# Provability and Countermodels in Gödel-Dummett Logics

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**Abstract.** Hypersequent calculi, that are a generalization of sequent calculi, have been studied for Gödel-Dummett logics $\mathsf{LC}$ and $\mathsf{LC}_n$. In this paper we propose a new characterization of validity in these logics from the construction of particular bi-colored graphs associated to hypersequents and the search of specific chains in such graphs. It leads to other contributions that are a new hypersequent calculus and a related tableau system for $\mathsf{LC}_n$. We mainly study the class of so-called basic hypersequents and then we generalize our approach to hypersequents.

## 1 Introduction

Gödel-Dummett logic $\mathsf{LC}$ and its finitary versions $(\mathsf{LC}_n)_{n>0}$ were introduced by Gödel and later axiomatized by Dummett in [8]. They are *intermediate* logics (between classical and intuitionistic logics) with semantics based on linear Kripke models. It has a Hilbert axiomatic system composed of axioms of intuitionistic logic and the axiom $A \to B \vee B \to A$. One of its interests lies in its relationship with fuzzy logics [12] and recently $\mathsf{LC}_n$ logics have been characterized as resource use bounding logics for some particular process calculus [14].
There exist various calculi dedicated to proof-search in $\mathsf{LC}$ like sequent calculi [9], sequent of relations calculi [6], tableau calculi [1], goal-directed calculi [17], decomposition systems [4] and also based-on bi-colored graphs calculi [16]. Hypersequent calculi, that generalize sequent calculi, have been also studied [3,5,10]. In order to propose decision procedures from sequent or hypersequent calculi an interesting approach consists in defining local and invertible proof rules, in reducing a (hyper)sequent into a set of irreducible (hyper)sequents and in defining an algorithm to decide such (hyper)sequents [13]. For instance [3] presents a decision procedure from a hypersequent calculus by using particular hypersequents, called *basic hypersequents*.
In this work we focus on hypersequent calculi mainly with countermodel search that is not developed in the above-mentioned works. Our alternative approach for deciding hypersequents is based on two main steps: the construction of a semantic graph called *bi-colored graph* and a characterization of validity based on detection of particular chains in this graph. The idea to characterize validity in a given logic from a semantic graph and its analysis has been already studied in non-classical logics, for instance in BI (logic of Bunched Implications) with resource graphs [11]. The notion of bi-colored graph has been recently defined to deal with formulae in $\mathsf{LC}$ and $\mathsf{LC}_n$ [13], but its possible use for

hypersequents is a non-trivial question to solve. In fact a hypersequent calculus incorporates a sequent calculus as a sub-calculus but adds an additional layer of information by considering a sequent to live in the context of finite multisets of sequents (called hypersequents) [5]. It includes rules for exchanging information between sequents that make it powerful but proof-search methods for sequents are not well-adapted to such structures.

A first contribution concerns the *basic hypersequents* [3], for which we define new characterizations of validity for $\mathsf{LC}$ and $\mathsf{LC}_n$. They are based on the construction of a specific bi-colored graph associated to a basic hypersequent and on the search of particular chains in such a graph. The detection of such chains and the generation of countermodels can be realized in linear time [13]. Using the result that for every hypersequent $G$ one can find a set $\mathcal{B}$ of basic hypersequents such that $G$ is valid if and only if every element of $\mathcal{B}$ is valid [3] we also provide new decision procedures for hypersequents in $\mathsf{LC}$ and $\mathsf{LC}_n$ with a focus on countermodel construction. The study of bi-colored graphs associated to hypersequents leads to other important contributions that complete the results of [3]: a new hypersequent calculus and a related tableau system for $\mathsf{LC}_n$. In the above mentioned results we start with a particular class of hypersequents but it seems important to study if we can directly deal with (general) hypersequents and define associated bi-colored graphs in order to characterize provability. From this perspective we can apply to a given hypersequent an indexing process defined in [13] and then associate a so-called indexed flat sequent from which a bi-colored graph can be built. Then we can define a procedure that decides validity from the detection of particular chains in all instances of such a graph. The key point is that, if one of these instances contains no particular chains, then we can extract a countermodel from this instance. The results are obtained first for mono-conclusioned hypersequents but also hold for (multi-conclusioned) hypersequents. As a sequent is a specific hypersequent, our procedure also provides by specialization a new procedure to decide sequents and generate countermodels in Gödel-Dummett Logics.

## 2 Gödel-Dummett logics

In this section, we consider the family of propositional Gödel-Dummett logics $\mathsf{LC}_n$. The value $n$ belongs to the set $\overline{\mathbb{N}}^* = \{1, 2, \ldots\} \cup \{\infty\}$ of strictly positive natural numbers with its natural order $\leqslant$, augmented with a greatest element $\infty$. In the case $n = \infty$, the logic $\mathsf{LC}_\infty$ is also denoted by $\mathsf{LC}$: this is the usual Gödel-Dummett logic.

Let us start by reminding the key points about semantics and proof theory. The set of propositional *formulae*, denoted $\mathsf{Form}$ is defined inductively, starting from a set of propositional *variables* denoted by $\mathsf{Var}$ with an additional bottom constant $\perp$ denoting *absurdity* and using the connectives $\wedge$, $\vee$ and $\rightarrow$. $\mathsf{IL}$ denotes the set of formulae that are provable in any intuitionistic propositional calculus and $\mathsf{CL}$ denotes the classically valid formulae. As usual an *intermediate propositional logic* [1] is a set of formulae $\mathcal{L}$ satisfying $\mathsf{IL} \subseteq \mathcal{L} \subseteq \mathsf{CL}$ and closed under the rule of modus ponens and under arbitrary substitution. In the case of $\mathsf{LC}$, the logic has a simple Hilbert axiomatic system: $(A \rightarrow B) \vee (B \rightarrow A)$ added to the axioms of $\mathsf{IL}$ but also a based-on sequent formulation.

