Sequent Calculi and Decidability for Intuitionistic Hybrid Logic

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Abstract. In this paper we study proof theory for the first constructive version of hybrid logic called IHL. In this perspective we propose a sequent-style natural deduction system and then the first sequent calculus for IHL. In addition to soundness and completeness, we show that this calculus has the cut-elimination property. Finally, we give the first decision procedure for IHL, that is based on this calculus, and therefore we prove its decidability.

1 Introduction

In the standard Kripke semantics for modal logics, a model is a transition system where the same formula may have different truth values at different worlds [4,9]. The hybrid logics were mainly introduced in order to express this relativity of truth [3,2] by adding to modal logics a new kind of propositional symbols called *nominals*, and also a new operator, called *satisfaction operator*, that allows one to jump to the world named by a nominal. There exist many works on hybrid logics, mainly on classical versions, about calculi, decidability and complexity [1,2,6,19].

In this work we aim at studying an intuitionistic version that is the first constructive version of hybrid logic, defined by Braüner and de Paiva in [7] and called here IHL. It has been designed from the intuitionistic modal logic IK introduced in [17], knowing that intuitionistic modal logics have some important applications in computer science, for instance for formal verification of computer hardware [11] and also definition of programming languages [10,14]. There exits a natural deduction system for IHL, extended with additional inference rules corresponding to conditions on the accessibility relation but in this logic proof-theory and decidability have not really been explored.

Let us mention another constructive version of hybrid logic [13] that is based on the intuitionistic modal logic IS5 [17] and later enriched with the disjunctive connective and the constant denoting absurdity in [8]. However, this logic cannot be seen as a complete hybridization of IS5 because the nominals (called *places* in the original paper) are only used with the satisfaction operator. We have recently studied proof-theory for this logic by defining sequent calculi dedicated to proof and countermodel construction [12]. Thus we have provided an alternative proof of decidability by proof-theoretical arguments and shown that that sequent calculus formalism can be a good formalism allowing an effective management of nominals in the proof-search process. Even if IHL is also an intuitionistic hybrid logic these results cannot be directly extended for this logic.

In this paper we consider the intuitionistic hybrid logic IHL for which as said before there only exists a natural deduction system [7] and decidability is still an open question. In order to solve this problem, we mainly propose a sequent calculus for IHL that is adapted to proof-search but also to the study of decidability. As mentioned before, there are many works dedicated to classical versions of hybrid logics but they cannot be directly adapted to propose a sequent calculus allowing to show decidability in such a constructive or intuitionistic version of hybrid logic. A key point to prove the decidability of IHL by using the sequent calculus is to solve the problem of the introduction of new nominals due to some rules. This problem is similar to the introduction of new labels in the labelled sequent calculi of the intuitionistic modal logics [17]. Let us note that the introduction of new nominals or labels is not a problem in the case of classical modal and hybrid logics because we can define proof systems with only invertible rules allowing terminating proof-search [5,15]. But it is a problem in the case of an intuitionistic version of hybrid logics and then one needs to introduce appropriate concepts to deal with it.
In this perspective the main contributions of this work are: the definition of a sequent calculus for IHL, the proofs of some of its properties like cut-elimination and the first proof of its decidability that is based this calculus. From these results we will study, in next works, the complexity of IHL [1], but also extensions of our sequent calculus with rules corresponding to conditions on the accessibility relations (geometric theories) like reflexivity, symmetry and transitivity, in order to obtain a modular system in which each condition on the accessibility relation has a corresponding rule and each combination of these rules is complete for the logic with the corresponding conditions.

In this context, Section 2 presents the first constructive version of intuitionistic hybrid logic IHL [7] and the known related results. In Section 3 we give a natural deduction system for IHL in a sequent-style in order to deal with validity in IHL. It is derived from the initial natural deduction system $\mathsf{N}_{\mathsf{IHL}}$ [7]. In Section 4 we propose a sequent calculus for IHL that is called $\mathsf{G}_{\mathsf{IHL}}$ and then prove its soundness from the semantics. In Section 5 we prove that the calculus satisfies the cut-elimination property.and shows its completeness by proving that if a sequent is derivable in $\mathsf{G}_{\mathsf{IHL}}$ then it is derivable in $\mathsf{NS}_{\mathsf{IHL}}$. In Section 6 we prove the decidability of IHL by using the sequent calculus $\mathsf{G}^{-}_{\mathsf{IHL}}$. The key point of the decision procedure we design is the use of the cut-elimination property in order to provide a suitable subformula property different from the usual property and called the *quasi-subformula property*. In this perspective we introduce a notion of *redundancy* on cut-free derivations in our calculus such that that any sequent that is valid has an irredundant proof. Then, by using the quasi-subformula property, we prove that there is no infinite proof which is not irredundant and then provide a decision procedure for IHL and then prove the decidability of this logic through proof-search using our sequent calculus.

2 Intuitionistic Hybrid Logic

Hybrid logics are logics obtained by adding to modal logics a new kind of propositional symbols, called *nominals*, which are used to refer to specific worlds in a model and also a new kind of operators called the *satisfaction operators* that allow us to jump to the worlds named by nominals. For more details about hybrid logics see [3]. In this paper, we study the first constructive version of intuitionistic hybrid logic IHL [7].

Let Prop be a countably set of propositional symbols and Nom be a countably set of nominals that is disjoint from Prop. We use $p, q, r, \ldots$ to range over Prop; and $a, b, c, \ldots$ to range over nominals. Moreover $Nom(S)$ denotes the set of nominals that appear in the syntactic object S.

The formulas of IHL are given by the following grammar:

$$\mathcal{F} ::= p \mid a \mid \bot \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \supset \mathcal{F} \mid \Box\mathcal{F} \mid \Diamond\mathcal{F} \mid a : \mathcal{F}$$

Definition 1. *A* IHL-Kripke model *is a tuple*

$$(W, \leqslant, \{D_w\}_{w\in W}, \{\sim_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$$

with
- W is a non-empty set (of 'worlds') partially ordered by $\leqslant$;
- for each $w \in W$, D_w is a non-empty set such that $w \leqslant w'$ implies $D_w \subseteq D_{w'}$;
- for each $w \in W$, $\sim_w$ is an equivalence relation on D_w such that $w \leqslant w'$ implies $\sim_w \subseteq \sim_{w'}$;
- for each $w \in W$, R_w is a binary relation on D_w such that $w \leqslant w'$ implies $R_w \subseteq R_{w'}$;
- for each $w \in W$, V_w is a function that to each $p \in$ Prop assigns a subset of D_w such that $w \leqslant w'$ implies $V_w(p) \subseteq V_{w'}(p)$.
It is assumed that if $d \sim_w d'$, $e \sim_w e'$ and $R_w(d,e)$ then $R_w(d',e')$, and similarly, if $d \sim_w d'$ and $d \in V_w(p)$ then $d' \in V_w(p)$.

Given a IHL-Kripke model $(W, \leqslant, \{D_w\}_{w\in W}, \{\sim_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ and an element $w \in W$, a *w-assignment* is a function which assigns to each nominal an element of D_w.

Definition 2. *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{\sim_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a* IHL-*Kripke model, $w \in W$, g be a w-assignment, $d \in D_w$ and A be a formula, $\mathcal{M}, g, w, d \vDash A$ is inductively defined as follows:*
- $\mathcal{M}, g, w, d \vDash p$ iff $d \in V_w(p)$;
- $\mathcal{M}, g, w, d \vDash a$ iff $g(a) \sim_w d$;
- $\mathcal{M}, g, w, d \vDash \bot$ never;
- $\mathcal{M}, g, w, d \vDash A \wedge B$ iff $\mathcal{M}, g, w, d \vDash A$ and $\mathcal{M}, g, w, d \vDash B$;
- $\mathcal{M}, g, w, d \vDash A \vee B$ iff $\mathcal{M}, g, w, d \vDash A$ or $\mathcal{M}, g, w, d \vDash B$;
- $\mathcal{M}, g, w, d \vDash A \supset B$ iff for all $v \geqslant w$, $\mathcal{M}, g, v, d \vDash A$ implies $\mathcal{M}, g, v, d \vDash B$;
- $\mathcal{M}, g, w, d \vDash \Box A$ iff for all $v \geqslant w$, $e \in D_v$, $R_v(d,e)$ implies $\mathcal{M}, g, v, e \vDash A$;
- $\mathcal{M}, g, w, d \vDash \Diamond A$ iff there exists $e \in D_w$ s. t. $R_w(d,e)$ and $\mathcal{M}, g, w, e \vDash A$;
- $\mathcal{M}, g, w, d \vDash a : A$ iff $\mathcal{M}, g, w, g(a) \vDash A$.

$$\frac{a:A \qquad a:B}{a:A\wedge B}\,[\wedge_I] \qquad \frac{a:A\wedge B}{a:A}\,[\wedge_E^1] \qquad \frac{a:A\wedge B}{a:B}\,[\wedge_E^2]$$

$$\frac{a:A}{a:A\vee B}\,[\vee_I^1] \qquad \frac{a:B}{a:A\vee B}\,[\vee_I^1] \qquad \frac{a:A\vee B \quad \begin{array}{c}[a:A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[a:B]\\ \vdots\\ C\end{array}}{C}\,[\vee_E]$$

$$\frac{\begin{array}{c}[a:A]\\ \vdots\\ a:B\end{array}}{a:A\supset B}\,[\supset_I] \qquad \frac{a:A\supset B \qquad a:A}{a:B}\,[\supset_E]$$

$$\frac{a:A}{c:a:A}\,[:_I] \qquad \frac{c:a:A}{a:A}\,[:_E]$$

$$\frac{e:A \qquad a:\Diamond e}{a:\Diamond A}\,[\Diamond_I] \qquad \frac{a:\Diamond A \quad \begin{array}{c}[c:A][a:\Diamond c]\\ \vdots\\ C\end{array}}{C}\,[\Diamond_E *]$$

$$\frac{\begin{array}{c}[a:\Diamond c]\\ \vdots\\ c:A\end{array}}{a:\Box A}\,[\Box_I **] \qquad \frac{a:\Box A \qquad a:\Diamond e}{e:A}\,[\Box_E]$$

$$\frac{a:\bot}{C}\,[\bot_E] \qquad \frac{}{a:a}\,[Ref] \qquad \frac{a:c \qquad a:A}{c:A}\,[Nom]$$

$*$ c does not occur in $a:\Diamond A$, in C, or in any undischarged assumption other than the specified occurrences of $c:A$ and $a:\Diamond c$.
$**$ c does not occur in $a:\Box A$ or in any undischarged assumption other than the specified occurrences of $a:\Diamond c$.

