Calculi for an Intuitionistic Hybrid Modal Logic

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Abstract

In this paper we study proof-search in an intuitionistic hybrid modal logic (for places), denoted *IHML*^{P}, whose modalities allow us to validate properties taking into account the notion of *place*. In this context we propose different sequent calculi for this logic and also tableau rules in the perspective of proof-search and countermodel generation. As this logic can be seen as an instance of *Hybrid IS5* we can derive new calculi and procedures for this logic. Finally we define a terminating calculus for the $\Box$-free fragment of *IHML*^{P} and then propose a decision procedure with countermodel generation.

Keywords: Intuitionistic modal logic, sequent calculus, proof-search, semantics, countermodels.

1 Introduction

In order to model heterogeneous environments, among them distributed systems, recent works provide logical foundations tuned to programming in such environments, like an intuitionistic modal logic with an operational interpretation of logical proofs as distributed programs [9]. Such a logic allows us to deal with systems seen as a set of different nodes, called *places*, that may have different properties and may contain different resources. It has been recently enriched with the disjunctive connective $\vee$ and the constant $\perp$ in order to obtain an intuitionistic hybrid modal logic (for places) [4], denoted here *IHML*^{P}, that is suitable for reasoning about distribution of resources. In this context we can mention related works based on separation logics [13] and resource logics like BI [12] and their extensions with modalities [1,11]. Our general aim consists in studying such modal logics dealing with notions of *locations* or *places* in both perspectives of expressiveness and proof and countermodel search.

Here we aim at focusing on the modal logic *IHML*^{P} for which several results have been proposed in [4]: two semantics, namely a Kripke semantics and a birelational semantics, both proved sound and complete; the finite model property w.r.t. birelational semantics and then decidability of the logic. The formulae in this logic include names, called *places* and assertions are associated with places and validated in places. A key point is that we are not

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only interested in whether a formula is true but also in where a formula is true. Therefore modalities allow us to express a property to be validated in a specific place p (@p), or in a unspecified place ($\Diamond$) or in any place ($\Box$). As the first modality internalizes the model in the logic, this modal logic can also be classified as a hybrid logic [2].
In this paper we aim at studying this logic in the perspective of proof and countermodel search by defining sequent calculi as an alternative to existing natural deduction systems. A first contribution consists of a sequent calculus for $IHML^P$ and its refinements in which contraction and weakening rules are absorbed in the axioms and logical rules. We prove the cut-elimination property and then the soundness and completeness of this calculus. From these results we derive a multi-conclusioned calculus for $IHML^P$ and then tableaux rules for this logic in which the so-called COPY rule is absorbed. As this logic can be seen as the hybridisation of the intuitionistic modal system $IS5$ [3] we can deduce, from our results, new calculi and decision procedures for this logic. Another contribution is the definition of a terminating calculus for the $\Box$-free fragment of $IHML^P$. Its completeness proof provides a way to build countermodels in case of non-validity. Moreover we show, for this fragment, the finite model property w.r.t. the Kripke semantics and thus derive the same result for IS5 without $\Box$.

2 An Intuitionistic Modal Logic

In this section, we summarize the key notions about an hybrid intuitionistic modal logic (of places) that we denote $IHML^P$ [4]. It is designed to reason about places with assertions of the form $(G \mathtt{~at~} p)$ meaning that the formula G is valid at place p. In such an assertion G does not contain any occurrence of the construct at but use modalities $@p$, one for each place, to cast the meta-linguistic at at the language level. The logic also has other modalities for reasoning about properties valid at different locations. It corresponds to the logic introduced in [9] enriched with the connectives $\vee$ and $\bot$.
The set of *pure formulae*, denoted $\mathsf{Form}(\mathsf{PL})$, is defined inductively from a set of propositional *variables*, denoted Var, with $\bot$ constant and from a countable set of *places* PL:
$\mathcal{F} ::= V \mid \bot \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \supset \mathcal{F} \mid \mathcal{F}\,@\,p \mid \Box\mathcal{F} \mid \Diamond\mathcal{F}$ where $V \in \mathsf{Var}$ and $p \in \mathsf{PL}$.
The assertions of the form $G \mathtt{~at~} p$ are called *sentences*. The sequents are of the following form: $\Gamma;\Delta \vdash^P G \mathtt{~at~} p$. Γ is a finite multiset of *pure formulae* called the *global context*, and contains assumptions that are valid everywhere; Δ is a finite multiset of *sentences* called the *local context*, and contains assumptions that are valid locally; $G \mathtt{~at~} p$ is a sentence called the *conclusion* and P is a finite set of places.
Let us mention that $P+q$ denotes the disjoint union of P and $\{q\}$ and that $PL(S)$ denotes the set of places that appear in the syntactic object S.
The sequent $\Gamma;\Delta \vdash^P G \mathtt{~at~} p$ is said to be defined iff the set of places $PL(\Gamma) \cup PL(\Delta) \cup PL(G \mathtt{~at~} p)$ is a subset of P. It has been proved in [4] that for $P = PL(\Gamma) \cup PL(\Delta) \cup PL(G \mathtt{~at~} p)$, if $P \subseteq P'$ then $\Gamma;\Delta \vdash^P G \mathtt{~at~} p$ is valid iff $\Gamma;\Delta \vdash^{P'} G \mathtt{~at~} p$ is valid. Therefore, by assuming that P is finite, there is no loss of generality.

A birelational semantics, similar to the one proposed in [14], have been defined in [4]. It allows to show the finite model property and the decidability of this logic. In this paper, we focus on the Kripke semantics that is similar to the one given for the intuitionistic system IS5 knowing that the logic corresponds to $IS5$ extended with the @ operator [3].

$$\frac{}{A \text{ at } p \vdash^P A \text{ at } p}\ [ID] \qquad \frac{}{\bot \text{ at } p \vdash^P G \text{ at } p'}\ [\bot]$$

$$\frac{\Gamma;\Delta \vdash^P G \text{ at } p'}{\Gamma;\Delta, A \text{ at } p \vdash^P G \text{ at } p'}\ [W_L^1] \qquad \frac{\Gamma;\Delta \vdash^P G \text{ at } p'}{\Gamma, A;\Delta \vdash^P G \text{ at } p'}\ [W_L^2]$$

$$\frac{\Gamma;\Delta, A \text{ at } p, A \text{ at } p \vdash^P G \text{ at } p'}{\Gamma;\Delta, A \text{ at } p \vdash^P G \text{ at } p'}\ [C_L] \qquad \frac{\Gamma, A;\Delta, A \text{ at } p \vdash^P G \text{ at } p'}{\Gamma, A;\Delta \vdash^P G \text{ at } p'}\ [COPY]$$

$$\frac{\Gamma;\Delta \vdash^P A \text{ at } p \qquad \Gamma;\Delta, A \text{ at } p \vdash^P G \text{ at } p'}{\Gamma;\Delta \vdash^P G \text{ at } p'}\ [CUT_1] \qquad \frac{\Gamma;\Delta \vdash^{P+q} A \text{ at } q \qquad \Gamma, A;\Delta \vdash^P G \text{ at } p'}{\Gamma;\Delta \vdash^P G \text{ at } p'}\ [CUT_2]$$

Figure 1. Axioms and Structural Rules of $SC_1^@$

Definition 2.1 [Kripke model] $\mathcal{K} \equiv (K, \leqslant, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ is a *Kripke model* iff
- K is a non-empty set partially ordered by $\leqslant$;
- for every $k \in K$, P_k is a set of places such that for all $k \leqslant l$, $P_k \subseteq P_l$;
- for every $k \in K$, $I_k : \mathsf{Var} \to 2^{P_k}$ is such that for all $k \leqslant l$ and $V \in \mathsf{Var}$ we have $I_k(V) \subseteq I_l(V)$.

The set of places $\bigcup_{k\in K} P_k$ is denoted by $Pls(\mathcal{K})$.

Definition 2.2 [Kripke semantics] Let $\mathcal{K} \equiv (K, \leqslant, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ be a Kripke model, $k \in K$, $p \in P_k$ and a pure formula A with $PL(A) \subseteq P_k$, we define $(k, p) \vDash A$ as follows:
$(k, p) \vDash X$ iff $p \in I_k(X)$ for $X \in \mathsf{Var}$;
$(k, p) \vDash \bot$ never;
$(k, p) \vDash A \wedge B$ iff $(k, p) \vDash A$ and $(k, p) \vDash B$;
$(k, p) \vDash A \vee B$ iff $(k, p) \vDash A$ or $(k, p) \vDash B$;
$(k, p) \vDash A \supset B$ iff for all $l \geqslant k$, if $(l, p) \vDash A$ then $(l, p) \vDash B$;
$(k, p) \vDash A@q$ iff $q \in P_k$ and $(k, q) \vDash A$;
$(k, p) \vDash \Box A$ iff for all $l \geqslant k$ and for all $q \in P_l$, $(l, q) \vDash A$;
$(k, p) \vDash \Diamond A$ iff there exists $q \in P_k$, $(k, q) \vDash A$.

