitting can be adapted to biRGs.

2 Bi-intuitionistic Propositional Logic

The language of BiInt consists of a countable set $\mathcal{V}$ of propositional letters $\mathrm{P}, \mathrm{Q} \ldots$ and the logical symbols $\bot$ (bottom), $\vee$ (disjunction), $\wedge$ (conjunction), $\rightarrow$ (implication) and $\prec$ (co-implication). Formulas are inductively built from propositional letters as follows:

$$\mathrm{A} ::= \mathrm{P} \mid \bot \mid \mathrm{A} \vee \mathrm{A} \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \rightarrow \mathrm{A} \mid \mathrm{A} \prec \mathrm{A}.$$

We write $\mathcal{F}$ to denote the set of all formulas of BiInt. Negation $\neg\mathrm{A}$ is defined as syntactic sugar for $\mathrm{A} \rightarrow \bot$ and is therefore not considered as primitive in our setting. Similarly, the conjunctive unit $\top$ is defined as a shorthand for $\bot \rightarrow \bot$. Some presentations of bi-intuitionistic logic also consider co-negation $\sim\mathrm{A}$ which is dual to negation and can be defined as a shorthand for $\top \prec \mathrm{A}$.

The Kripke semantics of BiInt is a straightforward extension of that of intuitionistic logic.

Definition 1 A *Kripke model* $\mathcal{M}$ is a triple $\langle M, \sqsubseteq, [\![\cdot]\!]\rangle$, where M is a non-empty set of elements called the worlds of $\mathcal{M}$, $\sqsubseteq$ is a partial order on M and $[\![\cdot]\!]$ is a function from worlds to sets of propositional letters satisfying the following *Kripke monotonicity* condition:

$$(\forall \mathrm{P} \in \mathcal{V})(\forall m \in M)(\forall n \in M)(\text{if } \mathrm{P} \in [\![m]\!] \text{ and } m \sqsubseteq n \text{ then } \mathrm{P} \in [\![n]\!]).$$

The *Kripke forcing relation* $\models$ associated with a Kripke model $\mathcal{M}$ is defined as the least relation between the worlds of $\mathcal{M}$ and formulas such that:

$$\frac{}{\Gamma, \bot \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\,\bot_{\mathrm{R}}$$

$$\frac{}{\Gamma, \mathrm{A} \vdash \mathrm{A}, \Delta}\,\mathrm{ax} \qquad \frac{\Gamma \vdash \mathrm{A}, \Delta \quad \Gamma, \mathrm{A} \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{cut}$$

$$\frac{\Gamma, \mathrm{A}, \mathrm{B} \vdash \Delta}{\Gamma, \mathrm{A} \wedge \mathrm{B} \vdash \Delta}\,\wedge_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}, \Delta \quad \Gamma \vdash \mathrm{B}, \Delta}{\Gamma \vdash \mathrm{A} \wedge \mathrm{B}, \Delta}\,\wedge_{\mathrm{R}}$$

$$\frac{\Gamma, \mathrm{A} \vdash \Delta \quad \Gamma, \mathrm{B} \vdash \Delta}{\Gamma, \mathrm{A} \vee \mathrm{B} \vdash \Delta}\,\vee_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}, \mathrm{B}, \Delta}{\Gamma \vdash \mathrm{A} \vee \mathrm{B}, \Delta}\,\vee_{\mathrm{R}}$$

$$\frac{\Gamma, \mathrm{A} \to \mathrm{B} \vdash \mathrm{A}, \Delta \quad \Gamma, \mathrm{B} \vdash \Delta}{\Gamma, \mathrm{A} \to \mathrm{B} \vdash \Delta}\,\to_{\mathrm{L}} \qquad \frac{\Gamma, \mathrm{A} \vdash \mathrm{B}}{\Gamma \vdash \mathrm{A} \to \mathrm{B}, \Delta}\,\to_{\mathrm{R}}$$

$$\frac{\mathrm{A} \vdash \mathrm{B}, \Delta}{\Gamma, \mathrm{A} \prec \mathrm{B} \vdash \Delta}\,\prec_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}, \Delta \quad \Gamma, \mathrm{B} \vdash \mathrm{A} \prec \mathrm{B}, \Delta}{\Gamma \vdash \mathrm{A} \prec \mathrm{B}, \Delta}\,\prec_{\mathrm{R}}$$

Fig. 1 Dragalin-style sequent calculus for Bilnt

- $m \models \bot$ never;
- $m \models \mathrm{P}$ iff $\mathrm{P} \in [\![m]\!]$;
- $m \models \mathrm{A} \vee \mathrm{B}$ iff $m \models \mathrm{A}$ or $m \models \mathrm{B}$;
- $m \models \mathrm{A} \wedge \mathrm{B}$ iff $m \models \mathrm{A}$ and $m \models \mathrm{B}$;
- $m \models \mathrm{A} \to \mathrm{B}$ iff for all $n \in M$ such that $m \sqsubseteq n$, $n \not\models \mathrm{A}$ or $n \models \mathrm{B}$;
- $m \models \mathrm{A} \prec \mathrm{B}$ iff for some $n \in M$ such that $n \sqsubseteq m$, $n \models \mathrm{A}$ and $n \not\models \mathrm{B}$.

Kripke monotonicity lifts from propositional letters to formulas as in intuitionistic logic. As usual, a formula A is *satisfied in* $\mathcal{M}$ iff $m \models \mathrm{A}$ for all worlds m in M, *satisfiable* if it is satisfied in some Kripke model $\mathcal{M}$, and *valid* if it is satisfied in all Kripke models.

Figure 1 depicts the standard (Dragalin-style) multi-conclusioned sequent calculus for Bilnt which can be found in [12]. We observe that the rules for co-implication simply behave as duals for the ones dealing with implication. Let us recall that in Dragalin's multi-conclusioned sequent calculus for intuitionistic logic, intuitionism is not obtained by restricting the right-hand side of the sequents to a single formula, but by discarding Δ in a backward application of the rule $\to_{\mathrm{R}}$ from the conclusion to the premiss. Co-implication acting as a dual to implication, it is then natural that in a backward application of the rule $\prec_{\mathrm{L}}$, the context that should disappear is Γ.

The price to pay for the easy dual formulation of the calculus of Figure 1 is the failure of cut-elimination. Indeed, although valid in the Kripke semantics, the sequent $\mathrm{P} \vdash \mathrm{R} \to ((\mathrm{P} \prec \mathrm{Q}) \wedge \mathrm{R}), \mathrm{Q}$ is not provable without cut as illustrated by the following proof attempt:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{P}, \mathrm{R} \vdash \mathrm{P}} \quad \dfrac{\dfrac{}{\mathrm{P}, \mathrm{R}, \mathrm{Q} \vdash \mathrm{P}} \quad \mathrm{P}, \mathrm{R}, \mathrm{Q} \vdash \mathrm{P} \prec \mathrm{Q}}{\mathrm{P}, \mathrm{R}, \mathrm{Q} \vdash \mathrm{P} \prec \mathrm{Q}}\,\prec_{\mathrm{R}}}{\mathrm{P}, \mathrm{R} \vdash \mathrm{P} \prec \mathrm{Q}}\,\prec_{\mathrm{R}} \quad \dfrac{}{\mathrm{P}, \mathrm{R} \vdash \mathrm{R}}}{\mathrm{P}, \mathrm{R} \vdash (\mathrm{P} \prec \mathrm{Q}) \wedge \mathrm{R}}\,\wedge_{\mathrm{R}}}{\mathrm{P} \vdash \mathrm{R} \to ((\mathrm{P} \prec \mathrm{Q}) \wedge \mathrm{R}), \mathrm{Q}}\,\to_{\mathrm{R}}$$

The proof fails because Q immediately disappears in the first step of the derivation when the rule $\to_{\mathrm{R}}$ is applied. Unfortunately, that rule is the only one applicable at

α	α_1	α_2	β	β_1	β_2
$(A \wedge B)^+$	A^+	B^+	$(A \wedge B)^-$	A^-	B^-
$(A \vee B)^-$	A^-	B^-	$(A \vee B)^+$	A^+	B^+
$(A \rightarrow B)^-$	A^+	B^-	$(A \rightarrow B)^+$	A^-	B^+
$(A \prec B)^+$	A^+	B^-	$(A \prec B)^-$	A^-	B^+

	itype
$(A \rightarrow B)^+$	ϕ
$(A \rightarrow B)^-$	ψ
$(A \prec B)^+$	$\overline{\psi}$
$(A \prec B)^-$	$\overline{\phi}$

Table 1 Signs, principal, secondary and intuitionistic types

the beginning of the derivation. However, using cut on the formula $P \prec Q$, one can construct the following proof:

$$\dfrac{\dfrac{\dfrac{}{P \vdash P, Q} \quad \dfrac{}{P, Q \vdash P \prec Q, Q}}{P \vdash P \prec Q, Q}\,{\prec_R} \quad \dfrac{\dfrac{\dfrac{}{P, P \prec Q, R \vdash P \prec Q} \quad \dfrac{}{P, P \prec Q, R \vdash R}}{P, P \prec Q, R \vdash (P \prec Q) \wedge R}\,{\wedge_R}}{P, P \prec Q \vdash R \rightarrow ((P \prec Q) \wedge R), Q}\,{\rightarrow_R}}{P \vdash R \rightarrow ((P \prec Q) \wedge R), Q}\,\text{cut}$$

Let us remark that if one mostly considers cut-elimination as a way to achieve analytical calculi, then BiInt falls into the category of logics for which analytical calculi can be obtained even when cut-elimination is not admissible [2]. Cut-elimination can be achieved using nested sequents [8] or deep inference rules [13], but the resulting calculi are not always well suited for automated backward proof-search. Moreover, ensuring completeness can also prove itself tricky because of the interactions between implication and its dual. In a recent work, such interactions are dealt with using a cut-free calculus in which derivations and refutations are considered as first class citizens [9]. Another recent solution is based on a cut-free labelled sequent calculus where labels correspond to worlds in Kripke structures [12].

In this paper we consider an approach based on labels and constraints arranged as specific directed graphs, called R-Graphs, already used to characterize provability in logics like BI (logic of Bunched Implications) or separation logics [5]. It allows us to derive a new labelled calculus with free variables that enjoys variable splitting.

