Label-free Natural Deduction Systems for Intuitionistic and Classical Modal Logics

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Abstract. In this paper we study natural deduction for the intuitionistic and classical modal logics obtained from the combinations of the axioms T, B, 4 and 5. They are based on a multi-contextual structure, called tree-sequent, that allows to design simple label-free systems. We show that they are sound and complete and that they satisfy the normalization property and also the subformula property in the intuitionistic case.

1 Introduction

Classical modal logics (we are interested in normal modal logics) extend classical logic by two operators, called modalities, that allow to express notions like necessity and possibility [4,9]. In the possible world (Kripke) semantics, the modalities are interpreted in a set of worlds with an accessibility relation. In this context, logics differ by the properties associated to the accessibility relation (for instance reflexivity (T), symmetry (B), transitivity (4), euclidness (5)). Intuitionistic modal logics can be considered as logics obtained by replacing in classical modal logics the reasoning principles with intuitionistic ones [24]. These logics have important applications in computer science, like formal verification [13] and definition of programming languages [12,16].
Our aim in this work consists in defining label-free natural deduction systems having good properties for several intuitionistic and classical modal logics. There exist many natural deduction systems in the classical case [14,20,23], but they are rare in the intuitionistic case because of the difficulty to deal with the modality $\Diamond$ [24]. As far as we know, the only approach that provide deduction systems satisfying normalization for all the intuitionistic modal logics we consider is the one of A. Simpson [24]. It is based on labels that explicitly integrate some semantic information, like the accessibility relation, into the systems. It allows to define simple systems for large number of modal logics, but they do not satisfy some properties, for instance the subformula property, because of the use of labels that are not in the logic language. Recently works based on a structure, called deep sequent, has provided label-free sequent calculi for several classical modal logics in a modular way and with good properties like cut-elimination and subformula properties [5]. It can be seen as a generalization of the approach based on the hypersequent structure for the modal logic $\mathsf{S5}$ [1]. However, deep sequent and hypersequent structures are not adapted to deal with natural deduction formalism and also the intuitionnistic modal logics. The key idea to solve this problem consists in defining a multi-contextual structure appropriate to deal with the logics we consider. In this perspective we have recently defined such a multi-contextual sequent structure that allows to defined a label-free natural deduction system and a sequent calculus for the intuitionistic modal logic $\mathsf{IS5}$ [15].

In this paper we define new natural deduction systems for the intuitionistic and classical modal logics obtained from the combinations of the axioms T, B, 4 and 5 that satisfy the normalization property but also the subformula property in the intuitionistic case. In order to design them we first define a multi-contextual structure, in the spirit of our previous work [15], that is adapted to deal with logics in both intuitionistic and classical cases. Thus our first contribution is the definition of a new structure, called Tree-sequent (T-sequent), that is different from the one of deep sequent [5]: firstly, in a T-sequent we have an explicit difference between some formulas with hypotheses and a conclusion; secondly, the definition of the formula associated to a T-sequent uses both operators $\Box$ and $\Diamond$. In fact the absence of inter-definissability between $\Diamond$ and $\Box$ (one from the other) in the intuitionistic modal logics (classical case: $\Diamond A = \neg\Box\neg A$, $\Box A = \neg\Diamond\neg A$) makes essential the use of a structure with a corresponding formula using both operators. Intuitively, a T-sequent can be seen as a mono-conclusion version of a deep sequent. Using this structure, we first focus on the the intuitionistic logic IK and define a natural deduction system for this logic, that is proved sound and complete. Moreover we show that it satisfies the normalization property. Then we generalize this work by defining natural deduction systems for the intuitionistic modal logics that are obtained from all the combinations of T, B, 4 and 5. We prove that they satisfy the normalization property but also the subformula property. To complete these contributions we naturally derive natural deduction systems for the classical modal logics obtained from the combinations of T, B, 4 and 5 and prove that they satisfy the normalization property.

In Section 2 we briefly present the key points about modal logics and their related deduction systems. In Section 3 we introduce our new multi-contextual structure, called Tree-sequent, that is similar but different from the deep (or nested) sequent structure [5,6,19]. In Section 4 we first focus on the intuitionistic modal logic IK and define a new natural deduction system DN_{IK} based on Tree-sequents. In order to prove the soundness we introduce two key notions (predecessor, chain) to express if a Tree-sequent has a countermodel or not. The completeness is proved in a standard way w.r.t. the Hilbert system axiomatization. In Section 5 we study the normalization in the system DN_{IK}. For that we define a set of notions and concepts (indexed formula, discharging rule, normal derivation) in order to describe the normalization procedure in a clear and concise way. After this work on the intuitionistic modal logic IK with a sound and complete label-free natural deduction system satisfying normalization as main contributions, we extend these results in Section 6 by defining natural deduction systems, based on T-sequents, for the intuitionistic modal logics obtained from the combinations of T, B, 4 and 5. We prove that all our systems satisfy normalization but also the subformula and separation properties. Having first focused on the intuitionistic modal logics we show, in Section 7, how to define in a simple way natural deduction systems for all classical modal logics obtained from the combinations of T, B, 4 and 5. For each logic, the system is obtained by the replacement of the rule associated to $\perp$ (absurdity) by a new rule in the corresponding intuitionistic system. We prove that they are sound and complete and that they satisfy normalization.

2 Classical and Intuitionistic Modal Logics

The language of modal logics is obtained from the language of propositional logic by adding two unary operators $\Box$ and $\Diamond$. Let Prop be a set of propositional variables, denoted by letters $p,q,r,\ldots$ The formulas are defined by the following grammar: $A ::= p \mid \perp \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A$ where the symbol $\perp$ represents the absurdity (constant false). The negation, denoted $\neg$, can be defined by using $\perp$ and the operator $\supset$ as follows: $\neg A \triangleq A \supset \perp$. The constant true is defined by $\top \triangleq \perp \supset \perp$.

2.1 Classical Modal Logics

Let us recall some key points about semantics and proof systems in classical modal logics. The Kripke semantics, that is related to the definition of truth w.r.t. possible worlds, includes a relation of accessibility between worlds. Thus "$\Box A$ is true in a world w" means that A is true in all worlds accessible from w and "$\Diamond A$ is true in a world w" means that A is true in at least one world accessible from w.

Definition 1. A classical modal model *is a triple* (W,R,V) *where* W *is a non-empty set of worlds,* R *is a binary relation on worlds, called accessibility relation, and* V *is a function from* W *to* 2^{Prop} *(the set of subsets of* Prop*).*

We associate to each model $\mathcal{M} = (W,R,V)$ a relation $\vDash_{\mathcal{M}}$, called satisfaction relation, between W and the set of formulas, that is inductively defined as follows:

- $w \vDash_{\mathcal{M}} p$ iff $p \in V(w)$;
- $w \vDash_{\mathcal{M}} \perp$ never;
- $w \vDash_{\mathcal{M}} A \wedge B$ iff $w \vDash_{\mathcal{M}} A$ and $w \vDash_{\mathcal{M}} B$;
- $w \vDash_{\mathcal{M}} A \vee B$ iff $w \vDash_{\mathcal{M}} A$ or $w \vDash_{\mathcal{M}} B$;
- $w \vDash_{\mathcal{M}} A \supset B$ iff if $w \vDash_{\mathcal{M}} A$ then $w \vDash_{\mathcal{M}} B$;
- $w \vDash_{\mathcal{M}} \Box A$ iff for any w' in W, if $R(w,w')$ then $w' \vDash_{\mathcal{M}} A$;
- $w \vDash_{\mathcal{M}} \Diamond A$ iff there exists w' in W such that $R(w,w')$ and $w' \vDash_{\mathcal{M}} A$.

The expression $w \vDash_{\mathcal{M}} A$ means that in a model $\mathcal{M}$ the formula A is satisfied in the world w. A formula A is valid in $\mathcal{M} = (W,R,V)$ if $w \vDash_{\mathcal{M}} A$ for any world w in W. The classical modal models define the validity in the minimal modal logic K: a formula A is valid in K iff A is valid in all classical modal models [10]. The other modal logics built from combinations of the axioms T, B, 4 and 5 are defined by classes of classical modal models. Each axiom corresponds to a property of the accessibility relation in each model:

(T) Reflexivity: $\forall w.R(w,w)$; (B) Symmetry: $\forall w,w'.R(w,w') \supset R(w',w)$; (4) Transitivity: $\forall w,w',w''.(R(w,w') \wedge R(w',w'')) \supset R(w,w'')$; (5) Euclidness: $\forall w,w',w''.(R(w,w') \wedge R(w,w'')) \supset R(w',w'')$.
For $\mathsf{Th} \subseteq \{T,B,4,5\}$ the class of models defining the logics KTh, denoted $\mathcal{C}_{\mathsf{Th}}$, corresponds to models in which the accessibility relations satisfies the properties associated to axioms in Th.
Let us note that the logic $\mathsf{K}\{T,4\}$ (resp. $\mathsf{K}\{T,5\}$) is the classical modal logic denoted $\mathsf{S4}$ (resp. $\mathsf{S5}$).

Theorem 1. *A formula A is valid in* KTh *iff A is valid in all models in $\mathcal{C}_{\mathsf{Th}}$.*

Proof. See [10].

Each logic KTh satisfies the finite model property w.r.t. the Kripke semantics and thus are decidable [4]. About proof systems for such a logic we recall the Hilbert system for the classical minimal modal logic K, called $\mathcal{H}_{\mathsf{K}}$, that is given by the following axioms:

1. Tautologies of propositional classical logic.
2. $\Box(A \supset B) \supset (\Box A \supset \Box B)$.
3. $\Diamond A \leftrightarrow \neg\Box\neg A$.

and the following rules:

$$\frac{A \supset B \qquad A}{B}\ [mp] \qquad\qquad \frac{A}{\Box A}\ [nec]$$

Hilbert systems for other classical modal logics are obtained by adding to $\mathcal{H}_{\mathsf{K}}$ axioms among the following ones: (T) $\Box A \supset A$; (B) $A \supset \Box\Diamond A$; (4) $\Box A \supset \Box\Box A$; (5) $\Diamond A \supset \Box\Diamond A$. For any subset Th of $\{T,B,4,5\}$ we call KTh the logic corresponding to the Hilbert system $\mathcal{H}_{\mathsf{Th}}$ obtained by adding axioms in Th to $\mathcal{H}_{\mathsf{K}}$.

2.2 Intuitionistic Modal Logics

Intuitionistic modal logics that we consider are the intuitionistic versions of the classical modal logics [24]. They have important applications in computer science like formal verification [13] and definition of programming languages [12,16]. For any $\mathsf{Th} \subseteq \{T,B,4,5\}$ we call IKTh the intuitionistic modal logic corresponding to the classical modal logic KTh. In this case the semantics and proof systems are different with key points summarized here. Let us note that we use the names IT, $\mathsf{IB4}$, $\mathsf{IS4}$ and $\mathsf{IS5}$ for the intuitionistic versions of respectively T, $\mathsf{K}\{B,4\}$, $\mathsf{S4}$, $\mathsf{S5}$.

Definition 2. *A* modal intuitionistic model *is a quadruple $(W, \leqslant, \{D_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ where*

- *W is a non-empty set of* Kripke worlds*;*
- *$\leqslant$ is a partial order relation on W;*
- *for any $w \in W$, D_w is a non-empty set of* modal worlds *such that if $w \leqslant w'$ then $D_w \subseteq D_{w'}$;*
- *for any $w \in W$, R_w is a binary relation on D_w, called* w-accessibility relation*, such that if $w \leqslant w'$ then $R_w \subseteq R_{w'}$;*
- *for any $w \in W$, V_w is a function from D_w to 2^{Prop} such that if $w \leqslant w'$ then $V_w(p) \subseteq V_{w'}(p)$.*

Let us note that there are two kinds of worlds: the Kripke worlds that correspond to the intuitionistic basis and the modal worlds that capture the modal aspects. As in the classical case, we associate to each modal intuitionistic logic a satisfaction (or forcing) relation.

Definition 3. *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ be a modal intuitionistic model, $w \in W$, $d \in D_w$ and F be a formula, the* forcing relation*, denoted $w,d \vDash_{\mathcal{M}} F$, is inductively defined as follows:*

- *$w,d \vDash_{\mathcal{M}} p$ iff $d \in V_w(p)$;*
- *$w,d \vDash_{\mathcal{M}} \bot$ never;*
- *$w,d \vDash_{\mathcal{M}} A \wedge B$ iff $w,d \vDash_{\mathcal{M}} A$ and $w,d \vDash_{\mathcal{M}} B$;*
- *$w,d \vDash_{\mathcal{M}} A \vee B$ iff $w,d \vDash_{\mathcal{M}} A$ or $w,d \vDash_{\mathcal{M}} B$;*
- *$w,d \vDash_{\mathcal{M}} A \supset B$ iff for all $w' \geqslant w$, if $w',d \vDash_{\mathcal{M}} A$ then $w',d \vDash_{\mathcal{M}} B$;*
- *$w,d \vDash_{\mathcal{M}} \Box A$ iff if for all $w' \geqslant w$ and for all $d' \in D_{w'}$, if $R_{w'}(d,d')$ then $w',d' \vDash_{\mathcal{M}} A$;*
- *$w,d \vDash_{\mathcal{M}} \Diamond A$ iff there exists $d' \in D_w$ such that $R_w(d,d')$ and $w,d' \vDash_{\mathcal{M}} A$.*

A formula A is valid in a model $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ if and only if $w, d \vDash_{\mathcal{M}} A$ for all $w \in W$ and for all $d \in D_w$. For $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, a model $(W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ is in the class of models $\mathcal{IC}_{\mathsf{Th}}$ if and only if for all $w \in W$, R_w satisfies the properties associated to the axioms in Th.

Theorem 2. *A formula A is valid in* IKTh *if and only if A is valid in all the models in* $\mathcal{IC}_{\mathsf{Th}}$.

Proof. See [24].

The satisfaction relation verifies the property of Kripke monotonicity like in intuitionistic logic.

Proposition 1 (Monotonicity). *If $w, d \vDash_{\mathcal{M}} A$ and $w \leqslant w'$ then we have $w', d \vDash_{\mathcal{M}} A$.*

Proof. By structural induction on A.

Let us note that these logics do not satisfy the finite model property w.r.t. Kripke semantics [17,24]. But some of them satisfy the property w.r.t. other semantics. The property has been proved for $\mathsf{IS5}$ [21] w.r.t. the algebraic semantics proposed in [7]. For the logics IK, $\mathsf{IK}\{B\}$ and $\mathsf{IK}\{T, B\}$ the finite model property has been proved w.r.t. the bi-relational semantics [24]. Concerning the proof systems we can mention a Hilbert system for IK, denoted $\mathcal{H}_{\mathsf{IK}}$, that is given by

- Tautologies of propositional intuitionistic logic.
- $\Box(A \supset B) \supset (\Box A \supset \Box B)$.
- $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$.
- $\Diamond\bot \supset \bot$.
- $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$.
- $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$.

with the rules

$$\frac{A \supset B \qquad A}{B}\ [mp] \qquad\qquad \frac{A}{\Box A}\ [nec]$$

It has been first proposed in [18] and another one can be found in [22]. For any $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, a Hilbert system for the logic IKTh, denoted $\mathcal{H}_{\mathsf{IKTh}}$, is obtained by the addition to $\mathcal{H}_{\mathsf{IK}}$ of axioms corresponding to the elements of Th among the following axioms [24]: (T) $(\Box A \supset A) \wedge (A \supset \Diamond A)$; (B) $(\Diamond\Box A \supset A) \wedge (A \supset \Box\Diamond A)$; (4) $(\Box A \supset \Box\Box A) \wedge (\Diamond\Diamond A \supset \Diamond A)$; (5) $(\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)$.
Let us consider the axiom T used in classical modal logic that is $\Box A \supset A$. Then we have $\Box\neg A \supset \neg A$. As we have $\Box A \leftrightarrow \neg\Diamond\neg A$ and $\neg\neg A \leftrightarrow A$, we obtain $\neg\Diamond A \supset \neg A$ that is the contraposition of $A \supset \Diamond A$. But the addition of $A \supset \Diamond A$ to the axiom T in the intuitionistic case comes from the fact that the two operators $\Box$ and $\Diamond$ are independent.

Before to consider natural deduction systems we give the definitions of two useful notions:
Let A be a formula, the *complexity measure* of a formula A, denoted $\mid A \mid$, is defined as follows:
- $\mid p \mid = \mid \bot \mid = 1$;
- $\mid A \otimes B \mid = \mid A \mid + \mid B \mid + 1$ where $\otimes \in \{\wedge, \vee, \supset\}$;
- $\mid \boxtimes A \mid = \mid A \mid + 1$ where $\boxtimes \in \{\Box, \Diamond\}$.
Let A be a formula, the *nesting degree* of a formula A, denoted $nest(A)$, is defined as follows:
- $nest(p) = nest(\bot) = 0$;
- $nest(A \otimes B) = max(nest(A), nest(B))$ where $\otimes \in \{\wedge, \vee, \supset\}$;
- $nest(\boxtimes A) = 1 + nest(A)$ where $\boxtimes \in \{\Box, \Diamond\}$.

2.3 Natural Deduction Systems and Modal Logics

Natural deduction systems have been defined for the logics $\mathsf{S4}$ and $\mathsf{S5}$ and their intuitionistic versions [20]. Other formulations improve these systems for the classical and intuitionistic versions of $\mathsf{S4}$ [3,11]. Let us note that the Prawitz approach is difficult to extend to other modal logics [8], for instance for the logic K [2]. Moreover, using Fitch's approach, natural deduction systems have been provided for several classical modal logics [14,23]. Unlike

Gentzen-style where the derivations have a tree form, in Fitch's approach the derivations are linear and thus, in this case, the accessibility relation is implicitly integrated into systems by a nesting of derivations.
Natural deduction systems for intuitionistic modal logics are rare because of the difficulty to deal with the modality $\Diamond$ [24]. As far as we know, the only approach allowing to provide systems satisfying normalization for all the intuitionistic modal logics we consider is the one of A. Simpson [24]. It explicitly integrates some semantic information, like the accessibility relation, into the systems by using labels. It allows to define simple systems for large number of modal logics, but they do not satisfy the subformula property.

In this paper we focus on the proof theory in the classical and intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5 via the natural deduction formalism. Here we aim at defining label-free systems that satisfy normalization and also subformula property. In order to solve this key question we need to introduce a multi-contextual structure in the spirit of [5,15] but it was done for the sequent calculus formalism and only for classical modal logics. In the next section we present our tree-sequent structure that is central in this work.

3 The Tree-sequent Structure

In this section we introduce a new structure, called Tree-sequent, denoted T-sequent, that can be seen as a kind of mono-conclusion version of a deep (or nested) sequent [5,6,19], but it is clearly different. In a deep sequent the formulas are not explicitely considered as hypotheses or conclusion and the definition of the formula corresponding to a deep sequent only uses the modal operator $\Box$. In the T-sequent structure all formulas are considered as hypotheses except one that is called a conclusion and the definition of the formula corresponding to a T-sequent uses the two modal operators $\Diamond$ and $\Box$.

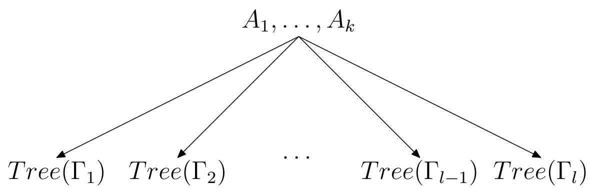
Definition 4 (T-context). *A* T-context *is a structure of the form $A_1,\ldots,A_k,\langle\Gamma_1\rangle,\ldots,\langle\Gamma_l\rangle$ where $\{A_1,\ldots,A_k\}$ is a multiset of formulas and $\{\Gamma_1,\ldots,\Gamma_k\}$ is a multiset of T-contexts.*

A *marked formula* is of the form $A^{\vdash}$ where A is a formula.