Fig. 1. The Natural Deduction System $\mathsf{N}_{\mathsf{IHL}}$

Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{\sim_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a IHL-Kripke model, a formula A is *valid in* $\mathcal{M}$, denoted $\mathcal{M} \vDash A$, if and only if $\mathcal{M}, g, w, d \vDash A$ for every $w \in W$, every w-assignment g and every $d \in D_w$. A formula is *valid in* IHL, denoted $\mathsf{IHL} \vDash A$, if and only if $\mathcal{M} \vDash A$ for every IHL-Kripke model $\mathcal{M}$.

Moreover, let us note that for any formula A and any Kripke model $\mathcal{M}$, $\mathcal{M} \vDash A$ if and only if $\mathcal{M} \vDash a : A$ where $a \notin Nom(A)$. Therefore, there is no loss of generality by considering only *satisfaction statements* that are statements of the form $a : A$.

Proposition 1 (Monotonicity). *If $\mathcal{M}, g, w, d \vDash A$ and $w \leqslant w'$, then $\mathcal{M}, g, w', d \vDash A$.*

Proof. By structural induction on A.

Proposition 2 (Equivalence). *If $\mathcal{M}, g, w, d \vDash A$ and $d \sim_w d'$, then $\mathcal{M}, g, w, d' \vDash A$.*

Proof. By structural induction on A.

The first results for IHL deal with some proof-theoretical aspects that are based on a natural deduction system [7] given in Figure 1. No other alternative calculi like sequent calculi have been proposed and the decidability of IHL is an open question.
The main goal of this paper is to study this question and to present the first proof of decidability for IHL through a decision procedure based on a sequent calculus.

3 A Sequent-style Natural Deduction System for IHL

In this section, we give a new natural deduction system for IHL in a sequent-style. It is obtained from the natural deduction system $\mathsf{N}_{\mathsf{IHL}}$ given in [7] (see Figure 1). Our main point here consists in defining a new system in order to deal with validity in IHL. It is a first step towards the new sequent calculus for this logic.

Definition 3 (Sequent). *A sequent is a structure of the form $\Gamma \vdash C$ where Γ is a possibly empty finite multiset of satisfaction statements and C is a satisfaction statement.*

A sequent $\Gamma \vdash C$ corresponds to the formula $(\bigwedge \Gamma) \supset C$. We use the notation $\bigwedge \Gamma$ as a shorthand for $a_1 : A_1 \wedge \ldots \wedge a_k : A_k$ when $\Gamma = a_1 : A_1, \ldots, a_k : A_k$. If Γ is empty, we identify $\bigwedge \Gamma$ with $\top$. We note $\mathcal{M}, g, w \vDash \Gamma$ if $\mathcal{M}, g, w, d \vDash \bigwedge \Gamma$ for $d \in D_w$ (the choice of d is not important).

The natural deduction system $\mathsf{NS}_{\mathsf{IHL}}$ is given in Figure 2. In fact $\mathsf{NS}_{\mathsf{IHL}}$ is nothing more than the natural deduction system $\mathsf{N}_{\mathsf{IHL}}$ with contexts. The use of the sequent structure can be seen as a syntactic way to denote particular formulas. Indeed the comma in left-hand side is interpreted as a conjunction and the symbol $\vdash$ as an implication.
Let us note that, like in the system $\mathsf{N}_{\mathsf{IHL}}$, we use the formulas of the form $a : \Diamond c$ to represent the accessibility relation. We can easily see that $\mathcal{M}, g, w, d \vDash a : \Diamond c$ if and only if $R(g(a), g(c))$.

Let us consider the formula $F = a : ((A \vee B \supset c : C) \supset (A \supset c : C))$. A proof of this formula in $\mathsf{NS}_{\mathsf{IHL}}$ is

$$\frac{}{\Gamma \vdash a : a}\,[Ref] \qquad \frac{}{\Gamma, a : A \vdash a : A}\,[Id]$$

$$\frac{\Gamma \vdash a : \bot}{\Gamma \vdash C}\,[\bot_E] \qquad \frac{\Gamma \vdash a : c \qquad \Gamma \vdash a : A}{\Gamma \vdash c : A}\,[Nom]$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash a : B}{\Gamma \vdash a : A \wedge B}\,[\wedge_I] \qquad \frac{\Gamma \vdash a : A \wedge B}{\Gamma \vdash a : A}\,[\wedge_E^1] \qquad \frac{\Gamma \vdash a : A \wedge B}{\Gamma \vdash a : B}\,[\wedge_E^2]$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a : A \vee B}\,[\vee_I^1] \qquad \frac{\Gamma \vdash a : B}{\Gamma \vdash a : A \vee B}\,[\vee_I^1]$$

$$\frac{\Gamma \vdash a : A \vee B \quad \Gamma, a : A \vdash C \quad \Gamma, a : B \vdash C}{\Gamma \vdash C}\,[\vee_E]$$

$$\frac{\Gamma, a : A \vdash a : B}{\Gamma \vdash a : A \supset B}\,[\supset_I] \qquad \frac{\Gamma \vdash a : A \supset B \qquad \Gamma \vdash a : A}{\Gamma \vdash a : B}\,[\supset_E]$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash c : a : A}\,[:_I] \qquad \frac{\Gamma \vdash c : a : A}{\Gamma \vdash a : A}\,[:_E]$$

$$\frac{\Gamma \vdash c : A \qquad \Gamma \vdash a : \Diamond c}{\Gamma \vdash a : \Diamond A}\,[\Diamond_I] \qquad \frac{\Gamma \vdash a : \Diamond A \qquad \Gamma, a : \Diamond c, c : A \vdash C}{\Gamma \vdash C}\,[\Diamond_E *]$$

$$\frac{\Gamma, a : \Diamond c \vdash c : A}{\Gamma \vdash a : \Box A}\,[\Box_I **] \qquad \frac{\Gamma \vdash a : \Box A \qquad \Gamma \vdash a : \Diamond c}{\Gamma \vdash c : A}\,[\Box_E]$$

$*$ c does not occur in $a : \Diamond A$, in C or in any assumption in Γ.
$**$ c does not occur in $a : \Box A$ or in any assumption in Γ.

Fig. 2. The Natural Deduction System $\mathsf{NS}_{\mathsf{IHL}}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a : (A \vee B \supset c : C), a : A \vdash a : (A \vee B \supset c : C)}\,[Id] \qquad \dfrac{\dfrac{}{a : (A \vee B \supset c : C), a : A \vdash a : A}\,[Id]}{a : (A \vee B \supset c : C), a : A \vdash a : (A \vee B)}\,[\vee_I^1]}{a : (A \vee B \supset c : C), a : A \vdash a : c : C}\,[\supset_E]}{a : (A \vee B \supset c : C) \vdash a : (A \supset c : C)}\,[\supset_I]}{\vdash a : ((A \vee B \supset c : C) \supset (A \supset c : C))}\,[\supset_I]$$

In order to illustrate the differences with the initial system $\mathsf{N}_{\mathsf{IHL}}$ we give the following proof of F in $\mathsf{N}_{\mathsf{IHL}}$:

$$\dfrac{\dfrac{\dfrac{[a:A\vee B\supset c:C] \quad \dfrac{[a:A]}{a:A\vee B}[\vee_I^1]}{c:C}[\supset_E]}{a:A\supset c:C}[\supset_I]}{a:((A\vee B\supset c:C)\supset(A\supset c:C))}[\supset_I]$$

Let us show now that our new sequent-style natural deduction system $\mathsf{NS}_{\mathsf{IHL}}$ is sound and complete.

Theorem 1 (Soundness). *The system* $\mathsf{NS}_{\mathsf{IHL}}$ *is sound.*

Proof. Proceeding contrapositively, for every rule, we suppose that its conclusion is not valid and prove that one of its premises is not valid. Here, we only show the case $[\Box_I]$. Let $\mathcal{S} = \Gamma \vdash a : \Box A$ be a sequent that is not valid and let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{\sim_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a countermodel of $\mathcal{S}$. Then, there exist $w_0 \in W$ and a w_0-assignment g such that $\mathcal{M}, g, w_0 \vDash \Gamma$ and $\mathcal{M}, g, w_0 \nvDash a : \Box A$. Since $\mathcal{M}, g, w_0 \nvDash a : \Box A$, we know that there exist $w_1 \geqslant w_0$ and $d \in D_{w_1}$ such that $R_{w_1}(g(a), d)$ and $\mathcal{M}, g, w_1, d \nvDash A$. Let c be a nominal not in $Nom(\mathcal{S})$. We define the w_1-assignment g' by $g'(c) = d$ and for any nominal b, different from c, $g'(b) = g(b)$. Using Proposition 1, $\mathcal{M}, g', w_1 \vDash \Gamma$ holds and as $R_{w_1}(g(a), d)$ holds, we have $\mathcal{M}, g', w_1 \vDash a : \Diamond c$. As we have $\mathcal{M}, g, w_1, d \nvDash A$ and $\mathcal{M}, g, w_1 \nvDash c : A$, we deduce that $\mathcal{M}$ is a countermodel of $\Gamma, a : \Diamond c \vdash c : A$.