Let us remind that the relation $\vDash$ satisfies the Kripke monotonicity property, i.e., if $l \geqslant k$ then $(k, p) \vDash A$ implies $(l, p) \vDash A$ [4]. We write $k \vDash \Gamma;\Delta$ for Γ a global context and Δ a local context iff for every $A \in \Gamma$ and $p \in P_k$, $(k, p) \vDash \Box A$; and for every B at $q \in \Delta$, and $q \in P_k$, $(k, q) \vDash B$. The sequent $\Gamma;\Delta \vdash^P C$ at p is valid in the Kripke model $\mathcal{K} = (K, \leqslant, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ iff $PL(\Gamma) \cup PL(\Delta) \cup PL(C) \cup \{p\} \subseteq P$; and for every $k \in K$ such that $P \subseteq P_k$, if $k \vDash \Gamma;\Delta$ then $(k, p) \vDash C$. We say that $\Gamma;\Delta \vdash^P A$ at p is valid iff it is valid in every Kripke model.

3 Sequent Calculi for $IHML^P$

In this section, we define a first sequent calculus $SC_1^@$ for $IHML^P$. It is obtained by extending a calculus for the fragment without $\vee$ and $\bot$ [8]. Axioms and structural rules are given in Figure 1 and logical rules are given in Figure 2. Let $\mathcal{S}$ be a sequent ($\Gamma;\Delta \vdash^P G$ at p) we write $\rhd_{SC_1^@} \mathcal{S}$ to express that $\mathcal{S}$ is derivable in $SC_1^@$.

Theorem 3.1 (Soundness) *Let $\mathcal{S}$ be a sequent, if $\rhd_{SC_1^@} \mathcal{S}$ then $\mathcal{S}$ is valid.*

Proof For every rule, we suppose that its premises are valid in every Kripke model, and we prove that its conclusion is valid in every Kripke model. Here, we show only the case

$$\frac{\Gamma;\Delta,A \text{ at } p,B \text{ at } p\vdash^P C \text{ at } p'}{\Gamma;\Delta,A\wedge B \text{ at } p\vdash^P C \text{ at } p'}\ [\wedge_L] \qquad \frac{\Gamma;\Delta\vdash^P A \text{ at } p \qquad \Gamma;\Delta\vdash^P B \text{ at } p}{\Gamma;\Delta\vdash^P A\wedge B \text{ at } p}\ [\wedge_R]$$

$$\frac{\Gamma;\Delta\vdash^P A \text{ at } p}{\Gamma;\Delta\vdash^P A\vee B \text{ at } p}\ [\vee_{R_1}] \qquad \frac{\Gamma;\Delta\vdash^P B \text{ at } p}{\Gamma;\Delta\vdash^P A\vee B \text{ at } p}\ [\vee_{R_2}] \qquad \frac{\Gamma;\Delta,A \text{ at } p\vdash^P C \text{ at } p' \qquad \Gamma;\Delta,B \text{ at } p\vdash^P C \text{ at } p'}{\Gamma;\Delta,A\vee B \text{ at } p\vdash^P C \text{ at } p'}\ [\vee_L]$$

$$\frac{\Gamma;\Delta\vdash^P A \text{ at } p \qquad \Gamma;\Delta,B \text{ at } p\vdash^P C \text{ at } p'}{\Gamma;\Delta,A\supset B \text{ at } p\vdash^P C \text{ at } p'}\ [\supset_L] \qquad \frac{\Gamma;\Delta,A \text{ at } p\vdash^P B \text{ at } p}{\Gamma;\Delta\vdash^P A\supset B \text{ at } p}\ [\supset_R]$$

$$\frac{\Gamma;\Delta,A \text{ at } p\vdash^P C \text{ at } p''}{\Gamma;\Delta,A@p \text{ at } p'\vdash^P C \text{ at } p''}\ [@_L] \qquad \frac{\Gamma;\Delta\vdash^P A \text{ at } p}{\Gamma;\Delta\vdash^P A@p \text{ at } p'}\ [@_R]$$

$$\frac{\Gamma,A;\Delta\vdash^P C \text{ at } p'}{\Gamma;\Delta,\Box A \text{ at } p\vdash^P C \text{ at } p'}\ [\Box_L] \qquad \frac{\Gamma;\Delta\vdash^{P+q} A \text{ at } q}{\Gamma;\Delta\vdash^P \Box A \text{ at } p}\ [\Box_R]$$

$$\frac{\Gamma;\Delta,A \text{ at } q\vdash^{P+q} C \text{ at } p'}{\Gamma;\Delta,\Diamond A \text{ at } p\vdash^P C \text{ at } p'}\ [\Diamond_L] \qquad \frac{\Gamma;\Delta\vdash^P A \text{ at } p}{\Gamma;\Delta\vdash^P \Diamond A \text{ at } p'}\ [\Diamond_R]$$

Figure 2. Logical Rules of $SC_1^@$

of $[\Box_R]$ rule. Let $\mathcal{K}$=$(K,\leqslant,\{P_k\}_{k\in K},\{I_k\}_{k\in K})$ be a countermodel of $\Gamma\vdash^P \Box A$ at p. Then, $\exists k_0\in K$ such that $P\subseteq P_{k_0}$, $k_0\vDash\Gamma;\Delta$ and $(k_0,p)\nvDash\Box A$. Thus, $\exists l_0\geqslant k_0$ and $\exists p_0\in P_{l_0}$ such that $(l_0,p_0)\vDash A$, and from the Kripke monotonicity, $l_0\vDash\Gamma;\Delta$. Let $\mathcal{K}'$=$(K,\leqslant,\{P_k\cap\{q\}\}_{k\in K},\{I'_k\}_{k\in K})$ where for every $A\in\mathsf{Var}$, $I'_k(A)=I_k(A)\cap\{q\}$ if $p_0\in I_k(A)$ and $I'_k(A)=I_k(A)$ otherwise. In $\mathcal{K}'$, the new place q duplicated p_0. Hence for all formulae $\mathcal{F}$, $(l_0,p_0)\vDash\mathcal{F}$ if and only if $(l_0,q)\vDash\mathcal{F}$. It is easy to show that $\mathcal{K}'$ is a Kripke model. Since q duplicating p_0, we have $l_0\vDash\Gamma;\Delta$ and $(l_0,q)\vDash A$ in $\mathcal{K}'$. Therefore, $\mathcal{K}'$ is a countermodel of $\Gamma;\Delta\vdash^{P+q} A$ at q. As $\Gamma;\Delta\vdash^{P+q} A$ at q is valid, we get a contradiction and we deduce that $\Gamma\vdash^P\Box A$ at p is valid. □

Theorem 3.2 (Completeness) *Let $\mathcal{S}$ be a sequent, if $\mathcal{S}$ is valid then $\rhd_{SC_1^@}\mathcal{S}$.*

Proof We consider the validity through the natural deduction system introduced in [8] and extended in [4]. Let $\mathcal{S}=\Gamma;\Delta\vdash^P G$ at p'. We suppose that $\mathcal{S}$ is derivable in the natural deduction system, and we prove by induction on the depth of the given derivation in the natural deduction system that $\rhd_{SC_1^@}\mathcal{S}$. We only prove it when the derivation ends with $\Diamond$-elimination:

$$\frac{\Gamma;\Delta\vdash^P \Diamond A \text{ at } p \qquad \Gamma;\Delta,A \text{ at } q\vdash^{P+q} G \text{ at } p'}{\Gamma;\Delta\vdash^P G \text{ at } p'}\ [\Diamond_E]$$

By induction hypothesis, we have derivations for $\Gamma;\Delta\vdash^P\Diamond A$ at p and $\Gamma;\Delta,A$ at $q\vdash^{P+q}$ G at p'. Then, by using $[\Diamond_L]$ and $[CUT_1]$, we obtain a derivation for $\Gamma;\Delta\vdash^P G$ at p':

$$\frac{\Gamma;\Delta\vdash^P \Diamond A \text{ at } p \qquad \dfrac{\Gamma;\Delta,A \text{ at } q\vdash^{P+q} G \text{ at } p'}{\Gamma;\Delta,\Diamond A \text{ at } p\vdash^P G \text{ at } p'}\,[\Diamond_L]}{\Gamma;\Delta\vdash^P G \text{ at } p'}\,[CUT_1]$$

□

Theorem 3.3 (Cut-elimination) *Let $\mathcal{S}$ be a sequent. If $\mathcal{S}$ has a proof in $SC_1^@$ then $\mathcal{S}$ has a proof in $SC_1^@$ without using the cut rules.*