Definition 5 (T-sequent). *A* T-sequent *is a structure inductively defined as follows:*

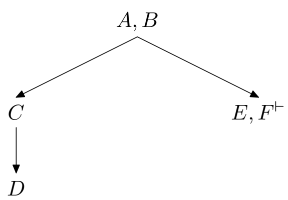
- *If Γ is a T-context and $A^{\vdash}$ is a marked formula then $\Gamma, A^{\vdash}$ is a T-sequent.*
- *If $\mathcal{S}$ is a T-sequent and Γ is a T-context then $\Gamma,\langle\mathcal{S}\rangle$ is a T-sequent.*

A T-sequent has the same form as a T-context, i.e., $A_1,\ldots,A_k,\langle\Gamma_1\rangle,\ldots,\langle\Gamma_l\rangle$ and it can be seen as a T-context with in addition only one occurrence of a marked formula, that is called the conclusion. T-sequents can be presented graphically as follows:



where $Tree(\Gamma_1),\ldots,Tree(\Gamma_k)$ are the trees respectively corresponding to $\Gamma_1,\ldots,\Gamma_k$.

Let us note that we do not distinguish the T-sequents and T-contexts and their associated trees. Then when we mention the *root*, the *leaf*, the *depth* or a *subtree* of a T-sequent or a T-context, we refer to its associated tree. In order to illustrate this point we give the tree associated to the T-sequent $A,B\langle C,\langle D\rangle\rangle,\langle E,F^{\vdash}\rangle$:



Definition 6 **(nT-context).** *A nT-context, with $n \geqslant 0$, is a T-context or a T-sequent with n occurrences of the symbol $\{\}$, that is called a T-hole.*

The nT-contexts are denoted $\Gamma\overbrace{\{\}\cdots\{\}}^{n\ times}$ by considering that there is a bijection that maps an occurrence of $\{\}$ in the nT-context to each occurrence of the symbol $\{\}$ following this notation. The structure $\Gamma\{\Delta_1\}\cdots\{\Delta_n\}$ is obtained by the substitution of the T-hole associated to the ith occurrence of $\{\}$ in $\Gamma\{\}\cdots\{\}$ by Δ_i, for all $i \in [1,n]$. For instance, any T-sequent has the form $\Gamma\{C^{\vdash}\}$ where $\Gamma\{\}$ is a $1T$-context. From now we call the T-context of $\Gamma\{C^{\vdash}\}$ the T-context $\Gamma\{\emptyset\}$. In general the T-holes are substituted by T-contexts, T-sequents or nT-contexts.

The T-sequent $\Box(A \supset B), \Diamond A, \langle A, B^{\vdash}\rangle$. It corresponds to $\Gamma\{B^{\vdash}\}$ such that $\Gamma\{\} = \Box(A \supset B), \Diamond A, \langle A, \{\}\rangle$.

The T-sequent structure can be seen as a multi-contextual structure because the truth value of a T-sequent can change w.r.t. the position (context) of its conclusion in the tree associated to its T-context.

Definition 7 **(Depth).** *The depth of a $1T$-context $\Gamma\{\}$, denoted $depth(\Gamma\{\})$, is defined as follows:*

- $depth(\Gamma, \{\}) = 0$;
- $depth(\Gamma, \langle\Delta\{\}\rangle) = 1 + depth(\Delta\{\})$.

Let $\mathcal{S}$ be a T-sequent, $sp(\mathcal{S})$ is a relation that is satisfied if and only if the depth of the tree corresponding to $\mathcal{S}$ is greater than 0. We define $nest(\mathcal{S})$ by $nest(\mathcal{S}) = max\{nest(A) \mid A \in \mathcal{S}\}$ where *max* means the maximum and $nest(A)$ the nesting degree of A previously defined.

The $\mathcal{F}$ fonction that associates a formula to each T-context is defined as follows:

- $\mathcal{F}(\emptyset) = \top$;
- $\mathcal{F}(A_1, \ldots, A_k, \langle\Gamma_1\rangle, \ldots, \langle\Gamma_l\rangle) = A_1 \wedge \ldots \wedge A_k \wedge \Diamond(\mathcal{F}(\Gamma_1)) \wedge \ldots \wedge \Diamond(\mathcal{F}(\Gamma_k))$.

It is extended to T-sequents in the following way:

- $\mathcal{F}(\Gamma, A^{\vdash}) = \mathcal{F}(\Gamma) \supset A$ (Γ is a T-context);
- $\mathcal{F}(\Gamma, \langle\mathcal{S}\rangle) = \mathcal{F}(\Gamma) \supset \Box(\mathcal{F}(\mathcal{S}))$ (Γ is a T-context and $\mathcal{S}$ is a T-sequent).

Thus for example we have $\mathcal{F}(\Box(A \supset B), \Diamond A, \langle A, B^{\perp}\rangle) = (\Box(A \supset B) \wedge (\Diamond A)) \supset \Box(A \supset B)$.

We note that the validity of a T-sequent $\mathcal{S}$ in a modal logic L is defined by the validity of $\mathcal{F}(\mathcal{S})$ in L.

4 A Natural Deduction System for IK

In this section we define a natural deduction system for IK, called DN_{IK}, that is based on the T-sequent structure and we prove the key properties of soundness and completeness.

4.1 The DN_{IK} System

The natural deduction system DN_{IK} is given in Figure 1. We observe that its rules are all of the following form:

$$\frac{\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\} \quad \cdots \quad \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}}{\Gamma\{\Delta_1\}\cdots\{\Delta_k\}}\,[R]$$

It means that each premisse is obtained by the transformation of some subtrees of the conclusion. Let us comment now the rules of DN_{IK}. The rules for the intuitionistic operators are defined as the ones of DN_{IPL} (the natural deduction system for intuitionistic logic [25]) by taking into account the existence of several contexts. For instance the rule $[\perp_E]$ expresses that if the hypotheses of the premisse imply absurdity then they also imply any formula, in any context. This idea is captured by the use of a $2T$-context.
Let us focus now on the modal rules. We say that a context $\mathcal{C}'$ is accessible from a context $\mathcal{C}$ in a T-sequent $\mathcal{S}$ if

$$\frac{}{\Gamma\{A, A^{\vdash}\}}\ [id] \qquad \frac{\Gamma\{\bot^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\bot_E]$$

$$\frac{\Gamma\{A^{\vdash}\} \qquad \Gamma\{B^{\vdash}\}}{\Gamma\{A \wedge B^{\vdash}\}}\ [\wedge_I] \qquad \frac{\Gamma\{A \wedge B^{\vdash}\}}{\Gamma\{A^{\vdash}\}}\ [\wedge_E^1] \qquad \frac{\Gamma\{A \wedge B^{\vdash}\}}{\Gamma\{B^{\vdash}\}}\ [\wedge_E^2]$$

$$\frac{\Gamma\{A^{\vdash}\}}{\Gamma\{A \vee B^{\vdash}\}}\ [\vee_I^1] \qquad \frac{\Gamma\{B^{\vdash}\}}{\Gamma\{A \vee B^{\vdash}\}}\ [\vee_I^2] \qquad \frac{\Gamma\{A \vee B^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A\}\{C^{\vdash}\} \qquad \Gamma\{B\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [\vee_E]$$

$$\frac{\Gamma\{A, B^{\vdash}\}}{\Gamma\{A \supset B^{\vdash}\}}\ [\supset_I] \qquad \frac{\Gamma\{A \supset B^{\vdash}\} \qquad \Gamma\{A^{\vdash}\}}{\Gamma\{B^{\vdash}\}}\ [\supset_E]$$

$$\frac{\Gamma\{\langle \Delta, A^{\vdash} \rangle\}}{\Gamma\{\langle \Delta \rangle, \Diamond A^{\vdash}\}}\ [\Diamond_I] \qquad \frac{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{\langle A \rangle\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [\Diamond_E]$$

$$\frac{\Gamma\{\langle A^{\vdash} \rangle\}}{\Gamma\{\Box A^{\vdash}\}}\ [\Box_I] \qquad \frac{\Gamma\{\langle \Delta \rangle, \Box A^{\vdash}\}}{\Gamma\{\langle \Delta, A^{\vdash} \rangle\}}\ [\Box_E]$$

Fig. 1. The Natural Deduction System DN_{IK}

$\mathcal{C}'$ is a son of $\mathcal{C}$ in the T-context of $\mathcal{S}$. The rule $[\Box_E]$ means that if a formula A is true in an empty context that is accessible from a context $\mathcal{C}$, then the formula $\Box A$ is true in the context $\mathcal{C}$. The rule $[\Box_I]$ means that if a formula $\Box A$ is true in a context $\mathcal{C}$ and $\mathcal{C}'$ is a context accessible from it then the formula A is true in $\mathcal{C}'$. The rule $[\Diamond_I]$ means that if a formula A is true in a context $\mathcal{C}'$ accessible from $\mathcal{C}$, then the formula $\Diamond A$ is true in w. Finally the rule $[\Diamond_E]$ is similar to a cut rule. If the formula $\Diamond A$ is true in a context $\mathcal{C}$, we cannot necessarily know in which context, accessible from $\mathcal{C}$, the formula A is true.

In the case of an application of an elimination rule we call *major premisse* the premisse that contains the eliminated operator and the other premisses are called *minor premisses*. Let us introduce two relations, denoted $\rightarrow_w$ et $\rightarrow_m$, that allow us to capture the notions of *weakening* and *merge* on the T-sequents. They correspond to the following structural rules on T-sequents:

$$\frac{\Gamma\{C^{\vdash}\}\{\emptyset\}}{\Gamma\{C^{\vdash}\}\{A\}}\ [W] \qquad\qquad \frac{\Gamma\{\langle \Delta_1 \rangle, \langle \Delta_2 \rangle\}}{\Gamma\{\langle \Delta_1, \Delta_2 \rangle\}}\ [M]$$

We define the relation $\rightarrow_w$ on the T-sequents by $\Gamma\{C^{\vdash}\}\{\emptyset\} \rightarrow_w \Gamma\{C^{\vdash}\}\{\Sigma\}$ where Σ is a T-context and we denote $\rightarrow_w^*$ its reflexive and transitive closure.

Definition 8. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two T-sequents such that $\mathcal{S} \rightarrow_w \mathcal{S}'$ and $\mathcal{D}$ be a proof of $\mathcal{S}$ in DN_{IK}. The proof $\mathcal{D}[\mathcal{S}']_w$ of $\mathcal{S}'$ is defined by induction $\mathcal{D}$ as follows:*

$$\mathcal{D}[\mathcal{S}']_w = \left\{ \frac{\begin{array}{ccc} \mathcal{D}_1[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\Sigma\}]_w & & \mathcal{D}_l[\Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\Sigma\}]_w \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\Sigma\} & \cdots & \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\Sigma\} \end{array}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Sigma\}}\ [R] \right.$$

where $\mathcal{S}' = \Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Sigma\}$ and

$$\mathcal{D} = \left\{ \frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_l \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\} & \cdots & \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\emptyset\} \end{array}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\emptyset\}}\ [R] \right.$$

We can extend this definition to the relation $\to_w^*$. Let $\mathcal{S}$ and $\mathcal{S}'$ be two T-sequents such that $\mathcal{S} \to_w^n \mathcal{S}'$ and $\mathcal{D}$ be a proof of $\mathcal{S}$ in DN_{IK}. The tree $\mathcal{D}[\mathcal{S}']_w$ is defined as follows:
- if $n = 0$ then $\mathcal{D}[\mathcal{S}']_w = \mathcal{D}$;
- otherwise $\mathcal{S} \to_w \mathcal{S}'' \to_w^{n-1} \mathcal{S}'$ and $\mathcal{D}[\mathcal{S}']_w = (\mathcal{D}[\mathcal{S}'']_w)[\mathcal{S}']_w$.

Let us consider the following example:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\dfrac{}{\Box(A\supset B), \Box A, \langle\,\rangle, \Box(A\supset B)^{\vdash}}[Id]}{\Box(A\supset B), \Box A, \langle A\supset B^{\vdash}\rangle}[\Box_E] \quad \dfrac{\dfrac{}{\Box(A\supset B), \Box A, \langle\,\rangle, \Box A^{\vdash}}[Id]}{\Box(A\supset B), \Box A, \langle A^{\vdash}\rangle}[\Box_E]}{\Box(A\supset B), \Box A, \langle B^{\vdash}\rangle}[\supset_E] \right.$$

We have $\Box(A\supset B), \Box A, \langle B^{\vdash}\rangle \to_w^* C, \Box(A\supset B), \Box A, \langle D, B^{\vdash}\rangle$ and the proof $\mathcal{D}[C, \Box(A\supset B), \Box A, \langle D, B^{\vdash}\rangle]_w$ is the following:

$$\dfrac{\dfrac{\dfrac{}{C, \Box(A\supset B), \Box A, \langle D\rangle, \Box(A\supset B)^{\vdash}}[Id]}{C, \Box(A\supset B), \Box A, \langle D, A\supset B^{\vdash}\rangle}[\Box_E] \quad \dfrac{\dfrac{}{C, \Box(A\supset B), \Box A, \langle D\rangle, \Box A^{\vdash}}[Id]}{C, \Box(A\supset B), \Box A, \langle D, A^{\vdash}\rangle}[\Box_E]}{C, \Box(A\supset B), \Box A, \langle D, B^{\vdash}\rangle}[\supset_E]$$

Similarly we define the relation $\to_m$ on T-sequents by $\Gamma\{\langle\Delta_1\rangle, \langle\Delta_2\rangle\} \to_m \Gamma\{\langle\Delta_1, \Delta_2\rangle\}$ and we denote $\to_m^*$ its reflexive and transitive closure.

Definition 9. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two T-sequents such that $\mathcal{S} \to_m \mathcal{S}'$ and $\mathcal{D}$ a proof of $\mathcal{S}_0$ in DN_{IK}. The proof $\mathcal{D}[\mathcal{S}']_m$ is defined by induction on $\mathcal{D}$ as follows:*

$$\mathcal{D}[\mathcal{S}']_m = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma_1^1, \Sigma_2^1\rangle\}]_m \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma_1^1, \Sigma_2^1\rangle\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l[\Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma_1^l, \Sigma_2^l\rangle\}]_m \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma_1^l, \Sigma_2^l\rangle\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma_1^0, \Sigma_2^0\rangle\}}[R] \right.$$

where $\mathcal{S}' = \Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma_1^0, \Sigma_2^0\rangle\}$ and

$$\mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma_1^1\rangle, \langle\Sigma_2^1\rangle\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma_1^l\rangle, \langle\Sigma_2^l\rangle\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma_1^0\rangle, \langle\Sigma_2^0\rangle\}}[R] \right.$$

We can extend this definition to the relation $\to_m^*$ as it is previously done for $\to_w^*$.
Let us consider the following example:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\dfrac{}{\langle A, A^{\vdash}\rangle, \langle B\rangle}[Id]}{\langle A\rangle, \langle B\rangle, \Diamond A^{\vdash}}[\Diamond_I] \quad \dfrac{\dfrac{}{\langle A\rangle, \langle B, B^{\vdash}\rangle}[Id]}{\langle A\rangle, \langle B\rangle, \Diamond B^{\vdash}}[\Diamond_I]}{\langle A\rangle, \langle B\rangle, \Diamond A \wedge \Diamond B^{\vdash}}[\wedge_I] \right.$$

We have $\langle A\rangle, \langle B\rangle, \Diamond A \wedge \Diamond B^{\vdash} \to_m \langle A, B\rangle, \Diamond A \wedge \Diamond B^{\vdash}$. The proof $\mathcal{D}[\langle, A, B\rangle, \Diamond A \wedge \Diamond B^{\vdash}]_m$ is

$$\dfrac{\dfrac{\dfrac{}{\langle A, B, A^{\vdash}\rangle}[Id]}{\langle A, B\rangle, \Diamond A^{\vdash}}[\Diamond_I] \quad \dfrac{\dfrac{}{\langle A, B, B^{\vdash}\rangle}[Id]}{\langle A, B\rangle, \Diamond B^{\vdash}}[\Diamond_I]}{\langle A, B\rangle, \Diamond A \wedge \Diamond B^{\vdash}}[\wedge_I]$$

Now we will consider some properties that are important to prove the completeness of DN_{IK}.

Proposition 2. $\Gamma\{A \supset B^{\vdash}\}$ *is provable in DN_{IK} if and only if* $\Gamma\{A, B^{\vdash}\}$ *is provable in DN_{IK}*

Proof. From the rule $[\supset_I]$ we know that if $\Gamma\{A, B^{\vdash}\}$ is provable in DN_{IK} then $\Gamma\{A \supset B^{\vdash}\}$ is provable in DN_{IK}.
Let us assume that $\Gamma\{A \supset B^{\vdash}\}$ is provable in DN_{IK}. We have $\Gamma\{A \supset B^{\vdash}\} \to_w \Gamma\{A, A \supset B^{\vdash}\}$ and then $\Gamma\{A, A \supset B^{\vdash}\}$ is also provable in DN_{IK}. Thus $\Gamma\{A, B^{\vdash}\}$ is provable in DN_{IK}:

$$\dfrac{\Gamma\{A, A \supset B^{\vdash}\} \qquad \dfrac{}{\Gamma\{A, A^{\vdash}\}}\,[Id]}{\Gamma\{A, B^{\vdash}\}}\,[\supset_E]$$

Proposition 3. *$\Gamma\{\Box A^{\vdash}\}$ is provable in DN_{IK} if and only if $\Gamma\{\langle A^{\vdash}\rangle\}$ is provable in DN_{IK}.*

Proof. From the rule $[\Box_I]$ we know that if $\Gamma\{\langle A^{\vdash}\rangle\}$ is provable in DN_{IK} then $\Gamma\{\Box A^{\vdash}\}$ is provable in DN_{IK}.
Let us assume that $\Gamma\{\Box A^{\vdash}\}$ is provable in DN_{IK}. We have $\Gamma\{\langle A^{\vdash}\rangle\} \to_w \Gamma\{\langle\ \rangle, \Box A^{\vdash}\}$ and then $\Gamma\{\langle\ \rangle, \Box A^{\vdash}\}$ is also provable in DN_{IK}. By using the rule $[\Box_E]$ we show that $\Gamma\{\langle A^{\vdash}\rangle\}$ is provable in DN_{IK}:

$$\dfrac{\Gamma\{\langle\ \rangle, \Box A^{\vdash}\}}{\Gamma\{\langle A^{\vdash}\rangle\}}\,[\Box_E]$$

Proposition 4. *If $\Gamma\{A \wedge B\}\{C^{\vdash}\}$ is provable in DN_{IK} then $\Gamma\{A, B\}\{C^{\vdash}\}$ is provable in DN_{IK}.*

Proof. By structural induction on the proof of $\Gamma\{A \wedge B\}$.
We only develop the case where $\Gamma\{A \wedge B\}$ is an instance of $[Id]$.
1. There exists $\Gamma'\{\}\{\}$ such that $\Gamma\{A \wedge B\}\{C^{\vdash}\} = \Gamma'\{A \wedge B\}\{C, C^{\vdash}\}$. We see that $\Gamma\{A, B\}\{C^{\vdash}\}$ is an instance of $[Id]$.
2. $C = A \wedge B$ and $\Gamma\{A \wedge B\}\{C^{\vdash}\} = \Gamma\{A \wedge B, A \wedge B^{\vdash}\}\{\emptyset\}$. A proof of $\Gamma\{A, B, A \wedge B^{\vdash}\}\{\emptyset\}$ is given by:

$$\dfrac{\dfrac{}{\Gamma\{A, B, A^{\vdash}\}\{\emptyset\}}\,[Id] \qquad \dfrac{}{\Gamma\{A, B, B^{\vdash}\}\{\emptyset\}}\,[Id]}{\Gamma\{\Delta, A, B, A \wedge B^{\vdash}\}\{\emptyset\}}\,[\wedge_I]$$

In the other cases the proof is obtained by induction.