Theorem 2 (Completeness). *The system* $\mathsf{NS}_{\mathsf{IHL}}$ *is complete.*

Proof. The proof is obtained from the system $\mathsf{N}_{\mathsf{IHL}}$ by using the approach proposed in [18]. Intuitively, the open assumptions in a derivation tree in $\mathsf{N}_{\mathsf{IHL}}$ are represented in the left-hand side of the corresponding sequent. We see that if we define a natural deduction system similar to $\mathsf{NS}_{\mathsf{IHL}}$ where we only replace the *discharge* of only one assumption with the discharge of all the assumptions of the same form (*Complete Discharge Convention*), then we obtain a system equivalent to $\mathsf{N}_{\mathsf{IHL}}$. For example the rule $[\supset_I]$ becomes

$$\dfrac{\Gamma, a : A \vdash a : B}{\Gamma' \vdash a : A \supset B}\ [\supset_I]$$

where $\Gamma' = \Gamma \setminus \{a : A\}$ (there is no occurrence of $a : A$ in Γ'). Now, to prove the completeness of $\mathsf{NS}_{\mathsf{IHL}}$, we only have to show that if a sequent has a derivation in the latter system then it has a derivation in $\mathsf{NS}_{\mathsf{IHL}}$.

4 The Sequent Calculus $\mathsf{G}_{\mathsf{IHL}}$

In this section, we propose a sequent calculus for IHL, called $\mathsf{G}_{\mathsf{IHL}}$. Its soundness is proved by using the semantics and its completeness is proved by showing that if a sequent is derivable in $\mathsf{G}_{\mathsf{IHL}}$ then it is derivable in $\mathsf{NS}_{\mathsf{IHL}}$.

$$\frac{}{\Gamma, a : p \vdash a' : p}\ [Id](a \sim a' \texttt{ and } p \texttt{ atomic}) \qquad \frac{}{\Gamma, a : \bot \vdash C}\ [\bot] \qquad \frac{}{\Gamma \vdash a' : a}\ [Ref](a \sim a')$$

$$\frac{\Gamma, a : A, a : B \vdash C}{\Gamma, a : A \wedge B \vdash C}\ [\wedge_L] \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash a : B}{\Gamma \vdash a : A \wedge B}\ [\wedge_R]$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a : A \vee B}\ [\vee_{R_1}] \qquad \frac{\Gamma \vdash a : B}{\Gamma \vdash a : A \vee B}\ [\vee_{R_2}] \qquad \frac{\Gamma, a : A \vdash C \qquad \Gamma, a : B \vdash C}{\Gamma, a : A \vee B \vdash C}\ [\vee_L]$$

$$\frac{\Gamma, a : A \supset B \vdash a : A \qquad \Gamma, a : B \vdash C}{\Gamma, a : A \supset B \vdash C}\ [\supset_L] \qquad \frac{\Gamma, a : A \vdash a : B}{\Gamma \vdash a : A \supset B}\ [\supset_R]$$

$$\frac{\Gamma, a : A \vdash C}{\Gamma, c : a : A \vdash C}\ [:_L] \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash c : a : A}\ [:_R]$$

$$\frac{\Gamma, a' : \Diamond c, a : \Box A, c : A \vdash C}{\Gamma, a' : \Diamond c, a : \Box A \vdash C}\ [\Box_L](a \sim a') \qquad \frac{\Gamma, a : \Diamond c \vdash c : A}{\Gamma \vdash a : \Box A}\ [\Box_R **]$$

$$\frac{\Gamma, a : \Diamond c, c : A \vdash C}{\Gamma, a : \Diamond A \vdash C}\ [\Diamond_L *] \qquad \frac{\Gamma, a' : \Diamond c \vdash c : A}{\Gamma, a' : \Diamond c \vdash a : \Diamond A}\ [\Diamond_R](a \sim a')$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma, a : A \vdash C}{\Gamma \vdash C}\ [Cut]$$

$*$ c does not occur in $\Gamma, a : \Diamond A \vdash C$ and $A \notin \mathsf{Nom}$.

$**$ c does not occur in $\Gamma \vdash a : \Box A$.

Fig. 3. The Sequent Calculus $\mathsf{G}_{\mathsf{IHL}}$

We observe that even if there exist works on the design of sequent calculi in some classical hybrid logics [16] we cannot follow a similar approach in the case of the intuitionistic IHL logic. In our work we use a sequent structure that contains only satisfaction statements because it allows easily to *absorb* the structural rules in the axioms, logical and modal rules. Moreover, as in [7] the premisses and the conclusion of each rule are formulas of the form $a : A$, we can relate our calculi construction with the initial systems provided for IHL. We will see that it facilitates the study of relationships between the cut-elimination and the normalization like in the case of intuitionistic logic [18]. Let us recall that a proof of normalization is given in [7].

The *principal formula* of an application of a rule is defined to be any formula which is introduced by that rule except the cases of $[\supset_L]$ and $[\Box_L]$ where the principal formulas are respectively $a : A \supset B$ and $a : \Box A$. We call *derivation* of a sequent $\mathcal{S}$ in $\mathsf{G}_{\mathsf{IHL}}$ any

tree labelled with sequents such that the root node is labelled with $\mathcal{S}$ and the labels at the immediate successors of a node n are the premises of a rule of $\mathsf{G_{IHL}}$ having the label at n as conclusion.
A sequent $\mathcal{S}$ has a *proof* in $\mathsf{G_{IHL}}$, denoted $\vdash_{\mathsf{G_{IHL}}} \mathcal{S}$, if and only if $\mathcal{S}$ has a finite derivation in $\mathsf{G_{IHL}}$ where any leaf node is labelled with an axiom. Moreover we write $\vdash^{n}_{\mathsf{G_{IHL}}} \mathcal{S}$ if $\mathcal{S}$ has a proof in $\mathsf{G_{IHL}}$ of depth smaller or equal to n.

Let $\mathcal{S} = \Gamma \vdash C$ be a sequent and R be the relation on $Nom(\mathcal{S})$ defined by aRb if and only if $a : b$ is an element of Γ. We note $\sim$ the reflexive, transitive, symmetric closure of R. When we associate a condition of the form $a \sim b$ to a rule, this means that $\sim$ is the reflexive, transitive, symmetric closure of the relation obtained from the conclusion of this rule. The problem to verify the conditions of this form is decidable.

The rules and axioms of $\mathsf{G_{IHL}}$ are given in Figure 3. The approach we used to obtain this calculus is similar to the one used to obtain the calculus $G3i$ for the intuitionistic logic from the calculus LJ by absorbing weakening and contraction into the axioms and the logical rules (see [18]). There are conditions of the form $a \sim b$ associated to some axioms and rules of $\mathsf{G_{IHL}}$ that are due from the absorption of the rules $[Nom]$ of $\mathsf{NS_{IHL}}$.

Let us note that $\mathsf{G_{IHL}}$ is sound and complete without the restriction on $[Id]$ that the principal formula must be atomic. However, without this restriction, $\mathsf{G_{IHL}}$ fails the *depth-preserving admissibility of contraction* property necessary in our approach to prove the cut-elimination property.

We illustrate the use of $\mathsf{G_{IHL}}$ by giving a proof of $a : \Box(b \supset c) \vdash a : \Diamond b \supset \Diamond c$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a : \Box(b \supset c), b : (b \supset c), a : \Diamond b \vdash b : b}\,[Ref] \quad \dfrac{}{a : \Box(b \supset c), a : \Diamond b, d : b, b : c \vdash b : c}\,[Ref]}{a : \Box(b \supset c), b : (b \supset c), a : \Diamond b \vdash b : c}\,[\supset_L]}{a : \Box(b \supset c), b : (b \supset c), a : \Diamond b \vdash a : \Diamond c}\,[\Diamond_R]}{a : \Box(b \supset c), a : \Diamond b \vdash a : \Diamond c}\,[\Box_L]}{a : \Box(b \supset c) \vdash a : \Diamond b \supset \Diamond c}\,[\supset_R]$$

Proposition 3. *Let $\mathcal{S} = \Gamma \vdash C$ be a sequent and $a, a' \in Nom(\mathcal{S})$ such that $a \sim a'$. If $\mathcal{M}, g, w \vDash \Gamma$ then $g(a) \sim_w g(a')$.*

Proof. It comes from the fact that if $\mathcal{M}, g, w \vDash a : b$ then $g(a) \sim_w g(b)$.

Theorem 3 (Soundness). *If $\vdash_{\mathsf{G_{IHL}}} \mathcal{S}$ then $\mathcal{S}$ is valid.*

Proof. Proceeding contrapositively, for every rule, we suppose that its conclusion is not valid and prove that one of its premises is not valid.
Here, we only show the case of $[\Box_L]$ rule. The other cases being similar.

Let us assume that $\mathcal{S} = \Gamma, a' : \Diamond c, a : \Box A \vdash C$ $(a \sim a')$ is not valid and $\mathcal{M} = (W, \leqslant, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ be a countermodel of $\mathcal{S}$. Then, there exist $w_0 \in W$ and a w_0-assignment g such that $\mathcal{M}, g, w_0 \vDash \Gamma, a' : \Diamond c, a : \Box A$ and $\mathcal{M}, g, w_0 \nvDash C$. Since $a \sim a'$, we have $g(a) \sim_{w_0} g(a')$ (Proposition 3). Thus, by using Proposition 2, $\mathcal{M}, g, w_0 \vDash a' : \Box A$ holds. We know that $\mathcal{M}, g, w_0 \vDash a' : \Diamond c$ iff $R_{w_0}(g(a'), g(c))$. Therefore, we obtain $\mathcal{M}, g, w_0 \vDash c : A$ and we deduce that $\mathcal{M}$ is a countermodel of $\Gamma, a' : \Diamond c, a : \Box A, c : A \vdash C$ (the premise of $[\Box_L]$).

We show in the next section that $\mathsf{G_{IHL}}$ has the cut-elimination property. It means that if a sequent $\mathcal{S}$ is provable in $\mathsf{G_{IHL}}$ then there exists a proof of $\mathcal{S}$ in $\mathsf{G_{IHL}}$ without $[Cut]$.

5 Cut-elimination and Completeness of $\mathsf{G_{IHL}}$

The cut-elimination is one of the most important property of a sequent calculus. Indeed, the cut-elimination generally results in the (*quasi-*)*subformula property*: in any proof of a sequent $\mathcal{S}$, only the (quasi-)subformulas of the formulas of $\mathcal{S}$ appear in this proof.