3 Indexing BiInt Formulas

In this section, we recall some basic terminology of connection-based characterizations of validity as we shall heavily rely on it in the forthcoming sections.

The *signed formula tree* for a formula C is a representation of its syntactic structure as a tree the nodes of which are decorated with positive and negative *signs* respectively denoted with the symbols "+" and "−". The root node of a signed formula tree is always associated with a negative sign. The sign of all other nodes is then determined inductively as prescribed by Table 1. Let us remark that conjunction and disjunction simply pass their sign to their subformulas leaving it unchanged. On the contrary, implication (respectively co-implication) changes its sign to the opposite before passing it to its left (respectively right) subformula. If one forgets about the indexes written as subscripts, Figure 2 gives an example of a signed formula tree.

Fig. 2 Indexed formula tree for $(((R \vee P) \prec Q) \wedge (P \rightarrow \bot)) \rightarrow ((P \rightarrow R) \prec Q)$

The notion of sign gives rise to the notion of *signed formula*[1], which is defined as a formula with a sign or, more formally, a pair (C, S), written C^S, where C is a BiInt formula and S is a sign. Given a formula C, the *signed formula for* C is defined as the signed formula C^- induced by the signed formula tree for C. Figure 2 shows that the signed formula for $(((R \vee P) \prec Q) \wedge (P \rightarrow \bot)) \rightarrow ((P \rightarrow R) \prec Q)$ is

$$(((R^+ \vee^+ P^+) \prec^+ Q^-) \wedge^+ (P^- \rightarrow^+ \bot^+)) \rightarrow^- ((P^+ \rightarrow^- R^-) \prec^- Q^+).$$

Depending on its principal connective and sign, a signed formula is given a *principal type (ptype)* α or β. If α (respectively β) is the principal type of a signed formula C, then, its left subformula A is of *secondary type (stype)* α_1 (respectively β_1) and its right subformula B is of *secondary type* α_2 (respectively β_2). Signed formulas the principal connective of which belongs to the set $\{\rightarrow, \prec\}$ also admit an additional *intuitionistic type (itype)* ϕ, $\overline{\phi}$, ψ or $\overline{\psi}$. Table 1 describes how principal, secondary and intuitionistic types are determined w.r.t. principal connectives and signs. For readability, we often simply speak of the type of a signed formula each time the context makes it clear what type (ptype, stype or itype) is actually intended; we also write "t-formula" as a shorthand for "formula of type t". Since a node in a signed formula tree for a formula C is in one-to-one correspondence with a unique subformula (signed subformula) of C (C^-), we shall often abusively apply to nodes all the terminology defined for formulas. For example, we shall simply write "a t-node in C" instead of "a node in the signed formula tree for C for which the corresponding subformula in C is of type t".

Let us now explain the notion of *indexed formula trees*, which are signed formula trees the nodes of which are not only decorated with signs, but also with *indexes* (often called "positions" in the matrix terminology). Indexes are written using an alphabet that contains all natural numbers enriched with two disjoint and denumerable sets of symbols Φ and Ψ respectively called variable and constant symbols. We shall use the letters ranging from a to d (possibly subscripted) to denote constant symbols. Similarly, we shall use the letters from x to z to denote variable symbols. For convenience, let us also assume that Ψ always contains the particular symbol ϵ and that if s is a constant (respectively variable) symbol, then so is $\overline{s}$.

[1] The reader more familiar with sequent calculi should think of positive (respectively negative) formulas as formulas that should be introduced in the left-hand (respectively right-hand) side context of sequents in a bottom-up sequent proof.

Given a formula C, an *indexed formula tree for* C is obtained from the signed formula tree for C by assigning an index to every node of the tree so that distinct nodes have distinct indexes, ψ- and $\overline{\psi}$-nodes are indexed with constant symbols in Ψ, ϕ- and $\overline{\phi}$-nodes are indexed with variable symbols in Φ, and all other nodes are indexed with natural numbers. Let us remark that we do not require any particular order (prefix, postfix...) on the indexes of an indexed formula tree in this paper. Therefore, a formula C admits infinitely many distinct legitimate indexed formula trees. Moreover, indexed formula trees straightforwardly induce the notion of *indexed formulas*. Figure 2 shows an indexed formula tree that induces the indexed formula

$$(((\mathrm{R}^+_2 \vee^+_1 \mathrm{P}^+_3) \prec^+_{\overline{a}} \mathrm{Q}^-_4) \wedge^+_0 (\mathrm{P}^-_5 \rightarrow^+_x \bot^+_6)) \rightarrow^-_a ((\mathrm{P}^+_7 \rightarrow^-_b \mathrm{R}^-_8) \prec^-_{\overline{x}} \mathrm{Q}^+_9).$$

Since indexes of an indexed formula tree for a formula C are in one-to-one correspondence with the (signed) subformulas of C, we shall use indexes and (signed) formulas interchangeably throughout the paper.

Given an indexed formula tree for a formula C, we write $f(i)$ (respectively $sf(i)$) to denote the unique subformula (respectively signed subformula) associated with the index i. A formula tree induces a strict partial ordering $\ll$ on indexes, called the *domination ordering*, which is such that the index of the root is the least element and if $i \ll j$ then i is encountered before j on a path from the root to j. If i is a non-atomic index in C (*i.e.*, if $f(i)$ is a non-atomic formula) then the two indexes j and k such that $f(j)$ and $f(k)$ are the immediate (left and right) subformulas of $f(i)$ are called *dual* and we write $j \triangle k = i$. The domination ordering allows us to define the *label* of an index i (or, equivalently, its corresponding (signed) formula) as the greatest index ℓ such that ℓ is not a natural number and $\ell \ll i$. Intuitively, the label of an index i is the last index ℓ which is either a variable, or a constant, encountered on a path from the root to i. For example, writing the labels in square brackets, Figure 2 induces the following labelled formula

$$(((\mathrm{R}^+_{2[\overline{a}]} \vee^+_{1[\overline{a}]} \mathrm{P}^+_{3[\overline{a}]}) \prec^+_{\overline{a}[a]} \mathrm{Q}^-_{4[\overline{a}]}) \wedge^+_{0[a]} (\mathrm{P}^-_{5[x]} \rightarrow^+_{x[a]} \bot^+_{6[x]})) \rightarrow^-_{a[\epsilon]} ((\mathrm{P}^+_{7[b]} \rightarrow^-_{b[\overline{x}]} \mathrm{R}^-_{8[b]}) \prec^-_{\overline{x}[a]} \mathrm{Q}^+_{9[\overline{x}]}).$$

In the sequent calculus of Figure 1, contraction is internalized by the repetition of the principal formula $\mathrm{A} \rightarrow \mathrm{B}$ (respectively $\mathrm{A} \prec \mathrm{B}$) in the left (respectively right) premiss of the $\rightarrow_{\mathrm{L}}$ (respectively $\prec_{\mathrm{R}}$) rule so that there is no need for an explicit contraction rule. In a connection-based setting contraction is usually handled via the notion of *multiplicity*. Given a formula C of BiInt, a multiplicity for C is a function μ which assigns a strictly positive integer to each ϕ- or $\overline{\phi}$-subformula in C. The formula $\mu(\mathrm{C})$ is then defined as the formula obtained from C by replacing all ϕ-subformulas A in C with the conjunction $\mathrm{A} \wedge \mathrm{A} \wedge \ldots \wedge \mathrm{A}$, where A occurs exactly $\mu(\mathrm{A})$ times, and replacing all $\overline{\phi}$-subformulas B in C with the disjunction $\mathrm{B} \vee \mathrm{B} \vee \ldots \vee \mathrm{B}$, where B occurs exactly $\mu(\mathrm{B})$ times. For example, if $\mathrm{C} = (\mathrm{P} \rightarrow \mathrm{Q}) \rightarrow (\mathrm{R} \prec \mathrm{S})$, $\mu(\mathrm{P} \rightarrow \mathrm{Q}) = 2$ and $\mu(\mathrm{R} \prec \mathrm{S}) = 3$, then

$$\mu(\mathrm{C}) = ((\mathrm{P} \rightarrow \mathrm{Q}) \wedge (\mathrm{P} \rightarrow \mathrm{Q})) \rightarrow ((\mathrm{R} \prec \mathrm{S}) \vee (\mathrm{R} \prec \mathrm{S}) \vee (\mathrm{R} \prec \mathrm{S})).$$

Intuitively, a multiplicity function encodes the number of occurrences that would be allowed (via contraction) for each ϕ- or $\overline{\phi}$-formula in a sequent-style derivation. For convenience, when dealing with indexed formulas, we use superscripted indexes to distinguish between the copies of ϕ- and $\overline{\phi}$-formulas. For the previous example, if we assume that x is the index of $\mathrm{R} \prec \mathrm{S}$ in C, then $\mu(\mathrm{R} \prec \mathrm{S}) = 3$ implies that x^2 and x^3

should respectively be the indexes of the second and third occurrences (the first and second additional copies) of $\mathrm{R} \prec \mathrm{S}$ in $\mu(\mathrm{C})$.

The previous notions are fairly common to most connection-based characterizations of validity. If we were to follow the standard recipe for such characterizations, the next step would be the introduction of the key notions of (atomic) matrix paths, connections and spanning sets of connections together with admissible substitutions leading to irreflexive reduction orderings. However, we shall not follow the standard approach for the upcoming sections and rather introduce the concept of R-graphs since it allows us to reformulate all the standard notions on the same graphical structure. Moreover, R-graphs can easily be turned into Kripke models when dealing with non valid formulas.

4 Bi-intuitionistic R-graphs

From a very general point of view, R-graphs for a given logic are directed graphs in which vertices are meant to represent worlds in the underlying Kripke semantics of the logic [6]. Let us first define the general notion of R-graph before restricting it to match the bi-intuitionistic case.

Definition 2 (R-graph) A *R-graph* (RG) is a directed graph $G(V, E)$ with vertices V and edges E. V is required to contain a distinguished vertex ϵ called the *ϵ-vertex*. The vertices are named with elements of $\Psi \cup \Phi$. Moreover, all vertices u are associated with a set $\mathcal{F}(G, u)$ of signed formulas the elements of which are referred to as the *tags* of u and all edges e are tagged with a letter $\mathcal{T}(G, e)$ from the set $\mathcal{T} = \{\psi, \phi, \bar{\psi}, \bar{\phi}, \sigma, \kappa\}$ of edge-tags.