From the semantic point of view Gödel-Dummett logic is characterized by the linear Kripke models of size $n$ (see [8].) The following strictly increasing sequence holds: $\mathsf{IL} \subset \mathsf{LC} = \mathsf{LC}_\infty \subset \cdots \subset \mathsf{LC}_n \subset \cdots \subset \mathsf{LC}_1 = \mathsf{CL}$ Moreover there exists an algebraic semantics characterization of $\mathsf{LC}_n$ [3]. Let $n \in \overline{\mathbb{N}}^*$, the algebraic model is the set $\overline{[0,n)} = [0,\ldots,n[\cup\{\infty\}$ composed of $n+1$ elements. An interpretation of propositional variables $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ is inductively extended to formulae: $\perp$ interpreted by 0, the conjunction $\wedge$ is interpreted by the *minimum* function denoted $\wedge$, the disjunction $\vee$ by the *maximum* function $\vee$ and the implication $\to$ by the operator $\twoheadrightarrow$ defined by $a \twoheadrightarrow b =$ if $a \leqslant b$ then $\infty$ else $b$. Then we have $[\![A \wedge B]\!] = [\![A]\!] \wedge [\![B]\!]$, $[\![\perp]\!] = 0$, $[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!]$, $[\![A \to B]\!] = [\![A]\!] \twoheadrightarrow [\![B]\!]$. A formula $D$ is *valid* for the interpretation $[\![\cdot]\!]$ if the equality $[\![D]\!] = \infty$ holds. This interpretation is complete for $\mathsf{LC}$. A countermodel of a formula $D$ is an interpretation $[\![\cdot]\!]$ such that $[\![D]\!] < \infty$.
For a sequent $\Gamma \vdash \Delta$, with $\Gamma, \Delta$ multisets of formulae, and a given interpretation $[\![\cdot]\!]$ we interpret $\Gamma \equiv A_1, \ldots, A_n$ and $\Delta \equiv B_1, \ldots, B_p$ by: $\llfloor \Gamma \rrfloor = [\![A_1]\!] \wedge \cdots \wedge [\![A_p]\!]$, $\llfloor \emptyset \rrfloor = \infty$ and $\llceil \Delta \rrceil = [\![B_1]\!] \vee \cdots \vee [\![B_q]\!]$, $\llceil \emptyset \rrceil = 0$. Then a sequent is *valid* for the interpretation $[\![\cdot]\!]$ if $\llfloor \Gamma \rrfloor \leq \llceil \Delta \rrceil$. Moreover $[\![\cdot]\!]$ is a countermodel of $\Gamma \vdash \Delta$ if $\llceil \Delta \rrceil < \llfloor \Gamma \rrfloor$.

From the proof-theoretic point of view there exist various calculi in $\mathsf{LC}$ mainly based on sequent calculi [6,9] but we consider here hypersequent calculi introduced as a natural generalization of Gentzen's sequent calculi [3,17]. A hypersequent calculus is defined by incorporating a sequent calculus as a sub-calculus and adding an additional layer of information. It allows to define rules for information exchange between sequents [5].
A *hypersequent* is a structure $\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ldots \mid \Gamma_n \vdash \Delta_n$ in which $\Gamma_i \vdash \Delta_i$ is a sequent, called a *component* of the hypersequent. Let us remark that sequents (resp. hypersequents) are multisets of formulae (resp. sequents). A hypersequent is mono-conclusioned if the $\Delta_i$'s consist of at most one formula. The symbol $\mid$ denotes a disjunction at the meta-level. Hypersequent calculi consists of axioms, structural and logical rules like in sequent calculi but structural rules are divided into internal and external rules. The first ones deal with formulae within components and the other manipulate whole components of a hypersequent.
The hypersequent calculus $HG$ (Figure 1) for $\mathsf{LC}$ is an extension of the hypersequent calculus for intuitionistic logic $HIL$ [9] with the *communication* rule $[com]$. A hypersequent can be seen as a multi-processor [2] and from this perspective the $(com)$ rule fixes the way of exchanging information between processes. As an illustration we prove the axiom $(A \to B) \vee (B \to A)$ in $HG$ that is not valid in intuitionistic logic.

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \vdash B \mid B \vdash A}[com]}{\vdash A \to B \mid \vdash B \to A}[\to_R]}{\vdash (A \to B) \vee (B \to A)}[\vee_R]$$

We observe that it is not possible to derive $(A \to B) \vee (B \to A)$ in $HG$ without using the $[com]$ rule. Let $\mathcal{H} = \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ldots \mid \Gamma_m \vdash \Delta_m$ be a hypersequent and $[\![\cdot]\!] : \overline{[0,n)}$ be an interpretation. $\mathcal{H}$ is *valid* for the interpretation $[\![\cdot]\!]$ iff there exists $i \in [1,m]$ such that $\llfloor \Gamma_i \rrfloor \leq \llceil \Delta_i \rrceil$. Then $[\![\cdot]\!]$ is a *countermodel* of $\mathcal{H}$ iff $\forall i \in [1,m]$, $\llceil \Delta_i \rrceil < \llfloor \Gamma_i \rrfloor$.

*Axioms*

$$A \vdash A$$

*Cut rule*

$$\frac{G \mid \Gamma' \vdash A \qquad G' \mid A, \Gamma \vdash C}{G \mid G' \mid \Gamma, \Gamma' \vdash C}\ [cut]$$

*External structural rules*

$$\frac{G}{G \mid \Gamma \vdash A}\ [ew] \qquad \frac{G \mid \Gamma \vdash A \mid \Gamma \vdash A}{G \mid \Gamma \vdash A}\ [ec]$$

*Internal structural rules*

$$\frac{G \mid \Gamma \vdash C}{G \mid \Gamma, A \vdash C}\ [w,l] \qquad \frac{G \mid \Gamma, A, A \vdash C}{G \mid \Gamma, A \vdash C}\ [c,l]$$

*Logical rules*

$$\frac{G \mid \Gamma, A, B \vdash C}{G \mid \Gamma, A \wedge B \vdash C}\ [\wedge_L] \qquad \frac{G \mid \Gamma \vdash A \qquad G \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \wedge B}\ [\wedge_R]$$

$$\frac{G \mid \Gamma, A \vdash C \qquad G \mid \Gamma, B \vdash C}{G \mid \Gamma, A \vee B \vdash C}\ [\vee_L] \qquad \frac{G \mid \Gamma \vdash A \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \vee B}\ [\vee_R]$$

$$\frac{G \mid \Gamma \vdash A \qquad G' \mid \Gamma, B \vdash C}{G \mid G' \mid \Gamma, A \to B \vdash C}\ [\to_L] \qquad \frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \to B}\ [\to_R]$$

*Special structural rule*

$$\frac{G \mid \Gamma, \Gamma' \vdash A \qquad G' \mid \Gamma_1, \Gamma'_1 \vdash A'}{G \mid G' \mid \Gamma, \Gamma'_1 \vdash A \mid \Gamma', \Gamma_1 \vdash A'}\ [com]$$

**Fig. 1.** The Hypersequent Calculus *HG* for LC

Proof-search in LC and in some intermediate logics is based on different calculi: a contraction-free calculus derived from intuitionistic logic [1,9], sequent or hypersequent of relations calculi in LC [6,7] and more generally in many-valued logics and hypersequent calculi [3,18]. Some refinements, based on local and invertible rules, have been proposed for sequents or hypersequents with semantic criteria to decide irreducible sequents or hypersequents [3]. Here we aim at studying validity and proof-search in LC and $\mathsf{LC}_n$ with hypersequent calculi in a new perspective based on countermodel construction from so-called bi-colored graph introduced in [13,15].