5.1 Depth-preserving Admissibility

Let us recall the notion of *depth-preserving admissibility*.

A rule $[R]$ is said to be *admissible* for a calculus $\mathcal{C}$, if for all instances $\dfrac{H_1 \ \dots \ H_k}{C}\ [R]$ of $[R]$, if for all $i \in [1,k]$ $\vdash_{\mathcal{C}} H_i$, then $\vdash_{\mathcal{C}} C$.

A rule $[R]$ is said to be *depth-preserving admissible* for $\mathcal{C}$, if for all n, if for all $i \in [1,k]$ $\vdash^n_{\mathcal{C}} H_i$, then $\vdash^n_{\mathcal{C}} C$.

We note $\mathsf{G^-_{IHL}}$ the sequent calculus $\mathsf{G_{IHL}}$ without the $[Cut]$ rule. The following proposition corresponds to the depth-preserving admissibility property of weakening.

Proposition 4. *If* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma \vdash C$ *then* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A \vdash C$.

Proof. By induction on n.

The following proposition is used to prove the depth-preserving admissibility of contraction. It is similar to the inversion lemma given in [18], knowing that for some rules of $\mathsf{G^-_{IHL}}$, if the conclusion has a proof of a depth n then some of its premises has a proof of a depth smaller or equal to n.

Proposition 5.
1. *If* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A \wedge B \vdash C$ *then* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A, a : B \vdash C$.
2. *If* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A_1 \vee A_2 \vdash C$ *then* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A_i \vdash C$, *for* $i = 1, 2$.
3. *If* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma \vdash a : A_1 \wedge A_2$ *then* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma \vdash a : A_i$, *for* $i = 1, 2$.
4. *If* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma \vdash a : A \supset B$ *then* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A \vdash a : B$.
5. *If* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : A \supset B \vdash C$ *then* $\vdash^n_{\mathsf{G^-_{IHL}}} \Gamma, a : B \vdash C$.

6. *If* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, c : a : A \vdash C$ *then* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : A \vdash C$.
7. *If* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma \vdash c : a : A$ *then* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma \vdash a : A$.
8. *If* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma \vdash a : \Box A$ *then* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : \Diamond c \vdash c : A$ $(c \notin Nom(\Gamma \vdash a : \Box A))$.
9. *If* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : \Diamond A \vdash C$ *then* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : \Diamond c, c : A \vdash C$, $c \notin Nom(\Gamma, a : \Diamond A \vdash C)$.

Proof. By induction on n. Here we only develop the case of 8.
- If $n = 0$ then $\Gamma \vdash a : \Box A$ is an axiom. Thus there is a formula of the form $d : \bot$ in Γ and $\vdash^{0}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : \Diamond c \vdash c : A$ holds.
- Let us assume that $\vdash^{n+1}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma \vdash a : \Box A$ by a derivation $\mathcal{D}$. If $a : \Box A$ is not principal in the last rule applied in $\mathcal{D}$, then by applying induction hypothesis to the premise(s) and using the same rule, $\vdash^{n+1}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : \Diamond c \vdash c : A$ holds. Otherwise, $a : \Box A$ is principal and $\mathcal{D}$ ends with

$$\frac{\Gamma, a : \Diamond c \vdash c : A}{\Gamma \vdash a : \Box A}\ [\Box^{\star}_{R}]$$

By taking the immediate subdeduction of the premise, $\vdash^{n+1}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : \Diamond c \vdash c : A$ holds.

In order to prove the depth-preserving admissibility of contraction, we need to prove that if a sequent $\mathcal{S}$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of a depth n then the sequent obtained from $\mathcal{S}$ by renaming some nominals has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of a depth smaller or equal to n.

Definition 4. *A renaming function f is a function from N to M where $N, M \subset \mathsf{Nom}$. It is inductively extended to the formulas having nominals in N as follows:*
- $f(p) = p$ *where* $p \in \mathsf{Prop} \cup \{\bot\}$
- $f(A \circ B) = f(A) \circ f(B)$ *where* $\circ \in \{\wedge, \vee, \supset\}$
- $f(\bigcirc A) = \bigcirc f(A)$ *where* $\bigcirc \in \{\Box, \Diamond\}$
- $f(a : A) = f(a) : f(A)$

We use the notation $f(\Gamma)$ for $f(a_1 : A_1), \ldots, f(a_k : A_k)$ when $\Gamma = a_1 : A_1, \ldots, a_k : A_k$. Moreover, the notation $f(\Gamma \vdash C)$ corresponds to $f(\Gamma) \vdash f(C)$.

Proposition 6. *Let $\mathcal{S}$ be a sequent and $f : Nom(\mathcal{S}) \to M$ be a renaming function. If* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \mathcal{S}$ *then* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} f(\mathcal{S})$.

Proof. It is sufficient to prove by induction on n that if $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \mathcal{S}$ then $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \mathcal{S}[c/a]$ for any c and a elements of Nom, where $\mathcal{S}[c/a]$ corresponds to the renaming of a by c in $\mathcal{S}$.

The following propositions correspond to the depth-preserving admissibility of contraction.

Proposition 7. *If* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : A, a : A \vdash C$ *then* $\vdash^{n}_{\mathsf{G}^{-}_{\mathsf{IHL}}} \Gamma, a : A \vdash C$.

Proof. By induction on n.
If $n = 0$ then $\Gamma, a : A, a : A \vdash C$ is an instance of an axiom. Thus, it is easy to see that $\Gamma, a : A \vdash C$ is also an instance of the same axiom.
Let us assume that $\vdash^{n+1}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : A, a : A \vdash C$ by a derivation $\mathcal{D}$. If $a : A$ is not principal in the last rule applied in $\mathcal{D}$, then by applying induction hypothesis to the premise(s) and using the same rule, we have $\vdash^{n+1}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : A \vdash C$. Otherwise, $a : A$ is principal in the last rule applied in $\mathcal{D}$. We have to distinguish the cases of this rule.
Here, we only develop the case of $[\Diamond_L]$ ($A \equiv \Diamond B$):

$$\frac{\Gamma, a : \Diamond B, a : \Diamond c, c : B \vdash C}{\Gamma, a : \Diamond B, a : \Diamond B \vdash C}\ [\Diamond_L]$$

Using Proposition 5, $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : \Diamond c, a : \Diamond c', c : B, c'B \vdash C$ holds. Next, using Proposition 6, $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : \Diamond c, a : \Diamond c, c : B, c : B \vdash C$ holds (c and c' are new nominals). Finally, using the induction hypothesis twice, $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : \Diamond c, c : B \vdash C$ holds and then $\vdash^{n+1}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : \Diamond B \vdash C$ is obtained using $[\Diamond_L]$.

5.2 Cut-elimination of $\mathsf{G}_{\mathsf{IHL}}$

Now, we give a proposition stronger than the depth-preserving admissibility of contraction. However, it does not cover all satisfaction statements. This proposition is useful for the proof of cut-elimination.

Proposition 8. *If* $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : A, a' : A \vdash C$ *and* $a \sim a'$ *in* $\Gamma, a : A \vdash C$, *then* $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : A \vdash C$.

Proof. By induction on n, similarly to the proof of Proposition 7.

The two following propositions are used in the proof of cut-elimination.

Proposition 9. *Let* $\mathcal{S} = \Gamma \vdash C$ *be a sequent and* $a, a' \in Nom(\mathcal{S})$ *such that* $a \sim a'$. *If* $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a' : a \vdash C$ *then* $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \mathcal{S}$.

Proof. By induction on n.

Proposition 10. *If* $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash a : A$ *and* $a \sim a'$, *then* $\vdash^{n}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash a' : A$.

Proof. By induction on n.

Let $\mathcal{D}$ be a derivation of $\Gamma \vdash C$, we denote $\mathcal{D}[\Gamma']$ the derivation of $\Gamma, \Gamma' \vdash C$ obtained from $\mathcal{D}$ by applying depth-preserving admissibility of weakening.

Theorem 4 (Cut-elimination). *Let be a* $\mathcal{S}$ *a sequent, if* $\vdash_{\mathsf{G}_{\mathsf{IHL}}} \mathcal{S}$ *then* $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \mathcal{S}$.

Proof. To prove the cut-elimination property, we use a variant of Gentzen's original proof of this property for classical and intuitionistic logic [18]. This proof consists in transforming the applications of cut rules to applications of cut rules on smaller formulae or applications of less height. It suffices to assume that $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash a : A$ and $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a : A \vdash C$ and to prove that $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash C$ by induction on the structure of A, on the depth of the derivation of $\Gamma \vdash a : A$, $\mathcal{D}_1$, and on the depth of the derivation of $\Gamma, a : A \vdash C$, $\mathcal{D}_2$. n_1 and n_2 are respectively the depths of $\mathcal{D}_1$ and $\mathcal{D}_2$.
There are four main cases we have to consider:

1. $\Gamma \vdash a : A$ is an axiom;
2. $\Gamma, a : A \vdash C$ is an axiom;
3. $\Gamma \vdash a : A$ and $\Gamma, a : A \vdash C$ are not axioms and $a : A$ is not principal in the last rule of $\mathcal{D}_1$ or $\mathcal{D}_2$;
4. $a : A$ is principal of the both last rules of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Case 1. $\Gamma \vdash a : A$ is an axiom.

- 1a. If $\Gamma \vdash a : A$ is an instance of $[Id]$, then it is of the form $\Gamma', a' : p \vdash a : p$ and $\Gamma, a : A \vdash C$ is of the form $\Gamma', a' : p, a : p \vdash C$ where $a \sim a'$ in all these sequents. By using Proposition 8, we have if $\vdash^n_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a' : p, a : p \vdash C$ with $a \sim a'$, then $\vdash^n_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a' : p \vdash C$. Thus, $\vdash^{n_2}_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma, a' : p \vdash C$ holds.
- 1b. If $\Gamma \vdash a : A$ is an instance of $[\bot]$, then it is of the form $\Gamma', a' : \bot \vdash a : A$. Thus, $\Gamma \vdash C$ is of the form $\Gamma', a' : \bot \vdash C$ and it is instance of $[\bot]$.
- 1c. If $\Gamma \vdash a : A$ is an instance of $[Ref]$, then it is of the form $\Gamma \vdash a : a'$ and $\Gamma, a : A \vdash C$ is of the form $\Gamma, a : a' \vdash C$ where $a \sim a'$ in these sequents. Using Proposition 9, we obtain $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash C$.