A vertex named with a constant symbol (respectively variable symbol) is called a ψ-vertex (respectively ϕ-vertex). The set of ψ-vertices (respectively ϕ-vertices) is written V^{Ψ} (respectively V^{Φ}). We use the letters u, v and w to range over arbitrary vertices and we write $u[\tau]v$ to denote the edge, tagged with the letter $\tau \in \mathcal{T}$, that goes from u to v; we then call this edge a τ-edge and say that u and v respectively are its *source* and *target*. Given a subset $T \subseteq \mathcal{T}$, the set of all τ-edges such that $\tau \in T$ is written E^T.

Definition 3 (bi-intuitionistic RG) A bi-intuitionistic RG (biRG) is a R-graph $G(V, E)$ satisfying the following structural conditions:

- every ψ-edge has a ψ-vertex as its target;
- every $\bar{\psi}$-edge has a ψ-vertex as its source;
- every ϕ-edge has a ϕ-vertex as its target;
- every $\bar{\phi}$-edge has a ϕ-vertex as its source;
- every σ-edge induces a "bidirectional" link between a ϕ-vertex and a ψ-vertex, more formally, $u[\sigma]v \in E$ if and only if $v[\sigma]u \in E$, with either $u \in V^{\Psi}$ and $v \in V^{\Phi}$, or $u \in V^{\Phi}$ and $v \in V^{\Psi}$;
- every κ-edge $u[\kappa]v$ is a link between two (arbitrary) vertices such that there exists at least one formula occurring positively (with a "+" sign) in $\mathcal{F}(G, u)$ and negatively (with a "−" sign) in $\mathcal{F}(G, v)$.

Definition 4 (purity) A biRG is *pure* if it does not contain any κ- or σ-edges. The *pure part* of a biRG $G(V, E)$ is the biRG obtained from G in which all κ- and σ-edges have been removed.

Fig. 3 Bi-intuitionistic R-graphs

Figure 3 gives some examples of biRGs. Let us now explain how such graphs can be associated with formulas. Standard matrix characterizations heavily rely on the notion of (atomic) matrix paths. In our setting, such atomic matrix paths are replaced with the notion of pure irreducible biRGs.

Definition 5 Let A be a BiInt formula. A (biRG-)reduction through A is a sequence $\mathcal{R} = R_0\rho_1 \dots \rho_{i-1}\rho_i \dots$ in which R_0 is a collection (of biRGs) containing the single biRG $G_0(V_0, E_0)$, where $V_0 = \{\epsilon\}$, $E_0 = \emptyset$ and $\mathcal{F}(G_0, \epsilon) = \{\mathrm{A}^-\}$, and each ρ_i is a reduction step that transforms the collection R_{i-1} inherited from the previous reduction step into a new collection R_i by applying one of the reduction rules given in Figure 4.

In order to apply a reduction rule to a biRG $G(V, E)$, one first needs to choose a vertex u in V and a signed formula A^{S} in $\mathcal{F}(G, u)$.

- If A^{S} has principal type α and has no intuitionistic type, $G(V, E)$ is reduced to the new biRG $G_1(V_1, E_1)$ such that $V_1 = V$, $E_1 = E$, $\mathcal{F}(G_1, v) = \mathcal{F}(G, v)$ for all $v \neq u$ and $\mathcal{F}(G_1, u) = \mathcal{F}(G, u) \cup \{\alpha_1(\mathrm{A}^{\mathrm{S}}), \alpha_2(\mathrm{A}^{\mathrm{S}})\}$, where $\alpha_1(\mathrm{A}^{\mathrm{S}})$ and $\alpha_2(\mathrm{A}^{\mathrm{S}})$ respectively are the first and second signed subformulas of A^{S} (of secondary type α_1 and α_2).
- If A^{S} has principal type β and has no intuitionistic type, $G(V, E)$ is reduced to two new biRGs $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ such that for $i \in \{1, 2\}$, $V_i = V$, $E_i = E$, $\mathcal{F}(G_i, v) = \mathcal{F}(G, v)$ for all $v \neq u$ and $\mathcal{F}(G_i, u) = \mathcal{F}(G, u) \cup \{\beta_i(\mathrm{A}^{\mathrm{S}})\}$.

The previous reduction rules are the reformulation of the standard α and β matrix-path reduction rules. The next two reduction rules are specific to BiInt and depend on the index (a, $\overline{a}$, x or $\overline{x}$) and intuitionistic type of the signed formula under reduction (principal formula).

a (resp. $\overline{a}$, x, $\overline{x}$) is the index of the $\alpha\psi$- (resp. $\alpha\overline{\psi}$-, $\beta\phi$-, $\beta\overline{\phi}$-) formula under reduction.

Fig. 4 Reduction rules

- If A^{S} has principal type α and has intuitionistic type ψ (respectively $\overline{\psi}$), $G(V, E)$ is reduced to the new biRG $G_1(V_1, E_1)$ such that $V_1 = V \cup \{a\}$ and $E_1 = E \cup \{u[\psi]a\}$ (respectively $V_1 = V \cup \{\overline{a}\}$ and $E_1 = E \cup \{\overline{a}[\bar{\psi}]u\}$). Moreover, $\mathcal{F}(G_1, v) = \mathcal{F}(G, v)$ for all $v \neq a$ (respectively $v \neq \overline{a}$), and $\mathcal{F}(G_1, a) = \{\alpha_1(\mathrm{A}^{\mathrm{S}}), \alpha_2(\mathrm{A}^{\mathrm{S}})\}$ (respectively $\mathcal{F}(G_1, \overline{a}) = \{\alpha_1(\mathrm{A}^{\mathrm{S}}), \alpha_2(\mathrm{A}^{\mathrm{S}})\}$).
- If A^{S} has principal type β and has intuitionistic type ϕ (respectively $\overline{\phi}$), $G(V, E)$ is reduced to two new biRGs $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ such that for $i \in \{1, 2\}$, $V_i = V \cup \{x\}$ and $E_i = E \cup \{u[\phi]x\}$ (respectively $V_i = V \cup \{\overline{x}\}$ and $E_i = E \cup \{\overline{x}[\bar{\phi}]u\}$). Moreover, $\mathcal{F}(G_i, v) = \mathcal{F}(G, v)$ for all $v \neq x$ (respectively $v \neq \overline{x}$) and $\mathcal{F}(G_i, x)$ (respectively $\mathcal{F}(G_i, \overline{x})) = \{\mathrm{A}^{\mathrm{S}}\} \cup \{\beta_i(\mathrm{A}^{\mathrm{S}})\}$.

Let us remark that we have chosen to prevent the reduction rules from discarding their principal formula from the tags of u (although we shall forget about them in graphical representations to increase readability). This is not a strict requirement but it makes the counter-model construction process (*e.g.*, the saturation relation in Definition 15) easier to define.

Definition 6 (irreducibility) A biRG is *irreducible* if it is stable under the reduction rules; it is *reducible* otherwise. Accordingly, a collection R of biRGs is *irreducible* if and only if all biRGs in R are irreducible.

Let $\mathcal{R}$ be a reduction through A, we say that $\mathcal{R}$ is *finished* if and only if for some natural number n, the collection R_n in $\mathcal{R}$ is irreducible and for all R_m in $\mathcal{R}$ such that $m < n$, R_m is not irreducible. We then say that n is the length of the reduction $\mathcal{R}$. Since we consider formulas indexed w.r.t. a given multiplicity, an inspection of the reduction rules of Figure 4 shows that all finished reductions through A lead to the same irreducible biRG-collection denoted R_f and called the *final* biRG-collection through A. Every biRG in R_f is then called an *irreducible biRG through* A. Let us remark that all biRGs in R_f are pure. Figure 3 gives examples of irreducible biRGs[2], the pure parts of which also are examples of irreducible biRGs through $(((\mathrm{R}_2^+ \vee_1^+ \mathrm{P}_3^+) \prec_{\overline{a}}^+ \mathrm{Q}_4^-) \wedge_0^+ (\mathrm{P}_5^- \rightarrow_x^+ \bot_6^+)) \rightarrow_a^- ((\mathrm{P}_7^+ \rightarrow_b^- \mathrm{R}_8^-) \prec_{\overline{x}}^- \mathrm{Q}_9^+)$.

[2] For readability, non-atomic signed formulas are not mentioned in the vertex tags.

5 Characterizing BiInt Validity through R-graphs

Before stating our connection-based characterization of BiInt validity, we need to define the notions of slice, concrete path and admissible R-graphs.

Definition 7 (slice) Let $G(V, E)$ be a biRG and S be a subset of V, the S-*slice* of G is defined as the smallest (w.r.t. the number of vertices and edges) biRG $G^S(V^S, E^S)$ such that $S \subseteq V^S$ and for all vertices $u \in V^S$ and $v \in V$,

- if $v[\tau]u \in E$ and $\tau \in \{\psi, \phi, \sigma\}$, then $v \in V^S$ and $v[\tau]u \in E^S$;
- if $u[\tau]v \in E$ and $\tau \in \{\bar{\psi}, \bar{\phi}, \sigma\}$, then $v \in V^S$ and $u[\tau]v \in E^S$.

The purpose of a slice is to capture only the essential information, *i.e.*, the minimal portion of a biRG, that is necessary to establish the validity of a BiInt formula. Let us remark that in the construction of a slice $\bar{\psi}$- and $\bar{\phi}$-edges are traversed forward, from their source to their target, while ψ- and ϕ-edges are traversed backward, from their target to their source.

Definition 8 (path) Given two vertices u, v in a biRG $G(V, E)$, a path in G from u to v is a sequence $u_0\tau_1u_1 \ldots u_{p-1}\tau_pu_p$ such that $u_0 = u$, $u_p = v$ and for all $1 \leqslant i \leqslant p$, u_i is a vertex in V, τ_i is an edge-tag in $\mathcal{T}$ and there exists a τ_i-edge $u_{i-1}[\tau_i]u_i$ in E. A cycle is a path such that $u_0 = u_p$.