## 3 A new procedure for basic hypersequents

Before starting to study a particular class of hypersequents, namely the basic hypersequents [3], let us remind what is a *bi-colored graph* in this context.

**Definition 3.1.** *A (conditional)* bi-colored graph *is a finite oriented graph with two kinds of arrows, the green ones represented by* $\to$ *and the red ones represented by* $\Rightarrow$*,* *that are indexed by boolean formulae. The boolean variables of these formulae can be instantiated by* $\{0,1\}$ *through a valuation. Moreover an* instance *of the graph is the bi-colored graph with only the arrows indexed by an expression e with* $v(e) = 1$.

We use the symbols $\rightarrow$ and $\Rightarrow$ to denote the corresponding relation in the graph. For example $\rightarrow\Rightarrow$ represents the composition of two relations and $u \rightarrow\Rightarrow w$ means that there exists a path $u \rightarrow v \Rightarrow w$ in the graph. The relation $\rightarrow^\star$ is the reflexive and transitive closure of $\rightarrow$, i.e, the accessibility of the relation $\rightarrow$. Moreover $\rightarrow + \Rightarrow$ is the union of both relations and $\overline{x}$ denotes the negation of the boolean expression $x$.

**Definition 3.2.** *Let* $\mathcal{G}$ *be a bi-colored graph, a* $\Rightarrow$-cycle *of* $\mathcal{G}$ *is a chain of the form* $u(\rightarrow + \Rightarrow)^\star \Rightarrow u$ *and a* $k$-alternating chain *of* $\mathcal{G}$ *is a chain of the form* $(\rightarrow^\star\Rightarrow)^k$.

Therefore the key point of our approach consists in associating a bi-colored graph to a given hypersequent and in relating validity in the given logic with the existence of $\Rightarrow$-cycle or $k$-alternating chain. Let us start this study with particular hypersequents.

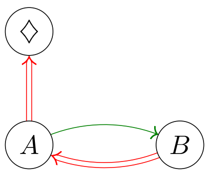
**Definition 3.3.** *A* basic hypersequent *is a hypersequent such that any component is either* $\Gamma \vdash p$ *where* $p$ *and any element of* $\Gamma$ *are atoms, or* $p \rightarrow q \vdash p$ *where* $p$ *and* $q$ *are atoms and* $p \neq q$, $p \neq \bot$.

Let $\mathcal{H} = S_1 \mid \ldots \mid S_k$ be a basic hypersequent, the bi-colored graph $\mathcal{G}_{\mathcal{H}}$ associated to $\mathcal{H}$ is built as follows:
- the *nodes* are: the variables of $\mathcal{H}$, a node denoted $\Diamond$ and a node $\bot$ if $\mathcal{H}$ contains $\bot$.
- the *arrows* are the union of the set $B$ and arrows $A_{i\in[1,k]}$ where

$$B = \begin{cases} \emptyset & \textit{if } \mathcal{H} \textit{ does not contain } \bot \\ \{\bot \rightarrow p, \text{ for any } p \in \mathsf{Var}\} & \textit{otherwise} \end{cases}$$

and $A_{i\in[1,k]}$ associated to the components $S_{i\in[1,k]}$ of $\mathcal{H}$ defined as follows: if $S_i = p \rightarrow q \vdash p$ then $A_i = \{p \rightarrow q, p \Rightarrow \Diamond\}$, else if $S_i = q_1, \ldots, q_m \vdash p$ then $A_i = \{p \Rightarrow q_1, \ldots, p \Rightarrow q_m\}$.

Let us illustrate this construction with the basic hypersequent $\mathcal{H}_1 \equiv A \rightarrow B \vdash A \mid A \vdash B$. The bi-colored graph associated to $\mathcal{H}_1$ is the following:




**Proposition 3.1.** *Let* $\mathcal{H}$ *be a basic hypersequent and* $\mathcal{G}_{\mathcal{H}}$ *be its associated bi-colored graph. Let* $[\![\cdot]\!]$ *be a countermodel of* $\mathcal{H}$ *in* $\mathsf{LC}_n$ *(extended with* $[\![\Diamond]\!] = \infty$*) and* $X_1 \rightarrow \ldots \rightarrow X_k \Rightarrow Y$ *be a chain in* $\mathcal{G}_{\mathcal{H}}$. *Then we have* $[\![X_1]\!] \leqslant \ldots \leqslant [\![X_k]\!] < [\![Y]\!]$.

*Proof.* Let $\mathcal{H} = S_1 \mid \ldots \mid S_k$ be a basic hypersequent. As $[\![\cdot]\!]$ is a countermodel of $\mathcal{H}$ then $[\![\cdot]\!]$ is a countermodel of all the components $S_{i\in[1,k]}$ of $\mathcal{H}$. If $Y \neq \Diamond$ then there exists a component $S_i$ such that $X_k$ is the conclusion of $S_i$ and $Y$ belongs to the multiset of $S_i$ hypotheses. Thus we have $[\![X_k]\!] < [\![Y]\!]$. If $Y = \diamondsuit$ then there exists a component $S_i$ having $X_k$ as conclusion. Therefore $X_k < [\![\diamondsuit]\!] = \infty$. Moreover for all $j \in [2,k]$ we have $X_{j-1} \to X_j \in \mathcal{G}_{\mathcal{H}}$ and there exists a component $S_i = X_{j-1} \to X_j \vdash X_{j-1}$ or $X_{j-1} = \bot$. Then we deduce that $[\![X_{j-1}]\!] \leqslant [\![X_j]\!]$ and $[\![X_1]\!] \leqslant \ldots \leqslant [\![X_k]\!] < [\![Y]\!]$.

Moreover we can define, from a bi-colored graph $\mathcal{G}$, the notion of *bi-height* that is a function $h : \mathcal{G} \to \mathbb{N}$ such that for any $u, v \in \mathcal{G}$, if $u \to v \in \mathcal{G}$ then $h(u) \leq h(v)$ and if $u \Rightarrow v \in \mathcal{G}$ then $h(u) < h(v)$ [13]. Then a countermodel can be generated from $\mathcal{G}$ by using the following results: if a bi-colored graph $\mathcal{G}$ does not contain a $\Rightarrow$-cycle (resp. a $n$-alternating chain) then there exists a bi-height $h$ (resp. that satisfies $h(v) < n$ for any $v \in \mathcal{G}$) [13]. Moreover it is known that we can decide if a graph instance contains or not a $\Rightarrow$-cycle and also compute the bi-height both in a linear time [15].

**Proposition 3.2.** *Let $\mathcal{H}$ be a basic hypersequent containing $\bot$. If there exists a bi-height $h$ for $\mathcal{G}_{\mathcal{H}}$ then the function $h'$ defined by: $h'(X) = 0$ if $h(X) = h(\bot)$ and $h'(X) = h(X)$ if $h(X) \neq h(\bot)$, is a bi-height for $\mathcal{G}_{\mathcal{H}}$.*

*Proof.* From the set of arrows $B$ defined in the construction of the bi-colored graphs associated to the basic hypersequents.