Proposition 5. *If $\Gamma\{\Diamond A\}\{C^{\vdash}\}$ is provable in DN_{IK} then $\Gamma\{A\}\{C^{\vdash}\}$ is provable in DN_{IK}.*

Proof. By structural induction on the proof of $\Gamma\{\Diamond A\}\{C^{\vdash}\}$. It is similar to the proof of Proposition 4.

4.2 Soundness of DN_{IK}

The soundness of the system ND_{IK} is proved by using the semantics of IK. The idea here consists in proving, for each rule, that if the conclusion is not valid (has a countermodel) then at least one premisse is not valid (has a countermodel). We introduce two notions of *predecessor* and of (w,k)*-chain* that we use in order to express the fact that a T-sequent has a countermodel. Then we propose some propositions in order to simplify the soundness proof.

Definition 10 (Predecessor). *Let $\Gamma\{\}$ be a 1T-context without marked formulas. The value of $pred^i(\Gamma\{\})$, with $i \in [0, depth(\Gamma\{\})]$, is defined by induction as follows:*

- *$pred^0(\Gamma\{\}) = \Delta$ such that $\Delta, \{\}$ is a subtree of $\Gamma\{\}$ (unique because $\Gamma\{\}$ has only one occurrence of $\{\}$).*
- *If $depth(\Gamma\{\}) > 0$ and $0 \leqslant j < depth(\Gamma\{\})$, then $pred^{j+1}(\Gamma\{\}) = pred^j(\Gamma'\{\})$ such that $\Gamma\{\} = \Gamma'\{\langle \Delta, \{\}\rangle\}$.*

This notion of predecessor can be described with the following figure:

$$\Gamma\{\} = \Delta_n, \langle\, \Delta_{n-1}, \langle\, \ldots\, \langle\, \Delta_1, \langle\, \Delta_0, \{\}\rangle\rangle\rangle\rangle$$

$$pred^n(\Gamma\{\}) \quad pred^{n-1}(\Gamma\{\}) \quad pred^1(\Gamma\{\}) \quad pred^0(\Gamma\{\})$$

Definition 11 **((w,k)-chain).** *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a Kripke model and $w \in W$. A* (w,k)-chain *in $\mathcal{M}$ is a sequence of the form $d_0 \to d_1 \to \cdots \to d_k$, where for all $m \in [0,k-1]$, we have $R_w(d_m, d_{m+1})$.*

Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a Kripke model, $w \in W$, $\Gamma\{\}$ un 1T-context without marked formulas and $c = d_1 \to \cdots \to d_k$ be a (w,k)-chain in $\mathcal{M}$ with $k = depth(\Gamma\{\})$. We note $w, c \vDash \Gamma\{\}$ if for all $i \in [0,k]$, $w, d_i \vDash \mathcal{F}(pred^{k-i}(\Gamma\{\}))$.

Proposition 6 **(Monotonicity).** *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a Kripke model, $w \in W$, $d \in D_w$, $\Gamma\{\}$ be a 1T-context without marked formulas and c be a (w,k)-chain such that $k = depth(\Gamma\{\})$. If $w, c \vDash \Gamma\{\}$ and $w \leqslant w'$, then $w', c \vDash \Gamma\{\}$.*

Proof. By induction on the length of c.

Proposition 7. *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a Kripke model, $w \in W$, $d \in D_w$, $\Gamma\{\}$ be a 1T-context without marked formulas. We have $w, d \vDash \mathcal{F}(\Gamma\{\emptyset\})$ if and only if there exists a (w,k)-chain $c = d \to d_1 \to \cdots \to d_k$ such that $k = depth(\Gamma\{\})$ and $w, c \vDash \Gamma\{\}$.*

Proof. We prove the "if part" by induction on k.
- If $k = 0$ then $c = d$ and there exists Δ such that $\Gamma = \Delta, \{\}$. As $w, d \vDash \mathcal{F}(\Delta)$ ($\Gamma\{\emptyset\} = \Delta$), we have $w, c \vDash \Gamma\{\}$.
- If $k = n+1$ with $n \geqslant 0$, then there exist $\Gamma'\{\}$ and Δ such that $\Gamma = \Delta, \langle\Gamma'\{\}\rangle$ and $depth(\Gamma'\{\}) = n$. As we have $w, d \vDash \mathcal{F}(\Gamma\{\emptyset\})$, there exists $d' \in D_w$ such that $R_w(d, d')$ and $w, d' \vDash \mathcal{F}(\Gamma'\{\emptyset\})$. By induction hypothesis there exists a (w,n)-chain $c = d' \to d'_1 \to \cdots \to d'_n$ such that $w, c \vDash \Gamma'\{\}$. Thus the $(w, n+1)$-chain $c' = d \to d' \to d'_1 \to \cdots \to d'_n$ satisfies the property $w, c' \vDash \Delta, \langle\Gamma'\{\}\rangle$. Then we have $w, c' \vDash \Gamma\{\}$.
We prove the "only if part" by induction on k, by using the definition of $\mathcal{F}(\Gamma\{\emptyset\})$.

Proposition 8. *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a Kripke model, $\Gamma\{C^{\vdash}\}$ be a T-sequent, $w \in W$ and $c = d_0 \to d_1 \to \cdots \to d_k$ be a (w,k)-chain in $\mathcal{M}$ such that $k = depth(\Gamma\{\})$. If $w, c \vDash \Gamma\{\}$ and $w, d_k \nvDash C$ then $w, d_0 \nvDash \mathcal{F}(\Gamma\{C^{\vdash}\})$.*

Proof. By induction on k.
- If $k = 0$ then there exists Δ such that $\Gamma\{C^{\vdash}\} = \Delta, C^{\vdash}$ and $c = d_0$. By using $w, c \vDash \Gamma\{\}$ and $w, d_k \nvDash C$, we have $w, d_0 \vDash \mathcal{F}(\Delta)$ and $w, d_0 \nvDash C$. Then $w, d_0 \nvDash \mathcal{F}(\Gamma\{C^{\vdash}\})$.
- If $k = n+1$ with $n \geqslant 0$ then there exist $\Gamma'\{\}$ and Δ such that $\Gamma\{C^{\vdash}\} = \Delta, \langle\Gamma'\{C^{\vdash}\}\rangle$ and $depth(\Gamma'\{\}) = n$. We suppose that $w, c \vDash \Gamma\{\}$ and $w, d_{n+1} \nvDash C$. Knowing that $w, c \vDash \Gamma\{\}$ and $\Gamma\{C^{\vdash}\} = \Delta, \langle\Gamma'\{C^{\vdash}\}\rangle$, we have $w, c' \vDash \Gamma'\{\}$ and $w, d_0 \vDash \Delta$ with $c' = d_1 \to \cdots \to d_{n+1}$ (a (w,n)-chain). By induction hypothesis we have $w, d_1 \nvDash \mathcal{F}(\Gamma'\{C^{\vdash}\})$. As $w, d_0 \vDash \Delta$, we obtain $w, d_0 \nvDash \mathcal{F}(\Delta) \supset \Box(\mathcal{F}(\Gamma'\{C^{\vdash}\}))$. Thus we have $w, d_0 \nvDash \mathcal{F}(\Gamma\{C^{\vdash}\})$.

Proposition 9. *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a Kripke model, $w \in W$, $d_0 \in D_w$ and $\Gamma\{C^{\vdash}\}$ be a T-sequent. If $w, d_0 \nvDash \mathcal{F}(\Gamma\{C^{\vdash}\})$ then there exist $w' \in W$ and a (w',k)-chain $c = d_0 \to d_1 \to \cdots \to d_k$ such that $w \leqslant w'$, $k = depth(\Gamma\{\})$, $w', c \vDash \Gamma\{\}$ and $w', d_k \nvDash C$.*

Proof. By induction on k.
- If $k = 0$ then there exists Δ such that $\Gamma\{C^{\vdash}\} = \Delta, C^{\vdash}$. Then we have $w, d_0 \vDash \mathcal{F}(\Delta)$ and $w, d_0 \nvDash C$. Thus $w, c \vDash \Gamma\{\}$ and $w, d_0 \nvDash C$ hold with $c = d_0$.
- If $k = n+1$ avec $n \geqslant 0$ then there exist $\Gamma'\{\}$ and Δ such that$\Gamma\{C^{\vdash}\} = \Delta, \langle\Gamma'\{C^{\vdash}\}\rangle$ and $depth(\Gamma'\{\}) = n$. Let us suppose that $w, d_0 \nvDash \mathcal{F}(\Delta, \langle\Gamma'\{C^{\vdash}\}\rangle)$. Then there exist $w_1 \in W$ and $d_1 \in D_{w_1}$ such that $w \leqslant w_1$, $R_{w_1}(d_0, d_1)$, $w_1, d_0 \vDash \mathcal{F}(\Delta)$ and $w_1, d_1 \nvDash \mathcal{F}(\Gamma'\{\Delta, C^{\vdash}\})$. By induction hypothesis and with $w, d_1 \nvDash \mathcal{F}(\Gamma'\{C^{\vdash}\})$, there exist $w' \in W$ and a (w', n)-chain $c' = d_1 \to d_1 \to \cdots \to d_{n+1}$ such that $w_1 \leqslant w'$, $w', c' \vDash \Gamma'\{\}$ and $w', d_{n+1} \nvDash C$. By the Kripke monotonicity and $w_1, d_0 \vDash \mathcal{F}(\Delta)$, we deduce that $w', d_0 \vDash \mathcal{F}(\Delta)$. Thus we have $w', c \vDash \Gamma\{\}$ and $w', d_{n+1} \nvDash C$ with $c = d_0 \to d_1 \to \cdots \to d_{n+1}$.

Theorem 3 **(Soundness).** *If a T-sequent is provable in DN_{IK} then it is valid in IK.*

Proof. We give the cases concerning $\bot$ and the modal operators, the other cases being similar.
- Case $[\bot]$. We suppose that $\Gamma\{\emptyset\}\{A^{\vdash}\}$ is not valid in IK. By Proposition 9, there exist a Kripke model $\mathcal{M}$, $w \in W$ and a (w,k)-chain $c = d_0 \to d_1 \to \cdots \to d_k$ such that $k = depth(\Gamma\{\emptyset\}\{\})$ and $w, c \vDash \Gamma\{\emptyset\}\{\}$. By Proposition 7 we have $w, d_0 \vDash \mathcal{F}(\Gamma\{\emptyset\}\{\emptyset\})$ and we know that there exists $c' = d_0 \to d'_1 \to \cdots \to d'_l$ such that $l = depth(\Gamma\{\}\{\emptyset\})$ and $w, c' \vDash \Gamma\{\}\{\emptyset\}$. By Proposition 8 and $w, d'_l \nvDash \bot$, we obtain $w, d_0 \nvDash \mathcal{F}(\Gamma\{\bot^{\vdash}\}\{\emptyset\})$.

- Case $[\Diamond_I]$. We suppose that $\Gamma\{\langle\Delta\rangle, \Diamond A^{\vdash}\}$ is not valid in IK. By Proposition 9 there exist a Kripke model $\mathcal{M}$, $w \in W$ and a (w,k)-chain $c = d_0 \to d_1 \to \cdots \to d_k$ such that $k = depth(\Gamma\{\langle\Delta\rangle, \{\}\})$, $w, c \vDash \Gamma\{\langle\Delta\rangle, \{\}\}$ and $w, d_k \nvDash \Diamond A$. Then there exists $d_{k+1} \in D_w$ such that $R_w(d_k, d_{k+1})$, $w, d_{k+1} \vDash \mathcal{F}(\Delta)$ and $w, d_{k+1} \nvDash A$. Thus we obtain $w, c' \vDash \Gamma\{\langle\Delta, \{\}\rangle\}$ and $w, d_{k+1} \nvDash A$ where c' is the $(w, k+1)$-chain defined by $c' = d_0 \to d_1 \to \cdots \to d_k \to d_{k+1}$. By Proposition 8 we have $w, d_0 \nvDash \mathcal{F}(\Gamma\{\langle\Delta, A^{\vdash}\rangle\})$.

- Case $[\Diamond_E]$. We suppose that $\Gamma\{\emptyset\}\{C^{\vdash}\}$ is not valid in IK. By Proposition 9 there exist a Kripke model $\mathcal{M}$, $w \in W$ and a (w,k)-chain $c = d_0 \to d_1 \to \cdots \to d_k$ such that $k = depth(\Gamma\{\emptyset\}\{\})$, $w, c \vDash \Gamma\{\emptyset\}\{\}$ and $w, d_k \nvDash C$. If $w, c \nvDash \Gamma\{\Diamond A\}\{\}$ then $\mathcal{M}$ is a countermodel of $\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}$. We have $w, c \vDash \Gamma\{\Diamond A\}\{\}$ and then $w, c \vDash \Gamma\{\langle A\rangle\}\{\}$. By Proposition 8 and $w, d_k \nvDash C$ we deduce that $w, d_0 \nvDash \mathcal{F}(\Gamma\{\langle A\rangle\}\{C^{\vdash}\})$.

- Case $[\Box_I]$. We suppose that $\Gamma\{\Box A^{\vdash}\}$ is not valid in IK. By Proposition 9 there exists a Kripke model $\mathcal{M}$, $w \in W$ and a (w,k)-chain $c = d_0 \to d_1 \to \cdots \to d_k$ such that $k = depth(\Gamma\{\})$, $w, c \vDash \Gamma\{\}$ and $w, d_k \nvDash \Box A$. From $w, d_k \nvDash \Box A$ we deduce that there exist $w' \in W$ and $d_{k+1} \in D_w$ such that $w \leqslant w'$, $R_{w'}(d_k, d_{k+1})$ and $w', d_{k+1} \nvDash A$. Let c' be the $(w', k+1)$-chain defined by $c' = d_0 \to d_1 \to \cdots \to d_k \to d_{k+1}$. From $w, c \vDash \Gamma\{\}$ we obtain $w, c' \vDash \Gamma\{\langle\{\}\rangle\}$. By Proposition 8 we have $w, d_0 \nvDash \mathcal{F}(\Gamma\{\langle A^{\vdash}\rangle\})$.

- Case $[\Box_E]$. We suppose that $\Gamma\{\langle\Delta, A^{\vdash}\rangle\}$ is not valid in IK. By Proposition 9 there exist a Kripke model $\mathcal{M}$, $w \in W$ and a $(w, k+1)$-chain $c = d_0 \to d_1 \to \cdots \to d_{k+1}$ such that $k = depth(\Gamma\{\})$, $w, c \vDash \Gamma\{\langle\Delta, \{\}\rangle\}$ and $w, d_{k+1} \nvDash A$. Moreover $w, d_{k+1} \nvDash A$ entails $w, d_k \nvDash \Box A$ and $w, c \vDash \Gamma\{\langle\Delta, \{\}\rangle\}$ entails $w, c' \vDash \Gamma\{\langle\Delta\rangle, \{\}\}$ where $c' = d_0 \to d_1 \to \cdots \to d_k$. By Proposition 8 we have $w, d_0 \nvDash \mathcal{F}(\Gamma\{\langle\Delta\rangle, \Box A^{\vdash}\})$.

4.3 Completeness of DN_{IK}

We prove the completeness of the system DN_{IK} from the Hilbert axiomatisation previously mentioned. We first show that the axioms are provable in DN_{IK} and that the rules (modus ponens, necessity) are admissible in DN_{IK}. We can see that the rules of natural deduction for intuitionistic logic (system DN_{IPL}) are particular cases of some rules of DN_{IK}. Moreover they are admissible in DN_{IK} and we obtain the following proposition:

Proposition 10. *If A is a theorem of propositional intuitionistic logic then $A^{\vdash}$ has a proof in DN_{IK}.*

Let us consider, for instance, the proof of $A \supset (B \supset (A \wedge B))$ in DN_{IPL}:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash A}[Id] \quad \dfrac{}{A, B \vdash B}[Id]}{A, B \vdash A \wedge B}[\wedge_I]}{A \vdash B \supset (A \wedge B)}[\supset_I]}{\vdash A \supset (B \supset (A \wedge B))}[\supset_I]$$

It can be translated in DN_{IK} as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, A^{\vdash}}[Id] \quad \dfrac{}{A, B, B^{\vdash}}[Id]}{A, B, A \wedge B^{\vdash}}[\wedge_I]}{A, B \supset (A \wedge B)^{\vdash}}[\supset_I]}{A \supset (B \supset (A \wedge B))^{\vdash}}[\supset_I]$$

Proposition 11. *The following T-sequents are provable in DN_{IK}:*

1. $\Box(A \supset B) \supset (\Box A \supset \Box B)^{\vdash}$
2. $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)^{\vdash}$
3. $\Diamond\bot \supset \bot^{\vdash}$

4. $\Diamond(A\vee B)\supset(\Diamond A\vee\Diamond B)^{\vdash}$
5. $(\Diamond A\supset\Box B)\supset\Box(A\supset B)^{\vdash}$

Proof. 1. $(\Box(A\supset B)\supset(\Box A\supset\Box B))^{\vdash}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box(A\supset B),\Box A,\langle\ \rangle,\Box(A\supset B)^{\vdash}}\,[\mathit{Id}]}{\Box(A\supset B),\Box A,\langle A\supset B^{\vdash}\rangle}\,[\Box_E] \qquad \dfrac{\dfrac{}{\Box(A\supset B),\Box A,\langle\ \rangle,\Box A^{\vdash}}\,[\mathit{Id}]}{\Box(A\supset B),\Box A,\langle A^{\vdash}\rangle}\,[\Box_E]}{\Box(A\supset B),\Box A,\langle B^{\vdash}\rangle}\,[\supset_E]}{\Box(A\supset B),\Box A,\Box B^{\vdash}}\,[\Box_I]}{\Box(A\supset B),\Box A\supset\Box B^{\vdash}}\,[\supset_I]}{\Box(A\supset B)\supset(\Box A\supset\Box B)^{\vdash}}\,[\supset_I]$$

2. $\Box(A\supset B)\supset(\Diamond A\supset\Diamond B)^{\vdash}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box(A\supset B),\Diamond A,\Diamond A^{\vdash}}\,[\mathit{Id}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Box(A\supset B),\Diamond A,\langle A\rangle,\Box(A\supset B)^{\vdash}}\,[\mathit{Id}]}{\Box(A\supset B),\Diamond A,\langle A,A\supset B^{\vdash}\rangle}\,[\Box_E] \quad \dfrac{}{\Box(A\supset B),\Diamond A,\langle A,A^{\vdash}\rangle}\,[\mathit{Id}]}{\Box(A\supset B),\Diamond A,\langle A,B^{\vdash}\rangle}\,[\supset_E]}{\Box(A\supset B),\Diamond A,\langle A\rangle,\Diamond B^{\vdash}}\,[\Diamond_I]}{\Box(A\supset B),\Diamond A,\Diamond B^{\vdash}}\,[\Diamond_E]}{\Box(A\supset B),\Diamond A\supset\Diamond B^{\vdash}}\,[\supset_I]}{\Box(A\supset B)\supset(\Diamond A\supset\Diamond B)^{\vdash}}\,[\supset_I]$$

3. $\Diamond\bot\supset\bot^{\vdash}$

$$\dfrac{\dfrac{\dfrac{}{\Diamond\bot,\Diamond\bot^{\vdash}}\,[\mathit{Id}] \qquad \dfrac{\dfrac{}{\Diamond\bot,\langle\bot,\bot^{\vdash}\rangle}\,[\mathit{Id}]}{\Diamond\bot,\langle\bot\rangle,\bot^{\vdash}}\,[\bot]}{\Diamond\bot,\bot^{\vdash}}\,[\Diamond_E]}{\Diamond\bot\supset\bot^{\vdash}}\,[\supset_I]$$