Given a subset T of $\mathcal{T}$, a T-path is a path $P = u_0\tau_1u_1 \ldots u_{p-1}\tau_pu_p$ such that for all $1 \leqslant i \leqslant p$, $\tau_i \in T$. In particular, when $T = \{\psi, \bar{\psi}, \sigma\}$, P is called a *concrete path* (in the graphical representation, a path using only solid edges). T-cycles and concrete cycles are defined accordingly. Using concrete paths, we then define the notion of *concrete reachability* as the relation ($___ \to _$) such that

$$Gu \to v \text{ if and only if } u = v \text{ or there exists a concrete path from } u \text{ to } v \text{ in } G.$$

It is not difficult to check that concrete reachability is a preordering relation on the vertices of a biRG which in turn gives rise to the notion of *mutual concrete reachability* as the equivalence ($___ \leftrightarrow _$) between vertices such that

$$Gu \leftrightarrow v \text{ if and only if } Gu \to v \text{ and } Gv \to u.$$

Definition 9 (admissibility) A biRG $G(V, E)$ is *admissible* if and only if

- all concrete cycles in G are σ-cycles that contain at most one ψ-vertex and
- for all τ-edges $u[\tau]v$ in E such that $\tau \in \{\phi, \bar{\phi}\}$, $Gu \to v$ (in other words, if u and v are distinct then there is a concrete path in G from u to v).

Definition 10 (consistency) A biRG $G(V, E)$ is *inconsistent* if and only if $(\exists u \in V)(\bot^+ \in \mathcal{F}(G, u))$; it is *consistent* otherwise.

Definition 11 (complementarity) A biRG $G(V, E)$ is *complementary* if and only if there is at least one κ-edge $u[\kappa]v$ in E such that the slice $G^{\{u,v\}}$ is admissible and $G^{\{u,v\}}u \to v$ (there is a concrete path in the slice from u to v).

A collection of biRGs is inconsistent (respectively admissible) if it contains at least one biRG which is inconsistent (respectively admissible). On the contrary, a collection is *complementary* if all of its biRGs are complementary.

Let us now consider the characterization of BiInt validity. Starting with the final biRG-collection R_f through a BiInt formula A, we define the notions of σ- and κ-bindings. A local σ-binding for a biRG $G(V,E)$ is a function σ that extends $G(V,E)$ by inserting σ-links (bidirectional σ-edges) in E.
More formally, $\sigma(G(V,E)) = G_\sigma(V_\sigma, E_\sigma)$ such that $V_\sigma = \sigma(V) = V$ and $E_\sigma = \sigma(E) = E \cup \Sigma$, where Σ is a set of σ-links between vertices of V^Φ and vertices of V^Ψ. Let us remark that a local σ-binding is completely determined by Σ.

Given two collections $R = \{G_1, \ldots, G_n\}$ and $S = \{\sigma_1, \ldots, \sigma_n\}$ such that for all $1 \leqslant i \leqslant n$, σ_i is a local σ-binding for the biRG G_i, the global σ-binding $\overline{\sigma}$ for R (induced by S) is defined as $\overline{\sigma}(G_i) = \sigma_i(G_i)$ for all $1 \leqslant i \leqslant n$. A global σ-binding induces a relation $\sqsubset$ on $\Psi \times \Phi$, called a *precedence relation*, such that $a \sqsubset x$ if there is a σ-link $a[\sigma]x$ in $\overline{\sigma}(G_i)$ for some $1 \leqslant i \leqslant n$.

A local σ-binding σ is admissible for a biRG G if $\sigma(G)$ is admissible. A global σ-binding $\overline{\sigma}$ is admissible for a biRG-collection R if for all G in R, $\overline{\sigma}(G)$ is admissible. Local and global κ-bindings are defined accordingly w.r.t. the structural conditions required for κ-edges in Definition 3.

Definition 12 (biRG-validity) A BiInt formula A is biRG-valid if and only if there exist a multiplicity μ, a (global) σ-binding $\overline{\sigma}$ and a (global) κ-binding $\overline{\kappa}$ for the final biRG-collection R_f of irreducible biRGs through $\mu(\mathrm{A})$ such that:

1. For all (not necessarily distinct) biRGs $G_1(V_1,E_1), G_2(V_2,E_2)$ in $\overline{\sigma} \circ \overline{\kappa}(R_f)$ and all ϕ-vertices x in $V_1 \cap V_2$, if $x[\sigma]u \in E_1$ and $x[\sigma]v \in E_2$ then $u = v$.
2. For all consistent biRGs $G(V,E)$ in $\overline{\sigma} \circ \overline{\kappa}(R_f)$, $G(V,E)$ is complementary.
3. The *reduction ordering* $\lhd = (\ll \cup \sqsubset)^+$ induced by $\overline{\sigma}$, where $(\cdot)^+$ stands for transitive closure, is irreflexive.

Let us illustrate the previous definition by showing that the (indexed) formula

$$\mathrm{C} = (((\mathrm{R}_2^+ \vee_1^+ \mathrm{P}_3^+) \prec_{\overline{a}}^{\pm} \mathrm{Q}_4^-) \wedge_0^+ (\mathrm{P}_5^- \rightarrow_x^+ \bot_6^+)) \rightarrow_a^- ((\mathrm{P}_7^+ \rightarrow_b^- \mathrm{R}_8^-) \prec_{\overline{x}}^{-} \mathrm{Q}_9^+)$$

is biRG-valid. Firstly, let us consider a multiplicity μ such that $\mu(x) = \mu(\overline{x}) = 1$. Let us then remark that the second condition of Definition 12 implies that one only needs to consider irreducible biRGs that are also consistent to establish the biRG-validity of a formula. Conveniently, it happens that the pure part of the four biRGs depicted in Figure 3 are the only consistent and irreducible biRGs through $\mu(\mathrm{C})$.

The first condition of Definition 12 requires that a global σ-binding should be found from local σ-bindings that are pairwise compatible in such a way that if some ϕ-vertex (*i.e.*, a variable) gets σ-linked with some ψ-vertex (*i.e.*, a constant) in some biRG through $\mu(\mathrm{C})$, then it cannot be σ-linked to a distinct ψ-vertex in the same or in any other biRG through $\mu(\mathrm{C})$. A straightforward way to build such a global σ-binding is to propagate any σ-link inserted in an irreducible and consistent biRG immediately to all other irreducible and consistent biRGs where that σ-link can be inserted. For the four biRGs of Figure 3, let us consider the global σ-binding $\overline{\sigma}$ built from the four local σ-bindings $\sigma_{i(1 \leqslant i \leqslant 4)}$ induced by the same set $\Sigma = \{\overline{x}[\sigma]\overline{a},\ x[\sigma]a\}$ of σ-links.

In order to satisfy the second condition of Definition 12, let us consider the following local κ-bindings for the four biRGs of Figure 3: $\kappa_1 = \{\overline{a}[\kappa]b\}$ for G_1, $\kappa_2 = \kappa_4 = \{\overline{x}[\kappa]\overline{a}\}$ for G_2 and G_4, $\kappa_3 = \{\overline{a}[\kappa]x\}$ for G_3, which leads to the global κ-binding $\overline{\kappa} = \{\overline{a}[\kappa]b,\ \overline{x}[\kappa]\overline{a},\ \overline{a}[\kappa]x\}$. Let us remark that once a global κ-binding has been determined, the formulas tagging the vertices of biRGs become useless since

Fig. 5 Slices under $\overline{\kappa} = \{\overline{a}[\kappa]b,\ \overline{x}[\kappa]\overline{a},\ \overline{a}[\kappa]x\}$ and $\overline{\sigma} = \{\overline{x}[\sigma]\overline{a},\ x[\sigma]a\}$

biRG-validity only relies on the structure (shape) of the biRGs. This fact opens the door for some factorizing optimizations as illustrated in Figure 5 where the slices $S_{i(1\leqslant i\leqslant 4)}$ generated by our global κ-binding $\overline{\kappa}$ are depicted. Indeed, since S_2 and S_4 share the same structure, only one of these two slices needs to be considered (the other may safely be discarded). The slices of Figure 5 should then be tested for complementarity according to Definition 11: on one hand, for all i such that $(1 \leqslant i \leqslant 4)$, S_i is admissible because the (non-solid) edges $a[\phi]x$ and $\overline{x}[\bar{\phi}]a$ are respectively covered by the concrete paths $a\sigma x$ and $\overline{x}\sigma\overline{a}\bar{\psi}a$, and on the other hand, $\overline{a}[\kappa]b$ is covered by the concrete path $\overline{a}\sigma\overline{x}\psi b$ in S_1, $\overline{x}[\kappa]\overline{a}$ is covered by the concrete path $\overline{x}\sigma\overline{a}$ in S_2 (and S_4) and $\overline{a}[\kappa]x$ is covered by the concrete path $\overline{a}\bar{\psi}a\sigma x$ in S_3.

Finally, the last thing to do in order to establish the biRG-validity of C is to check the third condition of Definition 12. The global σ-binding $\overline{\sigma} = \{\overline{x}[\sigma]\overline{a},\ x[\sigma]a\}$ induces a precedence relation $\sqsubset$ such that $\overline{a} \sqsubset \overline{x}$ and $a \sqsubset x$, which in turn implies the reduction ordering $\lhd = (\ll \cup \sqsubset)^+$ depicted in Figure 6 as a graph, called a *reduction graph for* C, where the solid and dotted arrows respectively stand for the domination ordering $\ll$ and the precedence relation $\sqsubset$. Since the reduction graph is acyclic, it follows that the reduction ordering $\lhd$ is irreflexive and thus, the formula C is biRG-valid.

The irreflexivity of $\lhd$ can also be checked directly on the biRGs of Figure 3 instead of resorting to a reduction graph. For that, one needs to discard all non-solid edges (since they are known to be covered by concrete paths), revert all $\bar{\psi}$-edges, orient all σ-links from their ψ-vertex (constant) to their ϕ-vertex (variable), merge all resulting graphs into a single one, called a *reduction biRG for* C, and check it for acyclicity. Doing so with the slices of Figure 5, leads to the reduction biRG of Figure 7, which is easily checked irreflexive. In general, a reduction biRG has fewer vertices than a reduction graph because biRGs only take into account vertices that correspond to constants and variables introduced by implications and co-implications, whereas an indexed formula tree has to take into account the whole syntactic structure of a formula (including atoms, conjunctions and disjunctions that introduce neither constant, nor variable indexes).