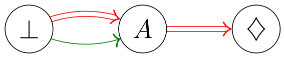
**Theorem 3.1** **($n < \infty$).** *A basic hypersequent $\mathcal{H}$ has a countermodel in $\mathsf{LC}_n$ if and only if its bi-colored graph $\mathcal{G}_{\mathcal{H}}$ does not contain a $(n+1)$-alternating chain.*

*Proof.* First we prove the *if* part. Let $\mathcal{H} = S_1 \mid \ldots \mid S_k$ be a basic hypersequent. We suppose that $\mathcal{G}_{\mathcal{H}}$ does not contain a chain of the form $(\to^\star\Rightarrow)^{n+1}$. Then there exists a bi-height $h : \mathcal{G}_{\mathcal{H}} \to [0,n]$. By Proposition 3.2, we define from $h$ a new bi-height $h' : \mathcal{G}_{\mathcal{H}} \to [0,n]$ by: $h'(X) = 0$ if $h(X) = h(\bot)$ and $h'(X) = h(X)$ if $h(X) \neq h(\bot)$ and then we define the semantic function $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ by: $[\![X]\!] = h'(X)$ if $h'(X) < n$ and $[\![X]\!] = \infty$ if $h'(X) = n$. We prove that $[\![\cdot]\!]$ is a countermodel of $\mathcal{H}$, i.e., a countermodel of any component $S_{i\in[1,k]}$.
(i) if $S_i = p \to q \vdash p$ then $p \to q \in \mathcal{G}_{\mathcal{H}}$ and $p \Rightarrow \diamondsuit \in \mathcal{G}_{\mathcal{H}}$. Thus, we have $h'(p) \leqslant h'(q)$ and $h'(p) < n$. We deduce that $[\![p]\!] \leqslant [\![q]\!]$ and $[\![p]\!] < \infty$. Thus we have $[\![p \to q]\!] = \infty$ and $[\![p]\!] < \infty = [\![p \to q]\!]$. Consequently $[\![\cdot]\!]$ is a countermodel of $S_i$.
(ii) if $S_i = q_1, \ldots, q_m \vdash p$ then $\forall i \in [1,m]$, $p \Rightarrow q_i \in \mathcal{G}_{\mathcal{H}}$ and $\forall i \in [1,m]$, $h'(p) < h'(q_i)$. We deduce that $\forall i \in [1,m]$, $[\![p]\!] < [\![q_i]\!]$ and $[\![\cdot]\!]$ is a countermodel of $S_i$.
From (i) and (ii) we deduce that $[\![\cdot]\!]$ is a countermodel of $\mathcal{H}$.
We now prove the *only if* part. Let $[\![\cdot]\!]$ be a countermodel of $\mathcal{H}$, we define a new interpretation $[\![\cdot]\!]'$ by: $[\![V]\!]' = [\![V]\!]$ for any variable $V$ of $\mathcal{H}$ and $[\![\diamondsuit]\!]' = \infty$. As $[\![\cdot]\!]'$ and $[\![\cdot]\!]$ have the same values for $\mathcal{H}$'s atoms, we deduce that $[\![\cdot]\!]'$ is a countermodel of $\mathcal{H}$. We suppose that there exists a chain of the form $(\to^\star\Rightarrow)^{n+1}$ in $\mathcal{G}_{\mathcal{H}}$: $X_0 \to^\star \Rightarrow X_1 \to^\star \Rightarrow X_2 \to^\star \Rightarrow \ldots \to^\star \Rightarrow X_n \to^\star \Rightarrow X_{n+1}$. Thus, by Proposition 3.1, the sequence $[\![X_0]\!] < [\![X_1]\!] < [\![X_2]\!] < \ldots < [\![X_n]\!] < [\![X_{n+1}]\!]$ is a strictly increasing sequence of $n+2$ elements in $\overline{[0,n)}$. As this set does contain only $n+1$ elements, that is contradictory.

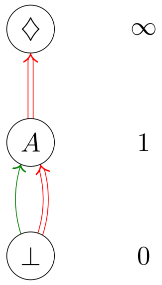
**Theorem 3.2** **($n = \infty$).** *A basic hypersequent $\mathcal{H}$ has a countermodel in $\mathsf{LC}$ if and only if its bi-colored graph $\mathcal{G}_{\mathcal{H}}$ does not contain a $\Rightarrow$-cycle.*

*Proof.* For the *if* part: if $\mathcal{G}_{\mathcal{H}}$ does not contain a $\Rightarrow$-cycle, we know that there exists a bi-height $h : \mathcal{G}_{\mathcal{H}} \to \mathbb{N}$. By Proposition 3.2, we define from $h$ a new bi-height $h' : \mathcal{G}_{\mathcal{H}} \to \mathbb{N}$ by: $h'(X) = 0$ if $h(X) = h(\perp)$ and $h'(X) = h(X)$ if $h(X) \neq h(\perp)$. By defining $[\![X]\!] \in \mathbb{N} \cup \{\infty\}$ by and $[\![X]\!] = h'(X)$ we obtain a countermodel of $\mathcal{H}$ in $\mathsf{LC}$. For the *only if* part: the existence of a chain $X \to^{\star} \Rightarrow \to^{\star} \ldots \to^{\star} \Rightarrow \to^{\star} \Rightarrow X$ implies $[\![X]\!] < [\![X]\!]$ and then we have a contradiction.

Coming back to our example we observe that the bi-colored graph associated to $\mathcal{H}_1$ contains a $\Rightarrow$-cycle: $A \to B \Rightarrow A$. Then we conclude that $\mathcal{H}_1$ has no countermodel in $\mathsf{LC}$. Let us give another example with the basic hypersequent $\mathcal{H}_2 \equiv \vdash A \mid A \vdash \perp$. The bi-colored graph associated to $\mathcal{H}_2$ is the following:




This graph has one instance and does not contain a $\Rightarrow$-cycle. In order to extract a countermodel we modify the previous graph in such a way that red arrows always go up and greens arrows never go down.




Then $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ such that $[\![A]\!] = 1$ is a countermodel of $\mathcal{H}_2$ in $\mathsf{LC}_n$ for $n \geqslant 2$.

A decision procedure for basic hypersequents has been already provided but only for $\mathsf{LC}$ [3]. It is based on the generation of constraints and some criteria for solving them. Here we define new criteria, based on bi-colored graphs, to decide the basic hypersequents in $\mathsf{LC}$ but also in $(\mathsf{LC}_n)_{n>0}$. Moreover, the extraction of countermodels can be realized from the bi-colored graphs associated to hypersequents.