4. $\Diamond(A\vee B)\supset(\Diamond A\vee\Diamond B)^{\vdash}$

$$\dfrac{\dfrac{\dfrac{}{\Diamond(A\vee B),\Diamond(A\vee B)^{\vdash}}\,[\mathit{Id}] \qquad \dfrac{\dfrac{}{\Diamond(A\vee B),\langle A\vee B,A\vee B^{\vdash}\rangle}\,[\mathit{Id}] \quad \mathcal{D}_1 \quad \mathcal{D}_2}{\Diamond(A\vee B),\langle A\vee B\rangle,\Diamond A\vee\Diamond B^{\vdash}}\,[\vee_E]}{\Diamond(A\vee B),\Diamond A\vee\Diamond B^{\vdash}}\,[\Diamond_E]}{\Diamond(A\vee B)\supset(\Diamond A\vee\Diamond B)^{\vdash}}\,[\supset_I]$$

with

$$\mathcal{D}_1=\left\{\dfrac{\dfrac{\dfrac{}{\Diamond(A\vee B),\langle A\vee B,A,A^{\vdash}\rangle}\,[\mathit{Id}]}{\Diamond(A\vee B),\langle A\vee B,A\rangle,\Diamond A^{\vdash}}\,[\Diamond_I]}{\Diamond(A\vee B),\langle A\vee B,A\rangle,\Diamond A\vee\Diamond B^{\vdash}}\,[\vee_I^1]\right. \qquad \mathcal{D}_2=\left\{\dfrac{\dfrac{\dfrac{}{\Diamond(A\vee B),\langle A\vee B,B,B^{\vdash}\rangle}\,[\mathit{Id}]}{\Diamond(A\vee B),\langle A\vee B,B\rangle,\Diamond B^{\vdash}}\,[\Diamond_I]}{\Diamond(A\vee B),\langle A\vee B,B\rangle,\Diamond A\vee\Diamond B^{\vdash}}\,[\vee_I^2]\right.$$

5. $(\Diamond A\supset\Box B)\supset\Box(A\supset B)^{\vdash}$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\Diamond A \supset \Box B, \langle A \rangle, \Diamond A \supset \Box B^{\vdash}}[Id] \qquad \cfrac{\cfrac{}{\Diamond A \supset \Box B, \langle A, A^{\vdash} \rangle}[Id]}{\Diamond A \supset \Box B, \langle A \rangle, \Diamond A^{\vdash}}[\Diamond_I]}{\Diamond A \supset \Box B, \langle A \rangle, \Box B^{\vdash}}[\supset_E]}{\Diamond A \supset \Box B, \langle A, B^{\vdash} \rangle}[\Box_E]}{\Diamond A \supset \Box B\vdash, \langle , A \supset B^{\vdash} \rangle}[\supset_I]}{\Diamond A \supset \Box B, \Box(A \supset B)^{\vdash}}[\Box_I]}{(\Diamond A \supset \Box B) \supset \Box(A \supset B)^{\vdash}}[\supset_I]
$$

We can easily show that the rule *modus ponens* is admissible in DN_{IK} as it is a particular case of the rule $[\supset_E]$. In the next proposition we show the admissibility of the necessity rule.

Proposition 12. *If $A^{\vdash}$ is provable in DN_{IK} then $\Box A^{\vdash}$ is provable in DN_{IK}.*

Proof. We first show that if a T-sequent $\mathcal{S}$ is provable in DN_{IK} then $\langle \mathcal{S} \rangle$ is provable in DN_{IK}, by structural induction on the proof of $\mathcal{S}$. Then from the proof of $\langle A^{\vdash} \rangle$ we apply the rule $[\Box_I]$ and thus we obtain a proof of $\Box A^{\vdash}$.

Theorem 4. *If A is valid in IK then $A^{\vdash}$ is provable in DN_{IK}.*

Proof. The validity in IK is given through the axiomatisation and the proof is by structural induction on the proof of A. By Propositions 10 and 11 if A is an axiom then $A^{\vdash}$ has a proof in ND_{IK}.
Then we consider the two cases of the last rule applied, namely $[mp]$ and $[nec]$.

- If the last rule is $[mp]$: $\dfrac{A \supset B \qquad A}{B}$, then by induction hypothesis $A \supset B^{\vdash}$ and $A^{\vdash}$ are provable in DN_{IK}. By the rule $[\supset_E]$ we deduce that $B^{\vdash}$ is provable in DN_{IK}.
- If the last rule is $[nec]$: $\dfrac{A}{\Box A}$, then by induction hypothesis we have $A^{\vdash}$ provable in DN_{IK}. By Proposition 12, $\Box A^{\vdash}$ is also provable in DN_{IK}.

Theorem 5 (Completeness). *If a T-sequent is valid in IK then it is provable in DN_{IK}.*

Proof. Let $\mathcal{S}$ be a T-sequent. If $\mathcal{S}$ is valid in IK, then its associated formula $\mathcal{F}(\mathcal{S})$ is valid in IK. By Theorem 4 $\mathcal{F}(\mathcal{S})^{\vdash}$ is a T-sequent provable in DN_{IK}. Then by Propositions 2, 3, 4 and 5 we deduce that $\mathcal{S}$ is provable in DN_{IK}.

5 Normalization in DN_{IK}

Having proved that the natural deduction system DN_{IK} is sound and complete we now study the property of normalization in this system and the related properties. First we define a set of notions and concepts in order to describe the normalization procedure in a clear and concise way. Then we prove the normalization theorem, i.e., any derivation can be transformed in a derivation in normal form.

5.1 Indexed Formulas

Let us note that the T-context of a T-sequent can contain several occurrences of the same formula and one needs to differenciate these ones. In this perspective each formula occurrence in a T-context is indexed by a variable such that if $x : A$ and $y : B$ are two different occurrences in a T-context then $x \neq y$. We note $Var(\mathcal{S})$ the set of variables in the T-sequent $\mathcal{S}$. Let us mention some problems related to this indexation.
Let $\mathcal{S} = A \supset A^{\vdash}$ and $\mathcal{S}' = x : B, A \supset A^{\vdash}$ be two T-sequents and $\mathcal{D}$ be the following proof of $\mathcal{S}$:

$$
\cfrac{\cfrac{}{x : A, A^{\vdash}}[Id]}{A \supset A^{\vdash}}[\supset_I]
$$

The proof $\mathcal{D}[\mathcal{S}']_w$ is given by:

$$\dfrac{\dfrac{}{x : B, x : A, A^{\vdash}}\,[Id]}{x : B, A \supset A^{\vdash}}\,[\supset_I]$$

We remark that $\mathcal{D}[\mathcal{S}']_w$ contains a T-sequent with a T-context having two different formula occurrences indexed by the same variable x. Then in order to avoid this problem we associate a renaming fonction to the weakening.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two T-sequents such that $\mathcal{S} \rightarrow_w^* \mathcal{S}'$ and $\mathcal{D}$ be a proof of $\mathcal{S}$. We note $\mathcal{D}[\mathcal{S}']_r$ any proof of $\mathcal{S}$ obtained from $\mathcal{D}$ by renaming all variables of $Var(\mathcal{S}') \backslash Var(\mathcal{S})$ by fresh variables knowing that one cannot rename two different variables with the same variable. Thus we define $\mathcal{D}[\mathcal{S}']_{rw}$ as being any proof corresponding to $(\mathcal{D}[\mathcal{S}']_r)[\mathcal{S}']_w$. Let us illustrate this point with the previous example:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{}{x : A, A^{\vdash}}\,[Id]}{A \supset A^{\vdash}}\,[\supset_I] \right. \quad \mathcal{D}[\mathcal{S}']_r = \left\{ \dfrac{\dfrac{}{y : A, A^{\vdash}}\,[Id]}{A \supset A^{\vdash}}\,[\supset_I] \right. \quad \mathcal{D}[\mathcal{S}']_{rw} = \left\{ \dfrac{\dfrac{}{x : B, y : A, A^{\vdash}}\,[Id]}{x : B, A \supset A^{\vdash}}\,[\supset_I] \right.$$

Let $\mathcal{D}$ be a proof of $\Gamma\{\Delta\}\{C^{\vdash}\}$. We denote $\mathcal{D} - \{\Delta\}$ the tree obtained from $\mathcal{D}$ by arising Δ from all T-sequents of $\mathcal{D}$. Such a tree is a proof of $\Gamma\{\emptyset\}\{C^{\vdash}\}$ when its leaves are labelled by instances of the axiom $[id]$.

Let $\mathcal{D}$ and $\mathcal{D}'$ be proofs of respectively $\mathcal{S} = \Gamma\{x : A\}\{C^{\vdash}\}$ and $\mathcal{S}' = \Gamma\{A^{\vdash}\}\{\emptyset\}$. We observe that any leaf of $\mathcal{D}$ is labelled by a T-sequent that is either of the form $\Gamma'\{x : A, A^{\vdash}\}$ such that $\Gamma\{A^{\vdash}\}\{\emptyset\} \rightarrow_w \Gamma'\{A^{\vdash}\}$, or of the form $\Gamma'\{x : A\}\{y : D, D^{\vdash}\}$. We denote $\mathcal{D}[\mathcal{D}'/x]$ the tree built as follows:
(i) replace all the leaves labelled by a T-sequent of the form $\Gamma'\{x : A, A^{\vdash}\}$ by the deduction $\mathcal{D}'[\Gamma'\{A^{\vdash}\}]_{rw}$;
(ii) suppress $x : A$ in all T-sequents of the resulting tree.

We can show that $\mathcal{D}[\mathcal{D}'/x]$ is a proof of $\Gamma\{\emptyset\}\{C^{\vdash}\}$. It comes from the fact that the instances of $[id]$ of the form $\Gamma'\{x : A\}\{y : C, C^{\vdash}\}$ remain axioms even after that $x : A$ is suppressed.

5.2 Discharging Rules

From now we use the expression *discharging rules* in order to refer to the rules $[\supset_I]$, $[\vee_E]$ and $[\Diamond_E]$. There is no discharge of hypotheses like in standard natural deduction systems but the rules $[\supset_I]$, $[\vee_E]$ and $[\Diamond_E]$ internalize hypothesis discharges such that a rule application discharges the T-sequents that are concerned by the introduction of formulas appearing in T-contexts of some premisses but not in the T-context of the conclusion.

Definition 12. *Let $\mathcal{D}$ be a proof in* IK*, f be a leaf of $\mathcal{D}$, $\mathcal{S}$ be the T-sequent labelling f and α be an application of a discharging rule in $\mathcal{D}$. The T-sequent $\mathcal{S}$ is discharged by α if $\mathcal{S}$ is not discharged by another rule applied before α, f and the conclusion of α being in the same branch and one of the following properties is satisfied:*

1. *α is an application of $[\supset_I]$ and there exist two $1T$-contexts $\Gamma\{\}$ and $\Gamma'\{\}$, two formulas A and B, and a variable x such that $\mathcal{S} = \Gamma'\{x : A, A^{\vdash}\}$, $\Gamma\{x : A, B^{\vdash}\}$ is the premisse of α and $\Gamma\{A \supset B^{\vdash}\}$ is the conclusion of α.*
2. *α is an application of $[\vee_E]$ and there exist a $2T$-context $\Gamma\{\}\{\}$, a $1T$-context $\Gamma'\{\}$, three formulas A, B and C, and a variable x such that $\mathcal{S} = \Gamma'\{x : A, A^{\vdash}\}$, $\Gamma\{F^{\vdash}\}\{\emptyset\}$ is the main premiss of α with F equal to $A \vee B$ or $B \vee A$ and $\Gamma\{x : A\}\{C^{\vdash}\}$ is one of the minor premisses of α (it belongs to the same branch as f).*
3. *α is an application of $[\Diamond_E]$ and there exist a $2T$-context $\Gamma\{\}\{\}$, a $1T$-context $\Gamma'\{\}$, two formulas A and C, and a variable x such that $\mathcal{S} = \Gamma'\{x : A, A^{\vdash}\}$, $\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}$ is the major premisse of α and $\Gamma\{\langle x : A\rangle\}\{C^{\vdash}\}$ is the minor premisse of α (it belongs to the same branch as f).*

We illustrate this definition with an example. Let $\mathcal{D}$ be the following proof:

$$
\dfrac{
\dfrac{}{x:\Box(A\supset B),y:\Diamond A,\Diamond A^{\vdash}}\,[Id]
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{x:\Box(A\supset B),y:\Diamond A,\langle z:A\rangle,\Box(A\supset B)^{\vdash}}\,[Id]
}{x:\Box(A\supset B),y:\Diamond A,\langle z:A,A\supset B^{\vdash}\rangle}\,[\Box_E]
\quad
\dfrac{}{x:\Box(A\supset B),y:\Diamond A,\langle z:A,A^{\vdash}\rangle}\,[Id]
}{x:\Box(A\supset B),y:\Diamond A,\langle z:A,B^{\vdash}\rangle}\,[\supset_E]
}{x:\Box(A\supset B),y:\Diamond A,\langle z:A\rangle,\Diamond B^{\vdash}}\,[\Diamond_I]
}{
\dfrac{
\dfrac{x:\Box(A\supset B),y:\Diamond A,\Diamond B^{\vdash}}{x:\Box(A\supset B),\Diamond A\supset\Diamond B^{\vdash}}\,[\supset_I]
}{\Box(A\supset B)\supset(\Diamond A\supset\Diamond B)^{\vdash}}\,[\supset_I]
}\,[\Diamond_E]
$$

The T-sequent $x:\Box(A\supset B),y:\Diamond A,\Diamond A^{\vdash}$ is discharged by the first application of $[\supset_I]$. The T-sequent $x:\Box(A\supset B),y:\Diamond A,\langle z:A\rangle,\Box(A\supset B)^{\vdash}$ is discharged by the second application of $[\supset_I]$. Moreover the application of $[\Diamond_E]$ discharges the T-sequent $x:\Box(A\supset B),y:\Diamond A,\langle z:A,A^{\vdash}\rangle$.

5.3 Normalization

Let us recall that a detour in a natural deduction proof corresponds to an application of a rule that introduces a logical operator followed by an application of a rule that eliminates it. The main goal of the normalization property is the elimination of all detours in a proof. In order to prove this property for the system DN_{IK} we consider an approach similar to the one of Prawitz [20,25].
We introduce first the notion of segment for this system, then we define the notion of cut that is a particular case of segment and propose the rules of our normalization procedure. Finally we prove the normalization theorem.

Definition 13. *A* segment *of length n in a proof $\mathcal{D}$ in DN_{IK} is a sequence $\Gamma_1\{A^{\vdash}\},\ldots,\Gamma_n\{A^{\vdash}\}$ of consecutive occurrences of T-sequents in $\mathcal{D}$ such that:*
- for $n>1$ and $n>i$, $\Gamma_i\{A^{\vdash}\}$ is a minor premisse of an application of $[\vee_E]$ or $[\Diamond_E]$ in $\mathcal{D}$ with the conclusion $\Gamma_{i+1}\{A^{\vdash}\}$,
- $\Gamma_1\{A^{\vdash}\}$ is not the conclusion of an application of $[\vee_E]$ or of $[\Diamond_E]$,
- $\Gamma_n\{A^{\vdash}\}$ is not a minor premisse of an application of $[\vee_E]$ or $[\Diamond_E]$.

We note that the T-sequents of a segment σ have the same conclusion, called the conclusion of the segment and denoted $C(\sigma)$. A segment is a premisse (resp. the conclusion) of a rule application if its last element (resp. first element) is a premisse (resp. the conclusion) of this application. A segment σ is a subformula of a segment σ' if $C(\sigma)$ is a subformula of $C(\sigma')$.

Definition 14. *A segment is a* cut *if $\Gamma_n\{A^{\vdash}\}$ is the main premisse of the application of an elimination rule, and either $n>1$ or $n=1$ and $\Gamma_1\{A^{\vdash}\}$ is the conclusion of an introduction rule or of the rule $[\bot]$.*
The cutrank σ*, denoted $cr(\sigma)$, is defined by the complexity of its conclusion $\mid C(\sigma)\mid$.*
The cutrank *of a proof $\mathcal{D}$, denoted $cr(\mathcal{D})$, is the maximum of the cutranks in $\mathcal{D}$ (*0 *if $\mathcal{D}$ does not contain a cut).*
A critical cut *in a proof $\mathcal{D}$ is a cut the cutrank of which is the cutrank of $\mathcal{D}$.*
A proof is in normal form *if it does not contain a cut.*

Now we give the rules used to prove the normalization property. First we present the *reduction* rules that allow to eliminate the cuts of length 1 (detours) and the *permutation* and *simplification* rules.

Reduction rules:

– $\wedge$-reduction:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma\{A_1^{\vdash}\}\end{matrix}\quad\begin{matrix}\mathcal{D}_2\\ \Gamma\{A_2^{\vdash}\}\end{matrix}}{\Gamma\{A_1\wedge A_2^{\vdash}\}}\,[\wedge_I]}{\Gamma\{A_i^{\vdash}\}}\,[\wedge_E^i]
\quad\rightsquigarrow\quad
\begin{matrix}\mathcal{D}_i\\ \Gamma\{A_i^{\vdash}\}\end{matrix}
\qquad \text{for } i\in\{1,2\}.
$$

– $\vee$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma\{A_i^{\vdash}\}\{\emptyset\}\end{matrix}}{\Gamma\{A_1\vee A_2^{\vdash}\}\{\emptyset\}}[\vee_I^i] \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma\{x:A_1\}\{C^{\vdash}\}\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma\{x:A_2\}\{C^{\vdash}\}\end{matrix}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}[\vee_E] \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_i[x/\mathcal{D}]\\ \Gamma\{\emptyset\}\{C^{\vdash}\}\end{matrix} \text{ for } i\in\{1,2\}.$$

– $\supset$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma\{x:A,B^{\vdash}\}\end{matrix}}{\Gamma\{A\supset B^{\vdash}\}}[\supset_I] \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma\{A^{\vdash}\}\end{matrix}}{\Gamma\{B^{\vdash}\}}[\supset_E] \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_1[x/\mathcal{D}_2]\\ \Gamma\{B^{\vdash}\}\end{matrix}$$

– $\Box$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma\{\langle\Delta\rangle,\langle A^{\vdash}\rangle\}\end{matrix}}{\Gamma\{\langle\Delta\rangle,\Box A^{\vdash}\}}[\Box_I]}{\Gamma\{\langle\Delta,A^{\vdash}\rangle\}}[\Box_E] \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}[\Gamma\{\langle\Delta,A^{\vdash}\rangle\}]_m\\ \Gamma\{\langle\Delta,A^{\vdash}\rangle\}\end{matrix}$$

– $\Diamond$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma\{\langle\Delta,A^{\vdash}\rangle\}\{\emptyset\}\end{matrix}}{\Gamma\{\langle\Delta\rangle,\Diamond A^{\vdash}\}\{\emptyset\}}[\Diamond_I] \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma\{\langle\Delta\rangle,\langle x:A\rangle\}\{C^{\vdash}\}\end{matrix}}{\Gamma\{\langle\Delta\rangle\}\{C^{\vdash}\}}[\Diamond_E] \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_2'[x/\mathcal{D}_1]\\ \Gamma\{\langle\Delta\rangle\}\{C^{\vdash}\}\end{matrix}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{\langle\Delta,x:A\rangle\}\{C^{\vdash}\}]_m$

Permutation rules:

– $\vee$-permutation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma\{A\vee B^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma\{x:A\}\{C^{\vdash}\}\{\emptyset\}\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma\{y:B\}\{C^{\vdash}\}\{\emptyset\}\end{matrix}}{\Gamma\{\emptyset\}\{C^{\vdash}\}\{\emptyset\}}[\vee_E] \qquad \mathcal{D}'}{\Gamma\{\emptyset\}\{\emptyset\}\{D^{\vdash}\}}[R_E]$$

$$\rightsquigarrow\quad \dfrac{\begin{matrix}\mathcal{D}\\ \Gamma\{A\vee B^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma\{x':A\}\{C^{\vdash}\}\{\emptyset\}\end{matrix} \qquad \mathcal{D}'_1}{\Gamma\{x':A\}\{\emptyset\}\{D^{\vdash}\}}[R_E] \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Gamma\{y':B\}\{C^{\vdash}\}\{\emptyset\}\end{matrix} \qquad \mathcal{D}'_2}{\Gamma\{y':B\}\{\emptyset\}\{D^{\vdash}\}}[R_E]}{\Gamma\{\emptyset\}\{\emptyset\}\{D^{\vdash}\}}[\vee_E]$$

where $[R_E]$ is an elimination rule, $\mathcal{D}'$ is a sequence of proofs (possibly empty) and $\mathcal{D}'_1$ (resp. $\mathcal{D}'_2$) is the sequence of proofs obtained by adding of $x':A$ (resp. $y':B$) to proofs of $\mathcal{D}'$ where x' (resp. y') is a fresh variable that is not in $\mathcal{D}'$.