Fig. 6 Reduction graph for $(((R_2 \vee_1 P_3) \prec_{\overline{a}} Q_4) \wedge_0 (P_5 \to_x \bot_6)) \to_a ((P_7 \to_b R_8) \prec_{\overline{x}} Q_9)$

Fig. 7 Reduction biRG for $(((R_2 \vee_1 P_3) \prec_{\overline{a}} Q_4) \wedge_0 (P_5 \to_x \bot_6)) \to_a ((P_7 \to_b R_8) \prec_{\overline{x}} Q_9)$

6 A New Labelled Calculus

Definition 12 can be used to extract a labelled calculus, called RGBiInt, the rules of which generate biRGs. Such a calculus is given in Figure 8 and generates one biRG per initial sequent[3] in a derivation, which is induced by the edges (written as side conditions) introduced by the inference rules for implication, negation, co-implication and co-negation along the branch that leads to this initial sequent. The rules for negation and co-negation are only given for convenience as they make derivations more compact, but let us remark that they are easily derivable from the rules for implication and co-implication using the weakening rules w_L and w_R, which is also the only real technical reason why we introduce such weakening rules[4]. Moreover, we choose to handle contraction through the notion of multiplicity instead of providing explicit contraction rules in order to cling to the notions introduced so far, and on which Definition 12 relies.

Reduction rules of RGBiInt are a simple reformulation of the reduction rules of Figure 4 in terms of unsigned labelled formulas, knowing that a positive (respectively negative) labelled formula should occur on the left-hand (respectively right-hand) side of a labelled sequent. More precisely, a labelled formula $A[v]$ on the left-hand (respectively right-hand) side of a labelled sequent simply means that A^+ (respectively A^-) appears in the tags of the vertex u of the biRG associated with the branch of the derivation where that labelled sequent occurs.

A derivation in RGBiInt is *finished* if all of its initial sequents are irreducible. Figure 9 gives an example of a finished derivation in RGBiInt for our well-known formula $C = (((R_2 \vee_1 P_3) \prec_{\overline{a}} Q_4) \wedge_0 (P_5 \to_x \bot_6)) \to_a ((P_7 \to_b R_8) \prec_{\overline{x}} Q_9)$, more

[3] A sequent that is not introduced in a derivation as the conclusion of an inference rule.

[4] Weakening rules are only there to clean-up all useless occurrences of $\bot$ on the right-hand side of labelled sequents that would otherwise bloat the rules for negation and co-negation.

Axioms and structural rules:

$$\frac{}{\Gamma, \mathrm{A}[u] \vdash \mathrm{A}[u], \Delta}\,\mathrm{ax} \qquad \frac{}{\Gamma, \bot[u] \vdash \Delta}\,\bot_{\mathrm{L}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\,\mathrm{w_L} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Delta'}\,\mathrm{w_R}$$

Reduction rules:

$$\frac{\Gamma, \mathrm{A}[u], \mathrm{B}[u] \vdash \Delta}{\Gamma, (\mathrm{A} \wedge \mathrm{B})[u] \vdash \Delta}\,\wedge_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}[u], \Delta \qquad \Gamma \vdash \mathrm{B}[u], \Delta}{\Gamma \vdash (\mathrm{A} \wedge \mathrm{B})[u], \Delta}\,\wedge_{\mathrm{R}}$$

$$\frac{\Gamma, \mathrm{A}[u] \vdash \Delta \qquad \Gamma, \mathrm{B}[u] \vdash \Delta}{\Gamma, (\mathrm{A} \vee \mathrm{B})[u] \vdash \Delta}\,\vee_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}[u], \mathrm{B}[u], \Delta}{\Gamma \vdash (\mathrm{A} \vee \mathrm{B})[u], \Delta}\,\vee_{\mathrm{R}}$$

$$\frac{\Gamma \vdash \mathrm{A}[x], \Delta \qquad \Gamma, \mathrm{B}[x] \vdash \Delta}{\Gamma, (\mathrm{A} \to \mathrm{B})[u] \vdash \Delta}\,u[\phi]x \qquad \frac{\Gamma, \mathrm{A}[a] \vdash \mathrm{B}[a], \Delta}{\Gamma \vdash (\mathrm{A} \to \mathrm{B})[u], \Delta}\,u[\psi]a$$

$$\frac{\Gamma, \mathrm{A}[\overline{a}] \vdash \mathrm{B}[\overline{a}], \Delta}{\Gamma, (\mathrm{A} \prec \mathrm{B})[u] \vdash \Delta}\,\overline{a}[\bar{\psi}]u \qquad \frac{\Gamma \vdash \mathrm{A}[\overline{x}], \Delta \qquad \Gamma, \mathrm{B}[\overline{x}] \vdash \Delta}{\Gamma \vdash (\mathrm{A} \prec \mathrm{B})[u], \Delta}\,\overline{x}[\bar{\phi}]u$$

Derivable reduction rules:

$$\frac{\Gamma \vdash \mathrm{A}[x], \Delta}{\Gamma, \neg\mathrm{A}[u] \vdash \Delta}\,u[\phi]x \qquad \frac{\Gamma, \mathrm{A}[a] \vdash \Delta}{\Gamma \vdash \neg\mathrm{A}[u], \Delta}\,u[\psi]a$$

$$\frac{\Gamma \vdash \mathrm{A}[\overline{a}], \Delta}{\Gamma, \sim\mathrm{A}[u] \vdash \Delta}\,\overline{a}[\bar{\psi}]u \qquad \frac{\Gamma, \mathrm{A}[\overline{x}] \vdash \Delta}{\Gamma \vdash \sim\mathrm{A}[u], \Delta}\,\overline{x}[\bar{\phi}]u$$

Fig. 8 Labelled calculus RGBiInt for BiInt

precisely, for $\mu(\mathrm{C})$ with $\mu(x) = \mu(\mathrm{P} \to \bot) = 1$ and $\mu(\overline{x}) = \mu((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q}) = 1$. The derivation has four irreducible initial sequents $s_{i(1 \leqslant i \leqslant 4)}$ for which we collect the sets E_i of all the edges that occur (as side notes to inference lines) along the branch of the derivation that leads to s_i. We get $E_2 = E_4 = \{\epsilon[\psi]a,\ a[\phi]x,\ \overline{a}[\bar{\psi}]a,\ \overline{x}[\bar{\phi}]a\}$ for s_2 and s_4, and $E_1 = E_3 = E_2 \cup \{\overline{x}[\psi]b\}$ for s_1 and s_3. As easily seen, each E_i induces a biRG the structure of which exactly corresponds to the pure part of the irreducible biRG G_i of Figure 3. The discussion in the previous paragraph also makes it clear that the atomic labelled formulas of each s_i exactly correspond to the atomic signed formulas tagging the vertices of G_i: *e.g.*, s_1 implies that in G_1, R^+ and Q^- are tags for the vertex $\overline{a}$, P^- is a tag for the vertex x, and P^+ and R^- are tags for the vertex b.

The easiest way to define the notion of RGBiInt-provability is to relate it to the notion biRG-validity.

Definition 13 (RGBiInt provability) A BiInt formula A is RGBiInt-provable if and only if there exist a multiplicity μ and a finished derivation $\mathcal{D}$ for $\mu(\mathrm{A})$ such that the biRG collection induced by the initial sequents of $\mathcal{D}$ is biRG-valid.

Let us remark that the first condition of Definition 12 ensures that a σ-binding is functional and can therefore actually be translated into the more standard notion of substitution, *i.e.*, $\sigma(x) = a$ if and only if there is a (uniquely determined) σ-link between x and a in some irreducible biRG associated with some irreducible initial sequent of a derivation. Moreover, the second condition of Definition 12 ensures that the notion of κ-binding can actually be translated into the more standard notion of spanning set of connections, *i.e.*, for all consistent biRGs G through $\mu(\mathrm{A})$, there is at least one κ-edge $u[\kappa]v$ and that κ-edge corresponds to a pair of labelled formulas $\mathrm{A}[u]$ and $\mathrm{A}[v]$ that respectively occur on the left-hand and right-hand side of the initial sequent s associated with G in the derivation $\mathcal{D}$ for $\mu(\mathrm{A})$.

$$\dfrac{\dfrac{\overset{s_1}{\mathrm{R}[\overline{a}], \mathrm{P}[b] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], \mathrm{R}[b]}}{\mathrm{R}[\overline{a}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], (\mathrm{P} \to \mathrm{R})[\overline{x}]}\,\overline{x}[\psi]b \qquad \overset{s_2}{\mathrm{Q}[\overline{x}], \mathrm{R}[\overline{a}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x]}}{\mathrm{R}[\overline{a}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}\,\overline{x}[\bar{\phi}]a}{\mathcal{D}_1}$$

$$\dfrac{\dfrac{\overset{s_3}{\mathrm{P}[\overline{a}], \mathrm{P}[b] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], \mathrm{R}[b]}}{\mathrm{P}[\overline{a}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], (\mathrm{P} \to \mathrm{R})[\overline{x}]}\,\overline{x}[\psi]b \qquad \overset{s_4}{\mathrm{P}[\overline{a}], \mathrm{Q}[\overline{x}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x]}}{\mathrm{P}[\overline{a}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}\,\overline{x}[\bar{\phi}]a}{\mathcal{D}_2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{(\mathrm{R} \vee \mathrm{P})[\overline{a}] \vdash \mathrm{Q}[\overline{a}], \mathrm{P}[x], ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}\,\vee_{\mathrm{L}}}{((\mathrm{R} \vee \mathrm{P}) \prec \mathrm{Q})[a] \vdash \mathrm{P}[x], ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}\,\overline{a}[\bar{\psi}]a \qquad \dfrac{}{\bot[a] \vdash ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}}{((\mathrm{R} \vee \mathrm{P}) \prec \mathrm{Q})[a], (\mathrm{P} \to \bot)[a] \vdash ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}\,a[\phi]x + \mathrm{w}_{\mathrm{L}}}{(((\mathrm{R} \vee \mathrm{P}) \prec \mathrm{Q}) \wedge (\mathrm{P} \to \bot))[a] \vdash ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q})[a]}\,\wedge_{\mathrm{L}}}{\vdash ((((\mathrm{R} \vee \mathrm{P}) \prec \mathrm{Q}) \wedge (\mathrm{P} \to \bot)) \to ((\mathrm{P} \to \mathrm{R}) \prec \mathrm{Q}))[\epsilon]}\,\epsilon[\psi]a$$

Fig. 9 RGBiInt-derivation for $(((\mathrm{R}_2 \vee_1 \mathrm{P}_3) \prec_{\overline{a}} \mathrm{Q}_4) \wedge_0 (\mathrm{P}_5 \to_x \bot_6)) \to_a ((\mathrm{P}_7 \to_b \mathrm{R}_8) \prec_{\overline{x}} \mathrm{Q}_9)$

6.1 Soundness

Let us discuss the soundness of the characterization, which relies on the labelled calculus RGBiInt given in Figure 8. The proof follows the standard pattern of proving that every inference rule of the calculus preserves a standard notion of realizability in BiInt Kripke models.