## 4 New results for $\mathsf{LC}$ and $\mathsf{LC}_n$

In order to propose new results for (general) hypersequents from the above results on basic hypersequents, we can relate them to the system $GLC^*$ and some results of [3].

### 4.1 Decision procedures for hypersequents

The main one is that for every hypersequent $G$ one can effectively find a set $\mathcal{B}$ of basic hypersequents, so that $G$ is valid if and only if $\mathcal{H}$ is valid for every $\mathcal{H} \in \mathcal{B}$.

$$\frac{G \mid \Gamma, A, B \vdash C}{G \mid \Gamma, A \wedge B \vdash C}\ [\wedge_L] \qquad \frac{G \mid \Gamma \vdash A \qquad G \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \wedge B}\ [\wedge_R]$$

$$\frac{G \mid \Gamma, A \vdash C \qquad G \mid \Gamma, B \vdash C}{G \mid \Gamma, A \vee B \vdash C}\ [\vee_L] \qquad \frac{G \mid \Gamma \vdash A \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \vee B}\ [\vee_R]$$

$$\frac{G \mid \Gamma, A \to B, A \to C \vdash D}{G \mid \Gamma, A \to (B \wedge C) \vdash D}\ [\to\wedge_L] \qquad \frac{G \mid \Gamma, A \to B \vdash D \qquad G \mid \Gamma, A \to C \vdash D}{G \mid \Gamma, A \to (B \vee C) \vdash D}\ [\to\vee_L]$$

$$\frac{G \mid \Gamma, A \to C \vdash D \qquad G \mid \Gamma, B \to C \vdash D}{G \mid \Gamma, (A \wedge B) \to C \vdash D}\ [\wedge\to_L] \qquad \frac{G \mid \Gamma, A \to C, B \to C \vdash D}{G \mid \Gamma, (A \vee B) \to C \vdash D}\ [\vee\to_L]$$

$$\frac{G \mid \Gamma, A \to C \vdash D \qquad G \mid \Gamma, B \to C \vdash D}{G \mid \Gamma, A \to (B \to C) \vdash D}\ [\to(\to)_L] \qquad \frac{G \mid A \vdash B \mid \Gamma, B \to C \vdash D \qquad G \mid \Gamma, C \vdash D}{G \mid \Gamma, (A \to B) \to C \vdash D}\ [(\to)\to_L]$$

$$\frac{G \mid \Gamma \vdash r \mid p \to q \vdash p \qquad G \mid \Gamma, q \vdash r}{G \mid \Gamma, p \to q \vdash r}\ [\to_L] \qquad \frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \to B}\ [\to_R]$$

**Fig. 2.** The Rules of $GLC^*$ for $\mathsf{LC}$

**Proposition 4.1.** *Let* $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ *be an interpretation,*
$[\![\cdot]\!]$ *is a countermodel of* $G \mid \Gamma, \bot \to A \vdash B$ *in* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*) iff* $[\![\cdot]\!]$ *is countermodel of* $G \mid \Gamma \vdash B$ *in* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*);*
$[\![\cdot]\!]$ *is countermodel of* $G \mid \Gamma, A \to A \vdash B$ *in* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*) iff* $[\![\cdot]\!]$ *is countermodel of* $G \mid \Gamma \vdash B$ *in* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*);*
$[\![\cdot]\!]$ *is countermodel of* $G \mid \Gamma, p \to q \vdash p$ *in* $\mathsf{LC}_n$ *(resp.*$\mathsf{LC}$*) iff* $[\![\cdot]\!]$ *is countermodel of* $G \mid \Gamma \vdash p \mid p \to q \vdash p$ *in* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*).*

From these results we now consider the $GLC^*$ system the rules of which are given in Figure 2. The axioms of $GLC^*$ are the generalized axioms defined as follows [3]: A generalized axiom is a basic hypersequent of one of the following forms: a) $p_1 \prec p_2 \mid p_2 \prec p_3 \mid \ldots \mid p_{n-1} \prec p_n \mid p_n \vdash p_1$ where $n \geqslant 1$, $p_1, \ldots, p_n$ are $n$ distinct propositional variables, and for all $1 \le i \le n-1$, $p_i \prec p_{i+1}$ is either $p_i \vdash p_{i+1}$ or $(p_{i+1} \to p_i) \vdash p_{i+1}$; b) $(p_1 \to \bot) \vdash p_1 \mid (p_2 \to p_1) \vdash p_2 \mid \ldots \mid (p_{n-1} \to p_{n-2}) \vdash p_{n-1} \mid p_{n-1} \vdash p_n$ where $n \geqslant 1$, $p_1, \ldots, p_n$ are $n$ distinct propositional variables (in the case $n = 1$ we take $p_0$ to be $\bot$).

Let us recall some useful definitions. Knowing that a proof rule is composed of premises $H_i$ with a conclusion $C$, it is *strongly sound* if, for any instance of the rule and any interpretation $[\![\cdot]\!]$, if $[\![\cdot]\!]$ is a model of all the $H_i$ then it is a model of $C$. Moreover it is *strongly invertible* if, for any instance of the rule and any interpretation $[\![\cdot]\!]$, if $[\![\cdot]\!]$ is a countermodel of at least one $H_i$ then it is a countermodel of $C$.

**Theorem 4.1.** *The rules of* $GLC^*$ *are strongly sound for* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*).*

*Proof.* We consider the $(\to)\to$ rule. The other cases are similar. Let $[\![\cdot]\!]$ be an interpretation which is a model of both premises. Thus, $[\![\cdot]\!]$ is a model of $G$ or both

$\llbracket\Gamma\rrbracket \wedge \llbracket C\rrbracket \leq \llbracket D\rrbracket$ and either $\llbracket A\rrbracket \leq \llbracket B\rrbracket$ or $\llbracket\Gamma\rrbracket \wedge \llbracket B \rightarrow C\rrbracket \vdash \llbracket D\rrbracket$ hold:
- if $\llbracket\cdot\rrbracket$ is a model of $G$ then $\llbracket\cdot\rrbracket$ is a model of $G \mid \Gamma,(A \rightarrow B) \rightarrow C \vdash D$, conclusion of the $(\rightarrow)\rightarrow$ rule;
- if $\llbracket\Gamma\rrbracket \wedge \llbracket C\rrbracket \leq \llbracket D\rrbracket$ and $\llbracket A\rrbracket \leq \llbracket B\rrbracket$. Since $\llbracket A\rrbracket \leq \llbracket B\rrbracket$, we have $\llbracket(A \rightarrow B) \rightarrow C\rrbracket = \llbracket C\rrbracket$ and we conclude that $\llbracket\Gamma\rrbracket \wedge \llbracket(A \rightarrow B) \rightarrow C\rrbracket \leq \llbracket D\rrbracket$; - if $\llbracket\Gamma\rrbracket \leq D$ then $\llbracket\Gamma\rrbracket \wedge \llbracket(A \rightarrow B) \rightarrow C\rrbracket \leq \llbracket D\rrbracket$ holds;
- if $\llbracket C\rrbracket \leq \llbracket D\rrbracket$ and $\llbracket B \rightarrow C\rrbracket \leq \llbracket D\rrbracket$ then if $\llbracket A\rrbracket > \llbracket B\rrbracket$ then $\llbracket(A \rightarrow B) \rightarrow C\rrbracket = \llbracket B \rightarrow C\rrbracket$ and $\llbracket\Gamma\rrbracket \wedge \llbracket(A \rightarrow B) \rightarrow C\rrbracket \leq \llbracket D\rrbracket$ holds. Else, $\llbracket(A \rightarrow B) \rightarrow C\rrbracket = \llbracket C\rrbracket$ and we deduce that $\llbracket\Gamma\rrbracket \wedge \llbracket(A \rightarrow B) \rightarrow C\rrbracket \leq \llbracket D\rrbracket$.