– $\Diamond$-permutation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma\{\langle x:A\rangle\}\{C^{\vdash}\}\{\emptyset\}\end{matrix}}{\Gamma\{\emptyset\}\{C^{\vdash}\}\{\emptyset\}}[\Diamond_E] \qquad \mathcal{D}'}{\Gamma\{\emptyset\}\{\emptyset\}\{D^{\vdash}\}}[R_E] \quad\rightsquigarrow\quad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Gamma\{\langle x:A\rangle\}\{C^{\vdash}\}\{\emptyset\}\end{matrix} \qquad \mathcal{D}''}{\Gamma\{\langle x:A\rangle\}\{\emptyset\}\{D^{\vdash}\}}[R_E]}{\Gamma\{\emptyset\}\{\emptyset\}\{D^{\vdash}\}}[\Diamond_E]$$

where $[R_E]$ is an elimination rule, $\mathcal{D}'$ is a sequence of proofs (possibly empty) and $\mathcal{D}''$ is the sequence of proofs obtained by adding $x':A$ to proofs of $\mathcal{D}'$ where x' is a fresh variable that is not in $\mathcal{D}'$.

– $\perp$-permutation:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\perp^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{array}}{\Gamma\{\emptyset\}\{A^{\vdash}\}\{\emptyset\}}[\perp] \qquad \mathcal{D}'}{\Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}}[R_E] \rightsquigarrow \cfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\perp^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{array}}{\Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}}[\perp]$$

where $[R_E]$ is an elimination rule.

Simplification rules:

– $\vee$-simplification:

$$\cfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{A_1 \vee A_2^{\vdash}\}\{\emptyset\}\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma\{x : A_1\}\{C^{\vdash}\}\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{x : A_2\}\{C^{\vdash}\}\end{array}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}[\vee_E] \rightsquigarrow \begin{array}{c}\mathcal{D}_i - \{x : A_i\}\\ \Gamma\{\emptyset\}\{C^{\vdash}\}\end{array}$$

where there is no T-sequent discharged by $[\vee_E]$ in $\mathcal{D}_i$. The absence of discharged T-sequents reflects that $x : A_i$ is not necessary in the T-sequents of $\mathcal{D}_i$.

– $\Diamond$-simplification:

$$\cfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}\end{array} \qquad \begin{array}{c}\mathcal{D}'\\ \Gamma\{\langle x : A_1\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}[\Diamond_E] \rightsquigarrow \begin{array}{c}\mathcal{D}' - \{\langle x : A\rangle\}\\ \Gamma\{\emptyset\}\{C^{\vdash}\}\end{array}$$

where there is no T-sequent in $\mathcal{D}'$ that contains a subtree of the form $x : A, \Delta$ o $\Delta \neq \emptyset$. In the case of disjunction the simplification rule only eliminates a formula occurrence but in the case of $\Diamond$ it eliminates the T-context $\langle x : A\rangle$. The previous condition on $\mathcal{D}'$ in order to apply simplification expresses that the formula occurrence $x : A$ and also the T-context $\langle x : A\rangle$ are not necessary.

Theorem 6 (Normalization). *Any proof $\mathcal{D}$ in DN_{IK} can be reduced to a proof in normal form.*

Proof. By induction on the value of the pair (n,m) where $n = cr(\mathcal{D})$ and m are the sum of the lengths of all critical cuts of $\mathcal{D}$. We say that the pair (n',m') is less than (n,m) if either $n' < n$, or $n' = n$ and $m' < m$. This proof is similar to the one of Prawitz.
Let σ be the rightmost critical cut having no other critical cut above it in $\mathcal{D}$. The application of a rule of reduction, permutation or simplification to σ in $\mathcal{D}$ gives a proof $\mathcal{D}'$ where either $cr(\mathcal{D}') < n$ or the sum of the lengths of all critical cuts of $\mathcal{D}'$ is less than m. If σ is the unique critical cut in $\mathcal{D}$ of length 1, then by application of the corresponding reduction rule we obtain a proof of rank less than n. Else the application of a permutation or simplification rule to σ in $\mathcal{D}$ gives a proof in which the sum of the lengths of all critical cuts is less than m.

For instance we illustrate the case of $\Diamond$-reduction:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{\langle\Delta, A^{\vdash}\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\langle\Delta\rangle, \Diamond A^{\vdash}\}\{\emptyset\}}[\Diamond_I] \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\langle\Delta\rangle, \langle x : A\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\langle\Delta\rangle\}\{C^{\vdash}\}}[\Diamond_E] \rightsquigarrow \begin{array}{c}\mathcal{D}_2'[x/\mathcal{D}_1]\\ \Gamma\{\langle\Delta\rangle\}\{C^{\vdash}\}\end{array}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{\langle\Delta, x : A\rangle\}\{C^{\vdash}\}]_m$.

As σ is the mostright critical cut then $\mathcal{D}_2$ does not contain a critical cut and consequently also $\mathcal{D}_2'$. Moreover, $\mathcal{D}_1$ has no critical cut because there is no critical cut above σ. We also observe that if a cut is introduced by application of a $\Diamond$-reduction then this one has A as conclusion. Then the value of its cutranks is less than n. We then deduce that $\mathcal{D}_2'[x/\mathcal{D}_1]$ does not contain any cutrank greater or equal to n.

$$\frac{\Gamma\{\Box A^{\vdash}\}}{\Gamma\{A^{\vdash}\}}\ [\Box_E^T] \qquad \frac{\Gamma\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}}\ [\Diamond_I^T]$$

$$\frac{\Gamma\{\langle\Delta,\Box A^{\vdash}\rangle\}}{\Gamma\{\langle\Delta\rangle,A^{\vdash}\}}\ [\Box_E^B] \qquad \frac{\Gamma\{\langle\Delta\rangle,A^{\vdash}\}}{\Gamma\{\langle\Delta,\Diamond A^{\vdash}\rangle\}}\ [\Diamond_I^B]$$

$$\frac{\Gamma\{\Delta\{\emptyset\},\Box A^{\vdash}\}}{\Gamma\{\Delta\{A^{\vdash}\}\}}\ [\Box_E^4](depth(\Delta\{\})>1) \qquad \frac{\Gamma\{\Delta\{A^{\vdash}\}\}}{\Gamma\{\Delta\{\emptyset\},\Diamond A^{\vdash}\}}\ [\Diamond_I^4](depth(\Delta\{\})>1)$$

$$\frac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\Box_E^5](depth(\Gamma\{\}\{\emptyset\})>0 \texttt{ et } depth(\Gamma\{\emptyset\}\{\})>0)$$

$$\frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_I^5](depth(\Gamma\{\}\{\emptyset\})>0 \texttt{ et } depth(\Gamma\{\emptyset\}\{\})>0)$$

$$\frac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\Box_E^{\mathsf{IB4}}](sp(\Gamma\{\Box A^{\vdash}\}\{\emptyset\})) \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_I^{\mathsf{IB4}}](sp(\Gamma\{\emptyset\}\{A^{\vdash}\}))$$

$$\frac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\Box_E^{\mathsf{IS5}}] \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_I^{\mathsf{IS5}}]$$

Fig. 2. Modal Rules

6 Quasi-modular Natural Deduction Systems

In this section we propose quasi-modular natural deduction systems for the intuitionistic modal logics obtained by combinations of the axioms T, B, 4 and 5. The modularity is based on the association of specific rules to the axioms. A system is modular if we have a system for IK such that for any subset Th of $\{T,B,4,5\}$, the addition of rules associated to axioms in Th leads to a system for the logic IKTh. For instance, in the case of classical modal logics based on these axioms, a modular calculus, based on deep sequents, has been recently defined [5]. Our system is said quasi-modular because the logics IB4 and IS5 are separately studied.

We associate to each logic IKTh, with $\mathsf{Th} \subseteq \{T,B,4,5\}$, the natural deduction system DN_{IKTh} obtained by using the rules described in Figure 2 as follows:

- if IKTh is IS5 then DN_{IKTh} is obtained from DN_{IK} by replacing the rules $[\Box_E]$ and $[\Diamond_I]$ by the rules $[\Box_E^{\mathsf{IS5}}]$ and $[\Diamond_I^{\mathsf{IS5}}]$;
- if IKTh is IB4 then DN_{IKTh} is obtained from DN_{IK} by replacing the rules $[\Box_E]$ and $[\Diamond_I]$ by the rules $[\Box_E^{\mathsf{IB4}}]$ and $[\Diamond_I^{\mathsf{IB4}}]$;
- otherwise DN_{IKTh} is obtained by adding to DN_{IK} the rules $[\Box_E^x]$ and $[\Diamond_I^x]$ for any $x \in \mathsf{Th}$.

The six rules $[\Box_E^4]$, $[\Diamond_I^4]$, $[\Box_E^5]$, $[\Diamond_I^5]$, $[\Box_E^{\mathsf{IB4}}]$ and $[\Diamond_I^{\mathsf{IB4}}]$ can be applied only if some conditions are satisfied. For instance we can only apply the rule $[\Box_E^{\mathsf{IB4}}]$ if it satisfies the condition $sp(\Gamma\{\emptyset\}\{A^{\vdash}\})$.

In the case of IK we consider that a context $\mathcal{C}'$ is accessible from another context $\mathcal{C}$ in a sequent $\mathcal{S}$ if $\mathcal{C}'$ is a son of $\mathcal{C}$ in the T-context of $\mathcal{S}$. The modal rules of Figure 2 internalize the properties of the accessibility relation associated to the axioms D, T, B, 4 and 5. For instance the pair of rules $[\Box_E^4]$ and $[\Diamond_I^4]$ internalize the transitivity property. If a context $\mathcal{C}'$ is the son of a context $\mathcal{C}$ in a T-context of a T-sequent $\mathcal{S}$, then all contexts being in the subtree St with $\mathcal{C}'$ as root are accessible from $\mathcal{C}$. If a formula $\Box A$ is true in $\mathcal{C}$ then, from the rules $[\Box_E]$ and $[\Box_E^4]$, the formula A is true in any context in St. Moreover, by using the rules $[\Box_I]$ and $[\Box_I^4]$, if a formula A is true in any context of St, then $\Diamond A$ is true in $\mathcal{C}$.

6.1 Soundness and Completeness

Let us note that, for any subset Th de $\{T,B,4,5\}$, the system obtained by adding to DN_{IK} the rules $[\Box_E^x]$ and $[\Diamond_I^x]$ for all $x \in \mathsf{Th}$ is a sound and complete system for the logic IKTh. But all the systems defined in such a way does not verifiy the normalization property and then the systems for $\mathsf{IB4}$ and $\mathsf{IS5}$ are built in a different way.
For instance let us consider the following proof with $\mathsf{Th} = \{T,5\}$:

$$\dfrac{\dfrac{\dfrac{}{\Box A, \Box A^{\vdash}}\,[Id]}{\Box A, \Diamond\Box A^{\vdash}}\,[\Diamond_I^T] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Box A, \langle \Box A, \Box A^{\vdash}\rangle, \langle\langle\ \rangle\rangle}\,[Id]}{\Box A, \langle \Box A\rangle, \langle\langle A^{\vdash}\rangle\rangle}\,[\Box_E^5]}{\Box A, \langle \Box A\rangle, \langle \Box A^{\vdash}\rangle}\,[\Box_I]}{\Box A, \langle \Box A\rangle, \Box\Box A^{\vdash}}\,[\Box_I]}{\dfrac{\Box A, \Box\Box A^{\vdash}}{\Box A \supset \Box\Box A^{\vdash}}\,[\supset_I]}\,[\Diamond_E]$$

We remark that there exists an application of an introduction rule ($[\Diamond_I^T]$) having as conclusion the main premisse of an application of an elimination rule ($[\Diamond_E]$). In the previous proof we cannot eliminate it by using only the rules associated to axioms T and 5.

Theorem 7 (Soundness). *If a T-sequent has a proof in DN_{IKTh} then it is valid in IKTh.*

Proof. The sound rules in IK are sound in all logics IKTh with $\mathsf{Th} \subseteq \{T,B,4,5\}$. Then it sufficient to prove the soundness of the other rules with the same approach used for Theorem 3. We only develop the cases of rules $[\Diamond_I^4]$ and $[\Box_E^4]$. Let L be one of the logics IKTh verifying the axiom (4). We observe that Propositions 7, 8 and 9 are also true for all logics based on combinations of axioms T, B, 4 and 5.
- Case $[\Diamond_I^4]$.
We suppose that $\Gamma\{\Delta\{\emptyset\}, \Diamond A^{\vdash}\}$ $(depth(\Delta\{\}) > 1)$ is not valid in L. There exist a model $\mathcal{M}$, $w \in W$ and a (w,k)-chain $c = d_0 \to d_1 \to \dots d_k$ such that $k = depth(\Gamma\{\Delta\{\emptyset\},\{\}\})$, $w,c \vDash \Gamma\{\Delta\{\emptyset\},\{\}\}$ and $w,d_k \nvDash \Diamond A$. If $w,c \vDash \Gamma\{\Delta\{\emptyset\},\{\}\}$ then $w,d_k \vDash \mathcal{F}(\Delta\{\emptyset\})$. Then there exists a (w,l)-chain $c' = d_k \to d_{k+1} \to \dots d_{k+l}$ such that $l = depth(\Delta\{\})$ and $w,c' \vDash \Delta\{\}$. Thus we have $w,c'' \vDash \Gamma\{\Delta\{\}\}$ where $c'' = d_0 \to d_1 \to \dots d_{k+l}$. Given a transitive relation R_w we have $w,d_{k+l} \nvDash A$ because $w,d_k \nvDash \Diamond A$. Then we deduce $w,d_0 \nvDash \mathcal{F}(\Gamma\{\Delta\{A^{\vdash}\}\})$.
- Case $[\Box_E^4]$.
We suppose that $\Gamma\{\Delta\{A^{\vdash}\}\}$ $(depth(\Delta\{\}) > 1)$ is not valid in L. There exist a model $\mathcal{M}$, $w \in W$ and a $(w,k+l)$-chain $c = d_0 \to d_1 \to \dots d_{k+l}$ such that $k = depth(\Gamma\{\})$, $l = depth(\Delta\{\})$, $w,c \vDash \Gamma\{\Delta\{\}\}$ and $w,d_{k+l} \nvDash A$. Given a transitive relation R_w if $w,d_{k+l} \nvDash A$ then $w,d_k \nvDash \Box A$. Moreover if $w,c \vDash \Gamma\{\Delta\{\}\}$ then $w,c' \vDash \Gamma\{\Delta\{\emptyset\},\{\}\}$ where $c' = d_0 \to d_1 \to \dots d_k$. Thus we have $w,d_0 \nvDash \mathcal{F}(\Gamma\{\Delta\{\emptyset\}, \Box A^{\vdash}\})$.

Theorem 8 (Completeness). *If a T-sequent is valid in IKTh, then it has a proof in DN_{IKTh}.*

Proof. As for the system DN_{IK} this proof of completeness of DN_{IKTh} can be obtained from the validity in IKTh through its Hilbert system. Let us note that the rule $[\Box_E^{\mathsf{IB4}}]$ (resp. $[\Diamond_I^{\mathsf{IB4}}]$) is a generalization of the rules $[\Box_E]$, $[\Box_E^B]$, $[\Box_E^4]$, $[\Box_E^5]$ (resp. $[\Diamond_I]$, $[\Diamond_I^B]$, $[\Diamond_I^4]$, $[\Diamond_I^5]$). Moreover the rules $[\Box_E^{\mathsf{IS5}}]$ and $[\Diamond_I^{\mathsf{IS5}}]$ generalize all the other rules that eliminate $\Box$ and introduce $\Diamond$. Then for all $\mathsf{Th} \subseteq \{T,B,4,5\}$, if a T-sequent has a proof in the system obtained by adding to DN_{IK} the rules $[\Box_E^x]$ and $[\Diamond_I^x]$ for all $x \in \mathsf{Th}$ then it has a proof in DN_{IKTh}. Then we do not consider the cases of the systems DN_{IS5} and DN_{IB4}.