Definition 14 (realizability) Let R be a biRG collection, $\overline{\sigma}$ be an admissible global σ-binding for R and $\mathcal{M}$ be a Kripke model $\langle M, \sqsubseteq, [\![\cdot]\!]\rangle$. R is *σ-realizable in $\mathcal{M}$* if and only if for some biRG $G(V, E)$ in R there exists a (total) function $\|\cdot\| : V^{\Psi} \to M$, called a *$\sigma$-realization* of G in $\mathcal{M}$, such that:

1. $(\forall u \in V^{\Phi})(\overline{\sigma}(u) = v$ implies $\|u\| = \|v\|)$;
2. $(\forall u \in V)(\forall v \in V)(Gu \to v$ implies $\|u\| \sqsubseteq \|v\|)$;
3. $(\forall u \in V)(\forall \mathrm{A}^{+} \in \mathcal{F}(G, u))(\|u\| \models \mathrm{A})$;
4. $(\forall u \in V)(\forall \mathrm{A}^{-} \in \mathcal{F}(G, u))(\|u\| \not\models \mathrm{A})$.

A biRG collection is σ-realizable if and only if it is σ-realizable in some Kripke model for some admissible global σ-binding.

Lemma 1 *A complementary biRG collection cannot be σ-realizable.*

Proof Let R be a complementary biRG collection and $G(V, E)$ be a biRG in R. By definition, G is complementary. Therefore, there is at least one κ-edge $u[\kappa]v$ in E such that the slice $G^{\{u,v\}}$ is admissible and $G^{\{u,v\}}u \to v$, which trivially implies $Gu \to v$. Moreover, by definition of κ-edges, there is some formula A such that $\mathrm{A}^{+} \in \mathcal{F}(G, u)$ and $\mathrm{A}^{-} \in \mathcal{F}(G, v)$. Let us suppose that R is σ-realizable, then for any σ-realization of G in a Kripke model $\mathcal{M}$, $\mathrm{A}^{+} \in \mathcal{F}(G, u)$ implies $\|u\| \models \mathrm{A}$, $\mathrm{A}^{-} \in \mathcal{F}(G, v)$ implies $\|v\| \not\models \mathrm{A}$ and $Gu \to v$ implies $u \sqsubseteq v$, which leads to the contradiction that $\|u\| \not\models \mathrm{A}$ by Kripke monotonicity. Therefore, R cannot be σ-realizable.

Lemma 2 *Reduction rules preserve the σ-realizability of biRG collections.*

Proof By case analysis of the rules of Figure 8.

Theorem 1 *Let* A *be a* BiInt *formula. If* A *is biRG-valid then* A *is valid in the Kripke semantics.*

Proof Let us suppose that A is not valid in the Kripke semantics, then, $m \not\models \mathrm{A}$ for some world m in some Kripke model $\mathcal{M} = \langle M, \sqsubseteq, [\![\cdot]\!] \rangle$. The initial biRG-collection R_0 of any reduction through A is therefore σ-realizable under the empty global σ-binding (since there are no ϕ-vertices in R_0) by setting $\|\epsilon\| = m$. Lemma 2 then implies that the final biRG-collection R_f is also σ-realizable, which by Lemma 1 implies that R_f cannot be complementary, thus contradicting the biRG-validity of A.

6.2 Completeness

Since completeness relies on counter-model construction, we first need to define a saturation relation which plays the same role for biRGs as Hintikka collections do for sets of formulas in intuitionistic logic.

Definition 15 (saturation) Let $G(V, E)$ be a biRG. The *saturation* relation on G is defined as the smallest relation between vertices and signed formulas such that:

- Base case: for all A in $\{\bot\} \cup \mathcal{V}$,
  - $Gu \Vdash \mathrm{A}^+$ iff $(\exists v \in V)(Gv \to u \text{ and } \mathrm{A}^+ \in \mathcal{F}(G, v))$;
  - $Gu \Vdash \mathrm{A}^-$ iff $(\exists v \in V)(Gu \to v \text{ and } \mathrm{A}^- \in \mathcal{F}(G, v))$;
- Induction:
  - $Gu \Vdash (\mathrm{A} \wedge \mathrm{B})^+$ iff $Gu \Vdash \mathrm{A}^+$ and $Gu \Vdash \mathrm{B}^+$;
  - $Gu \Vdash (\mathrm{A} \wedge \mathrm{B})^-$ iff $Gu \Vdash \mathrm{A}^-$ or $Gu \Vdash \mathrm{B}^-$;
  - $Gu \Vdash (\mathrm{A} \vee \mathrm{B})^+$ iff $Gu \Vdash \mathrm{A}^+$ or $Gu \Vdash \mathrm{B}^+$;
  - $Gu \Vdash (\mathrm{A} \vee \mathrm{B})^-$ iff $Gu \Vdash \mathrm{A}^-$ and $Gu \Vdash \mathrm{B}^-$;
  - $Gu \Vdash (\mathrm{A} \to \mathrm{B})^+$ iff $(\forall v \in V)(\text{if } Gu \to v \text{ and } Gv \Vdash \mathrm{A}^+ \text{ then } Gv \Vdash \mathrm{B}^+)$;
  - $Gu \Vdash (\mathrm{A} \to \mathrm{B})^-$ iff $(\exists v \in V)(Gu \to v \text{ and } Gv \Vdash \mathrm{A}^+ \text{ and } Gv \Vdash \mathrm{B}^-)$;
  - $Gu \Vdash (\mathrm{A} \prec \mathrm{B})^+$ iff $(\exists v \in V)(Gv \to u \text{ and } Gv \Vdash \mathrm{A}^+ \text{ and } Gv \Vdash \mathrm{B}^-)$;
  - $Gu \Vdash (\mathrm{A} \prec \mathrm{B})^-$ iff $(\forall v \in V)(\text{if } Gv \to u \text{ and } Gv \Vdash \mathrm{A}^+ \text{ then } Gv \Vdash \mathrm{B}^+)$.

$G(V, E)$ is *saturated* if and only if $(\forall u \in V)(\forall \mathrm{C}^{\mathrm{s}} \in \mathcal{F}(G, u))(Gu \Vdash \mathrm{C}^{\mathrm{s}})$.

Definition 16 (contradiction) A biRG $G(V, E)$ is *contradictory* if and only if $(\exists \mathrm{P} \in \mathcal{V})(\exists u \in V)(Gu \Vdash \bot^+ \text{ or } (Gu \Vdash \mathrm{P}^- \text{ and } Gu \Vdash \mathrm{P}^+))$.

Lemma 3 *If $G(V, E)$ is a non-contradictory saturated biRG then* $(\forall \mathrm{A} \in \mathcal{F})(\forall u \in V)(Gu \nVdash \mathrm{A}^- \text{ or } Gu \nVdash \mathrm{A}^+)$.

Proof By structural induction on A.

Definition 17 (H-model) Let $G(V, E)$ be a biRG. The *H-model* of $G(V, E)$ is defined as the triple $\mathcal{H}(G) = \langle M, \sqsubseteq, [\![\cdot]\!] \rangle$ where

- M is the quotient of V by the relation of mutual concrete reachability, *i.e.*

$$M = \{\dot{u} \mid u \in V\} \text{ with } \dot{u} = \{v \mid v \in V \text{ and } Gu \leftrightarrow v\}.$$

– $\sqsubseteq$ is defined as the binary relation on M such that

$$(\forall m \in M)(\forall n \in M)(m \sqsubseteq n \text{ if and only if } (\forall u \in m)(\forall v \in n)(Gu \to v)).$$

– $[\![\cdot]\!]$ is the function from M to sets of propositional letters such that

$$(\forall \mathrm{P} \in \mathcal{V})(\forall m \in M)(\mathrm{P} \in [\![m]\!] \text{ if and only if } (\forall u \in m)(Gu \Vdash \mathrm{P}^+)).$$

Lemma 4 *Let $G(V, E)$ be a biRG and $\mathcal{H}(G) = \langle M, \sqsubseteq, [\![\cdot]\!]\rangle$ be its H-model. If G is saturated and non-contradictory then $\mathcal{H}(G)$ is a Kripke model such that*

1. $(\forall m \in M)(\forall n \in M)(m \sqsubseteq n$ *if and only if* $(\exists u \in m)(\exists v \in n)(Gu \to v))$ *and*
2. $(\forall \mathrm{P} \in \mathcal{V})(\forall m \in M)(\mathrm{P} \in [\![m]\!]$ *if and only if* $(\exists u \in m)(Gu \Vdash \mathrm{P}^+))$.

Proof The fact that $\sqsubseteq$ is a partial order on M directly follows from the fact that $(G_ \to _)$ is a preorder on V. Moreover, Lemma 3 implies that $[\![\cdot]\!]$ is well defined while Definition 15 ensures that $[\![\cdot]\!]$ satisfies Kripke monotonicity. Therefore $\mathcal{H}(G)$ is a Kripke model. Let us show the two additional properties.

1. The "only if" direction is immediate since equivalence classes cannot be empty. For the "if" direction, for any two vertices u' and v' in m and n respectively, it directly follows from $Gu \leftrightarrow u'$ and $Gv \leftrightarrow v'$ that $Gu' \to v'$.
2. Immediate by the monotonicity of $(G_ \Vdash _)$ encapsulated in Definition 15.