Proof See Appendix A. □

By using the approach of [15], we provide another calculus $SC_2^@$ in which the contraction and weakening rules are *absorbed* in the axioms and logical rules. This version allows us to improve proof-search in our calculus. It is obtained by replacing the axioms by $\dfrac{}{\Gamma;\Delta,A \text{ at } p \vdash^P A \text{ at } p}[ID]$ and $\dfrac{}{\Gamma;\Delta,\bot \text{ at } p \vdash^P G \text{ at } p'}[\bot]$ and the rule $[\supset_L]$ by the following rule

$$\frac{\Gamma;\Delta,A \supset B \text{ at } p \vdash^P A \text{ at } p \qquad \Gamma;\Delta,B \text{ at } p \vdash^P C \text{ at } p'}{\Gamma;\Delta,A \supset B \text{ at } p \vdash^P C \text{ at } p'}[\supset_L] \; .$$

Theorem 3.4 (Soundness) *Let $\mathcal{S}$ be a sequent, if $\rhd_{SC_2^@}\mathcal{S}$ then $\mathcal{S}$ is valid.*

Proof By using Kripke semantics like in the proof of Theorem 3.1. □

Theorem 3.5 (Completeness) *Let $\mathcal{S}$ be a sequent, if $\rhd_{SC_1^@}\mathcal{S}$ then $\rhd_{SC_2^@}\mathcal{S}$.*

Proof We start by proving that:
1. If $\rhd_{SC_2^@}\Gamma;\Delta,A \text{ at } p,A \text{ at } p \vdash^P G \text{ at } p'$ then $\rhd_{SC_2^@}\Gamma;\Delta,A \text{ at } p \vdash^P G \text{ at } p'$;
2. If $\rhd_{SC_2^@}\Gamma;\Delta \vdash^P G \text{ at } p'$ then $\rhd_{SC_2^@}\Gamma;\Delta,A \text{ at } p \vdash^P G \text{ at } p'$;

It is done by structural induction on both the derivation of the assumption and A. Then, by structural induction over the given derivation, we can easily prove the result. Here, we consider for 1. the case when the derivation of the assumption ends with $[\supset_L]$ rule:

$$\frac{\dfrac{\mathcal{D}_1}{\Gamma;\Delta,A \supset B \text{ at } p,A \supset B \text{ at } p \vdash^P A \text{ at } p} \quad \dfrac{\mathcal{D}_2}{\Gamma;\Delta,A \supset B \text{ at } p,B \text{ at } p \vdash^P G \text{ at } p'}}{\Gamma;\Delta,A \supset B \text{ at } p,A \supset B \text{ at } p \vdash^P G \text{ at } p'}[\supset_L]$$

From $\rhd_{SC_2^@}\Gamma;\Delta,A \supset B$ at $p,A \supset B$ at $p \vdash^P A$ at p, by induction hypothesis, we have $\rhd_{SC_2^@}\Gamma;\Delta,A \supset B$ at $p \vdash^P A$ at p. Then, we show, by induction on the depth of the given derivation, that if $\rhd_{SC_2^@}\Gamma;\Delta,A \supset B$ at $p \vdash^P G$ at p' then $\rhd_{SC_2^@}\Gamma;\Delta,B$ at $p \vdash^P G$ at p'. Thus, $\rhd_{SC_2^@}\Gamma;\Delta,B$ at p,B at $p \vdash^P G$ at p' and by induction hypothesis we deduce that $\rhd_{SC_2^@}\Gamma;\Delta,B$ at $p \vdash^P G$ at p'. Therefore, $\rhd_{SC_2^@}\Gamma;\Delta,A \supset B$ at $p \vdash^P G$ at p'. □

Let us give the following example of proof in order to illustrate the use of this sequent calculus.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \supset B,A;A \text{ at } q \vdash^{\{p,q\}} A \text{ at } q \quad A \supset B,A;A \text{ at } q,B \text{ at } q \vdash^{\{p,q\}} B \text{ at } q}{A \supset B,A;A \text{ at } q,A \supset B \text{ at } q \vdash^{\{p,q\}} B \text{ at } q}[\supset_L]}{A \supset B,A;A \text{ at } q \vdash^{\{p,q\}} B \text{ at } q}[COPY]}{A \supset B,A;\vdash^{\{p,q\}} B \text{ at } q}[COPY]}{A \supset B,A;\vdash^{\{p\}} \Box B \text{ at } p}[\Box_R]}{A \supset B;\Box A \text{ at } p \vdash^{\{p\}} \Box B \text{ at } p}[\Box_L]}{A \supset B;\vdash^{\{p\}} \Box A \supset \Box B \text{ at } p}[\supset_R]$$

4 Tableaux Rules for $HIML^p$

In this section, we propose a multi-conclusioned variant of $SC_2^@$ calculus and then derive from it a tableau rules. A multi-conclusioned sequent has the form $\Gamma;\Delta \vdash^P \Pi$, where Γ and Δ are respectively the global and local contexts and Π is a multiset of sentences.

Definition 4.1 Let $\mathcal{K}$=$(K, \leqslant, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ be a Kripke model. The sequent $\Gamma;\Delta \vdash^P \Pi$ is *valid in* $\mathcal{K}$ iff $PL(\Gamma) \cup PL(\Delta) \cup PL(\Pi) \subseteq P$ and for every $k \in K$ such that $P \subseteq P_k$, if $k \vDash \Gamma;\Delta$ then there exists G at $p \in \Pi$ such that $(k, p) \vDash G$. The sequent $\Gamma;\Delta \vdash^P \Pi$ is *valid* iff it is valid in every Kripke model.

The multi-conclusioned version $SC^{@}_{2m}$ is obtained by adding a multiset Π of sentences to the right part of the sequents in each rule of $SC^{@}_{2}$ except $[\vee_{R_1}]$, $[\vee_{R_2}]$, $[\supset_R]$ and $[\Diamond_R]$. For example, the rule $[\wedge_R]$ is transformed into:

$$\frac{\Gamma;\Delta \vdash^P A \text{ at } p, \Pi \qquad \Gamma;\Delta \vdash^P B \text{ at } p, \Pi}{\Gamma;\Delta \vdash^P A \wedge B \text{ at } p, \Pi}\ [\wedge_R]$$

We replace $[\vee_{R_1}]$, $[\vee_{R_2}]$, $[\supset_R]$ and $[\Diamond_R]$ by the following three rules:

$$\frac{\Gamma;\Delta \vdash^P A \text{ at } p, B \text{ at } p, \Pi}{\Gamma;\Delta \vdash^P A \vee B \text{ at } p, \Pi}\ [\vee_R] \qquad \frac{\Gamma;\Delta, A \text{ at } p \vdash^P B \text{ at } p}{\Gamma;\Delta \vdash^P A \supset B \text{ at } p, \Pi}\ [\supset_R] \qquad \frac{\Gamma;\Delta \vdash^{\{p_1,\dots,p_n\}} A \text{ at } p_1, \dots, A \text{ at } p_n, \Pi}{\Gamma;\Delta \vdash^{\{p_1,\dots,p_n\}} \Diamond A \text{ at } p, \Pi}\ [\Diamond_R]$$

Theorem 4.2 (Soundness) *If* $\rhd_{SC^{@}_{2m}} \Gamma;\Delta \vdash^P \Pi$ *then* $\Gamma;\Delta \vdash^P \Pi$ *is valid.*

Proof The proof is similar to the one of Theorem 3.1, by using the Kripke semantics and Definition 4.1. □

Theorem 4.3 (Completeness) *If* $\Gamma;\Delta \vdash^P \Pi$ *is valid then* $\rhd_{SC^{@}_{2m}} \Gamma;\Delta \vdash^P \Pi$.

Proof We can see that a multi-conclusioned sequent $\Gamma;\Delta \vdash^P A_1$ at $p_1, \dots, A_n$ at p_n is valid iff $\Gamma;\Delta \vdash^P A_1 @ p_1 \vee \dots \vee A_n @ p_n$ at p, where $p \in P$, is valid. Thus, we can show that if $\rhd_{SC^{@}_{2}} \Gamma;\Delta \vdash^P A_1 @ p_1 \vee \dots \vee A_n @ p_n$ at p then $\rhd_{SC^{@}_{2m}} \Gamma;\Delta \vdash^P A_1$ at $p_1, \dots, A_n$ at p_n. The proof is done by structural induction on the given derivation of the assumption. We must prove the weakening property: if $\rhd_{SC^{@}_{2m}} \Gamma;\Delta \vdash^P \Pi$ then $\rhd_{SC^{@}_{2m}} \Gamma;\Delta \vdash^P \Pi, A$ at p. □

Having defined this multi-conclusioned calculus we derive a tableau calculus appropriate for proof-search because of the control of the COPY rule.

Definition 4.4 A *signed formula* is an expression of the form $S\,A$ where $S \in \{F, T\}$ and A is a pure formula.

Definition 4.5 A *tableau node* is an expression of the form $M_1; M_2; PL$ where M_1 is a multiset of pure formulae, M_2 is of the form $\{(S\,A, p) \mid S\,A$ is a signed formula and p is a place$\}$ and PL is a set of places.
A *tableau* is tree whose nodes are tableau nodes. The rules of branch expansion are displayed in Figure 3.