- $(\Box A \supset A) \wedge (A \supset \Diamond A)^{\vdash}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box A, \Box A^{\vdash}}\,[Id]}{\Box A, A^{\vdash}}\,[\Box_E^T]}{\Box A \supset A^{\vdash}}\,[\supset_I] \qquad \dfrac{\dfrac{\dfrac{}{A, A^{\vdash}}\,[Id]}{A, \Diamond A^{\vdash}}\,[\Diamond_I^T]}{A \supset \Diamond A^{\vdash}}\,[\supset_I]}{(\Box A \supset A) \wedge (A \supset \Diamond A)^{\vdash}}\,[\wedge_I]$$

– $(\Diamond\Box A \supset A) \wedge (A \supset \Box\Diamond A)^{\vdash}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond\Box A, \Diamond\Box A^{\vdash}}[Id] \quad \dfrac{\dfrac{}{\Diamond\Box A, \langle \Box A, \Box A^{\vdash}\rangle}[Id]}{\Diamond\Box A, \langle \Box A\rangle, A^{\vdash}}[\Diamond_I^B]}{\Diamond\Box A, A^{\vdash}}[\Diamond_E]}{\Diamond\Box A \supset A^{\vdash}}[\supset_I] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A, \langle\ \rangle, A^{\vdash}}[Id]}{A, \langle \Diamond A^{\vdash}\rangle}[\Diamond_I^B]}{A, \Box\Diamond A^{\vdash}}[\Box_I]}{A \supset \Box\Diamond A^{\vdash}}[\supset_I]}{(\Diamond\Box A \supset A) \wedge (A \supset \Box\Diamond A)^{\vdash}}[\wedge_I]$$

– $(\Box A \supset \Box\Box A) \wedge (\Diamond\Diamond A \supset \Diamond A)^{\vdash}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box A, \Box A^{\vdash}, \langle\langle\ \rangle\rangle}[Id]}{\Box A, \langle\langle A^{\vdash}\rangle\rangle}[\Box_E^4]}{\Box A, \langle \Box A^{\vdash}\rangle}[\Box_I]}{\Box A, \Box\Box A^{\vdash}}[\Box_I]}{\Box A \supset \Box\Box A^{\vdash}}[\supset_I] \qquad \mathcal{D}}{(\Box A \supset \Box\Box A) \wedge (\Diamond\Diamond A \supset \Diamond A)^{\vdash}}[\wedge_I]$$

with

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\dfrac{}{\Diamond\Diamond A, \Diamond\Diamond A^{\vdash}}[Id] \quad \dfrac{\dfrac{}{\Diamond\Diamond A, \langle \Diamond A, \Diamond A^{\vdash}\rangle}[Id] \quad \dfrac{\dfrac{}{\Diamond\Diamond A, \langle \Diamond A, \langle A, A^{\vdash}\rangle\rangle}[Id]}{\Diamond\Diamond A, \langle \Diamond A, \langle A\rangle\rangle, \Diamond A^{\vdash}}[\Diamond_I^4]}{\Diamond\Diamond A, \langle \Diamond A\rangle, \Diamond A^{\vdash}}[\Diamond_E]}{\Diamond\Diamond A, \Diamond A^{\vdash}}[\Diamond_E]}{\Diamond\Diamond A \supset \Diamond A^{\vdash}}[\supset_I] \right.$$

– $(\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)^{\vdash}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond\Box A, \Diamond\Box A^{\vdash}}[Id] \quad \dfrac{\dfrac{\dfrac{}{\Diamond\Box A, \langle \Box A, \Box A^{\vdash}\rangle, \langle\ \rangle}[Id]}{\Diamond\Box A, \langle \Box A\rangle, \langle A^{\vdash}\rangle}[\Box_E^5]}{\Diamond\Box A, \langle \Box A\rangle, \Box A^{\vdash}}[\Box_I]}{\Diamond\Box A, \Box A^{\vdash}}[\Diamond_E]}{\Diamond\Box A \supset \Box A^{\vdash}}[\supset_I] \quad \dfrac{\dfrac{\dfrac{}{\Diamond A, \Diamond A^{\vdash}}[Id] \quad \dfrac{\dfrac{\dfrac{}{\Diamond A, \langle A, A^{\vdash}\rangle, \langle\ \rangle}[Id]}{\Diamond A, \langle A\rangle, \langle \Diamond A^{\vdash}\rangle}[\Diamond_I^5]}{\Diamond A, \langle A\rangle, \Box\Diamond A^{\vdash}}[\Box_I]}{\Diamond A, \Box\Diamond A^{\vdash}}[\Diamond_E]}{\Diamond A \supset \Box\Diamond A^{\vdash}}[\supset_I]}{(\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)^{\vdash}}[\wedge_I]$$

6.2 Normalization and Properties

As DN_{IK} and DN_{IKTh} contain the same rules $[\supset_I]$, $[\vee_E]$ and $[\Diamond_E]$, we consider the notions of discharging rules and of discharged T-sequent in the case of DN_{IKTh} that are the same in the case of DN_{IK} (Definition 12). Concerning the notions of segment and of cut and the other related notions they are defined in the case of DN_{IKTh} like in the case of DN_{IK}. The rules of Figure 2 are introduction and elimination rules and we have to define reduction rules in order to eliminate the detours due to these rules. Here we only give the reduction rules for the system DN_{IB4} (the other cases are developed in Annexe A).

We define the relation $\rightarrow_{\mathsf{IB4}}$ as $\Gamma\{\langle\Delta\rangle\}\{\emptyset\} \rightarrow_{\mathsf{IB4}} \Gamma\{\emptyset\}\{\Delta\}$ where $sp(\Gamma\{\emptyset\}\{\Delta\})$ and we denote $\rightarrow^*_{\mathsf{IB4}}$ its reflexive and transitive closure. Let us show now how to build the proof $\mathcal{D}[\mathcal{S}']_{\mathsf{IB4}}$ of $\mathcal{S}'$ from a proof $\mathcal{D}$ of $\mathcal{S}$ where $\mathcal{S} \rightarrow \mathcal{S}'$. The definition of $\mathcal{D}[\mathcal{S}']_{\mathsf{IB4}}$ is extended to the relation $\rightarrow^n_x$, by induction on n, as follows:

– if $n = 0$ then $\mathcal{D}[\mathcal{S}']_{\mathsf{IB4}} = \mathcal{D}$;
– else $\mathcal{D}[\mathcal{S}']_{\mathsf{IB4}} = (\mathcal{D}[\mathcal{S}'']_w)[\mathcal{S}']_{\mathsf{IB4}}$ such that $\mathcal{S} \to_w \mathcal{S}'' \to^{n-1}_{\mathsf{IB4}} \mathcal{S}'$.

Like for relations $\to_w$ and $\to_m$, the construction of $\mathcal{D}[\mathcal{S}']_{\mathsf{IB4}}$ is done by structural induction on $\mathcal{D}$:

If $\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\langle\Delta^1\rangle\}\{\emptyset\}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_l\\ \Gamma\{\Delta^l_1\}\cdots\{\Delta^l_k\}\{\langle\Delta^l\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\Delta^0_1\}\cdots\{\Delta^0_k\}\{\langle\Delta\rangle\}\{\emptyset\}}\,[R] \right.$ then

$$\mathcal{D}[\mathcal{S}']_{\mathsf{IB4}} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1[\Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\emptyset\}\{\Delta^1\}]_{\mathsf{IB4}}\\ \Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\emptyset\}\{\Delta^1\}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_l[\Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\emptyset\}\{\Delta^l\}]_{\mathsf{IB4}}\\ \Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\emptyset\}\{\Delta^l\}\end{array}}{\Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\emptyset\}\{\Delta\}}\,[R] \right.$$

with $\mathcal{S}' = \Gamma\{\Delta^1_1\}\cdots\{\Delta^1_k\}\{\emptyset\}\{\Delta\}$.

Let us introduce the reduction rules used in case of detours due to applications of $[\Box^{\mathsf{IB4}}_E]$ and $[\Diamond^{\mathsf{IB4}}_I]$:

– $\Box^{\mathsf{IB4}}$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\langle A^{\vdash}\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}\,[\Box_I]}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\,[\Box^{\mathsf{IB4}}_E] \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}[\Gamma\{\emptyset\}\{A^{\vdash}\}]_{\mathsf{IB4}}\\ \Gamma\{\emptyset\}\{A^{\vdash}\}\end{array}$$

– $\Diamond^{\mathsf{IB4}}$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{A^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{array}}{\Gamma\{\emptyset\}\{\Diamond A^{\vdash}\}\{\emptyset\}}\,[\Diamond^{\mathsf{IB4}}_I] \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\langle x:A\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}}\,[\Diamond_E] \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}'_2[x/\mathcal{D}_1]\\ \Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}\end{array}$$

with $\mathcal{D}'_2 = \mathcal{D}[\Gamma\{x:A\}\{\emptyset\}\{C^{\vdash}\}]_{\mathsf{IB4}}$

Theorem 9 (Normalization). *Any proof in DN_{IKTh} can be reduced to a proof in normal form.*

Proof. The proof is similar to the one of Theorem 6, by induction on the value of (n,m) where n is the rank of the proof and m is the sum of lengths of all critical cuts.

We now study the structure of proofs in normal form. It leads to prove the subformula property, namely all formulas in a normal proof are subformulas of the root of this proof

First we define the notion of *path* that is a particular sequence of T-sequents belonging to a proof. The idea is that any T-sequent in a proof belongs to at least one path and we show that any path in a proof in a normal form can be decomposed in three particular parts. Such a decomposition allows to prove that all formulas of a path in a proof in normal form are subformulas of the formulas of the root and then to prove the subformula property. Then we prove some interesting properties of the system.

Definition 15 (Path). *A* path *in a proof $\mathcal{D}$ in DN_{IK} is a sequence of occurrences of T-sequents $\mathcal{S}_0,\ldots,\mathcal{S}_n$ such that:*

– *$\mathcal{S}_0$ is the label of a leaf of $\mathcal{D}$ that is not discharged by an application of $[\vee_E]$ or $[\Diamond_E]$;*
– *$\mathcal{S}_i$ for $i < n$ is not a minor premisse of an instance of $[\supset_E]$ and*
  (i) $\mathcal{S}_i$ is not a major premisse of an instance of $[\vee_E]$ or $[\Diamond_E]$ and $\mathcal{S}_{i+1}$ is the T-sequent that is directly below $\mathcal{S}_i$, or
  (ii) $\mathcal{S}_i$ is a major premisse of an instance of $[\vee_E]$ or $[\Diamond_E]$ and $\mathcal{S}_{i+1}$ is an occurrence of a T-sequent discharged by this instance;
– *$\mathcal{S}_n$ is a minor premisse of $[\supset_E]$, the root of $\mathcal{D}$, or a major premisse of an application of $[\vee_E]$ or $[\Diamond_E]$ that does not discharge a T-sequent.*

Proposition 13. *Any T-sequent of a proof $\mathcal{D}$ in DN_{IK} belongs to a path of $\mathcal{D}$.*

Proof. By structural induction on $\mathcal{D}$.
If $\mathcal{D}$ is an axiom (an instance of $[Id]$), then the T-sequent belongs to the path that only contains it.
Let us consider the cases of the last rule applied in $\mathcal{D}$:

- If the rule is different of $[\supset_E]$, $[\vee_E]$ and $[\Diamond_E]$, then its conclusion added to any path ending with one of its premisses (existence of such a path due to the induction hypothesis) is a path.
- If the rule is $[\supset_E]$, then its conclusion added to a path ending by its major premisse (induction hypothesis) is a path.
- If the rule is $[\vee_E]$:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{A_1\vee A_2^{\vdash}\}\{\emptyset\}\end{array}\qquad\begin{array}{c}\mathcal{D}_2\\ \Gamma\{x:A_1\}\{C^{\vdash}\}\end{array}\qquad\begin{array}{c}\mathcal{D}_3\\ \Gamma\{x:A_2\}\{C^{\vdash}\}\end{array}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\,[\vee_E]$$

  By induction hypothesis there exists a path π_2 in $\mathcal{D}_2$ ending with $\Gamma\{x:A_1\}\{C^{\vdash}\}$. If this path does not begin with an occurrence of a T-sequent discharged by the application of this rule then $\pi_2,\Gamma\{\emptyset\}\{C^{\vdash}\}$ is a path in $\mathcal{D}$. Otherwise by induction hypothesis there exists a path π_1 in $\mathcal{D}_1$ ending with $\Gamma\{A_1\vee A_2^{\vdash}\}\{\emptyset\}$. In this case $\pi_1,\pi_2,\Gamma\{\emptyset\}\{C^{\vdash}\}$ is a path in $\mathcal{D}$.
- The case of rule $[\Diamond_E]$ is similar to the one of $[\vee_E]$.

In the next proposition we describe some characteristics of paths in a proof in normal form.

Proposition 14. *Let $\mathcal{D}$ be a proof in normal form in DN_{IK} and $\pi=\sigma_0,\ldots,\sigma_n$ be a path in $\mathcal{D}$. There exists a segment σ_i in π, called the minimal segment, splitting π in two parts, called E-part and I-part, verifying the following properties:*

- *for each σ_j in the E-part ($j<i$), σ_j is the major premisse of an elimination rule (σ_{j+1} is a subformula of σ_j);*
- *for each σ_j in the I-part ($i<j$), if $j\neq n$ then σ_j is the premisse of an introduction rule (σ_j is a subformula of σ_{j+1});*
- *If $i\neq n$ then σ_i is a premisse of an introduction rule or a premisse of $[\bot]$ (σ_i is a subformula of σ_0).*

Proof. Let σ_i be the first segment that is not a premisse of an application of an elimination rule. If $i=n$ then we can see that the proposition is true. Otherwise σ_i is an application of an introduction rule or the premisse of an application of $[\bot]$. If σ_i is a premisse of $[\bot]$, then either $i+1=n$ or σ_i is a premisse of an application of an introduction rule ($\bot$-reduction). Moreover we know that π does not contain a segment that is the conclusion of an application of an introduction rule and the premisse of an application of an elimination rule or of $[\bot]$ (reduction and permutation rules). Thus for all $i<j<n$, σ_j is a premisse of an application of an introduction rule. The case where σ_i is a premisse of an application of an introduction rule is proved in a similar way.

Now we define the notion of *order of a path* that will be used to make a proof by induction on the paths of a normal proof.

Definition 16. *A* path *in a proof in normal form $\mathcal{D}$ is* of order 0 *if it satisfies the two following properties:*

- *it ends with the conclusion of $\mathcal{D}$;*
- *it begins with a T-sequent that is not discharged by any rule and ends with the major premisse of an application of $[\Diamond_E]$.*

A path *in a proof in normal form $\mathcal{D}$ is* of order $n+1$ *if it satisfies one of the following properties:*

- *it ends with the minor premisse of an application of $[\supset_E]$, with a major premisse in a path of order n;*
- *it ends with the major premisse of an application of $[\Diamond_E]$ and its begins with a T-sequent that is discharged by an application of $[\supset_I]$ belonging to a path of order n.*

We show now that if all conclusions of T-sequents in a proof are subformulas of the formulas of the root then all the formulas of the proof are subformulas are subformulas of the formulas of the root.

Proposition 15. *Let $\mathcal{D}$ be a proof of $\mathcal{S}=\Gamma\{C^{\vdash}\}$ dans DN_{IK}. For any T-sequent $\Gamma'\{C'^{\vdash}\}$ dans $\mathcal{D}$, if A is a formula in $\Gamma'\{\emptyset\}$, then A is in $\Gamma\{\emptyset\}$ or there exists a T-sequent $\Gamma''\{C''^{\vdash}\}$ in $\mathcal{D}$ such that A is a subformula of C''.*

Proof. By structural induction on $\mathcal{D}$.

Theorem 10 (Subformula property). *Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ be a proof in normal form of $\mathcal{S}$. Any formula of $\mathcal{D}$ is a subformula of a formula of $\mathcal{S}$.*

Proof. We show by induction on n that for all paths of order n the conclusions of its segments are subformulas of formulas of $\mathcal{S}$. By Proposition 15 we can extend this property to all formulas of $\mathcal{D}$.
Let $\pi = \sigma_0, \dots, \sigma_k$ a path of order n in $\mathcal{D}$ and σ_i the minimal segment of π.
- Case $n = 0$.
We know that σ_k ends either with $\mathcal{S}$ or with the major premisse of an application of $[\Diamond_E]$.
- If σ_k ends with $\mathcal{S}$ then $C(\sigma_k)$ is a subformula of a formula of $\mathcal{S}$. By Proposition 14, for all $j \in [i+1,k]$, $C(\sigma_j)$ is a subformula of a formula $\mathcal{S}$. Moreover we know that σ_0 begins with either a T-sequent that is not discharged or a T-sequent discharged by an application of $[\supset_I]$ of the I-part of π. Then $C(\sigma_0)$ is a subformula of a formula of $\mathcal{S}$. By Proposition 14 we deduce that for all $j \in [0,i]$, $C(\sigma_j)$ is a subformula of a formula of $\mathcal{S}$.
- If σ_k ends with the major premisse of an application of $[\Diamond_E]$. Then for all $j \in [0,k]$, σ_j is the main premisse of an elimination rule (Proposition 14). We know that σ_0 begins with a T-sequent that is not discharged. Thus $C(\sigma_0)$ is a subformula of a formula of $\mathcal{S}$. By Proposition 14 we deduce that for all $j \in [0,k]$, $C(\sigma_j)$ is a subformula of a formula of $\mathcal{S}$.
- Case $n = m+1$ ($m \geqslant 0$).
We know that σ_k ends with either the minor premisses of an application of $[\supset_E]$, or the major premisse of an application of $[\Diamond_E]$.
- If σ_k ends with the minor premisse of an application of $[\supset_E]$ then by induction hypothesis and Proposition 14, for all $j \in [i+1,k]$, $C(\sigma_j)$ is a subformula of a formula of $\mathcal{S}$. Moreover σ_0 begins with either a T-sequent that is not discharged by an application of $[\supset_I]$ of the I-part of π or a path of order less or equal than m. We deduce that, for all $j \in [0,i]$, $C(\sigma_j)$ is a subformula of a formula of $\mathcal{S}$.
- If σ_k is the major premisse of an application of $[\Diamond_E]$ then, for all $j \in [0,k]$, σ_j is the main premisse of an elimination rule (Proposition 14). We know that σ_0 begins with a T-sequent that is discharged by an application of $[\supset_I]$ that is in a path of order m. By induction hypothesis and Proposition 14, we deduce that for all $j \in [0,k]$, $C(\sigma_j)$ is a subformula of a formula of $\mathcal{S}$.

A direct consequence of the subformula property is the property of separation.

Proposition 16 (Separation Property). *A proof of a T-sequent only uses rules associated to the operators of the T-sequent.*

In the case of DN_{IKTh} for $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T,B\}\}$ another important property that can be deduced from the subformula property concerns the depth of the T-sequents in a proof in a normal form.

Proposition 17 (Depth Property). *Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ a proof in normal form of $\mathcal{S}$ in $DN_{\mathsf{IK}} + \mathsf{Th}$ for $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T,B\}\}$. If $\mathcal{S}'$ is a T-sequent in $\mathcal{D}$ then its depth is less or equal to depth($\mathcal{S}$)+ $nest(\mathcal{S})$.*

Proof. By structural induction on $\mathcal{D}$.
The only rules having premisses with a depth greater than the depth of conclusion are $[\Box_I]$ and $[\Diamond_E]$. If the depth of $\mathcal{S}'$ is greater than the depth of $\mathcal{S}$ plus $nest(\mathcal{S})$ then $\mathcal{D}$ contains a formula with $nest(A) > nest(\mathcal{S})$. It is contradictory because all the formulas in $\mathcal{D}$ are subformulas of formulas of $\mathcal{S}$.

This property is not satisfied in deduction systems like DN_{IS5} because of the elimination rule of $\Box$ and the introduction rule of $\Diamond$ for which there is no value bounding the differences of depths between the position of the formula marked in the premisse and its position in the conclusion. Let us illustrate this point with the following proof of the T-sequent $\mathcal{S} = \Box\Diamond A, \Diamond A^{\vdash}$ dans DN_{IS5}:

$$\dfrac{\mathcal{D}_3 \qquad \dfrac{\mathcal{D}_2 \qquad \dfrac{\mathcal{D}_1 \qquad \dfrac{\dfrac{}{\Box\Diamond A, \langle A, \langle A, \langle A\rangle\rangle\rangle, \Box\Diamond A^{\vdash}}\,[id]}{\Box\Diamond A, \langle A, \langle A, \langle A\rangle\rangle\rangle, \Diamond A^{\vdash}}\,[\Box_E^{\mathsf{IS5}}]}{\Box\Diamond A, \langle A, \langle A\rangle\rangle, \Diamond A^{\vdash}}\,[\Diamond_E]}{\Box\Diamond A, \langle A\rangle, \Diamond A^{\vdash}}\,[\Diamond_E]}{\Box\Diamond A, \Diamond A^{\vdash}}\,[\Diamond_E]$$

where

$$\mathcal{D}_1 = \left\{ \dfrac{\overline{\Box\Diamond A, \langle A, \langle A\rangle\rangle, \Box\Diamond A^{\vdash}}\ [id]}{\Box\Diamond A, \langle A, \langle A, \Diamond A^{\vdash}\rangle\rangle}\ [\Box_E^{\mathsf{IS5}}] \right. \quad \mathcal{D}_2 = \left\{ \dfrac{\overline{\Box\Diamond A, \langle A\rangle, \Box\Diamond A^{\vdash}}\ [id]}{\Box\Diamond A, \langle A, \Diamond A^{\vdash}\rangle}\ [\Box_E^{\mathsf{IS5}}] \right. \quad \mathcal{D}_3 = \left\{ \dfrac{\overline{\Box\Diamond A, \Box\Diamond A^{\vdash}}\ [id]}{\Box\Diamond A, \Diamond A^{\vdash}}\ [\Box_E^{\mathsf{IS5}}] \right.$$

This proof is in normal form. We have $nest(\mathcal{S}) = 2$ and the depth of the T-sequents $\Box\Diamond A, \langle A, \langle A, \langle A\rangle\rangle\rangle, \Box\Diamond A^{\vdash}$ is equal to 3. It is a counter-example of the depth property.