Lemma 5 *Let $G(V, E)$ be a biRG. If G is saturated and non-contradictory then for all formulas* A *and all vertices u in V, $Gu \Vdash \mathrm{A}^+$ if and only if $\dot{u} \models \mathrm{A}$ in $\mathcal{H}(G)$.*

Proof The proof is by induction on the structure of A. The base case where A is a propositional letter holds by definition of $\models$ and $[\![\cdot]\!]$. The base case where A is $\bot$ holds because on one hand, the fact that G is non-contradictory implies that $Gu \Vdash \bot^+$ can never hold and, on the other hand, $\bot$ can never be forced by definition of a forcing relation. For the induction step, we only show a few cases, the others being similar:

– Case $\mathrm{A} \equiv (\mathrm{B} \to \mathrm{C})$:
  1. For the "only if" direction, let us assume that $Gu \Vdash (\mathrm{B} \to \mathrm{C})^+$. Let v be some vertex in V such that $\dot{u} \sqsubseteq \dot{v}$ and $\dot{v} \models \mathrm{B}$. By definition of $\sqsubseteq$, $\dot{u} \sqsubseteq \dot{v}$ implies $Gu \to v$. By induction hypothesis, $n \models \mathrm{B}$ implies $Gv \Vdash \mathrm{B}$. Since G is saturated, if follows from $Gu \Vdash (\mathrm{B} \to \mathrm{C})^+$, $Gv \Vdash \mathrm{B}$ and $Gu \to v$ that $Gv \Vdash \mathrm{C}$, which leads to $\dot{v} \models \mathrm{C}$ by induction hypothesis. Therefore, $\dot{u} \models (\mathrm{B} \to \mathrm{C})$.
  2. For the "if" direction, let us assume that $\dot{u} \models (\mathrm{B} \to \mathrm{C})$ in $\mathcal{H}(G)$. Let v be a vertex in V such that $Gu \to v$ and $Gv \Vdash \mathrm{B}^+$. $Gu \to v$ implies $\dot{u} \sqsubseteq \dot{v}$ by Lemma 4 and $Gv \Vdash \mathrm{B}^+$ implies $\dot{v} \models \mathrm{B}$ by induction hypothesis. Since $\dot{u} \models (\mathrm{B} \to \mathrm{C})$, $\dot{v} \models \mathrm{B}$ implies $\dot{v} \models \mathrm{C}$, which leads to $Gv \Vdash \mathrm{C}^+$ by induction hypothesis. Therefore, $Gu \Vdash (\mathrm{B} \to \mathrm{C})^+$.
– Case $\mathrm{A} \equiv (\mathrm{B} \prec \mathrm{C})$:
  1. For the "only if" direction, let us assume that $Gu \Vdash (\mathrm{B} \prec \mathrm{C})^+$. It then follows that there is some vertex v in V such that $Gv \to u$, $Gv \Vdash \mathrm{B}^+$ and $Gv \Vdash \mathrm{C}^-$. By definition of $\sqsubseteq$, $Gv \to u$ implies $\dot{v} \sqsubseteq \dot{u}$. By induction hypothesis, $Gv \Vdash \mathrm{B}^+$ implies $\dot{v} \models \mathrm{B}$. By Lemma 3, $Gv \Vdash \mathrm{C}^-$ implies $Gv \nVdash \mathrm{C}^+$, which leads to $\dot{v} \not\models \mathrm{C}$ by induction hypothesis. Therefore, $\dot{u} \models (\mathrm{B} \prec \mathrm{C})$.
  2. For the "if" direction, let us assume that $\dot{u} \models (\mathrm{B} \prec \mathrm{C})$ in $\mathcal{H}(G)$. By definition, $\dot{u} \models (\mathrm{B} \prec \mathrm{C})$ implies $\dot{v} \models \mathrm{B}$ and $\dot{v} \not\models \mathrm{C}$ for some vertex in v in V such that $\dot{v} \sqsubseteq \dot{u}$. By induction hypothesis and by definition of $\sqsubseteq$, it follows that $Gv \Vdash \mathrm{B}$, $Gv \nVdash \mathrm{C}$ and $Gv \to u$. Therefore, $Gu \Vdash (\mathrm{B} \prec \mathrm{C})^+$.

Let us illustrate how to extract counter-models from saturated biRGs with a short example, the formula $\mathrm{D} = \mathrm{Q} \to ((\neg(\mathrm{P} \prec \mathrm{Q}) \prec \mathrm{P}) \vee \mathrm{P})$.
Up to σ- and κ-edges, the collection of irreducible biRGs through D is as follows:

Both biRGs G_{00} and G_{01} are admissible and consistent, but only G_{00} fails to be complementary and is therefore non-contradictory. Moreover, G_{00} is also saturated. In order to turn G_{00} into a counter-model of D, we need to build the H-model $\langle M, \sqsubseteq, [\![\cdot]\!]\rangle$ of G_{00}. We first calculate the quotient of G_{00} by the equivalence generated by the σ-edges which leads to the following set of vertex-classes:

$$M = \{\dot{\epsilon}, \dot{a}, \dot{b}, \dot{\overline{a}} \mid \dot{\epsilon} = \{\epsilon\}, \dot{a} = \dot{\overline{x}} = \{a, \overline{x}\}, \dot{b} = \{b\}, \dot{\overline{a}} = \{\overline{a}\}\}.$$

Then, we consider M as a set of worlds on which the accessibility relation $\sqsubseteq$ is induced by the reflexive and transitive closure of the solid-edges in the initial biRG. More precisely, a world n is accessible from a world m in M if and only if there is a concrete path in G_{00} from some vertex in m to some vertex in n. Finally, for all propositional letters P in $\mathcal{V}$ and all worlds m in M, the atomic Kripke interpretation $[\![\cdot]\!]$ is obtained by setting $\mathrm{P} \in [\![m]\!]$ if $\mathrm{P}^+ \in \mathcal{F}(G, m)$ and closing under Kripke monotonicity, which leads to the following bi-intuitionistic Kripke model[5]:

Theorem 2 *Let* A *be a* BiInt *formula. If* A *is valid in the Kripke semantics then* A *is biRG-valid.*

Proof The completeness proof follows from Lemma 5 and proceeds by counter-model construction from a saturated and non-contradictory biRG in the final biRG-collection of a finished reduction through A.

7 Solving Admissibility Constraints

In plain intuitionistic logic, we could use prefixes instead of labels and resort to T-string (prefix) unification to solve prefix constraints [11]. However we cannot do that in the case of BiInt because a prefix essentially is a way to encode the path to a given node in a Kripke tree. Since the Kripke semantics of BiInt deals with graphs instead of trees, there can be several distinct paths to a given node and taking care

[5] Kripke monotonicity is never explicity depicted in our graphical representations in order to improve their readability, but it is always implicitly assumed.

of that using prefixes (by encoding both successors and predecessors) would break the T-string property of such prefixes, which in turns prevents the use of T-string unification.

Given an admissible σ-binding, it is not difficult to check whether κ-edges are covered by a concrete path or not, the problem is to find such σ-bindings. A trivial but particularly inefficient solution would be to enumerate all possible σ-bindings and check whether they are admissible or not. As a first step toward more efficient solutions, we now sketch an algorithm that, given a pure biRG G and a set K of κ-edges, enumerates only those σ-bindings that are admissible for the slice of G induced by K by preserving the acyclicity of the underlying reduction ordering $\lhd$.

In the intuitionistic case, an edge $u[\tau]v$, with $\tau \in \{\psi, \phi\}$ has a double intuitive meaning. From a semantic standpoint, $u[\tau]v$ means that, whenever interpreted in a Kripke model $\mathcal{M}$, the world corresponding to v should be accessible in $\mathcal{M}$ from the world corresponding to u, as prescribed by the Kripke interpretation of implication given in Definition 1. Therefore, in the semantic interpretation, one reasons in terms of Kripke model accessibility. From a syntactic standpoint, $u[\tau]v$ means that the formula $f(v)$ associated with the index v should be introduced before the formula $f(u)$ associated with the index u in a sequent-style derivation (*e.g.*, in the sequent calculus of Figure 1). Therefore, in the syntactic interpretation, one reasons in terms of introduction precedence in sequent-style derivations.

In the bi-intuitionistic case, the semantic interpretation still holds, but the syntactic interpretation breaks down. Indeed, although oriented from u to v, the edge $u[\bar{\psi}]v$ implies that $f(u)$ is a subformula of $f(v)$ and thus has to be introduced after $f(v)$ in a sequent-style derivation. In order to recover the syntactic interpretation, one has to cross $\bar{\psi}$- and $\bar{\phi}$-edges backward (from their target to their source) when reasoning in terms of introduction precedence, which leads to the notion of *walks*.

A *walk* through a biRG is similar to a path as described in Definition 8 except that, in a walk, $\bar{\phi}$- and $\bar{\psi}$-edges must be crossed backward and σ-edges can only be crossed from their ψ-vertex to their ϕ-vertex. T-walks are defined accordingly as walks that only cross τ-edges such that $\tau \in T$ and, whenever $T = \{\psi, \bar{\psi}, \sigma\}$, a T-walk is called a *concrete walk*. Summing it up, we now have the notion of paths to reason in terms of Kripke model accessibility (predecessors/successors) and the notion of walks to reason in terms of introduction precedence (before/after).

Let u be an arbitrary vertex in a biRG, we define $A_t^n(T, u)$ (respectively $B_t^n(T, u)$) as the set of all vertices v with $itype(v)$[6] occurring in the set t that can be reached from u (respectively from which u can be reached) by a T-walk of length n. The two sets $S_t^n(T, u)$ and $P_t^n(T, u)$ are defined similarly using the notion of T-path instead of T-walk. The letters A and B are reminiscent of "After" and "Before" and shall handle constraints related to introduction precedence in the syntactic interpretation of biRGs, while the letters S and P are reminiscent of "Successor" and "Predecessor" and shall handle constraints related to Kripke model accessibility in the semantic interpretation of biRGs. For all $F \in \{S, P, A, B\}$, we define

$$F_t(T, u) = \bigcup_{i \in \mathbb{N}} F_t^i(T, u), \text{ with } F_t^0(T, u) = \begin{cases} \{u\} \text{ if } itype(u) \in t \\ \emptyset \text{ otherwise.} \end{cases}$$

[6] The intuitionistic type of the subformula the reduction of which is responsible for the introduction of the vertex v according to the rules of Figure 4.