A tableau node $M_1; M_2, VP$ is said to be *closed* if M_2 contains occurrences of both (TA, p) and (FA, p), or if M_2 contains $(T\bot, p)$. A branch is closed if it contains a closed tableau node. A tableau is closed if it only contains closed branches.

Theorem 4.6 (Soundness and completeness) *Let* $\mathcal{S} = \Gamma;\Delta \vdash^P \Pi$ *be a multi-conclusion sequent.* $\mathcal{S}$ *is valid iff there is a closed tableau with the initial tableau node* $\Gamma; M; P$*, where* $M = \{T(A, p) \mid A$ at $p \in \Delta\} \cup \{T(B, q) \mid B \in \Gamma$ *and* $q \in P\} \cup \{F(G, p) \mid G$ at $p \in \Pi\}$.

$$\frac{M_1;M_2,(T(A\wedge B),p);P}{M_1;M_2,(TA,p),(TB,p);P}\,[T\wedge] \qquad \frac{M_1;M_2,(F(A\wedge B),p);P}{M_1;M_2,(FA,p);P \mid M_1;M_2,(FB,p);P}\,[F\wedge]$$

$$\frac{M_1;M_2,(T(A\vee B),p);P}{M_1;M_2,(TA,p);P \mid M_1;M_2,(TB,p);P}\,[T\vee] \qquad \frac{M_1;M_2,(F(A\vee B),p);P}{M_1;M_2,(FA,p),(FB,p);P}\,[F\vee]$$

$$\frac{M_1;M_2,(T(A\supset B),p);P}{M_1;M_2,(T(A\supset B),p),(FA,p);P \mid M_1;M_2,(TB,p);P}\,[T\supset] \qquad \frac{M_1;M_2,(F(A\supset B),p);P}{M_1;(M_2)_T,(TA,p),(FB,p);P}\,[F\supset]$$

$$\frac{M_1;M_2,(T(A@p'),p);P}{M_1;M_2,(TA,p');P}\,[T@] \qquad \frac{M_1;M_2,(F(A@p'),p);P}{M_1;M_2,(FA,p');P}\,[F@]$$

$$\frac{M_1;M_2,(T(\Box A),p);P}{M_1,A;M_2,(TA,p_1),\ldots,(TA,p_K);P}\,[T\Box] \qquad \frac{M_1;M_2,(F(\Box A),p);P}{M_1;M_2,(TM_1,q),(FA,q);P+q}\,[F\Box]$$

$$\frac{M_1;M_2,(T(\Diamond A),p);P}{M_1;M_2,(TM_1,q),(TA,q);P+q}\,[T\Diamond] \qquad \frac{M_1;M_2,(F(\Diamond A),p);P}{M_1,A;M_2,(FA,p_1),\ldots,(FA,p_K);P}\,[F\Diamond]$$

Where $\{p_1,\ldots,p_k\} = P$ and $(TM_1,q) = \{(TF,q) \mid F \in M_1\}$.

Figure 3. The Tableau rules

Proof Soundness and completeness of this tableau method come from $SC^{@}_{2m}$ system. Intuitively, in this method, we associated the application of the COPY rule to the application of the rules where there is an introduction of a new place, i.e., the $[F\Box]$ and $[T\Diamond]$ rules. Because of a given pure formula $A \in \Gamma$ and a given place q, a single copy A at q, in the local context, in a derivation is enough. Since we do not use the COPY rule for the places in P, we copy the pure formulae of the global context with the places in P in the initial tableau node. □

5 A Terminating Calculus for the □-free Fragment

In this section, we propose a terminating calculus, called $SCT^{@}$, for the □-free fragment of this logic by using the approach used in [6] for intuitionistic logic.

We start by defining a particular class of Kripke models by using a structure called *Kripke trees*. For this, we use a similar approach to that given in [7].

Definition 5.1 A *node* is a set $\mathcal{N} = \{(p_1, S^{p_1}_{\mathcal{N}}), \ldots, (p_n, S^{p_n}_{\mathcal{N}})\}$ where $\forall i \in 1..n$, p_i is a place and $S^{p_i}_{\mathcal{N}}$ a finite set of logical variables. We note $Var_{\mathcal{N}}$ the set $S^{p_1}_{\mathcal{N}} \cup \ldots \cup S^{p_n}_{\mathcal{N}}$, $PL_{\mathcal{N}}$ the set $\{p_1,\ldots,p_n\}$ and $P^X_{\mathcal{N}}$ the set $\{p_i \mid X \in S^{p_i}_{\mathcal{N}}\}$.

Definition 5.2 [Kripke tree] A *Kripke tree* is a pair $\mathcal{T} = (\mathcal{N}_{\mathcal{T}}, [\mathcal{T}_1,\ldots,\mathcal{T}_p])$ where $\mathcal{N}_{\mathcal{T}}$ is a node And $[\mathcal{T}_1,\ldots,\mathcal{T}_p]$ is a finite list of Kripke trees. Moreover, for each i, $Var_{\mathcal{N}_{\mathcal{T}}} \subseteq Var_{\mathcal{N}_{\mathcal{T}_i}}$ and $\forall X \in Var_{\mathcal{N}_{\mathcal{T}}}$, $P^X_{\mathcal{N}_{\mathcal{T}}} \subseteq P^X_{\mathcal{N}_{\mathcal{T}_i}}$.

The concept of *subtree* is defined inductively by: $\mathcal{T}'$ is a subtree of $\mathcal{T} = (\mathcal{N}_{\mathcal{T}}, [\mathcal{T}_1,\ldots,\mathcal{T}_p])$ iff $\mathcal{T}' = \mathcal{T}$ or there exists $i \in \{1,\ldots,p\}$ such that $\mathcal{T}' = \mathcal{T}_i$ or $\mathcal{T}'$ is a *subtree* of $\mathcal{T}_i$.

Definition 5.3 Let $\mathcal{T} = (\mathcal{N}_{\mathcal{T}}, [\mathcal{T}_1,\ldots,\mathcal{T}_p])$ be a Kripke tree, the *subtree model associated to* $\mathcal{T}$, denoted $\mathcal{K}_{\mathcal{T}}$, is the quadruple: $(\mathcal{T}^*, \leqslant, \{PL_{\mathcal{N}_{\mathcal{T}'}}\}_{\mathcal{T}'\in\mathcal{T}^*}, \{I_{\mathcal{T}'}\}_{\mathcal{T}'\in\mathcal{T}^*})$ where
- $\mathcal{T}^*$ is the set of all subtrees of $\mathcal{T}$;
- $\leqslant$ is a partial order on $\mathcal{T}^*$ where $\mathcal{T}'' \leqslant \mathcal{T}'$ iff $\mathcal{T}''$ is a subtree of $\mathcal{T}'$;

$$\frac{\Gamma, A^{P_A - p}; \Delta, A \text{ at } p \vdash^P G \text{ at } p'}{\Gamma, A^{P_A}; \Delta \vdash^P G \text{ at } p'}\ [COPY] \qquad \frac{(\Gamma[\Diamond A/A@q, P])^{+q}; \Delta, A \text{ at } q \vdash^{P+q} C \text{ at } p'}{\Gamma; \Delta, \Diamond A \text{ at } p \vdash^P C \text{ at } p'}\ [\Diamond_L]$$

$$\frac{\Gamma; \Delta, X \text{ at } p, B \text{ at } p \vdash^P G \text{ at } p'}{\Gamma; \Delta, X \text{ at } p, X \supset B \text{ at } p \vdash^P G \text{ at } p'}\ [\supset_L^1] \qquad \frac{\Gamma; \Delta, A \supset (B \supset C) \text{ at } p \vdash^P G \text{ at } p'}{\Gamma; \Delta, (A \wedge B) \supset C \text{ at } p \vdash^P G \text{ at } p'}\ [(\wedge)\supset_L]$$

$$\frac{\Gamma; \Delta, A \supset C \text{ at } p, B \supset C \text{ at } p \vdash^P G \text{ at } p'}{\Gamma; \Delta, (A \vee B) \supset C \text{ at } p \vdash^P G \text{ at } p'}\ [(\vee)\supset_L] \qquad \frac{\Gamma; \Delta, A \supset B@p' \text{ at } p \vdash^P G \text{ at } p''}{\Gamma; \Delta, A@p \supset B \text{ at } p' \vdash^P G \text{ at } p''}\ [@\supset]$$

$$\frac{\Gamma, (A \supset B@p)^P; \Delta \vdash^P G \text{ at } p'}{\Gamma; \Delta, \Diamond A \supset B \text{ at } p \vdash^P G \text{ at } p'}\ [\Diamond\supset_L] \qquad \frac{\Gamma; \Delta \vdash^P G \text{ at } p'}{\Gamma; \Delta, \bot \supset A \text{ at } p \vdash^P G \text{ at } p'}\ [\bot\supset_L]$$

$$\frac{\Gamma; \Delta, A \text{ at } p, B \supset C \text{ at } p \vdash^P B \text{ at } p \qquad \Gamma; \Delta, C \text{ at } p \vdash^P G \text{ at } p'}{\Gamma; \Delta, (A \supset B) \supset C \text{ at } p \vdash^P G \text{ at } p'}\ [(\supset)\supset_L]$$

Figure 4. The $SCT^@$ calculus

- $\forall \mathcal{T}' \in \mathcal{T}^*$, $I_{\mathcal{T}'} : \mathsf{Var} \rightarrow 2^{PL_{\mathcal{N}_{\mathcal{T}'}}}$ such that for all $X \in \mathsf{Var}$ we have $I_{\mathcal{T}'}(X) = P^X_{\mathcal{N}_{\mathcal{T}'}}$.