7 T-sequents and Classical Modal Logics

In this section we propose natural deduction systems for all classical modal logics obtained by combinations of the axioms T, B, 4 and 5. For each logic the system is obtained by the replacement of the rule $[\bot]$ by a new rule in the corresponding intuitionistic system. Then for all $\mathsf{Th} \subseteq \{T, B, 4, 5\}$ we define the natural deduction DN_{KTh} as the system obtained by DN_{IKTh} from the replacement of the rule $[\bot]$ by the following rule:

$$\dfrac{\Gamma\{\neg A\}\{\bot^{\vdash}\}}{\Gamma\{A^{\vdash}\}\{\emptyset\}}\ [\bot_c]$$

We see that this rule is a generalization of the rule $[\bot]$: if $\Gamma\{\emptyset\}\{\bot^{\vdash}\}$ has a proof in DN_{KTh}, then $\Gamma\{A^{\vdash}\}\{\emptyset\}$ has also a proof. By adding $\neg A$ to all the T-sequents of a proof of $\Gamma\{\emptyset\}\{\bot^{\vdash}\}$ dans DN_{KTh} we obtain a proof of $\Gamma\{\neg A\}\{\bot^{\vdash}\}$. Then we apply the rule $[\bot_c]$ in order to have a proof of $\Gamma\{A^{\vdash}\}\{\emptyset\}$.

Theorem 11 (Soundness). *If a T-sequent has a proof in DN_{KTh} then it is valid in* KTh.

Proof. We know that any rule that is sound in IKTh is sound in KTh. Thus all common rules to DN_{KTh} and DN_{IKTh} are sound in KTh. Let us prove now the soundness of $[\bot_c]$ in KTh.
As the rule $[\bot]$ is sound in DN_{IKTh} it is also sound in KTh. Let $\mathcal{D}$ be a proof of $\Gamma\{\neg A\}\{\bot^{\vdash}\}$. By using the rules $[\bot]$ and $[\supset_I]$, we obtain from $\mathcal{D}$ a proof of $\Gamma\{\neg\neg A^{\vdash}\}\{\emptyset\}$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\neg A\}\{\bot^{\vdash}\}\end{array}}{\Gamma\{\neg A, \bot^{\vdash}\}\{\emptyset\}}\ [\bot]}{\Gamma\{\neg\neg A^{\vdash}\}\{\emptyset\}}\ [\supset_I]$$

$\neg\neg A$ being equivalent to A in KTh, we deduce that the T-sequent $\Gamma\{A^{\vdash}\}\{\emptyset\}$ is valid in KTh.

Theorem 12 (Completeness). *If a T-sequent is valid in* KTh*, then it has a proof in DN_{KTh}.*

Proof. In order to show the completeness of DN_{KTh} we observe that adding the axiom $\neg\neg A \supset A$ to IKTh gives KTh. As the rule $[\bot_c]$ is a generalization of $[\bot]$ it is sufficient to prove that $\neg\neg A \supset A$ has a proof in DN_{KTh}:

$$\dfrac{\dfrac{\dfrac{\overline{\neg\neg A, \neg A, \neg\neg A^{\vdash}}\ [id] \quad \overline{\neg\neg A, \neg A, \neg A^{\vdash}}\ [id]}{\neg\neg A, \neg A, \bot^{\vdash}}\ [\supset_E]}{\neg\neg A, A^{\vdash}}\ [\bot_c]}{\neg\neg A \supset A^{\vdash}}\ [\supset_I]$$

About the normalization the main problems were related to the two rules $[\vee_E]$ and $[\Diamond_E]$. In the classical case they can be solved with the De Morgan laws. As the operators $\vee$ and $\Diamond$ can be expressed with the operators $\wedge$, $\supset$ and $\Box$ we can only consider deduction systems with rules associated to these ones. A key point in the proof of normalization in the classical case is the restriction of $[\bot_c]$ to atomic formulas:

$$\dfrac{\Gamma\{\neg p\}\{\bot^{\vdash}\}}{\Gamma\{p^{\vdash}\}\{\emptyset\}}\ [\bot_c'](p \texttt{ is atomic})$$

For all $\mathsf{Th} \subseteq \{T,B,4,5\}$ we denote DN'_{KTh} the deduction system composed by the rule $[\bot_c']$ and the rules associated to the operators $\wedge$, $\supset$ and $\Box$ in DN_{KTh}. To show the completeness of DN'_{KTh} it is sufficient to show the admissibility of the rule $[\bot_c]$. For that we need to prove the admissibility of the cut rule:

Proposition 18. *The following rule is admissible in DN'_{KTh}:*

$$\dfrac{\Gamma\{A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [Cut]$$

Proof. Let $\mathcal{D}_1$ be a proof of $\Gamma\{A^{\vdash}\}\{\emptyset\}$ and $\mathcal{D}_2$ be a proof of $\Gamma\{x : A\}\{C^{\vdash}\}$. A proof of $\Gamma\{\emptyset\}\{C^{\vdash}\}$ is given by $\mathcal{D}_2[x/\mathcal{D}_1]$ where $\mathcal{D}_2[x/\mathcal{D}_1]$ is defined like in the case of DN_{IKTh}.

Theorem 13. *If $\Gamma\{\neg A\}\{\bot^{\vdash}\}$ has a proof in DN'_{KTh} then $\Gamma\{A^{\vdash}\}\{\emptyset\}$ has a proof in DN'_{KTh}.*

Proof. By structural induction on A.
If A is an atomic formula then a proof of $\Gamma\{A^{\vdash}\}\{\emptyset\}$ is obtained by the application of $[\bot_c']$ to $\Gamma\{\neg A\}\{\bot^{\vdash}\}$.
Let us consider the other cases:

– **Case** $A = B \wedge C$. By using weakening the two T-sequents $\Gamma\{\neg(B\wedge C),\neg B\}\{\bot^{\vdash}\}$ and $\Gamma\{\neg(B\wedge C),\neg C\}\{\bot^{\vdash}\}$ have proofs in DN'_{KTh}. The proof of $\Gamma\{B^{\vdash}\}\{\emptyset\}$ is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\{B\wedge C,\neg B,\neg B^{\vdash}\}\{\emptyset\}}[id] \quad \dfrac{\dfrac{}{\Gamma\{B\wedge C,\neg B,B\wedge C^{\vdash}\}\{\emptyset\}}[id]}{\Gamma\{B\wedge C,\neg B,B^{\vdash}\}\{\emptyset\}}[\wedge_E^1]}{\Gamma\{B\wedge C,\neg B,\bot^{\vdash}\}\{\emptyset\}}[\supset_E]}{\Gamma\{\neg B,\neg(B\wedge C)^{\vdash}\}\{\emptyset\}}[\supset_I] \qquad \Gamma\{\neg(B\wedge C),\neg B\}\{\bot^{\vdash}\}}{\Gamma\{\neg B\}\{\bot^{\vdash}\}}[Cut]}{\Gamma\{B^{\vdash}\}\{\emptyset\}}[H.I]$$

where $[H.I]$ corresponds to the application of the induction hypothesis. A proof of $\Gamma\{C^{\vdash}\}\{\emptyset\}$ is obtained in a similar way. Then by application of the rule $[\wedge_I]$ to the premisses $\Gamma\{B^{\vdash}\}\{\emptyset\}$ and $\Gamma\{C^{\vdash}\}\{\emptyset\}$ we obtain a proof of $\Gamma\{B\wedge C^{\vdash}\}\{\emptyset\}$.

– **Case** $A = B \supset C$. By using weakening the T-sequent $\Gamma\{\neg(B\supset C),B,\neg C\}\{\bot^{\vdash}\}$ has a proof in DN'_{KTh}. A proof of $\Gamma\{B\supset C^{\vdash}\}\{\emptyset\}$ is obtained as follows:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{}{\Gamma\{B\supset C,B,\neg C,B\supset C^{\vdash}\}\{\emptyset\}}[id] \quad \dfrac{}{\Gamma\{B\supset C,B,\neg C,B^{\vdash}\}\{\emptyset\}}[id]}{\Gamma\{B\supset C,B,\neg C,C^{\vdash}\}\{\emptyset\}}[\supset_E] \right.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\{B\supset C,B,\neg C,\neg C^{\vdash}\}\{\emptyset\}}[id] \quad \mathcal{D}}{\Gamma\{B\supset C,B,\neg C,\bot^{\vdash}\}\{\emptyset\}}[\supset_E]}{\Gamma\{B,\neg C,\neg(B\supset C)^{\vdash}\}\{\emptyset\}}[\supset_I] \qquad \Gamma\{\neg(B\supset C),B,\neg C\}\{\bot^{\vdash}\}}{\dfrac{\Gamma\{B\supset C,B,\neg C,\bot^{\vdash}\}\{\emptyset\}}{\Gamma\{B\supset C^{\vdash}\}\{\emptyset\}}[H.I]}[Cut]$$

– **Case** $A = \Box B$. By using weakening the T-sequent $\Gamma\{\neg\Box B, \langle\neg B\rangle\}\{\bot^{\vdash}\}$ has a proof in DN'_{KTh}. A proof of $\Gamma\{\Box B^{\vdash}\}\{\emptyset\}$ is obtained as follows:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\{\Box B, \langle\neg B, \neg B^{\vdash}\rangle, \neg\bot\}\{\emptyset\}}[id] \quad \dfrac{\dfrac{}{\Gamma\{\Box B, \langle\neg B\rangle, \Box B^{\vdash}, \neg\bot\}\{\emptyset\}}[id]}{\Gamma\{\Box B, \langle\neg B, B^{\vdash}\rangle, \neg\bot\}\{\emptyset\}}[\Box_E^*]}{\Gamma\{\Box B, \langle\neg B, \bot^{\vdash}\rangle, \neg\bot\}\{\emptyset\}}[\supset_E]}{\Gamma\{\Box B, \langle\neg B\rangle, \bot^{\vdash}\}\{\emptyset\}}[\bot'_c]}{\Gamma\{\langle\neg B\rangle, \Box B^{\vdash}\}\{\emptyset\}}[\supset_I] \right.$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D} \qquad \Gamma\{\neg\Box B, \langle\neg B\rangle\}\{\bot^{\vdash}\}}{\Gamma\{\langle\neg B\rangle\}\{\bot^{\vdash}\}}[Cut]}{\Gamma\{\langle B^{\vdash}\rangle\}\{\emptyset\}}[H.I]}{\Gamma\{\Box B^{\vdash}\}\{\emptyset\}}[\Box_I]$$

$[\Box_E^*]$ corresponds to the application of $[\Box_E]$, $[\Box_E^{\mathsf{IB4}}]$ or $[\Box_E^{\mathsf{IS5}}]$ depending on the one in DN'_{KTh}.

Theorem 14 (Normalization). *Any proof of DN'_{KTh} can be reduced into a proof in normal form.*

Proof. Let $\mathcal{D}$ be a proof in DN'_{KTh}. As DN'_{KTh} does not contain the rules $[\vee_E]$ and $[\Diamond_E]$, all cuts in $\mathcal{D}$ are of length 1. The restriction on application of rule $[\bot'_c]$ allows to deduce that all these cuts are detours. Then the proof of normalization is similar to the one of Theorem 6 by induction on the value of the pair (n,m) where $n = cr(\mathcal{D})$ and m is the sum of lengths of all critical cuts. In the case of DN'_{KTh}, the sum of lengths of all critical cuts corresponds to their number. The used rules are only the reduction rules associated to the operators $\wedge$, $\supset$ and $\Box$.

Let us note that the normalization does not allow here to obtain the subformula property. But we have the following property: *if $\mathcal{D}$ is a proof in normal form of a T-sequent $\mathcal{S}$ than any formula in $\mathcal{D}$ is a subformula of a formula of $\mathcal{S}$, $\bot$ or a formula of the form $\neg A$ such that A is a subformula of a formula of $\mathcal{S}$.*

8 Conclusions and Perspectives

In this paper we have defined new natural deduction systems for the intuitionistic and classical modal logics based on the combinations of axioms T, B, 4 and 5. They satisfy the normalization property but also the subformula property in the intuitionistic case. Compared to existing works on natural deduction in the intuitionistic modal logics we provide new label-free systems that are uniform and have important properties w.r.t. proof theory, i.e., normalization and subformula properties. The central notion, on which these results are based, is a multi-contextual structure, called T-sequent, that is appropriate to deal with such logics in both intuitionistic and classical cases.
A similar work can be done in the framework of sequent calculus and will provide uniform label-free sequent calculi for intuitionistic modal logics with the cut-elimination property. Further work will be also dedicated to the design of term calculi associated to these logics, the study of their properties and their impact on applications involving deductions in these logics.

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A Normalization rules

First we introduce the following relations on the T-sequents:

1. $\to_T$ defined by $\Gamma\{\langle\Delta\rangle\} \to_T \Gamma\{\Delta\}$;
2. $\to_B$ defined by $\Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle\} \to_B \Gamma\{\langle\Delta\rangle,\Delta'\}$;
3. $\to_4$ defined by $\Gamma\{\Gamma'\{\emptyset\},\langle\Delta\rangle\} \to_4 \Gamma\{\Gamma'\{\Delta\}\}$with $depth(\Gamma'\{\}) > 1$;
4. $\to_5$ defined by $\Gamma\{\langle\Delta\rangle\}\{\emptyset\} \to_5 \Gamma\{\emptyset\}\{\Delta\}$ with $depth(\Gamma\{\}\{\emptyset\}) \geqslant 1$ and $depth(\Gamma\{\emptyset\}\{\}) \geqslant 1$;
5. $\to_{\mathsf{IS5}}$ defined by $\Gamma\{\langle\Delta\rangle\}\{\emptyset\} \to_{\mathsf{IS5}} \Gamma\{\emptyset\}\{\Delta\}$.

Now we show that for any $x \in \{T,4,B,5,\mathsf{IS5}\}$, if $\mathcal{S} \to_x \mathcal{S}'$ and $\mathcal{S}$ has a proof in a system $DN_{\mathsf{IK+Th}}$ including $[\Box_E^x]$ and $[\Diamond_I^x]$, then $\mathcal{S}'$ has also a proof in $DN_{\mathsf{IK+Th}}$. For that we show how to rewrite a proof $\mathcal{D}$ of $\mathcal{S}$ into a proof of $\mathcal{S}'$, denoted$\mathcal{D}[\mathcal{S}']_x$.
We built $\mathcal{D}[\mathcal{S}']_x$ by structural induction on $\mathcal{D}$ as follows:

Construction of $\mathcal{D}[\mathcal{S}']_T$:

- **Case 1:**
If $\mathcal{D} = \left\{ \begin{array}{c} \mathcal{D}' \\ \dfrac{\Gamma\{\langle\Delta,A^\vdash\rangle\}}{\Gamma\{\langle\Delta\rangle,\Diamond A^\vdash\}}\,[\Diamond_I] \end{array} \right.$ then $\mathcal{D}[\mathcal{S}']_T = \left\{ \begin{array}{c} \mathcal{D}'[\Gamma\{\Delta,A^\vdash\}]_T \\ \dfrac{\Gamma\{\Delta,A^\vdash\}}{\Gamma\{\Delta,\Diamond A^\vdash\}}\,[\Diamond_I^T] \end{array} \right.$
with $\mathcal{S}' = \Gamma\{\Delta,\Diamond A^\vdash\}$.
- **Case 2:**
If $\mathcal{D} = \left\{ \begin{array}{c} \mathcal{D}' \\ \dfrac{\Gamma\{\langle\Delta\rangle,\Box A^\vdash\}}{\Gamma\{\langle\Delta,A^\vdash\rangle\}}\,[\Box_E] \end{array} \right.$ then $\mathcal{D}[\mathcal{S}']_T = \left\{ \begin{array}{c} \mathcal{D}'[\Gamma\{\Delta,\Box A^\vdash\}]_T \\ \dfrac{\Gamma\{\Delta,\Box A^\vdash\}}{\Gamma\{\Delta,A^\vdash\}}\,[\Box_E^T] \end{array} \right.$
with $\mathcal{S}' = \Gamma\{\Delta,A^\vdash\}$.
- **Case 3:**
If $\mathcal{D} = \left\{ \begin{array}{c} \mathcal{D}' \\ \dfrac{\Gamma\{\langle\Delta\rangle,A^\vdash\}}{\Gamma\{\langle\Delta,\Diamond A^\vdash\rangle\}}\,[\Diamond_I^B] \end{array} \right.$ then $\mathcal{D}[\mathcal{S}']_T = \left\{ \begin{array}{c} \mathcal{D}'[\Gamma\{\Delta,A^\vdash\}]_T \\ \dfrac{\Gamma\{\Delta,A^\vdash\}}{\Gamma\{\Delta,\Diamond A^\vdash\}}\,[\Diamond_I^T] \end{array} \right.$
with $\mathcal{S}' = \Gamma\{\Delta,\Diamond A^\vdash\}$.
- **Case 4:**
If $\mathcal{D} = \left\{ \begin{array}{c} \mathcal{D}' \\ \dfrac{\Gamma\{\langle\Delta,\Box A^\vdash\rangle\}}{\Gamma\{\langle\Delta\rangle,A^\vdash\}}\,[\Box_E^B] \end{array} \right.$ then $\mathcal{D}[\mathcal{S}']_T = \left\{ \begin{array}{c} \mathcal{D}'[\Gamma\{\Delta,\Box A^\vdash\}]_T \\ \dfrac{\Gamma\{\Delta,\Box A^\vdash\}}{\Gamma\{\Delta,A^\vdash\}}\,[\Box_E^T] \end{array} \right.$
with $\mathcal{S}' = \Gamma\{\Delta,A^\vdash\}$.
- **Case 5:**
If $\mathcal{D} = \left\{ \begin{array}{c} \mathcal{D}' \\ \dfrac{\Gamma\{\langle\Delta_1,\langle\Delta_2,A^\vdash\rangle\rangle\}}{\Gamma\{\langle\Delta_1,\langle\Delta_2\rangle\rangle,\Diamond A^\vdash\}}\,[\Diamond_I^4] \end{array} \right.$ then $\mathcal{D}[\mathcal{S}']_T = \left\{ \begin{array}{c} \mathcal{D}'[\mathcal{S}'']_T \\ \dfrac{\mathcal{S}''}{\mathcal{S}'}\,[\Diamond_I] \end{array} \right.$
where $\mathcal{S}'$ is equal to $\Gamma\{\langle\Delta_1,\Delta_2\rangle,\Diamond A^\vdash\}$ or $\Gamma\{\Delta_1,\langle\Delta_2\rangle,\Diamond A^\vdash\}$ with if $\mathcal{S}' = \Gamma\{\langle\Delta_1,\Delta_2\rangle,\Diamond A^\vdash\}$ then $\mathcal{S}'' = \Gamma\{\langle\Delta_1,\Delta_2,A^\vdash\rangle\}$ else $\mathcal{S}'' = \Gamma\{\Delta_1,\langle\Delta_2,A^\vdash\rangle\}$.
- **Case 6:**
If $\mathcal{D} = \left\{ \begin{array}{c} \mathcal{D}' \\ \dfrac{\Gamma\{\langle\Delta_1,\langle\Delta_2\rangle\rangle,\Box A^\vdash\}}{\Gamma\{\langle\Delta_1,\langle\Delta_2,A^\vdash\rangle\rangle\}}\,[\Box_E^4] \end{array} \right.$ then $\mathcal{D}[\mathcal{S}']_T = \left\{ \begin{array}{c} \mathcal{D}'[\mathcal{S}'']_T \\ \dfrac{\mathcal{S}''}{\mathcal{S}'}\,[\Box_E] \end{array} \right.$
where $\mathcal{S}'$ is equal to $\Gamma\{\langle\Delta_1,\Delta_2,A^\vdash\rangle\}$ or $\Gamma\{\Delta_1,\langle\Delta_2,A^\vdash\rangle\}$ with if $\mathcal{S}' = \Gamma\{\langle\Delta_1,\Delta_2,A^\vdash\rangle\}$ then $\mathcal{S}'' = \Gamma\{\langle\Delta_1,\Delta_2\rangle,\Box A^\vdash\}$ else $\mathcal{S}'' = \Gamma\{\Delta_1,\langle\Delta_2\rangle,\Box A^\vdash\}$.
- **Case 7:**
This case captures all cases not previously considered.

$$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma^1\rangle\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma^l\rangle\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma\rangle\}}\,[R] \right. \text{ then}$$

$$\mathcal{D}[\mathcal{S}']_T = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\Sigma^1\}]_T \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\Sigma^1\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l[\Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\Sigma^l\}]_T \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\Sigma^l\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Sigma\}}\,[R] \right.$$

with $\mathcal{S}' = \Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Sigma\}$.