Case 2. $\Gamma, a : A \vdash C$ is an axiom.

- 2a. If $\Gamma, a : A \vdash C$ is an instance of $[Id]$. If it is of the form $\Gamma, a : p \vdash a' : p$, then $\Gamma \vdash a : A$ is of the form $\Gamma \vdash a : p$ with $a \sim a'$. Using Proposition 10, we deduce that $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash a' : p$. Now, we study the case when $\Gamma, a : A \vdash C$ is of the form $\Gamma, a : A, a' : p \vdash a'' : p$ with $a' \sim a''$. If $a' \sim a''$ in $\Gamma \vdash C$, then $\vdash_{\mathsf{G}^-_{\mathsf{IHL}}} \Gamma \vdash C$ holds. Otherwise, $a : A$ is of the form $a : c$. If $\Gamma \vdash a : c$ is an axiom, then it is of the form $\Gamma', d : \bot \vdash a : c$ and we deduce that $\Gamma \vdash C$ is an instance of $[\bot]$. Now we distinguish the cases where the last rule applied in $\mathcal{D}_1$ is $[\supset_L]$ or not. If the last rule applied in $\mathcal{D}_1$ is $[\supset_L]$ then it is of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}'_1\\ \Gamma \vdash e : B\end{array} \quad \begin{array}{c}\mathcal{D}''_1\\ \Gamma', e : D \vdash a : c\end{array}}{\Gamma', e : B \supset D \vdash a : c}\,[\supset_L]$$

Since $\Gamma, a : c \vdash C$ is an instance of $[Id]$, we can easily see that $\Gamma', e : D, a : c \vdash C$ is also an instance of $[Id]$. Using the induction hypothesis, we obtain a derivation of $\Gamma \vdash C$ as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma\vdash e:B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1''\\ \Gamma',e:D\vdash a:c\end{array} \quad \dfrac{}{\Gamma',e:D,a:c\vdash C}\,[Id]}{\Gamma',e:D\vdash C}\,[Cut]}{\Gamma\vdash C}\,[\supset_L]$$

$[Cut]$ is where we apply the induction hypothesis.
Otherwise, the last rule applied in $\mathcal{D}_1$ is not $[\supset_L]$. We consider only the case where the last rule of $\mathcal{D}_1$ is a two-premises rule (the case of one-premise rule being simpler):

$$\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma'\vdash a:c\end{array} \quad \begin{array}{c}\mathcal{D}_1''\\ \Gamma''\vdash a:c\end{array}}{\Gamma\vdash a:c}\,[R]$$

It is easy to see that $\Gamma',a:c\vdash C$ and $\Gamma'',a:c\vdash C$ are instances of $[Id]$. By applying the induction hypothesis, we obtain a derivation of $\Gamma\vdash C$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma'\vdash a:c\end{array} \quad \dfrac{}{\Gamma',a:c\vdash C}\,[Id]}{\Gamma'\vdash C}\,[Cut] \qquad \dfrac{\begin{array}{c}\mathcal{D}_1''\\ \Gamma''\vdash a:c\end{array} \quad \dfrac{}{\Gamma'',a:c\vdash C}\,[Id]}{\Gamma''\vdash C}\,[Cut]}{\Gamma\vdash C}\,[R]$$

The proof in the subcases $\Gamma,a:A\vdash C$ is an instance of $[Ref]$ and $\Gamma,A@a\vdash C$ is an instance of $[\bot]$ is obtained using arguments similar to these of Subcase 2a.

Case 3. $\Gamma\vdash a:A$ and $\Gamma,a:A\vdash C$ are not axioms and $a:A$ is not principal in the last rule of $\mathcal{D}_1$ or $\mathcal{D}_2$.

- 3a. $\Gamma\vdash a:A$ and $\Gamma,a:A\vdash C$ are not axioms and $a:A$ is not principal in the last rule of $\mathcal{D}_1$. We consider only the case where the last rule of $\mathcal{D}_1$ is a two-premises rule:

$$\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma'\vdash C'\end{array} \quad \begin{array}{c}\mathcal{D}_1''\\ \Gamma''\vdash a:A\end{array}}{\Gamma\vdash a:A}\,[R]$$

If $[R]\neq[\supset_L]$ then $C'\equiv a:A$ holds. By applying the induction hypothesis and the depth-preserving admissibility of weakening and contraction (Proposition 4 and Proposition 7), we obtain a derivation of $\Gamma\vdash C$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1'[\Gamma]\\ \Gamma,\Gamma'\vdash a:A\end{array} \quad \begin{array}{c}\mathcal{D}_2[\Gamma']\\ \Gamma,\Gamma',a:A\vdash C\end{array}}{\Gamma,\Gamma'\vdash C}\,[Cut] \qquad \dfrac{\begin{array}{c}\mathcal{D}_1''[\Gamma]\\ \Gamma,\Gamma''\vdash a:A\end{array} \quad \begin{array}{c}\mathcal{D}_2[\Gamma'']\\ \Gamma,\Gamma'',a:A\vdash C\end{array}}{\Gamma,\Gamma''\vdash C}\,[Cut]}{\Gamma,\Gamma\vdash C}\,[R]$$

If $[R]=[\supset_L]$ then a derivation of $\Gamma\vdash C$ is obtained similarly by applying the induction hypothesis and the depth-preserving admissibility of weakening and contraction as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}_1'[\Gamma]\\ \Gamma,\Gamma'\vdash C'\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1''[\Gamma]\\ \Gamma,\Gamma''\vdash a:A\end{array} \quad \begin{array}{c}\mathcal{D}_2[\Gamma'']\\ \Gamma,\Gamma'',a:A\vdash C\end{array}}{\Gamma,\Gamma''\vdash C}\,[Cut]}{\Gamma,\Gamma\vdash C}\,[\supset_L]$$

- 3b. $\Gamma\vdash a:A$ and $\Gamma,a:A\vdash C$ are not axioms and $a:A$ is not principal in the last rule of $\mathcal{D}_2$. The proof in this case is similar to Subcase 3a.

Case 4. $a:A$ is principal of the both last rules of $\mathcal{D}_1$ and $\mathcal{D}_2$. Here, we only develop the case of $A\equiv\Box B$, The other cases are similar. $\Gamma\vdash a:A$ and $\Gamma,a:A\vdash C$ are respectively of the form $\Gamma',a':\Diamond c'\vdash a:\Box B$ and $\Gamma',a:\Box B,a':\Diamond c'\vdash C$ with $a\sim a'$ in these sequents and then $\mathcal{D}_1$ and $\mathcal{D}_2$ are respectively of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma',a:\Diamond c,a':\Diamond c'\vdash c:B\end{array}}{\Gamma',a':\Diamond c'\vdash a:\Box B}\,[\Box_R] \quad \text{and} \quad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Gamma',a:\Box B,a':\Diamond c',c':B\vdash C\end{array}}{\Gamma',a:\Box B,a':\Diamond c'\vdash C}\,[\Box_L]$$

Using Proposition 8 and Proposition 6 we obtain $\vdash^{n_1-1}_{\mathsf{G_{IHL}}}\Gamma',a':\Diamond c'\vdash c':B$. Thus, a derivation of $\Gamma\vdash C$ is obtained by using Proposition 4 and the induction hypothesis as follows:

$$\dfrac{\Gamma',a':\Diamond c'\vdash c':B \quad \dfrac{\begin{array}{c}\mathcal{D}_1[c':B]\\ \Gamma,a':\Diamond c',c':B\vdash a:\Box B\end{array} \quad \begin{array}{c}\mathcal{D}_2'\\ \Gamma',a:\Box B,a':\Diamond c',c':B\vdash C\end{array}}{\Gamma,a':\Diamond c',c':B\vdash C}\,[Cut]}{\Gamma,a':\Diamond c'\vdash C}\,[Cut]$$

5.3 Completeness of $\mathsf{G_{IHL}}$

The following propositions allow us to prove the completeness of $\mathsf{G_{IHL}}$.

Proposition 11. *If $\vdash^n_{\mathsf{G_{IHL}}}\Gamma,a:\Diamond c',c':c\vdash C$ with $c'\notin Nom(\Gamma,C)\cup\{a,c\}$, then we have $\vdash^n_{\mathsf{G_{IHL}}}\Gamma,a:\Diamond c\vdash C$.*

Proof. From $\vdash^n_{\mathsf{G_{IHL}}}\Gamma,a:\Diamond c',c:c\vdash C$ and Proposition 6, we deduce $\vdash^n_{\mathsf{G_{IHL}}}\Gamma,a:\Diamond c,c:c\vdash C$. Therefore, from Proposition 9, $\vdash^n_{\mathsf{G_{IHL}}}\Gamma,a:\Diamond c\vdash C$.

Proposition 12. *If $\vdash_{\mathsf{G_{IHL}}}\Gamma\vdash a:A$ then $\vdash_{\mathsf{G_{IHL}}}\Gamma,a:c\vdash c:A$.*

Proof. From Proposition 10.

Theorem 5 (Completeness). *The sequent calculus $\mathsf{G_{IHL}}$ is complete.*

Proof. We consider the validity through the natural deduction system $\mathsf{NS_{IHL}}$ (Figure 2). Let $\mathcal{S}$ be a sequent. We suppose that $\mathcal{S}$ has a proof $\mathcal{D}$ in $\mathsf{NS_{IHL}}$ and we prove by induction on the structure of $\mathcal{D}$ that $\vdash_{\mathsf{G_{IHL}}}\mathcal{S}$. We distinguish the cases of the last rule applied in $\mathcal{D}$. The cases of the axioms and the introduction rules are trivial.
Now, we develop some cases of the elimination rules and $[Nom]$.