Proposition 5.4 *For every Kripke tree $\mathcal{T} = (\mathcal{N}_{\mathcal{T}}, [\mathcal{T}_1, \ldots, \mathcal{T}_p])$, the subtree model $\mathcal{K}_{\mathcal{T}} = (\mathcal{T}^*, \leqslant, \{PL_{\mathcal{N}_{\mathcal{T}'}}\}_{\mathcal{T}' \in \mathcal{T}^*}, \{I_{\mathcal{T}'}\}_{\mathcal{T}' \in \mathcal{T}^*})$ is a Kripke model.*

Proof Let $\mathcal{T} = (\mathcal{N}_{\mathcal{T}}, [\mathcal{T}_1, \ldots, \mathcal{T}_p])$ be a Kripke tree. From Definition 2.1 and Definition 5.3, to show that $(\mathcal{T}^*, \leqslant, \{PL_{\mathcal{N}_{\mathcal{T}'}}\}_{\mathcal{T}' \in \mathcal{T}^*}, \{I_{\mathcal{T}'}\}_{\mathcal{T}' \in \mathcal{T}^*})$ is a Kripke model, we have only to show:
1. for all $\mathcal{T}', \mathcal{T}''$ in $\mathcal{T}^*$ such that $\mathcal{T}'' \leqslant \mathcal{T}'$, we have $PL_{\mathcal{N}_{\mathcal{T}''}} \subseteq PL_{\mathcal{N}_{\mathcal{T}'}}$;
2. for all $\mathcal{T}', \mathcal{T}''$ in $\mathcal{T}^*$ such that $\mathcal{T}'' \leqslant \mathcal{T}'$, we have for all $X \in \mathsf{Var}$, $P^X_{\mathcal{N}_{\mathcal{T}''}} \subseteq P^X_{\mathcal{N}_{\mathcal{T}'}}$.
These properties can be proved by structural induction on $\mathcal{T}$, namely with induction hypothesis for every subtree of $\mathcal{T}$. $\square$

In order to define the $SCT^@$ calculus, we replace the rules $[COPY]$ and $[\supset_L]$ of the $SC_2^@$ calculus by the set of rules of Figure 4 in which every formula of the global contexts is indexed by a set of places A^{ind}. By such indexes we limit the use of $[COPY]$. Let us note that in the $[COPY]$ rule we have $p \in P_A$. Moreover the expression $\Gamma[\Diamond A/A@q, P]$ in the $[\Diamond_L]$ rule means that one substitutes $A@q$ to $\Diamond A$ in all formulae of Γ and changes the index with P. In addition $(\Gamma)^{+q}$ means that one adds q to the indexes of the Γ formulae.

Definition 5.5 [Irreducible sequent] An *irreducible sequent* is a sequent of the form $\Gamma; X_1 \text{ at } p_1, \ldots, X_m \text{ at } p_m, Y_1 \supset C_1 \text{ at } q_1, \ldots, Y_n \supset C_n \text{ at } q_n \vdash^P F \text{ at } p$ where Γ is a multiset of pure formulae; for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, if $X_i \equiv Y_j$ then $p_i \neq q_j$; and $F \in \mathsf{Var} \cup \{\bot\}$ and if $F \in \{X_1, \ldots, X_m\}$ then $p \notin \{p_1, \ldots, p_m\}$; for all A^{ind} in Γ, $ind = \emptyset$.

Definition 5.6 [Inv-irreducible sequent] An *inv-irreducible sequent* is a sequent of the form $\Gamma; X_1 \text{ at } p_1, \ldots, X_k \text{ at } p_k, Y_1 \supset D_1 \text{ at } q_1, \ldots, Y_l \supset D_l \text{ at } q_l, (A_1 \supset B_1) \supset C_1 \text{ at } r_1, \ldots, (A_m \supset B_m) \supset C_m \text{ at } r_m \vdash^P F \text{ at } p$ where Γ is a multiset of pure formulae; for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, if $X_i \equiv Y_j$ then $p_i \neq q_j$; $F \in \mathsf{Var} \cup \{\bot\}$, $F \equiv A \vee B$ or $F \equiv \Diamond A$; if $F \in \{X_1, \ldots, X_k\}$ then $p \notin \{p_1, \ldots, p_m\}$; for all G^{ind} in Γ, $ind = \emptyset$.

Proposition 5.7 *The number of applications of the COPY rule in any derivation in the $SCT^@$ calculus is finite.*

Proof Let $\mathcal{S} \equiv \Gamma; \Delta \vdash^P G \text{ at } p'$ be a sequent and $\mathcal{D}$ be a derivation of $\mathcal{S}$ in $SCT^@$. For every A in Γ, the number of applications of the COPY rule is smaller than the size of P and the number of the new places introduced in $\mathcal{D}$. One can see that the number of the new places introduced in $\mathcal{D}$ is smaller than the number of the subformulae of the form $\Diamond F$ in $\mathcal{S}$. Since the size of P and the set of subformulae of the form $\Diamond F$ in $\mathcal{S}$ are finite, we deduce that the number of applications of the COPY rule is finite. $\square$

Proposition 5.8 *The application of the SCT$^@$ rules to a given sequent terminates with axioms or irreducible sequents.*

Proof See Appendix B. $\square$

Theorem 5.9 (Soundness) *The rules of the SCT$^@$ calculus are sound.*

Proof We consider the case for rule $[(\supset)\supset_L]$. We suppose that $\Gamma;\Delta, A \text{ at } p, B \supset C \text{ at } p \vdash^P B \text{ at } p$ and $\Gamma;\Delta, C \text{ at } p \vdash^P G \text{ at } p$ are valid. Let $\mathcal{K} = (K, \leqslant, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ be a countermodel of $\Gamma;\Delta, (A \supset B) \supset C \text{ at } p \vdash^P G \text{ at } p'$. Then, $\exists k \in K$ such that $P \subseteq P_k$, $k \vDash \Gamma;\Delta$, $(k, p) \vDash ((A \supset B) \supset C)$ and $(k, p') \nvDash G$. From $(k, p) \vDash \square((A \supset B) \supset C)$, we have $\forall l \geqslant k$, if $(l, p) \vDash A \supset B$ then $(l, p) \vDash C$. We suppose that there exists $l_0 \geqslant k$ such that $(l_0, p) \vDash A$ and $(l_0, p) \nvDash B$. From $l_0 \geqslant k$ and the Kripke monotonicity, we have $l_0 \vDash \Gamma;\Delta$. Moreover from $(l_0, p) \vDash A$ and $(l_0, p) \nvDash B$, we have $(l_0, p) \vDash B \supset C$, because for $l' > l_0$ if $(l', p) \vDash B$ then $(l', p) \vDash A \supset B$ and we deduce that $(l', p) \vDash C$. Therefore, $\mathcal{K}$ is countermodel of $\Gamma;\Delta, A \text{ at } p, B \supset C \text{ at } p \vdash^P B \text{ at } p$ and this is a contradiction. Thus, $\forall l \geqslant k$, we have $(l, p) \vDash A \supset B$ and thus $(l, p) \vDash C$. Since $(k, p') \nvDash G$, $\mathcal{K}$ is a countermodel of $\Gamma;\Delta, C \text{ at } p \vdash^P G \text{ at } p$. From this contradiction we deduce that $\Gamma;\Delta, (A \supset B) \supset C \text{ at } p \vdash^P G \text{ at } p'$ is valid. Proofs for other rules are similar. $\square$

Let us remind that a proof rule is *invertible* if, for any instance of the rule, the non-validity of at least one of its premises entails the non-validity of its conclusion. It is *strongly invertible* if, for any instance of the rule and any Kripke model $\mathcal{K}$, if $\mathcal{K}$ is a countermodel of at least one of its premises then it is a countermodel of its conclusion. We can observe that strong invertibility implies invertibility.