**Theorem 4.2.** *The rules of GLC* are strongly invertible for* $\mathsf{LC}_n$ *(resp.* $\mathsf{LC}$*).*

*Proof.* We consider the $(\rightarrow)\rightarrow$ rule. The other cases are similar. Let $\llbracket\cdot\rrbracket$ be a countermodel of $G \mid \Gamma, C \vdash D$ (the left premise). Then both $\llbracket\cdot\rrbracket$ is a countermodel of $G$ and $\llbracket\Gamma\rrbracket \wedge \llbracket C\rrbracket > \llbracket D\rrbracket$ hold. Since $\llbracket C\rrbracket \leq \llbracket(A \rightarrow B) \rightarrow C\rrbracket$, we deduce that $\llbracket\cdot\rrbracket$ is a countermodel of $G$ and $\llbracket\Gamma\rrbracket \wedge \llbracket(A \rightarrow B) \rightarrow C\rrbracket > \llbracket D\rrbracket$. Therefore, $\llbracket\cdot\rrbracket$ is a countermodel of the conclusion of the rule $(\rightarrow)\rightarrow$. Let $\llbracket\cdot\rrbracket$ be a countermodel of $G \mid \Gamma, B \rightarrow C \vdash D$ (the right premise). We have $\llbracket\cdot\rrbracket$ is a countermodel of $G$ and both $\llbracket A\rrbracket > \llbracket B\rrbracket$ and $\llbracket\Gamma\rrbracket \wedge \llbracket B \rightarrow C\rrbracket > \llbracket D\rrbracket$ hold. Since $\llbracket A\rrbracket > \llbracket B\rrbracket$, we have $\llbracket(A \rightarrow B) \rightarrow C\rrbracket = \llbracket B \rightarrow C\rrbracket$. Thus $\llbracket\Gamma\rrbracket \wedge \llbracket(A \rightarrow B) \rightarrow C\rrbracket > \llbracket D\rrbracket$ holds and we conclude that $\llbracket\cdot\rrbracket$ is a countermodel of the $(\rightarrow)\rightarrow$ rule conclusion.

Since all *GLC** rules are strongly invertible, we obtain, for any $\mathcal{H} \in \mathcal{B}$, if $\llbracket\cdot\rrbracket : \mathsf{Var} \rightarrow \overline{[0,n)}$ is countermodel of $\mathcal{H}$ in $\mathsf{LC}_n$ (resp. $\mathsf{LC}$) then $\llbracket\cdot\rrbracket$ is countermodel of $G$ in $\mathsf{LC}_n$ (resp. $\mathsf{LC}$) because $\mathcal{B}$ is obtained from the *GLC** rules and Proposition 4.1. Thus we get a decision procedure for hypersequents in $\mathsf{LC}_n$ (resp. $\mathsf{LC}$) which builds countermodels, by using the previous decision procedure in order to decide which basic hypersequents are valid in $\mathsf{LC}_n$ (resp. $\mathsf{LC}$) and eventually to build a countermodel. Moreover we can characterize the axioms of *GLC** as the basic hypersequents with associated bi-colored graphs that contain a $\Rightarrow$-cycle.
Therefore we have provided new decision procedures for $\mathsf{LC}$ but also $\mathsf{LC}_n$ with construction of countermodels and decision of irreducible hypersequents that can be realized in linear time. In comparison sequent of relations calculi provide a nice framework for proof search in $\mathsf{LC}$ [6,7] but cannot deal with the finitary versions $\mathsf{LC}_n$.

### 4.2 A new hypersequent calculus and a tableau system for $\mathsf{LC}_n$

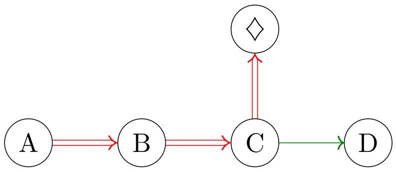
Having defined a new procedure for hypersequents in $\mathsf{LC}$ but mainly for $\mathsf{LC}_n$ by defining bi-colored graphs associated to hypersequents. In a dual approach we show how we can deduce, from our study of bi-colored graphs, a new hypersequent calculus for $\mathsf{LC}_n$ similar to system *GLC**, by providing a new class of axioms called *n*-generalized axioms.

**Definition 4.1** **(*n*-generalized axiom).** *A* $n$-generalized axiom *is either a generalized axiom or a basic hypersequent of the form:*
$p^1_{m_1} \vdash p^1_{m_1-1} \mid (p^2_1 \rightarrow p^2_2) \vdash p^2_1 \mid (p^2_2 \rightarrow p^2_3) \vdash p^2_2 \mid \ldots \mid (p^2_{m_2-2} \rightarrow p^2_{m_2-1}) \vdash p^2_{m_2-2} \mid p^2_{m_2} \vdash p^2_{m_2-1} \mid (p^3_1 \rightarrow p^3_2) \vdash p^3_1 \mid (p^3_2 \rightarrow p^3_3) \vdash p^3_2 \mid \ldots \mid (p^3_{m_3-2} \rightarrow p^3_{m_3-1}) \vdash p^3_{m_3-2} \mid p^3_{m_3} \vdash p^3_{m_3-1}$

...
$\mid (p^n_1 \to p^n_2) \vdash p^n_1 \mid (p^n_2 \to p^n_3) \vdash p^n_2 \mid \ldots \mid (p^n_{m_n-2} \to p^n_{m_n-1}) \vdash p^n_{m_n-2} \mid p^n_{m_n} \vdash p^n_{m_n-1} \mid p^n_{m_n} \vdash' p_f$
*where for all* $1 \le k \le n$, $m_k \geqslant 2$ *and* $p^i_{m_k} = p^{i+1}_1$. *Moreover, for all* $2 \le i \le n$ *and* $1 \le j \le m_i$, $p^i_j, p^1_{m_1}, p_f$ *are* $2 + m_2 + \ldots + m_n$ *distinct propositional variable, and* $p^1_{m_1-1}$ *is either a distinct propositional variable or* $\bot$ *and* $p \vdash' q$ *is either* $q \prec p$ *or* $q \to p \vdash q$.