- Case of $[\Diamond_E]$:

$$\frac{\Gamma \vdash a : \Diamond A \qquad \Gamma, a : \Diamond c, c : A \vdash C}{\Gamma \vdash C}\ [\Diamond_E]$$

We have two subcases:
- if A is a nominal, then from $\vdash_{\mathsf{G_{IHL}}} \Gamma, a : \Diamond c, c : A \vdash C$ we deduce $\vdash_{\mathsf{G_{IHL}}} \Gamma, a : \Diamond A \vdash C$ holds (Proposition 11);
- otherwise, by applying $[\Diamond_L]$ to $\Gamma, a : \Diamond c, c : A \vdash C$ we have $\vdash_{\mathsf{G_{IHL}}} \Gamma, a : \Diamond A \vdash C$ and a proof of $\Gamma \vdash C$ is obtained using $[Cut]$.

- Case of $[\Box_E]$:

$$\frac{\Gamma \vdash a : \Box A \qquad \Gamma \vdash a : \Diamond c}{\Gamma \vdash c : A}\ [\Box_E]$$

A proof of $\Gamma \vdash c : A$ in $\mathsf{G_{IHL}}$ is obtained as follows:

$$\dfrac{\Gamma \vdash a : \Diamond c \qquad \dfrac{\dfrac{\Gamma \vdash a : \Box A}{\Gamma, a : \Diamond c \vdash a : \Box A}\ [W_L] \qquad \dfrac{\dfrac{}{\Gamma, a : \Diamond c, a : \Box A, c : A \vdash c : A}\ [Id]}{\Gamma, a : \Diamond c, a : \Box A \vdash c : A}\ [\Box_L]}{\Gamma, a : \Diamond c \vdash c : A}\ [Cut]}{\Gamma \vdash c : A}\ [Cut]$$

- Case of $[Nom]$:

$$\frac{\Gamma \vdash a : c \qquad \Gamma \vdash a : A}{\Gamma \vdash c : A}\ [Nom]$$

Using Proposition 12 and $\vdash_{\mathsf{G_{IHL}}} \Gamma \vdash a : A$, $\vdash_{\mathsf{G_{IHL}}} \Gamma, a : c \vdash c : A$ holds. Thus $\vdash_{\mathsf{G_{IHL}}} \Gamma \vdash c : A$ is obtained using $[Cut]$.

5.4 Another sequent calculus $\mathsf{G^2_{IHL}}$

It is possible to define another sequent calculus without conditions of the form $a \sim b$. We call $\mathsf{G^2_{IHL}}$ the calculus obtained from $\mathsf{G_{IHL}}$ by adding the following structural rule:

$$\frac{\Gamma[a/c] \vdash C[a/c]}{\Gamma, c : a \vdash C}\ [S]$$

and by replacing $[Id]$, $[Ref]$, $[\Box_L]$ and $[\Diamond_R]$ by the following new rules:

$$\frac{}{\Gamma, a : A \vdash a : A}\ [Id] \qquad \frac{}{\Gamma \vdash a : a}\ [Ref]$$

$$\frac{\Gamma, a : \Diamond c, a : \Box A, c : A \vdash C}{\Gamma, a : \Diamond c, a : \Box A \vdash C}\ [\Box_L] \qquad \frac{\Gamma, a : \Diamond c \vdash c : A}{\Gamma, a : \Diamond c \vdash a : \Diamond A}\ [\Diamond_R]$$

Theorem 6. *The sequent calculus $\mathsf{G^2_{IHL}}$ is sound and complete.*

Proof. We show that a sequent is derivable in $\mathsf{G^2_{IHL}}$ if and only if it is derivable in $\mathsf{G_{IHL}}$.

6 Decidability of IHL

In this section, we prove the decidability of IHL by using the sequent calculus $\mathsf{G}^{-}_{\mathsf{IHL}}$. The key point of our decision procedure is the use of the cut-elimination property in order to provide a suitable subformula property different from the usual one, called the *quasi-subformula property*.
We introduce a notion of *redundancy* satisfying the fact that any sequent valid in IHL has an *irredundant proof*. Then we prove that there is no infinite derivation which is not irredundant and deduce the decidability result.

Introduction of New Nominals. In order to prove the decidability of IHL by using $\mathsf{G}^{-}_{\mathsf{IHL}}$, we must solve the problem of the introduction of new nominals using $[\Diamond_L]$ and $[\Box_R]$. This problem is similar to the one of the introduction of new labels in the labelled sequent calculi of the intuitionistic modal logics studied by Simpson in [17]. Let us note that the introduction of new nominals or labels is not a problem in the case of classical modal and hybrid logics because we can define proof systems with only invertible rules allowing terminating proof-search [5,15].
In the case of Simpson's calculi, the problem of the introduction of new labels was resolved using the following property:
for any derivation, there is a positive integer n such that there is no sequent containing a chain *of length greater than n. A chain is a sequence of the form* $x_0Rx_1,\ldots,x_{m-1}Rx_m$ *where R represents the accessibility relation,* x_0 *is not a new label and* x_{i+1} *is a new label for* $i \in [0,m-1]$.

We can say that any infinite derivation is redundant because there are necessarly two sequents where one can be obtained from the other by renaming some new nominals (for more details see [17]).
Similarly, for $\mathsf{G}_{\mathsf{IHL}}$ a chain is a sequence of the form $a_0 : \Diamond a_1, a_1 : \Diamond a_2, \ldots, a_{m-1} : \Diamond a_m$ where a_0 is not a new nominal and a_{i+1} is a new nominal for $i \in [0,m-1]$.

Let us give an example in order to show that the previous property fails in the case of $\mathsf{G}^{-}_{\mathsf{IHL}}$:

$$
\begin{array}{c}
\cdot \\
\cdot \\
a_0 : \Diamond a_1, \ldots, a_i : \Diamond a_{i+1}, a_1 : a_0, \ldots, a_i : a_0, a_2 : p, \ldots, a_{i+1} : p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot \\
\cdot \\
\cdot \\
\cdot \\
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a_0\Diamond a_1, a_1 : \Diamond a_2, a_2 : \Diamond a_3, a_1 : a_0, a_2a_0, a_2 : p, a_3 : p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}{a_0 : \Diamond a_1, a_1 : \Diamond a_2, a_1 : a_0, a_2 : a_0, a_2 : p, a_2 : \Diamond p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}\,[\Diamond_L]}{a_0 : \Diamond a_1, a_1 : \Diamond a_2, a_1 : a_0, a_2 : p, a_2 : a_0 \wedge \Diamond p, a_0\Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}\,[\wedge_L]}{a_0\Diamond a_1, a_1\Diamond a_2, a_1 : a_0, a_2 : p, a_0\Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}\,[\Box_L]}{a_0 : \Diamond a_1, a_1 : a_0, a_1 : \Diamond p, a_0\Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}\,[\Diamond_L]}{a_0 : \Diamond a_1, a_1 : a_0 \wedge \Diamond p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}\,[\wedge_L]}{a_0 : \Diamond a_1, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \bot}\,[\Box_L]
\end{array}
$$

Moreover, it is easy to see that, in this infinite derivation, there are no two distinct sequents such that one can be obtained from the other by renaming some new nominals. To solve this problem, we associate to every sequent $\mathcal{S}$ appearing in a given derivation a particular sequent, called the equivalid sequent of $\mathcal{S}$, satisfying the previous property and the fact that a sequent has a proof if and only if its equivalid sequent has a proof of the same size. The equivalid sequents are obtained by renaming some new nominals.

6.1 Quasi-subformula Property

Let us remind that the *subformulas* of a formula A are inductively defined as follows:
- A is a subformula of A;
- if $B \otimes C$ is a subformula of A then so are B,C, for $\otimes = \wedge, \vee, \supset$;
- if $\boxtimes B$ is a subformula of A the so is B, for $\boxtimes = \Box, \Diamond$;
- if $a : B$ is a subformula of A then so is B.

Now, we introduce the notion of *quasi-subformula*. It is similar to the weak subformula notion introduced in [15] and the quasi-subformula notion given in [7].

Definition 5 (Quasi-subformula). *Let A be a formula, the* quasi-subformulas *are inductively defined as follows:*
- for every $\circ \in \{\wedge, \vee, \supset\}$, the quasi-subformulas of $a : A \circ B$ are $a : A \circ B$ and all the quasi-subformulas of $a : A$ and $a : B$;
- the quasi-subformulas of $a : \bigcirc A$ for $\bigcirc \in \{\Diamond, \Box\}$ are $a : \bigcirc A$ and all the quasi-subformulas of $c : A$ for an arbitrary c.

Theorem 7. *Let $\mathcal{D}$ be a derivation of $\mathcal{S}$ in $\mathsf{G}^{-}_{\mathsf{IHL}}$. Any formula occurrence $a : A$ in $\mathcal{D}$ is either a quasi-subformula of a formula in $\mathcal{S}$ or of the form $a : \Diamond c$, c being a nominal.*

Proof. By induction on the depth of $\mathcal{D}$. We only have to distinguish the cases of the last rule applied.

We are interested in the size of derivations. Previously, we proved that weakening and contraction are depth-preserving admissible for $\mathsf{G}^{-}_{\mathsf{IHL}}$. Weakening and contraction are also size-preserving admissible for $\mathsf{G}^{-}_{\mathsf{IHL}}$. Moreover, if a sequent $\mathcal{S}$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n, then any sequent obtained from $\mathcal{S}$ by renaming some nominals with others has a proof of size smaller or equal to n.

Proposition 13. *If $\Gamma \vdash C$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n, then $\Gamma, a : A \vdash C$ has a proof of size smaller or equal to n.*

Proof. Similar to the proof of Proposition 4.

Proposition 14. *If $\Gamma, a : A, a : A \vdash C$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n, then $\Gamma, a : A \vdash C$ has a proof of size smaller or equal to n.*

Proof. Similar to the proof of Proposition 7.

Proposition 15. *Let $\Gamma \vdash C$ be a sequent and $f : Nom(\mathcal{S}) \to M$ be a renaming function. If $\Gamma \vdash C$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n, then $f(\Gamma \vdash C)$ has a proof of size smaller or equal to n.*

Proof. Similar to the proof of Proposition 6.