Theorem 5.10 *All the rules of the SCT$^@$ calculus, except the $[(\supset)_L\supset]$, $[\vee_R]$, $[\Diamond_R]$ rules, are strongly invertible.*

Proof We consider the case for rule $[\Diamond\supset_L]$. Let $\mathcal{K} = (K, \leqslant, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ be a countermodel of $\Gamma, (A \supset B@p)^P;\Delta \vdash^P G \text{ at } p'$. Then, $\exists k \in K$ such that $P_k \subseteq P$, $k \vDash \Gamma;\Delta$, $(k, q) \vDash \square(A \supset B@p)$ for $q \in P_k$ and $(k, p') \nvDash G$. Thus, from $(k, q) \vDash \square(A \supset B@p)$, we have $\forall l \geqslant k$ and $\forall r \in P_l$, if $(l, r) \vDash A$ then $(l, p) \vDash B$. Therefore, $(k, p) \vDash \Diamond A \supset B$, because $\forall l \geqslant k$, if $(l, p) \vDash \Diamond A$ then $(l, p) \vDash B$. We deduce that $\mathcal{K}$ is a countermodel of $\Gamma;\Delta, \Diamond A \supset B \text{ at } p \vdash^P G \text{ at } p'$. Thus, the rule $[\Diamond\supset_L]$ is strongly invertible. Proofs for other rules are similar. $\square$

A proof-refutation tree is a tree in which the nodes are indexed by sequents. Especially, the root node is indexed by a sequent in which the pure formulae of the global context are indexed by the set of all places belonging to this sequent. The rules of branch expansion are obtained from the rules of $SCT^@$: if the node is indexed by an inv-irreducible sequent then its children are indexed by the sequents which correspond to the premises of all rules that can be applied to its index. Else, the children correspond to premises of one of the strongly invertible rule which can be applied to its index.
From Proposition 5.8, we can deduce that a proof-refutation tree is finite and its leaf nodes are indexed by axioms and irreducible sequents. The formal definition is given below.

Definition 5.11 [Proof-refutation tree] A *proof-refutation tree* is a tree where the nodes are indexed by sequents and verifying the following properties:
1) The root node is indexed by a sequent of the form $\Gamma;\Delta\vdash^P G$ at p where the pure formulae of Γ are indexed by the set P of places.
2) For every internal node n indexed by a sequent $\mathcal{S}$ which is not an inv-irreducible sequent, n has a maximum of two children: if n has two children (resp. a single child) indexed by $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}$) then $\dfrac{\mathcal{H}_1 \quad \mathcal{H}_2}{\mathcal{S}}\,[R]$ (resp. $\dfrac{\mathcal{H}}{\mathcal{S}}\,[R]$) is an instance of a strongly invertible rule.
3) For every internal node n indexed by an inv-irreducible sequent $\Gamma;X_1$ at $p_1,\ldots,X_m$ at p_m, $Y_1\supset D_1$ at $q_1,\ldots,Y_n\supset D_n$ at $q_n,(A_1\supset B_1)\supset C_1$ at $r_1,\ldots,(A_l\supset B_l)\supset C_l$ at $r_l\vdash^P K$ at p, the set of children of n is obtained by: for every $i\in\{1,\ldots,l\}$, we have two children indexed respectively by $\Gamma;\Delta',\Delta''_i,C_i$ at $r_i\vdash^P K$ at p and by $\Gamma;\Delta',\Delta''_i,A_i$ at $r_i,B_i\supset C_i$ at $r_i\vdash^P B_i$ at r_i where $\Delta'=X_1$ at $p_1,\ldots,X_m$ at $p_m,Y_1\supset D_1$ at $q_1,\ldots,Y_n\supset D_n$ at q_n, $\Delta''=(A_1\supset B_1)\supset C_1$ at $r_1,\ldots,(A_l\supset B_l)\supset C_l$ at r_l and Δ''_i is Δ'' without $(A_i\supset B_i)\supset C_i$ at r_i. Moreover, if $K=A\vee B$ then we have two children indexed respectively by $\Gamma;\Delta',\Delta''\vdash^P A$ at p and $\Gamma;\Delta',\Delta''\vdash^P B$ at p. And if $K=\Diamond A$ then for every $pl\in P$, we have a child indexed by $\Gamma;\Delta',\Delta''\vdash^P A$ at pl.
4) The leaf nodes are indexed by axioms and irreducible sequents.

Proposition 5.12 *For a rule* $\dfrac{\Gamma;\Delta'\vdash^P G' \text{ at } p \qquad \Gamma;\Delta''\vdash^P G'' \text{ at } p}{\Gamma;\Delta\vdash^P G \text{ at } p}\,[R]$ *(resp.* $\dfrac{\Gamma';\Delta'\vdash^{P'} G' \text{ at } p'}{\Gamma;\Delta\vdash^P G \text{ at } p}\,[R]$ *) and* $\forall\mathcal{K}=(K,\leqslant,\{P_k\}_{k\in K},\{I_k\}_{k\in K})$ *and* $\forall k\in K$ *such that* $P\subseteq P_k$ *(resp.* $P'\subseteq P_k$*), if (*$k\vDash\Delta'$ *or* $k\vDash\Delta''$*) (resp.* $k\vDash\Delta'$*) then* $k\vDash\Delta$.

Proof We consider the rules $[(\supset)\supset_L]$, $[\Diamond_L]$ and $[(\wedge)\supset_L]$. We start with the rule $[(\supset)\supset_L]$. Let $\mathcal{K}=(K,\leqslant,\{P_k\}_{k\in K},\{I_k\}_{k\in K})$ be a Kripke model and $k\in K$ such that $P\subseteq P_k$. If $k\vDash\Delta,A$ at $p,B\supset C$ at p then by monotonicity $\forall k'\geqslant k$ we have $k'\vDash\Delta,A$ at $p,B\supset C$ at p. Thus, $\forall k'\geqslant k$, if $(k',p)\vDash A\supset B$ then $(k',p)\vDash B$ and we have $(k',p)\vDash C$. Therefore, $k\vDash\Delta,(A\supset B)\supset C$ at p. Otherwise, if $k\vDash\Delta,C$ at p then it is easy to see that $k\vDash\Delta,(A\supset B)\supset C$ at p.
We now consider the rule $[\Diamond_L]$. Let $\mathcal{K}=(K,\leqslant,\{P_k\}_{k\in K},\{I_k\}_{k\in K})$ be a Kripke model and $k\in K$ such that $P+q\subseteq P_k$. If $k\vDash\Delta,A$ at q then $k\vDash\Delta$ and there exists $pl\in P_k$ such that $(k,pl)\vDash A$. Thus, $k\vDash\Delta,\Diamond A$ at p. We now consider the rule $[(\wedge)\supset_L]$. Let $\mathcal{K}=(K,\leqslant,\{P_k\}_{k\in K},\{I_k\}_{k\in K})$ be a Kripke model and $k\in K$ such that $P\subseteq P_k$. If $k\vDash\Delta,A\supset(B\supset C)$ at p then $\forall k'\geqslant k$, if $(k',p)\vDash A$ then $(k',p)\vDash B\supset C$. Thus, if $(k',p)\vDash A\wedge B$ then $(k',p)\vDash B\supset C$ and $(k',p)\vDash C$ because $(k',p)\vDash B$. Therefore $k\vDash\Delta,(A\wedge B)\supset C$ at p. Other cases are treated by similar arguments. □

Theorem 5.13 (Completeness) *Let* $\mathcal{S}=\Gamma;\Delta\vdash^P G$ at p' *be a sequent where the formulae of* Γ *are indexed by the set of places* P. *If* $\mathcal{S}$ *does not have a proof in* $SCT^@$ *then it has a countermodel.*

Proof Let $\mathcal{S}=\Gamma;\Delta\vdash^P G$ at p' be a sequent where all the formulae of Γ are indexed by the set of places P. Let $\mathcal{PR}$ be a proof-refutation tree in which the root node is indexed by $\mathcal{S}$. We suppose that $\mathcal{S}$ has not a proof in $SCT^@$ and we show how to extract a countermodel of $\mathcal{S}$ from $\mathcal{PR}$. See Appendix C. □

First we show how to generate a countermodel for the sequent $\vdash^{\{p\}}(\Diamond A)\supset A$ at p. For this, we need to build the proof-refutation tree associated to this sequent:

$$\dfrac{\dfrac{A \text{ at } q \vdash^{\{p,q\}} A \text{ at } p}{\Diamond A \text{ at } p \vdash^{\{p\}} A \text{ at } p}\,[\Diamond_L]}{\vdash^{\{p\}} \Diamond A \supset A \text{ at } p}\,[\supset_R]$$