From the $n$-generalized axioms, we can derive all the basic hypersequents the bi-colored graphs of which contain a $(n+1)$-alternating chain, by using (internal and external) weakenings and permutations.

As an example $\mathcal{H} \equiv B \vdash A \mid C \vdash B \mid C \to D \vdash C$ is a 2-generalized axiom with the following bi-colored graph:




**Theorem 4.3.** *A basic hypersequent is valid in* $\mathsf{LC}_n$ *iff it is a basic hypersequent derived from some n-generalized axioms using weakenings and permutations.*

*Proof.* First we prove the if part. Let $\mathcal{H}$ be a basic hypersequent. We suppose that $\mathcal{H}$ is derived from a $n$-generalized axiom using weakenings and permutations. Then the bi-colored $\mathcal{G}_{\mathcal{H}}$ contain a $n+1$-alternating chain. By Theorem 3.1, we have $\mathcal{H}$ valid in $\mathsf{LC}_n$. We now prove the only if part. Let $\mathcal{H}$ be a basic hypersequent valid in $\mathsf{LC}_n$. We suppose that $\mathcal{H}$ is not derived from a $n$-generalized axiom using weakenings and permutations. Then the bi-colored $\mathcal{G}_{\mathcal{H}}$ does not contain a $(n+1)$-alternating chain. By Theorem 3.1, we deduce that $\mathcal{H}$ is not valid in $\mathsf{LC}_n$ and then we have a contradiction.

**Definition 4.2.** *We define the* $GLC^*_n$ *system as the hypersequent calculus having*
*- basic hypersequents derived from the n-generalized axioms using (internal and external) weakenings and permutations, as axioms;*
*- rules of* $GLC^*$ *as rules.*

The $GLC^*_n$ axioms are the basic hypersequents whose the bi-colored graphs contain $(n+1)$-alternating chains. Therefore, they are the basic hypersequents valid in $\mathsf{LC}_n$. Since for every hypersequent $\mathcal{G}$, one can find a set of basic hypersequents $\mathcal{B}$, so that $\mathcal{G}$ is valid in $\mathsf{LC}_n$ if and only if $\mathcal{H}$ is valid in $\mathsf{LC}_n$ for every $\mathcal{H} \in \mathcal{B}$, we conclude that a hypersequent $\mathcal{G}$ is valid in $\mathsf{LC}_n$ if and only if $\mathcal{G}$ has a proof in $GLC^*_n$.

**Theorem 4.4.** *A formula* $\mathcal{F}$ *is valid in* $\mathsf{LC}_n$ *iff the sequent* $\vdash \mathcal{F}$ *has a proof in* $GLC^*_n$.

A consequence is that a tableau system for finitary versions of Gödel-Dummett logic $(\mathsf{LC}_n)_{n>0}$ based on the hypersequent calculus $GLC^*_n$ can be obtained from the Avron's tableau system for $\mathsf{LC}$ based on $GLC^*$ [3]. We only have to change the definition of closed branchs by using the axioms of $GLC^*_n$ instead of the ones of $GLC^*$. This direct extension to $\mathsf{LC}_n$ is the result of the use of bi-colored graphs to decide the basic hypersequents. In order to check if a branch is closed, it seems simpler to verify the existence of a particular chain or cycle in the graph than to verify if a set of signed formulas (and links) represents or not an instance of an axiom (in $GLC^*_n$ or $GLC^*$).

## 5 Bi-colored graphs and hypersequents in $\mathsf{LC}$ and $\mathsf{LC}_n$

In this section we consider (general) hypersequents and aim at studying if the approach used in the case of a particular class of hypersequents can be generalized in the general case by defining adequate bi-colored graphs to to characterize provability.
Our approach consists in applying to a given hypersequent $\mathcal{H}$ an indexing process [13] and then to reduce it to a flat sequent $\mathcal{S}$ such that $\mathcal{H}$ is valid if and only if $\mathcal{S}$ is valid. Let us precise that $\mathcal{H}$ cannot include occurrences of special variables $\square$ and $\diamondsuit$ but can include occurrences of $\bot$. Such occurrences are eliminated during the flattening process. We remind that a formula is flat if it is implicational, of the form $X \rightarrow (Y \star Z)$ or $(X \star Y) \rightarrow Z$ with $X, Y, Z \in \mathsf{Var}$ and $\star \in \{\wedge, \vee, \rightarrow\}$. A $\diamondsuit$-context $\Delta_\diamondsuit$ is a non-empty multiset of implicational formulae such that if $A \rightarrow B \in \Delta_\diamondsuit$ then $\diamondsuit \rightarrow B \in \Delta_\diamondsuit$. Moreover $\Gamma \vdash \Delta_\diamondsuit$ is a flat sequent if the context $\Gamma$ contains only flat formulae and $\Delta_\diamondsuit$ is a $\diamondsuit$-context. A flat hypersequent is such that all its components are flat.
The indexing process is based on the two linear functions $\delta^+$ and $\delta^-$, that map occurrences of subformulae of a given formula $D$ to multisets. They are defined as follows:

$$
\begin{array}{ll}
\delta^+(\bot) & = \mathcal{X}_\bot \rightarrow \square \\
\delta^+(V) & = \mathcal{X}_V \rightarrow V, \square \rightarrow V \text{ with } V \text{ is a variable} \\
\delta^+(A * B) & = \delta^+(A), \delta^+(B), \mathcal{X}_{A*B} \rightarrow (\mathcal{X}_A * \mathcal{X}_B) \text{ with } * \in \{\wedge, \vee\} \\
\delta^+(A \rightarrow B) & = \delta^-(A), \delta^+(B), \mathcal{X}_{A\rightarrow B} \rightarrow (\mathcal{X}_A \rightarrow \mathcal{X}_B)
\end{array}
$$

$$
\begin{array}{ll}
\delta^-(\bot) & = \square \rightarrow \mathcal{X}_\bot \\
\delta^-(V) & = V \rightarrow \mathcal{X}_V, \square \rightarrow V \text{ with } V \text{ is a variable} \\
\delta^-(A * B) & = \delta^-(A), \delta^-(B), (\mathcal{X}_A * \mathcal{X}_B) \rightarrow \mathcal{X}_{A*B} \text{ with } * \in \{\wedge, \vee\} \\
\delta^-(A \rightarrow B) & = \delta^+(A), \delta^-(B), (\mathcal{X}_A \rightarrow \mathcal{X}_B) \rightarrow \mathcal{X}_{A\rightarrow B}
\end{array}
$$

The size of a formula is the number of occurrences of its subformulae that is the number of nodes in its decomposition tree. Let $D$ be a formula of size $n$, it has been proved that the cardinals of $\delta^+(D)$ and $\delta^-(D)$ are smaller than $2n$. Moreover the elements of these multisets are only flat formulae of size less than 5. Then the size of $\delta^-(D)$ and $\delta^+(D)$ are bounded by $5n$.