Construction of $\mathcal{D}[\mathcal{S}']_B$:

- **Case 1:**
  $$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma\{\langle\Delta,\langle\Delta',A^{\vdash}\rangle\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,\Diamond A,\langle\Delta'\rangle\rangle\}}\,[\Diamond_I] \right. \text{ then } \mathcal{D}[\mathcal{S}']_B = \left\{ \dfrac{\begin{matrix}\mathcal{D}'[\Gamma\{\langle\Delta\rangle,\Delta',A^{\vdash}\}]_B \\ \Gamma\{\langle\Delta\rangle,\Delta',A^{\vdash}\}\end{matrix}}{\Gamma\{\langle\Delta,\Diamond A^{\vdash}\rangle,\Delta'\}}\,[\Diamond_I^B] \right.$$
  where $\mathcal{S}' = \Gamma\{\langle\Delta,\Diamond A^{\vdash}\rangle,\Delta'\}$.
- **Case 2:**
  $$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma\{\langle\Delta,\Box A^{\vdash},\langle\Delta'\rangle\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,\langle\Delta',A^{\vdash}\rangle\rangle\}}\,[\Box_E] \right. \text{ then } \mathcal{D}[\mathcal{S}']_B = \left\{ \dfrac{\begin{matrix}\mathcal{D}'[\Gamma\{\langle\Delta,\Box A^{\vdash}\rangle,\Delta'\}]_B \\ \Gamma\{\langle\Delta,\Box A^{\vdash}\rangle,\Delta'\}\end{matrix}}{\Gamma\{\langle\Delta\rangle,\Delta',A^{\vdash}\}}\,[\Box_E^B] \right.$$
  where $\mathcal{S}' = \Gamma\{\langle\Delta\rangle,\Delta',A^{\vdash}\}$.
- **Case 3:**
  $$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma\{\langle\Delta,A^{\vdash},\langle\Delta'\rangle\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,\langle\Delta',\Diamond A^{\vdash}\rangle\rangle\}}\,[\Diamond_I^B] \right. \text{ then } \mathcal{D}[\mathcal{S}']_B = \left\{ \dfrac{\begin{matrix}\mathcal{D}'[\Gamma\{\langle\Delta,A^{\vdash}\rangle,\Delta'\}]_B \\ \Gamma\{\langle\Delta,A^{\vdash}\rangle,\Delta'\}\end{matrix}}{\Gamma\{\langle\Delta\rangle,\Delta',\Diamond A^{\vdash}\}}\,[\Diamond_I] \right.$$
  where $\mathcal{S}' = \Gamma\{\langle\Delta\rangle,\Delta',\Diamond A^{\vdash}\}$.
- **Case 4:**
  $$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma\{\langle\Delta,\langle\Delta',\Box A^{\vdash}\rangle\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,A^{\vdash},\langle\Delta'\rangle\rangle\}}\,[\Box_E^B] \right. \text{ then } \mathcal{D}[\mathcal{S}']_B = \left\{ \dfrac{\begin{matrix}\mathcal{D}'[\Gamma\{\langle\Delta\rangle,\Delta',\Box A^{\vdash}\}]_B \\ \Gamma\{\langle\Delta\rangle,\Delta',\Box A^{\vdash}\}\end{matrix}}{\Gamma\{\langle\Delta,A^{\vdash}\rangle,\Delta'\}}\,[\Box_E] \right.$$
  where $\mathcal{S}' = \Gamma\{\langle\Delta,A^{\vdash}\rangle,\Delta'\}$.
- **Case 5:**
  This case captures all the case not previously considered.
  $$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma_1^1,\langle\Sigma_2^1\rangle\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma_1^l,\langle\Sigma_2^l\rangle\rangle\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma_1,\langle\Sigma_2\rangle\rangle\}}\,[R] \right. \text{ then}$$
  $$\mathcal{D}[\mathcal{S}']_B = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma_1^1\rangle,\Sigma_2^1\}]_B \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Sigma_1^1\rangle,\Sigma_2^1\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l[\Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma_1^l\rangle,\Sigma_2^l\}]_B \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Sigma_1^l\rangle,\Sigma_2^l\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma_1\rangle,\Sigma_2\}}\,[R] \right.$$
  with $\mathcal{S}' = \Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Sigma_1\rangle,\Sigma_2\}$.

Construction of $\mathcal{D}[\mathcal{S}']_4$:

- **Case 1:**
  $$\text{If } \mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma\{\Gamma'\{\emptyset\},\langle\Delta,A^{\vdash}\rangle\}\end{matrix}}{\Gamma\{\Gamma'\{\emptyset\},\langle\Delta\rangle,\Diamond A^{\vdash}\}}\,[\Diamond_I] \right. \text{ then } \mathcal{D}[\mathcal{S}']_4 = \left\{ \dfrac{\begin{matrix}\mathcal{D}'[\Gamma\{\Gamma'\{\Delta,A^{\vdash}\}\}]_4 \\ \Gamma\{\Gamma'\{\Delta,A^{\vdash}\}\}\end{matrix}}{\Gamma\{\Gamma'\{\Delta\},\Diamond A^{\vdash}\}}\,[\Diamond_I^4] \right.$$
  with $\mathcal{S}' = \Gamma\{\Gamma'\{\Delta\},\Diamond A^{\vdash}\}$.

– **Case 2:**

If $\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma\{\Gamma'\{\emptyset\}, \langle\Delta\rangle, \Box A^{\vdash}\}\end{array}}{\Gamma\{\Gamma'\{\emptyset\}, \langle\Delta, A^{\vdash}\rangle\}} [\Box_E] \right.$ then $\mathcal{D}[\mathcal{S}']_4 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}'[\Gamma\{\Gamma'\{\Delta\}, \Box A^{\vdash}\}]_4 \\ \Gamma\{\Gamma'\{\Delta\}, \Box A^{\vdash}\}\end{array}}{\Gamma\{\Gamma'\{\Delta, A^{\vdash}\}\}} [\Box_E^4] \right.$

with $\mathcal{S}' = \Gamma\{\Gamma'\{\Delta, A^{\vdash}\}\}$.

– **Case 3:**

This case captures all the case not previously considered.

$$\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\Gamma'\{\Sigma_1^1\}\dots\{\Sigma_m^1\}\{\emptyset\}, \langle\Delta^1\rangle\}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_l \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\Gamma'\{\Sigma_1^l\}\dots\{\Sigma_m^l\}\{\emptyset\}, \langle\Delta^l\rangle\}\end{array}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Gamma'\{\Sigma_1^0\}\dots\{\Sigma_m^0\}\{\emptyset\}, \langle\Delta\rangle\}} [R] \right.$$

$$\mathcal{D}[\mathcal{S}']_4 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1[\mathcal{S}^1]_4 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\Gamma'\{\Sigma_1^1\}\dots\{\Sigma_m^1\}\{\Delta_1\}\}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_l[\mathcal{S}^l]_4 \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\Gamma'\{\Sigma_1^l\}\dots\{\Sigma_m^l\}\{\Delta_l\}\}\end{array}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Gamma'\{\Sigma_1^0\}\dots\{\Sigma_m^0\}\{\Delta\}\}} [R] \right.$$

where $\mathcal{S}' = \Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\Gamma'\{\Sigma_1^0\}\dots\{\Sigma_m^0\}\{\Delta\}\}$ and, for any $i \in [1,l]$, $\mathcal{S}^i = \Gamma\{\Delta_1^i\}\cdots\{\Delta_k^i\}$ $\{\Gamma'\{\Sigma_1^i\}\dots\{\Sigma_m^i\}\{\Delta^i\}\}$.

<u>Construction of $\mathcal{D}[\mathcal{S}]_5$:</u>

– **Case 1:**

If $\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma\{\langle\Delta, A^{\vdash}\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\langle\Delta\rangle, \Diamond A^{\vdash}\}\{\emptyset\}} [\Diamond_I] \right.$ then $\mathcal{D}[\mathcal{S}']_5 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}'[\Gamma\{\emptyset\}\{\Delta, A^{\vdash}\}]_5 \\ \Gamma\{\emptyset\}\{\Delta, A^{\vdash}\}\end{array}}{\Gamma\{\Diamond A^{\vdash}\}\{\Delta\}} [\Diamond_I^5] \right.$

with $\mathcal{S}' = \Gamma\{\Diamond A^{\vdash}\}\{\Delta\}$.

– **Case 2:**

If $\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma\{\langle\Delta\rangle, \Box A^{\vdash}\}\{\emptyset\}\end{array}}{\Gamma\{\langle\Delta, A^{\vdash}\rangle\}\{\emptyset\}} [\Box_E] \right.$ then $\mathcal{D}[\mathcal{S}']_5 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}'[\Gamma\{\Box A^{\vdash}\}\{\Delta\}]_5 \\ \Gamma\{\Box A\}\{\Delta\}\end{array}}{\Gamma\{\emptyset\}\{\Delta, A^{\vdash}\}} [\Box_E^5] \right.$

with $\mathcal{S}' = \Gamma\{\emptyset\}\{\Delta, A^{\vdash}\}$.

– **Case 3:**

If $\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma\{\Gamma'\{\langle\Delta, A^{\vdash}\rangle\}\}\{\emptyset\}\end{array}}{\Gamma\{\Gamma'\{\langle\Delta\rangle\}, \Diamond A^{\vdash}\}\{\emptyset\}} [\Diamond_I^4] \right.$ then $\mathcal{D}[\mathcal{S}']_5 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}'[\Gamma\{\Gamma'\{\emptyset\}\}\{\Delta, A^{\vdash}\}]_5 \\ \Gamma\{\Gamma'\{\emptyset\}\}\{\Delta, A^{\vdash}\}\end{array}}{\Gamma\{\Gamma'\{\emptyset\}, \Diamond A^{\vdash}\}\{\Delta\}} [\Diamond_I^5] \right.$

where $\mathcal{S}' = \Gamma\{\Gamma'\{\emptyset\}, \Diamond A^{\vdash}\}\{\Delta\}$, $depth(\Gamma\{\}\{\emptyset\}) \geqslant 1$ and $depth(\Gamma'\{\}) > 1$.

– **Case 4:**

If $\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma\{\Gamma'\{\langle\Delta\rangle\}, \Box A^{\vdash}\}\{\emptyset\}\end{array}}{\Gamma\{\Gamma'\{\langle\Delta, A^{\vdash}\rangle\}\}\{\emptyset\}} [\Box_E^4] \right.$ then $\mathcal{D}[\mathcal{S}']_5 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}'[\Gamma\{\Gamma'\{\emptyset\}, \Box A^{\vdash}\}\{\Delta\}]_5 \\ \Gamma\{\Gamma'\{\emptyset\}, \Box A^{\vdash}\}\{\Delta\}\end{array}}{\Gamma\{\Gamma'\{\emptyset\}\}\{\Delta, A^{\vdash}\}} [\Box_E^5] \right.$

with $\mathcal{S}' = \Gamma\{\Gamma'\{\emptyset\}\}\{\Delta, A^{\vdash}\}$, $depth(\Gamma\{\}\{\emptyset\}) \geqslant 1$ and $depth(\Gamma'\{\}) > 1$.

– **Case 5:**

This case captures all the cases not previously considered.

$$\mathcal{D} = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Delta^1\rangle\}\{\emptyset\}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_l \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Delta^l\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Delta\rangle\}\{\emptyset\}} [R] \right.$$

$$\mathcal{D}[\mathcal{S}']_5 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^1\}]_5 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^1\}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_l[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^l\}]_5 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^l\}\end{array}}{\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta\}} [R] \right.$$

with $\mathcal{S}' = \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta\}$.

Construction of $\mathcal{D}[\mathcal{S}']_{\mathsf{IS5}}$:

If $\mathcal{D} = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\langle\Delta^1\rangle\}\{\emptyset\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l \\ \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}\{\langle\Delta^l\rangle\}\{\emptyset\}\end{matrix}}{\Gamma\{\Delta_1^0\}\cdots\{\Delta_k^0\}\{\langle\Delta\rangle\}\{\emptyset\}}\,[R] \right.$

then $\mathcal{D}[\mathcal{S}']_{\mathsf{IS5}} = \left\{ \dfrac{\begin{matrix}\mathcal{D}_1[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^1\}]_{\mathsf{IS5}} \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^1\}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_l[\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^l\}]_{\mathsf{IS5}} \\ \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta^l\}\end{matrix}}{\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta\}}\,[R] \right.$

with $\mathcal{S}' = \Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\}\{\emptyset\}\{\Delta\}$.

For any $x \in \{T, B, 4, 5, \mathsf{IS5}\}$, we denote $\rightarrow_x^*$ the reflexive and transitive closure of $\rightarrow_x$. Let $\mathcal{S}$ and $\mathcal{S}'$ be two T-sequents such that $\mathcal{S} \rightarrow_x^n \mathcal{S}'$. We extend the definition of $\mathcal{D}[\mathcal{S}']_x$ to the relation $\rightarrow_x^*$ by induction on n as follows:

- If $n = 0$ alors $\mathcal{D}[\mathcal{S}']_x = \mathcal{D}$;
- else $\mathcal{D}[\mathcal{S}']_x = (\mathcal{D}[\mathcal{S}'']_x)[\mathcal{S}']_x$ such that $\mathcal{S} \rightarrow_w \mathcal{S}'' \rightarrow_w^{n-1} \mathcal{S}'$.

Reduction rules:

- $\Box^T$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \Gamma\{\langle A^\vdash\rangle\}\end{matrix}}{\Gamma\{\Box A^\vdash\}}\,[\Box_I]}{\Gamma\{A^\vdash\}}\,[\Box_E^T] \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}[\Gamma\{A^\vdash\}]_T \\ \Gamma\{A^\vdash\}\end{matrix}$$

- $\Diamond^T$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma\{A^\vdash\}\{\emptyset\}\end{matrix}}{\Gamma\{\Diamond A^\vdash\}\{\emptyset\}}\,[\Diamond_I^T] \quad \begin{matrix}\mathcal{D}_2 \\ \Gamma\{\langle x:A\rangle\}\{C^\vdash\}\end{matrix}}{\Gamma\{\emptyset\}\{C^\vdash\}}\,[\Diamond_E] \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}_2'[x/\mathcal{D}_1] \\ \Gamma\{\emptyset\}\{C^\vdash\}\end{matrix}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{x:A\}\{C^\vdash\}]_T$

- $\Box^B$-reduction :

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \Gamma\{\langle\Delta,\langle A^\vdash\rangle\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,\Box A^\vdash\rangle\}}\,[\Box_I]}{\Gamma\{\langle\Delta\rangle, A^\vdash\}}\,[\Box_E^B] \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}[\Gamma\{\langle\Delta\rangle, A^\vdash\}]_B \\ \Gamma\{A^\vdash\}\end{matrix}$$

- $\Diamond^B$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma\{\langle\Delta\rangle, A^\vdash\}\{\emptyset\}\end{matrix}}{\Gamma\{\langle\Delta,\Diamond A^\vdash\rangle\}\{\emptyset\}}\,[\Diamond_I^B] \quad \begin{matrix}\mathcal{D}_2 \\ \Gamma\{\langle\Delta,\langle x:A\rangle\rangle\}\{C^\vdash\}\end{matrix}}{\Gamma\{\langle\Delta\rangle\}\{C^\vdash\}}\,[\Diamond_E] \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}_2'[x/\mathcal{D}_1] \\ \Gamma\{\langle\Delta\rangle\}\{C^\vdash\}\end{matrix}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{\langle\Delta\rangle, x:A\}\{C^\vdash\}]_B$

- $\Box^4$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\Gamma'\{\emptyset\},\langle A^{\vdash}\rangle\}\end{array}}{\Gamma\{\Gamma'\{\emptyset\},\Box A^{\vdash}\}}\,[\Box_I]}{\Gamma\{\Gamma'\{A^{\vdash}\}\}}\,[\Box_E^4] \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{D}[\Gamma\{\Gamma'\{A^{\vdash}\}\}]_4\\ \Gamma\{A^{\vdash}\}\end{array}$$

– $\Diamond^4$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{\Gamma\{A^{\vdash}\}\}\{\emptyset\}\end{array}}{\Gamma\{\Gamma'\{\emptyset\},\Diamond A^{\vdash}\}\{\emptyset\}}\,[\Diamond_I^4] \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Gamma'\{\emptyset\},\langle x:A\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\Gamma'\{\emptyset\}\}\{C^{\vdash}\}}\,[\Diamond_E] \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{D}_2'[x/\mathcal{D}_1]\\ \Gamma\{\Gamma'\{\emptyset\}\}\{C^{\vdash}\}\end{array}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{\Gamma'\{x:A\}\}\{C^{\vdash}\}]_4$

– $\Box^5$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\langle A^{\vdash}\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}\,[\Box_I]}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\,[\Box_E^5] \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{D}[\Gamma\{\emptyset\}\{A^{\vdash}\}]_5\\ \Gamma\{\emptyset\}\{A^{\vdash}\}\end{array}$$

– $\Diamond^5$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{A^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{array}}{\Gamma\{\emptyset\}\{\Diamond A^{\vdash}\}\{\emptyset\}}\,[\Diamond_I^5] \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\langle x:A\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}}\,[\Diamond_E] \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{D}_2'[x/\mathcal{D}_1]\\ \Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}\end{array}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{x:A\}\{\emptyset\}\{C^{\vdash}\}]_5$

– $\Box^{\mathsf{IS5}}$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\{\langle A^{\vdash}\rangle\}\{\emptyset\}\end{array}}{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}\,[\Box_I]}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\,[\Box_E^{\mathsf{IS5}}] \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{D}[\Gamma\{\emptyset\}\{A^{\vdash}\}]_{\mathsf{IS5}}\\ \Gamma\{\emptyset\}\{A^{\vdash}\}\end{array}$$

– $\Diamond^{\mathsf{IS5}}$-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{A^{\vdash}\}\{\emptyset\}\{\emptyset\}\end{array}}{\Gamma\{\emptyset\}\{\Diamond A^{\vdash}\}\{\emptyset\}}\,[\Diamond_I^{\mathsf{IS5}}] \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\langle x:A\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}}\,[\Diamond_E] \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{D}_2'[x/\mathcal{D}_1]\\ \Gamma\{\emptyset\}\{\emptyset\}\{C^{\vdash}\}\end{array}$$

with $\mathcal{D}_2' = \mathcal{D}[\Gamma\{x:A\}\{\emptyset\}\{C^{\vdash}\}]_{\mathsf{IS5}}$