	A
p	q

As this logic can be seen as an hybridisation of $IS5$, we can provide, from the previous calculi, a new calculi for $IS5$. For example to prove the formula $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ we can prove $A \supset B; \vdash^{\{p\}} \Diamond A \supset \Diamond B$ at p by using $SCT^{@}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \supset B; A \text{ at } q, B \text{ at } q \vdash^{\{p,q\}} B \text{ at } q}{A \supset B; A \text{ at } q, A \supset B \text{ at } q \vdash^{\{p,q\}} \Diamond B \text{ at } p}\,[\Diamond_R]}{A \supset B; A \text{ at } q \vdash^{\{p\}} \Diamond B \text{ at } p}\,[COPY]}{A \supset B; \Diamond A \text{ at } p \vdash^{\{p\}} \Diamond B \text{ at } p}\,[\Diamond_L]}{A \supset B; \vdash^{\{p\}} \Diamond A \supset \Diamond B \text{ at } p}\,[\supset_R]$$

6 Conclusion and Perspectives

In this paper we propose a sequent calculus for $IHML^p$ and its variants that absorb weakening and contraction rules. Moreover tableaux rules are naturally designed from a derived multi-conclusioned sequent calculus. Knowing that this logic can be seen as an hybridisation of the intuitionistic modal system $IS5$, namely it corresponds to $IS5$ extended with a satisfaction operator (@), we can provide, from our calculi, new calculi and decision procedures for $IS5$. Further investigations will be devoted to the comparison with existing calculi for such a logic [14]. Moreover we define a terminating calculus for the $\Box$-free fragment of $IHML^p$ that allows to build (finite) countermodels in case of non-validity. A consequence of this study, not developed here, is the proof of the finite model property w.r.t. the Kripke semantics for this $\Box$-free fragment and thus of the same result for $IS5$ without $\Box$. Next studies will be devoted to the definition of specific rules for the $\Box$ modality and to the characterization of the logical fragment of the logic, including $\Box$, for which the finite model property w.r.t. the Kripke semantics is verified. Moreover we will focus on semantics and on the design of new tree-based structures allowing to build finite countermodels w.r.t. birelational semantics. Finally we will consider our approach for the extension of the logic with nominals in order to deal with a full intuitionistic hybrid logic like in [3].

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Appendix A: Proof of Theorem 3.3

Theorem 3.3. *Let $\mathcal{S}$ be a sequent. If $\mathcal{S}$ has a proof in $SC_1^@$ then $\mathcal{S}$ has a proof in $SC_1^@$ without using the cut rules.*

Proof We let $\rhd_{SC_1^@ -}\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'$ denote that $\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'$ has a derivation in $SC_1^@$ without using the cut rules. To make the proof we use the structural cut-elimination described in [10], by using a simple structural induction from the admissibility of the cut rules in the cut-free system.Thus, we have only to show:

(i) If $\rhd_{SC_1^@ -}\Gamma;\Delta\vdash^P A\ \mathsf{at}\ p$ and $\rhd_{SC_1^@ -}\Gamma;\Delta,A\ \mathsf{at}\ p\vdash^P G\ \mathsf{at}\ p'$ then $\rhd_{SC_1^@ -}\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'$.

(ii) If $\rhd_{SC_1^@ -}\Gamma;\Delta\vdash^{P+q} A\ \mathsf{at}\ q$ and $\rhd_{SC_1^@ -}\Gamma,A;\Delta\vdash^P G\ \mathsf{at}\ p'$ then $\rhd_{SC_1^@ -}\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'$.

The proof proceeds by mutual structural induction on the cut formula and the given derivations. For example, if we are in the case:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma;\Delta,A\ \mathsf{at}\ p\vdash^P B\ \mathsf{at}\ p}}{\Gamma;\Delta\vdash^P A\supset B\ \mathsf{at}\ p}[\supset_R] \quad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma;\Delta\vdash^P A\ \mathsf{at}\ p} \quad \dfrac{\mathcal{D}_3}{\Gamma;\Delta,B\ \mathsf{at}\ p\vdash^P G\ \mathsf{at}\ p'}}{\Gamma;\Delta,A\supset B\ \mathsf{at}\ p\vdash^P G\ \mathsf{at}\ p'}[\supset_L]}{\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'}[CUT_1]$$

It can be replaced by:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{\Gamma;\Delta\vdash^P A\ \mathsf{at}\ p} \quad \dfrac{\mathcal{D}_1}{\Gamma;\Delta,A\ \mathsf{at}\ p\vdash^P B\ \mathsf{at}\ p}}{\Gamma;\Delta\vdash^P B\ \mathsf{at}\ p}[CUT_1] \quad \dfrac{\mathcal{D}_3}{\Gamma;\Delta,B\ \mathsf{at}\ p\vdash^P G\ \mathsf{at}\ p'}}{\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'}[CUT_1]$$

Since A and B are structurally lower than $A\supset B$, we deduce, by the induction hypothesis, that $\rhd_{SC_1^@ -}\Gamma;\Delta\vdash^P G\ \mathsf{at}\ p'$. □

Appendix B: Proof of Proposition 5.8

Proposition 5.8. *The application of the $SCT^{@}$ rules to a given sequent terminates with axioms or irreducible sequents.*

Proof From Proposition 5.7, the number of the applications of the COPY rule in every derivation of a given sequent in $SCT^{@}$ is finite. Thus, to prove termination, we have only to prove that the application of the $SCT^{@}$ rules without the COPY rule to a given sequent terminates. For this, we will use the technic proposed in [6], by showing that for every rule, its conclusion is more complex than its premises by using a measures of complexity over the pure formulae and the sentences. Here, we use the measure α defined by:
$\alpha(A) = 1\ (A \in \mathsf{Var} \cup \{\top, \bot\})$, $\alpha(A \wedge B) = \alpha(A) + \alpha(B) + 1$, $\alpha(A \vee B) = \alpha(A) + \alpha(B) + 1$, $\alpha(A \supset B) = 2 * \alpha(A) + \alpha(B) + 1$, $\alpha(\Diamond A) = \alpha(A) + 1$, $\alpha(\Box A) = \alpha(A) + 1$, $\alpha(A @ p) = \alpha(A) + 1$, $\alpha(A \text{ at } p) = \alpha(A) + 1$.
From this definition, the order relation $>$ on pure formulae and sentences, with $A > B$ iff $\alpha(A) > \alpha(b)$, is well-founded. Now, we define an order relation on multisets of pure formulae and sentences: let M_1 and M_2 two multisets of pure formulae and sentences, $M_1 >_m M_2$ iff M_2 is obtained form M_1 by replacing one or more pure formulae and sentences by a finite number of pure formulae and sentences, such that if A is replaced by B then $\alpha(A) > \alpha(B)$. Since the relation order on pure formulae and sentences is well-fonded, the order relation $>_m$ is well-founded [5]. It is the order relation which is used to show that in every rule, the conclusion is greater than the premises. For example for the rule $[(\wedge)\supset_L]$, we have $\Gamma \cup \Delta \cup \{(A \wedge B) \supset C \text{ at } p\} \cup \{G \text{ at } p'\} >_m \Gamma \cup \Delta \cup \{A \supset (B \supset C) \text{ at } p\} \cup \{G \text{ at } p'\}$, because $\alpha((A \wedge B) \supset C) = 2 * \alpha(A) + 2 * \alpha(B) + \alpha(C) + 3 > 2 * \alpha(A) + 2 * \alpha(B) + \alpha(C) + 2 = \alpha(A \supset (B \supset C))$.
Since there is always a rule for any sequent which is not an axiom or an irreducible sequent, we deduce that the application of the $SCT^{@}$ rules to a given sequent terminates with axioms or irreducible sequents. $\square$

Appendix C: Proof of Theorem 5.13

Theorem 5.13. *Let $\mathcal{S} = \Gamma; \Delta \vdash^P G \text{ at } p'$ be a sequent where all the formulae of Γ are indexed by the set of places P. If $\mathcal{S}$ has not a proof in $SCT^{@}$ then it has a countermodel.*

Proof Let $\mathcal{S} = \Gamma; \Delta \vdash^P G \text{ at } p'$ be a sequent where all the formulae of Γ are indexed by the set of places P. Let $\mathcal{PR}$ be a proof-refutation tree in which the root node is indexed by $\mathcal{S}$. We suppose that $\mathcal{S}$ has not a proof in $SCT^{@}$ and we show how to extract a countermodel of $\mathcal{S}$ from $\mathcal{PR}$.