Let $\mathcal{S}'$ be a sequent appearing in a given derivation of $\mathcal{S}$, let $\sim$ be its associated equivalence relation and let c be an element of $Nom(\mathcal{S}')$ such that $c \notin Nom(\mathcal{S})$. We denote $N(c, \mathcal{S}')$ the set of nominals defined by $a \in N(c, \mathcal{S}')$ if and only if $a \in Nom(\mathcal{S})$ and $a \sim c$.

Definition 6. *Let $\mathcal{S}'$ be a sequent appearing in a given derivation of $\mathcal{S}$ and $N = Nom(\mathcal{S})$ (N is considered as a totally ordered set). We define $Eq(\mathcal{S}')$ as the sequent $f(\mathcal{S}')$ where f is a renaming function defined as follows:*

$$f(c) = \begin{cases} c & \text{if either } c \in N \text{ or } N(c, \mathcal{S}') = \emptyset \\ c' & \text{otherwise, with } c' = max(N(c, \mathcal{S}')) \end{cases}$$

where max denotes the maximum.

If the formulas of $\mathcal{S}'$ are quasi-subformulas of formulas in $\mathcal{S}$ then the formulas of $Eq(\mathcal{S}')$ are also quasi-subformulas of formulas in $\mathcal{S}$.

Proposition 16. *Let $\mathcal{S}'$ be a sequent appearing in a given derivation. $\mathcal{S}'$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n if and only if $Eq(\mathcal{S}')$ has a proof of size n.*

Proof. We only have to prove that for any sequent $\mathcal{S}$ and for all $a, b \in Nom(\mathcal{S})$ such that $a \sim b$, $\mathcal{S}$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n if and only if $\mathcal{S}[a/b]$ has a proof in $\mathsf{G}^{-}_{\mathsf{IHL}}$ of size n. The *if part* is proved by induction on n. The *only if part* comes from Proposition 15.

6.2 N-chains

The nesting degree of a formula A, denoted $nest(A)$, is inductively defined as follows:
- $nest(p) = 0$; $nest(a) = 0$; $nest(\bot) = 0$;
- $nest(A \circ B) = max(nest(A), nest(B))$ where $\circ \in \{\wedge, \vee, \supset\}$;
- $nest(a : A) = nest(A)$;
- $nest(\bigcirc A) = 1 + nest(A)$ where $\bigcirc \in \{\Box, \Diamond\}$.

The nesting degree of a sequent is the maximum of the nesting degrees of its formulas.

Now, we introduce the notion of *N-chain*. Intuitively, the N-chains correspond to sequences of formulas of the form $a : \Diamond c$ which will allow us to give a description of the arrangement of the new nominals introduced using the rules $[\Box_R]$ and $[\Diamond_L]$. The key point is that the length of these sequences, in any sequent equivalid to a sequent appearing in any derivation of $\mathcal{S}$ in $\mathsf{G}^{-}_{\mathsf{IHL}}$, is bounded by the nesting degree of $\mathcal{S}$.

Definition 7 (N-chain). *Let $\mathcal{S} = \Gamma \vdash C$ be a sequent and N a finite set of nominals. A N-chain is a sequence of the form $a_0 : \Diamond a_1, a_1 : \Diamond a_2, \ldots a_{k-1} :, \Diamond a_k$ (a_0 when $k = 0$) where*

- *$a_{i-1} : \Diamond a_i \in \Gamma$ for $i = 1, \ldots, k$;*

– $a_0 \in N$;
– $a_i \notin N$ *for* $i = 1, \ldots, k-1$;
– *if* $a_k \in N$ *and* $k \neq 0$ *then* $k > 1$; *and*
– *if* $a_k \notin N$ *then there is no nominal* a *such that* $a_k : \Diamond a \in \Gamma$.

Proposition 17. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two sequents such that $\mathcal{S}'$ is the equivalid sequent of a sequent appears in a derivation of $\mathcal{S}$. Then, for all $a : \Diamond b \in \Gamma$ ($\mathcal{S}' = \Gamma \vdash C$) such that either $a \notin Nom(\mathcal{S})$ or $b \notin Nom(\mathcal{S})$, $a : \Diamond b$ belongs to a $Nom(\mathcal{S})$-chain of $\mathcal{S}'$.*
Moreover, the length of any $Nom(\mathcal{S})$-chain in $\mathcal{S}'$ is smaller or equal to $nest(\mathcal{S}) + 1$.

Proof. The first property is obtained from the rules that introduce new nominals, namely $[\Box_L]$ and $[\Diamond_R]$, and also from the definition of the equivalid sequents where only some nominals introduced using the previous two rules are renamed by nominals in $Nom(\mathcal{S})$. The second property holds because if there exists a $Nom(\mathcal{S})$-chain with a length greater than $nest(\mathcal{S})$ in $\mathcal{S}'$ then there is a nominal $b \notin Nom(\mathcal{S})$ in this $Nom(\mathcal{S})$-chain and another nominal $a \in Nom(\mathcal{S})$ such that $a \sim b$ and b is renamed by a in order to build $Eq(\mathcal{S}')$ (from the definition of the rules $[\Box_L]$ and $[\Diamond_R]$).

A preorder, denoted $\lesssim_{\mathcal{S}}$, on the sequents appearing in the derivations of $\mathcal{S}$ is defined as follows: $\mathcal{S}_1 \lesssim_{\mathcal{S}} \mathcal{S}_2$ if and only if there exists a renaming function f such that $set(f(\Gamma_1)) \subseteq set(\Gamma_2)$ and $f(C_1) = C_2$ where $Eq(\mathcal{S}_1) = \Gamma_1 \vdash C_1$ and $Eq(\mathcal{S}_2) = \Gamma_2 \vdash C_2$.
Let us recall that we use $set(\Gamma)$ to denote the set underlying the multiset Γ (the set of the formulas of Γ). Moreover, we use the notation $set(\Gamma \vdash C)$ for $set(\Gamma) \vdash C$.

Proposition 18. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sequents in a derivation of $\mathcal{S}$. If $\mathcal{S}_1 \lesssim_{\mathcal{S}} \mathcal{S}_2$ then if $\mathcal{S}_1$ has a proof of size n then $\mathcal{S}_2$ has a proof of size smaller or equal to n.*

Proof. From the size-preserving admissibility of weakening and contraction (see Proposition 13 and Proposition 14) and also Proposition 15.

6.3 Trees and Skeletons

Now, we use the notion of N-chain to represent the equivalid sequents by sets of *trees*. Then we derive from such trees other trees called *skeletons*. Next, we prove that the numbers of nodes of the skeletons obtained from the equivalid sequents of the sequents in a given derivation are bounded. Using this property, we show that for any derivation of $\mathcal{S}$ having an infinite branch, there are two sequents $\mathcal{S}'$ and $\mathcal{S}''$, with $\mathcal{S}'$ strictly occurring above $\mathcal{S}''$ in this branch, and such that $\mathcal{S}' \lesssim_{\mathcal{S}} \mathcal{S}''$.

Definition 8 (Tree). *Let $\mathcal{S}$ and $\mathcal{S}'$ be two sequents such that $\mathcal{S}'$ is the equivalid sequent of a sequent in a derivation of $\mathcal{S}$ and a be an element of $Nom(\mathcal{S})$. We define the* tree associated to a in $\mathcal{S}'$, *denoted $T(a, \mathcal{S}')$, as follows:*

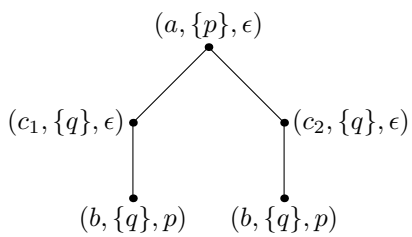
– *the nodes are labelled with triples $(b, \Gamma(b), \alpha(b))$ where $b \in Nom(\mathcal{S}')$; $\Gamma(b) = \{A \mid b : A \in \Gamma\}$; and*

$$\alpha(b) = \begin{cases} C & \text{if } b = c \\ \varepsilon & \text{otherwise} \end{cases}$$

where ε is a symbol not element of Nom *and* Prop *and $\mathcal{S}' = \Gamma \vdash c : C$.*

– *the root node is labelled with* $(a, \Gamma(a), \alpha(a))$.
– *a node labelled with* (b', Δ', α') *is an immediate successor of a node labelled with* (b, Δ, α) *iff* $b : \Diamond b'$ *belongs to a* $Nom(\mathcal{S})$*-chain of* $\mathcal{S}'$ *starting with* a*. label.*

Let us illustrate this notion with the following sequent $\mathcal{S}' = a : p, c_1 : q, c_2 : q, b : q, a : \Diamond c_1, a : \Diamond c_2, c_1 : \Diamond b, c_2 : \Diamond b \vdash b : p$ which is an equivalid sequent of a sequent appearing in a derivation of $\mathcal{S}$ such that $Nom(\mathcal{S}) = \{a, b\}$.
$T(a, \mathcal{S}')$ is the following tree:



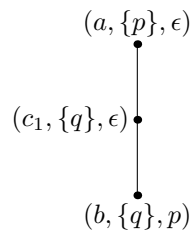
We characterize every sequent $\mathcal{S}' = \Gamma \vdash C$ which is equivalid to a sequent appearing in a derivation of $\mathcal{S}$ by the set of trees $T_{\mathcal{S}'} = \{T(a, \mathcal{S}') \mid a \in Nom(\mathcal{S})\}$. It is called the *tree set characterizing* $\mathcal{S}'$. From this set we can easily obtain the value of $set(\Gamma \vdash C)$.

We write $\cong_N$, with N is a finite set of nominals, the equivalence relation on the trees defined by $T_1 \cong_N T_2$ if and only if $T_1 = f_1(T_2)$ and $T_2 = f_2(T_1)$ where f_1 and f_2 are two renaming functions satisfying the property that for all $a \in N$ we have $f_1(a) = f_2(a) = a$. Intuitively, two trees are equivalent if and only if each tree can be obtained from the other by renaming some nominals which is not in N.