Rule	`selectRule(`u`,`c`,`v`)`	`applyRule(`u`,`c`,`v`)`
Bind	$R(v) = u$	$c,\ D(v) \cup M(c, v)$
Narrow	$c \in A(v)$	$R(v),\ D(v) - A(u)$
Widen	$c = R(v) \wedge (c \neq R(u) \vee \mathit{itype}(u) = \mathit{itype}(v))$	$R(v),\ D(v) \cup M(u, v)$

Table 2 Solving rules (assuming $v \neq u$)

Let u, v be two ϕ-vertices, we define the set $M_t(T, u, v)$ as follows:

$$M_t(T, u, v) = \begin{cases} S_t(T, u) - A_t(T, v) \text{ if } \mathit{itype}(v) = \phi \\ P_t(T, u) - A_t(T, v) \text{ if } \mathit{itype}(v) = \overline{\phi} \end{cases}$$

For readability, we shall omit the T parameter whenever $T = \{\phi, \bar{\phi}, \psi, \bar{\psi}\}$ and the t subscript whenever $t = \{\psi, \overline{\psi}\}$.

Our solving algorithm relies on two particular objects $R(u)$ and $D(u)$ that are initially computed for all ϕ-vertices u in the biRG $G(V, E)$ given as an input to the solving algorithm. In the initial step, for all $u \in V^{\Phi}$ we set

$$R(u) = B^1(u) \text{ and } D(u) = M(R(u), u).$$

The set $D(u)$, called the *domain* of u, is meant to represent all suitables instantiations for u, *i.e.*, ψ-vertices that would be suitable targets for a σ-link the source of which is u. In the initial step $R(u)$, called the *root* of u, is the vertex responsible for the introduction of u in the biRG reduction process. Domains and roots shall be rewritten as prescribed by the rewrite rules of Table 2 during the solving process.

In a second step, let us define the following notion of *variable dependency*.

Definition 18 (variable dependency) Let x and y be two ϕ-vertices in a biRG, x *depends on* y, written $x \propto y$, if and only if $x \in M_t(R(y), y)$ with $t = \{\psi, \overline{\psi}, \phi, \overline{\phi}\}$.

Intuitively, "x depends on y" means that y should be bound before x because some constant b may only become admissible for x after y gets bound to some particular constant a. Consequently, we arrange all variables in a pre-ordered list $X = x_1, \ldots, x_n$ so that if $i < j$ and $x_i \propto x_j$ then $x_j \propto x_i$, which means that traversing the list from left to right, if a variable x_i is encountered before the variable x_j then, either x_i does not depend on x_j, or x_i and x_j mutually depend on each other (so that a choice has to be made as to which one should be considered first).

The final step consists of the actual enumeration. For all variables $u = x_i$ in X, select a constant c in $D(u)$ and assign it to u (*i.e.*, add a σ-link $u[\sigma]c$ in the biRG). Then, for all subsequent variables v, select the first (top-down) applicable rewrite rule according to the firing conditions described in the second column of Table 2 and, if there is any such rule, apply it to rewrite $R(v)$ and $D(v)$ as prescribed by the third column of Table 2. More precisely, for all $v = x_j$ in X such that $j > i$, perform

$$\texttt{applyRule} := \texttt{selectRule(}u, c\texttt{,}v\texttt{)}$$

$$R(v), D(v) := \texttt{applyRule(}u, c\texttt{,}v\texttt{)}$$

If all variables in X can be assigned then we have an admissible σ-binding and we just check whether all the κ-edges in K (given as an input to the algorithm) are covered by a concrete path. If so, we are done, otherwise, we backtrack and try new assignments for the variables in X until a solution is found or all possible assignments have been exhausted.

8 Variable Splitting

In this section we briefly discuss how the technique of variable splitting recently developed for prefixes [1] can be adapted to our R-graph based setting.

Let us illustrate the main ideas with a short example. With a multiplicity $\mu(x) = \mu((\mathrm{P} \to \mathrm{P}) \to \mathrm{P}) = 1$, it is not possible to prove the validity of the formula for which a derivation in RGBiInt is given below (indexes are indicated as subscripts).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{P}_2[d] \vdash \mathrm{P}_3[d]}\,ax}{\vdash (\mathrm{P}_2 \to_d \mathrm{P}_3)[x]}\,x[\psi]d \qquad \dfrac{\overset{s_2}{\dfrac{\mathrm{P}_4[x], \mathrm{Q}_6[b] \vdash \mathrm{P}_7[b]}{\mathrm{P}_4[x] \vdash (\mathrm{Q}_6 \to_b \mathrm{P}_7)[a]}}\,a[\psi]b \quad \overset{s_1}{\dfrac{\mathrm{P}_4[x], \mathrm{R}_8[c] \vdash \mathrm{P}_9[c]}{\mathrm{P}_4[x] \vdash (\mathrm{R}_8 \to_c \mathrm{P}_9)[a]}}\,a[\psi]c}{\mathrm{P}_4[x] \vdash ((\mathrm{Q}_6 \to_b \mathrm{P}_7) \wedge_5 (\mathrm{R}_8 \to_c \mathrm{P}_9))[a]}\,\wedge_{\mathrm{R}}}{((\mathrm{P}_2 \to_d \mathrm{P}_3) \to_x \mathrm{P}_4)[a] \vdash ((\mathrm{Q}_6 \to_b \mathrm{P}_7) \wedge_5 (\mathrm{R}_8 \to_c \mathrm{P}_9))[a]}\,a[\phi]x}{(((\mathrm{P}_2 \to_d \mathrm{P}_3) \to_x \mathrm{P}_4) \to_a ((\mathrm{Q}_6 \to_b \mathrm{P}_7) \wedge_5 (\mathrm{R}_8 \to_c \mathrm{P}_9)))[\epsilon]}\,\epsilon[\psi]a$$

The initial sequent s_1 requires $\sigma_1 = \{x/c\}$, while s_2 requires $\sigma_2 = \{x/b\}$. The conflict on x thus makes it impossible to compute a global substitution from the two local substitutions σ_1 and σ_2. A first solution would be to increase multiplicity in order to have one copy x^2 of the variable x so as to set $\sigma_1 = \{x/c\}$ and $\sigma_2 = \{x^2/b\}$. The price to pay for this solution would be an unnecessary longer derivation because, in this example, assigning two distinct values to x would not harm soundness. The actual problem lies in that RGBiInt is *variable sharing*: the same ϕ- or $\overline{\phi}$-formula occurring in distinct branches leads to the introduction of the same variable in all branches since we use the index of that formula as the introduced variable.

Variable sharing leads to full permutability of the rules, but results in potentially longer derivations. Had we allowed the rules $\to_{\mathrm{L}}$ and $\prec_{\mathrm{R}}$ to introduce a fresh copy of the variable associated to its principal ϕ- or $\overline{\phi}$-formula for each of its occurrences in distinct branches of a derivation, expanding the β-formula $(\mathrm{Q} \to \mathrm{P}) \wedge (\mathrm{R} \to \mathrm{P})$ before the ϕ-formula $(\mathrm{P} \to \mathrm{P}) \to \mathrm{P}$ would have resulted in the introduction of x and of a fresh copy x^2 in the branches corresponding to the first and second premices of $\wedge_{\mathrm{R}}$ respectively. However, such a *variable pure* formulation of RGBiInt would break full permutability as β-formulas would need to be expanded before ϕ- and $\overline{\phi}$-formulas to enable as many copies of each variable as possible.

Variable splitting is a technique that allows variable sharing calculi to bind each shared variable to a distinct value in each branch it occurs in, which enables the computation of local substitutions and helps keeping derivations shorter. Let A be a BiInt formula, a *splitting set* for A is a set of dual-free indexes of secondary type β_1 or β_2 which is downward closed w.r.t. the tree ordering $\ll$. Adding variable splitting to RGBiInt requires that variables should be replaced with *colored variables*, *i.e.*, pairs x^X where x is a variable occurring as an index in A and X is a splitting set for A. Substitutions are replaced with *colored substitutions* accordingly. A colored substitution σ induces a *splitting ordering* which is the least relation between β- and ϕ- or $\overline{\phi}$-indexes such that if $\sigma(x^X) \neq \sigma(x^Y)$, then there are dual indexes $i \in X$ and $j \in Y$ such that $(i \triangle j) \prec x$. A splitting ordering encodes the fact that β-formulas should be introduced before ϕ- or $\overline{\phi}$-formulas in a variable pure bottom-up derivation. Finally, Definition 12 is extended to take care of splitting orderings by redefining the reduction ordering so that $\lhd = (\ll \cup \sqsubset \cup \prec)^+$. In the above derivation, $\{c\}$ and $\{b\}$ are splitting sets that give rise to two distinct colored substitutions $\sigma_1 = \{x^{\{c\}}/c\}$ and $\sigma_2 = \{x^{\{b\}}/b\}$ such that $(b \triangle c) = 5$, $5 \prec x$, $b \sqsubset x$ and $c \sqsubset x$. The induced reduction ordering $\lhd = (\ll \cup \sqsubset \cup \prec)^+$ is easily checked irreflexive.

9 Conclusion

The main contributions of this work are: a new connection-based characterization of validity in propositional bi-intuitionistic logic in terms of bi-intuitionistic R-graphs and a sound and complete cut-free free-variable labelled sequent calculus that enjoys variable splitting. Moreover, as a first step from the characterization toward an actual connection-based proof-search method, we define a new algorithm for solving reachability constraints in bi-intuitionistic R-graphs. Such an algorithm is required as encoding paths between vertices of a bi-intuitionistic R-graph would not lead to a set of prefixes satisfying the T-string property, thus preventing the use of T-string unification algorithms developed for intuitionistic logic [11]. Our algorithm has been implemented in Prolog and computes admissible σ-bindings more efficiently than the "enumerate all σ-bindings and check them for admissibility" strategy. Further work will be devoted to the study of the complexity of our constraint-solving algorithm and to the key problem of increasing multiplicities more wisely by finding tight upper bounds in order to obtain more efficient connection-based proof-search procedures.

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