We show how to decide if an index of a given node in $\mathcal{PR}$ is valid or not. We start by the leaf nodes. We know that the leaf nodes of $\mathcal{PR}$ are indexed by axioms and irreducible sequents. If a leaf node is indexed by an axiom then its index is valid. Now, we prove that the irreducible sequents are not valid. Let $\mathcal{L} = \Gamma; X_1 \text{ at } p_1, \ldots, X_m \text{ at } p_m, Y_1 \supset C_1 \text{ at } q_1, \ldots, Y_n \supset C_n \text{ at } q_n \vdash^P K \text{ at } p$ be an irreducible sequent. We denote by VP the set $\{X_1 \text{ at } p_1, \ldots, X_m \text{ at } p_m\}$. Let $\mathcal{T} = (\mathcal{N}, \emptyset)$ be a Kripke tree with a single node such that $PL_{\mathcal{N}} = P$, $Var_{\mathcal{N}} = \{X_1, \ldots, X_m\}$, $\forall r \in P$ we have $S^r_{\mathcal{N}} = \{X_k \mid X_k \text{ at } r \in VP\}$. We have $\forall i \in 1 \ldots, m$, $(\mathcal{T}, p_i) \vDash X_i$, and since $\forall i \in \{1, \ldots, m\}$ and $\forall j \in \{1, \ldots, n\}$ we have if $X_i \equiv Y_j$ then $p_i \neq q_j$, we obtain $\forall j \in 1 \ldots, n$, $(\mathcal{T}, q_j) \nvDash Y_j$, and thus, $\forall j \in 1 \ldots, n$, $(\mathcal{T}, q_j) \vDash Y_j \supset C_j \text{ at } q_j$. We can see that for every $A^{\emptyset} \in \Gamma$ and $p \in P$, there exists a derivation with a root sequent of the form $\Gamma'; \Delta', A \text{ at } p \vdash^{P'} G' \text{ at } q$ where $\mathcal{L}$ is one of its leaf sequents. Thus, by using

Proposition 5.12, we have $\mathcal{T} \vDash \Gamma$. It is easy to see that $(\mathcal{T}, p) \nvDash K$. From Proposition 5.4, we deduce that $\mathcal{K}_{\mathcal{T}}$ is a countermodel of $\mathcal{L}$.

Now, we see how, from the children of a given internal node, we can propagate the validity or build a countermodel. Let I be an index of an internal node. If I is not an inv-irreducible sequent then, from Definition 5.11, this node has a maximum of two children where if these children are indexed by $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}$) then $\dfrac{\mathcal{H}_1 \qquad \mathcal{H}_2}{\mathcal{S}}\,[R]$ (resp. $\dfrac{\mathcal{H}}{\mathcal{S}}\,[R]$) is an instance of a strongly invertible rule. Thus, if $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}$) are valid then I is valid because $[R]$ is a sound rule. Else, from the strong invertibility of $[R]$, I has the same countermodels of the non-valid premises of $[R]$.

Let us consider the case of the internal nodes indexed by inv-irreducible sequents. Let $I = \Gamma; X_1 \text{ at } p_1, \ldots, X_m \text{ at } p_m, Y_1 \supset D_1 \text{ at } q_1, \ldots, Y_n \supset D_n \text{ at } q_n, (A_1 \supset B_1) \supset C_1 \text{ at } r_1, \ldots, (A_l \supset B_l) \supset C_l \text{ at } r_l \vdash^P K \text{ at } p$ be an inv-irreducible sequent and the index of an internal node. We define $\Delta' \triangleq X_1 \text{ at } p_1, \ldots, X_m \text{ at } p_m, Y_1 \supset D_1 \text{ at } q_1, \ldots, Y_n \supset D_n \text{ at } q_n$ and $\Delta'' \triangleq (A_1 \supset B_1) \supset C_1 \text{ at } r_1, \ldots, (A_l \supset B_l) \supset C_l \text{ at } r_l \vdash^P K \text{ at } p$. Then, we define Δ''_i for $i \in \{1, \ldots, l\}$ by Δ'' without $(A_i \supset B_i) \supset C_i \text{ at } r_i$ and $VP \triangleq \{X_1 \text{ at } p_1, \ldots, X_m \text{ at } p_m\}$. Here, we start by studying the case where $k \equiv \Diamond F$. From Definition 5.11, the children of our internal node are indexed by the premises of the following rules:

$$\dfrac{\Gamma; \Delta', \Delta''_i, A_i \text{ at } r_i, B_i \supset C_i \text{ at } r_i \vdash^P B_i \text{ at } r_i \qquad \Gamma; \Delta', \Delta''_i, C_i \text{ at } r_i \vdash^P \Diamond F \text{ at } p}{\Gamma; \Delta', \Delta''_i, (A_i \supset B_i) \supset C_i \text{ at } r_i \vdash^P \Diamond F \text{ at } p}\,[(\supset)\supset_L]$$

and

$$\dfrac{\Gamma; \Delta', \Delta'' \vdash^P F \text{ at } pl}{\Gamma; \Delta', \Delta'' \vdash^P \Diamond F \text{ at } p}\,[\Diamond_L]$$

where $i \in \{1, \ldots, l\}$ and $pl \in P$. If there exists $i \in \{1, \ldots, l\}$ such that $\Gamma; \Delta', \Delta''_i, C_i \text{ at } r_i \vdash^P \Diamond F \text{ at } p$ is not valid, then, it has a countermodel $\mathcal{T}_{C_i}$. Therefore, $\mathcal{T}_{C_i}$ is a countermodel of I because the premiss $\Gamma; \Delta', \Delta''_i, C_i \text{ at } r_i \vdash^P \Diamond F \text{ at } p$ in $[(\supset)\supset_L]$ is strongly invertible. Else, if there exists $i \in \{1, \ldots, l\}$ such that $\Gamma; \Delta', \Delta''_i, A_i \text{ at } r_i, B_i \supset C_i \text{ at } r_i \vdash^P B_i \text{ at } r_i$ is valid or there exists $pl \in P$ such that $\Gamma; \Delta', \Delta'' \vdash^P F \text{ at } pl$ is valid, then I is valid because the rules $[(\supset)\supset_L]$ and $[\Diamond_L]$ are sound. Now we deal with the last case, $\forall i \in \{1, \ldots, l\}$, $\Gamma; \Delta', \Delta''_i, A_i \text{ at } r_i, B_i \supset C_i \text{ at } r_i \vdash^P B_i \text{ at } r_i$ has a countermodel $\mathcal{K}_{\mathcal{T}_i}$; and for all $pl \in P$ we have $\Gamma; \Delta', \Delta'' \vdash^P F \text{ at } pl$ has a countermodel $\mathcal{K}_{\mathcal{T}_{pl}}$. We define $\mathcal{T} \triangleq (\mathcal{N}, \{\mathcal{T}_1, \ldots, \mathcal{T}_l, \mathcal{T}_{pl_1}, \ldots, \mathcal{T}_{pl_k}\})$ where $P = \{pl_1, \ldots, pl_k\}$, $PL_{\mathcal{N}} = P$, $Var_{\mathcal{N}} = \{X_1, \ldots, X_m\}$, $\forall r \in P$ we have $S^r_{\mathcal{N}} = \{X_k \mid X_k \text{ at } r \in VP\}$. It is easy to see that $\mathcal{T}$ is a Kripke tree.

Now we prove that $\mathcal{T} \vDash \Gamma; \Delta', \Delta''$ and $(\mathcal{T}, p) \nvDash \Diamond F$ in $\mathcal{K}_{\mathcal{T}}$. By using Proposition 5.12, we have for all $i \in \{1, \ldots, l\}$ and for all $j \in \{1, \ldots, k\}$, $\mathcal{T}_i \vDash \Gamma; \Delta', \Delta''$ and $\mathcal{T}_{pl_j} \vDash \Gamma; \Delta', \Delta''$. We have for all $i \in \{1, \ldots, l\}$, $\mathcal{T}_i \vDash A$ and $\mathcal{T}_i \nvDash B$. Thus, $\mathcal{T}_i \nvDash A \supset B$ and by Kripke monotonicity we obtain $\mathcal{T} \nvDash A \supset B$. Therefore, $\mathcal{T} \vDash \Gamma; \Delta', \Delta''$ holds. As $\forall j \in \{1, \ldots, k\}$ we have $(\mathcal{T}_{pl_j}, pl_j) \nvDash F$ at pl, and we obtain by monotonicity $(\mathcal{T}, p) \nvDash \Diamond F$ because in $\mathcal{K}_{\mathcal{T}}$ we have $P_{\mathcal{T}} = P$. Thus, $\mathcal{K}_{\mathcal{T}}$ is a countermodel of $\Gamma; \Delta', \Delta'' \vdash^P \Diamond F \text{ at } p$. For the case $K \equiv A \vee B$, the Kripke model is $\mathcal{T} \triangleq (\mathcal{N}, \{\mathcal{T}_1, \ldots, \mathcal{T}_l, \mathcal{T}_A, \mathcal{T}_B\})$ where $\mathcal{T}_A$ (resp. $\mathcal{T}_B$) is a countermodel of $\Gamma; \Delta', \Delta'' \vdash^P A \text{ at } p$ (resp. $\Gamma; \Delta', \Delta'' \vdash^P B \text{ at } p$). For the case $K \in \mathsf{Var} \cup \bot$, we use the Kripke model $\mathcal{T} \triangleq (\mathcal{N}, \{\mathcal{T}_1, \ldots, \mathcal{T}_l\})$. The proofs of these two cases are similar to the previous proof. We can see that if $\mathcal{S}$ is valid then it has a proof in $SCT^{@}$ and we get a contradiction. Therefore, $\mathcal{S}$ has a countermodel built by the previous method. □