We note that any tree can be represented by the expression (r, L) where r is the root node and L is a list of trees. The set of the *subtrees* of a tree $\mathcal{T}$ is inductively defined as follows: $\mathcal{T}$ is a subtree of $\mathcal{T}$; if (r, L) is a subtree of $\mathcal{T}$ then so are the elements of L.
We note $dp(T)$ the depth of the tree T.

Definition 9 (Skeleton). *Let* $\mathcal{S}$ *and* $\mathcal{S}'$ *be two sequents such that* $\mathcal{S}'$ *is the equivalid sequent of a sequent in a derivation of* $\mathcal{S}$*,* $\mathcal{T}$ *be an element of the tree set characterizing* $\mathcal{S}'$ *and* $N = Nom(\mathcal{S})$*.*
A skeleton of $\mathcal{T}$*, denoted* $Sk(\mathcal{T})$*, is a subtree of* $\mathcal{T}$ *built in a following way:*
- Step 0*: we initialize* $Sk(\mathcal{T})$ *with* $\mathcal{T}$*.*
- Step i+1*: for all subtrees* $\mathcal{ST} = (r, L)$ *of* $Sk(\mathcal{T})$ *of depth equal to* $(i+1)$*, we replace* $\mathcal{ST}$ *in* $Sk(\mathcal{T})$ *by* (r, L') *where* L' *is a sublist of* L *obtained as follows: we start with* $L' = [\,]$*; for all* $\mathcal{T}_0 \in L$*, if there is no tree* $\mathcal{T}_1$ *in* L' *such that* $\mathcal{T}_0 \cong_N \mathcal{T}_1$*, then we add* $\mathcal{T}'$ *to* L'*.*

A skeleton of the tree given in the previous example is:



We can see that there is not always a single skeleton associated to a tree. However, all the skeletons associated to any tree are equivalent. If $\{T_1, \dots, T_k\}$ is the tree set characterizing $\mathcal{S} = \Gamma \vdash C$ and $\mathcal{S}' = \Gamma' \vdash C$ is the sequent obtained from $\{Sk(T_1), \dots, Sk(T_k)\}$ then $\Gamma' \subseteq \Gamma$ and $set(\mathcal{S}')$ can be obtained from $set(\mathcal{S})$ by renaming some new nominals.

Proposition 19. *Let $\mathcal{S}$ be a sequent and $\mathcal{D}$ be a derivation of $\mathcal{S}$. Then there exists a constant K such that for any sequent $\mathcal{S}'$ equivalid to a sequent appearing in $\mathcal{D}$, if $\mathcal{T}$ is in the tree set characterizing $\mathcal{S}'$ then the number of nodes of $Sk(\mathcal{T})$ is smaller or equal to K.*

Proof. We know that the depth of $\mathcal{T}$ is equal to the depth of $\mathcal{T}' = Sk(\mathcal{T})$ and is smaller or equal to $nest(\mathcal{S}) + 1$ (Proposition 17). We note $dp(\mathcal{T}')$ the depth of $\mathcal{T}'$. Let Φ be the set of the subformulas of the formulas of $\mathcal{S}$ and ϕ its size. The size of the set of the subsets of Φ is 2^{ϕ}. Using the quasi-subformula property (Theorem 7), we prove that for all n a node of $\mathcal{T}'$ of depth $dp(\mathcal{T}') - 1$, n has at most $K_1 = (N+1) \times 2^{\phi} \times 2$ successors where N is the size of $Nom(S)$.
Similarly, the number of the successors of any node in $\mathcal{T}$ of depth $dp(\mathcal{T}') - 2$ is at most equal to $K_2 = (N+1) \times 2^{K_1} \times 2$. We continue until the root node ($K_{dp(\mathcal{T})}$). Thus, we can take the constant K equal to

$$1 + \sum_{i=0}^{dp(\mathcal{T})-1} \prod_{j=0}^{i} K_{dp(\mathcal{T})-j}$$

.

Proposition 20. *Let $\mathcal{S}$ be a sequent and $\mathcal{D}$ be a derivation of $\mathcal{S}$. The set of skeletons obtained from the equivalid sequents of the sequents in $\mathcal{D}$ is partitioned into a finite set of equivalence classes by $\cong_{Nom(\mathcal{S})}$.*

Proposition 21. *Let $\mathcal{D}$ be a derivation of a sequent $\mathcal{S}$ with an infinite branch $\mathcal{B} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k, \dots)$. Then, there exist i and j such that $i < j$ and $\mathcal{S}_j \lesssim_{\mathcal{S}} \mathcal{S}_i$.*

Proof. Let $\mathcal{B}' = (\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_1, \dots)$ where $\mathcal{S}'_i = Eq(\mathcal{S}_i)$ for $i = 1, 2, \dots$. We associate to every sequent in $\mathcal{B}'$ the set of tree $T_{\mathcal{S}'_i} = \{Sk(T(a, \mathcal{S}'_i)) \mid a \in Nom(\mathcal{S})\}$. Using Proposition 20, we deduce that there exist two sequents $\mathcal{S}'_i = \Gamma'_i \vdash C'_i$ and $\mathcal{S}'_j = \Gamma'_j \vdash C'_j$ such that $i < j$ and for all $T_j \in T_{\mathcal{S}'_j}$, there is $T_i \in T_{\mathcal{S}'_i}$ satisfying $T_i \cong_{Nom(\mathcal{S})} T_j$. If $\Gamma''_i \vdash C''_i$ and $\Gamma''_j \vdash C''_j$ are the two sequents obtained respectively from $T_{\mathcal{S}'_i}$ and $T_{\mathcal{S}'_j}$, then there is a renaming function f such that $f(\Gamma''_j) = \Gamma''_i$ and $f(C''_j) = C''_i$. Moreover, we have $\Gamma''_i \subseteq \Gamma'_i$ and $C''_i = C'_i$. Since there is a renaming function g such that $set(g(\Gamma'_j)) = \Gamma''_j$ and $g(C'_j) = C''_j$, $set((f \circ g)(\Gamma'_j)) \subseteq set(\Gamma'_i)$ and $(f \circ g)(C'_j) = C'_i$ hold. Therefore, we deduce that $\mathcal{S}_j \lesssim_{\mathcal{S}} \mathcal{S}_i$.

6.4 A Decision Procedure for IHL

Now we introduce a notion of *redundancy* on cut-free derivations in our calculus such that any sequent that is valid has an irredundant proof. Then, by using the quasi-subformula property, we prove that there is no infinite proof which is redundant and then provide a decision procedure for IHL and then prove the decidability of this logic through proof-search using our sequent calculus.

Definition 10. *A derivation of $\mathcal{S}$ is said to be* redundant *if it contains two sequents $\mathcal{S}'$ and $\mathcal{S}''$, with $\mathcal{S}'$ occurring strictly above $\mathcal{S}''$ in the same branch, such that $\mathcal{S}' \lesssim_{\mathcal{S}} \mathcal{S}'$. A derivation is* irredundant *if it is not redundant.*

Proposition 22. *If $\mathcal{S}$ is derivable in $\mathsf{G}^{-}_{\mathsf{IHL}}$ then it has an irredundant derivation.*

Proof. By induction on the size s of the derivation of $\mathcal{S}$.
If $s = 1$ then it is an irredundant derivation.
Now, assume that for any sequent, if it has a derivation of size smaller or equal to n ($n \geqslant 1$), then it has an irredundant derivation (the induction hypothesis).
Let $\mathcal{D}$ be a derivation of $\mathcal{S}$ of size $(n+1)$. If $\mathcal{D}$ is irredundant then we have the result. Otherwise it has a branch containing two sequents $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1$ occurring above $\mathcal{S}_2$ and $\mathcal{S}_1 \lesssim_{\mathcal{S}} \mathcal{S}_2$. Let n' be the size of the subderivation of $\mathcal{S}_1$ in $\mathcal{D}$. It is easy to see that the size of the subderivation of $\mathcal{S}_2$ in $\mathcal{D}$ is strictly greater than n'. Using Proposition 18, we know that $\mathcal{S}_2$ has a derivation $\mathcal{D}_2$ of size smaller or equal to n'. So by replacing in $\mathcal{D}$ the subderivation of $\mathcal{S}_2$ with $\mathcal{D}_2$ we obtain a derivation of $\mathcal{S}$ of size smaller or equal to n. Therefore, by applying the induction hypothesis, we deduce that $\mathcal{S}$ has an irredundant derivation.

Now, we provide a decision procedure for the sequents in IHL based on the redundancy notion similar to this proposed in [17] for the intuitionistic modal logics. It consists of an exhaustive search for an irredundant derivation.

Let $\mathcal{S}$ be a sequent.
- Step 1: we start with the derivation containing only $\mathcal{S}$ which is the unique irredundant derivation of size 1. If this derivation is a proof then we return it. Otherwise we move to the next step.
- Step $i+1$: we build the set of all the irredundant derivations of size $i+1$. If this set contains a proof of $\mathcal{S}$ then we return it. Otherwise if this set is empty then the decision algorithm fails, else we move to the next step.

There are only a finite number of possible rule applications (the choice of the new nominals introduced by the rules $[\Box_R]$ and $[\Diamond_L]$ is not essential). Thus, the set of the irredundant derivations of size $i+1$ is finite. Moreover, this set can be built in a finite time because the $\lesssim_{\mathcal{S}}$ relation is decidable.

Theorem 8. IHL *is decidable.*

Proof. Using Proposition 21, we know that there is no infinite irredundant derivation. Thus, we deduce that our algorithm terminates. Therefore, IHL is decidable.

7 Conclusion

In this work, we provide the first sequent calculus for the hybrid intuitionistic logic IHL [7] that is appropriate for proof-search thanks to the absence of structural rules. After proving the main properties of this calculus that are soundness, completeness and cut-elimination, we define a decision procedure and then we propose the first proof of decidability of this logic. The study of complexity of IHL [1] will be the next step developed in further works but we will also consider extensions of our calculi with rules corresponding to conditions on the accessibility relations (geometric theories) like reflexivity, symmetry and transitivity, in order to obtain a system in which each condition on the accessibility relation has a corresponding rule and each combination of these rules is complete for the logic with the corresponding conditions.

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