**Proposition 5.1.** *Let $D$ be a formula, if $[\![\cdot]\!]$ is an interpretation such that $[\![\square]\!] = 0$ and $[\![\mathcal{X}_K]\!] = [\![K]\!]$ for any occurrence of subformula $K$ of $D$ then $\delta^+(D) = \delta^-(D) = \infty$.*

This proposition has been proved in [15]. The next step consists, using this indexing process, in transforming a given hypersequent $\mathcal{H}$ into an indexed flat sequent $\mathcal{S}$ and in building a bi-colored graph from this sequent. Before to give this construction we study the preservation of validity and countermodels through such a transformation.

### 5.1 Hypersequents and flat sequents

For the presentation we consider mono-conclusioned hypersequents but we finally show how and why results are valid for (multi-conclusioned) hypersequents.
Let $\mathcal{H} = A^1_1, \ldots, A^1_{n_1} \vdash B_1 \mid \ldots \mid A^p_1, \ldots, A^p_{n_p} \vdash B_p$ be a hypersequent of $\mathsf{LC}$. We associate to $\mathcal{H}$ a particular flat sequent $\mathcal{S} = FS(\mathcal{H}) = \delta^+(A^1_1), \ldots, \delta^+(A^p_{n_p}),\ \delta^-(B_1), \ldots, \delta^-(B_p)$

$\vdash X_{A_1^1} \to X_{B_1}, \ldots, X_{A_{n_1}^1} \to X_{B_1}, X_{A_1^2} \to X_{B_2}, \ldots, X_{A_{n_2}^2} \to X_{B_2}, \ldots, X_{A_1^p} \to X_{B_p}, \ldots, X_{A_{n_p}^p} \to X_{B_p}, \Diamond \to X_{B_1}, \ldots, \Diamond \to X_{B_p}$.

**Theorem 5.1.** *Let $\mathcal{H}$=$A_1^1, \ldots, A_{n_1}^1 \vdash B_1 \mid \ldots \mid A_1^p, \ldots, A_{n_p}^p \vdash B_p$. If the sequent $FS(\mathcal{H})$ is valid in $\mathsf{LC}_n$ then the hypersequent $\mathcal{H}$ is valid in $\mathsf{LC}_n$.*

*Proof.* Let $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ be an interpretation. We define a new interpretation $[\![\cdot]\!]'$ by $[\![V]\!]' = [\![V]\!]$ for any variable $V$ of $\mathcal{H}$, $[\![X_K]\!]' = [\![K]\!]$ for any $K$ subformula of $\mathcal{H}$ formulae, $[\![\Diamond]\!]' = \infty$ and $[\![\Box]\!]' = 0$. As $[\![\cdot]\!]'$ and $[\![\cdot]\!]$ have the same values for $\mathcal{H}$'s atoms , for any subformula $K$ of $\mathcal{H}$ formulae, $[\![K]\!]' = [\![K]\!]$. Therefore $[\![X_K]\!]' = [\![K]\!]'$ and $[\![\Box]\!]' = 0$. By Proposition 5.1, we obtain $\forall i \in [1,p]\ \forall j \in [1,n_i]\ [\![\delta^+(A_j^i)]\!] = [\![\delta^-(B_i)]\!] = \infty$. As $\mathcal{S}$ is valid in $\mathsf{LC}_n$, $[\![\cdot]\!]'$ is a model of $\mathcal{S}$ and consequently $\exists i \in [1,p]\ \exists j \in [1,n_i]\ [\![X_{A_j^i} \to X_{B_i}]\!]' = \infty$ or $[\![\Diamond \to X_{B_i}]\!]' = \infty$. But $[\![\Diamond]\!] = \infty$ and then $\exists i \in [1,p]\ \exists j \in [1,n_i]\ [\![X_{A_j^i}]\!]' \leqslant [\![X_{B_i}]\!]'$ or $[\![X_{B_i}]\!]' = \infty$. Thus $\exists i \in [1,p]\ \exists j \in [1,n_i]\ [\![A_j^i]\!] \leqslant [\![B_i]\!]$ or $[\![B_i]\!] = \infty$. As we have proved that $\exists i \in [1,p]\ \exists j \in [1,n_i]\ [\![A_j^i]\!] \leqslant [\![B_i]\!]$ or $[\![B_i]\!] = \infty$, for any interpretation $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$, we deduce that $\mathcal{H}$ is valid in $\mathsf{LC}_n$.

Let $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ be an interpretation and $\alpha \in \overline{[0,n)}$, we define the translated interpretation by:

$$[\![X]\!]_{-\alpha} = \begin{cases} \infty & \text{if } [\![X]\!] = \infty \\ [\![X]\!] - \alpha & \text{if } [\![X]\!] \geq \alpha \\ 0 & \text{if } [\![X]\!] < \alpha \end{cases}$$

**Theorem 5.2.** *Let $\mathcal{H}$=$A_1^1, \ldots, A_{n_1}^1 \vdash B_1 \mid \ldots \mid A_1^p, \ldots, A_{n_p}^p \vdash B_p$, if $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ is a countermodel of the sequent $FS(\mathcal{H})$ in $\mathsf{LC}_n$ then $[\![\Box]\!] < \infty$ and for $\alpha = [\![\Box]\!]$, the translated interpretation $[\![\cdot]\!]_{-\alpha}$ is a countermodel of the hypersequent $\mathcal{H}$ in $\mathsf{LC}_n$.*

*Proof.* Given in appendix A.

### 5.2 Bi-colored graphs

Let $\mathcal{H}$ be a given hypersequent of $\mathsf{LC}$. As shown in the previous section we can associate to $\mathcal{H}$ an equivalent flat sequent $FS(\mathcal{H}) = \mathcal{S} = \delta^+(A_1^1), \ldots, \delta^+(A_{n_p}^p), \delta^-(B_1), \ldots, \delta^-(B_p) \vdash X_{A_1^1} \to X_{B_1}, \ldots, X_{A_{n_1}^1} \to X_{B_1}, X_{A_1^2} \to X_{B_2}, \ldots, X_{A_{n_2}^2} \to X_{B_2}, \ldots, X_{A_1^p} \to X_{B_p}, \ldots, X_{A_{n_p}^p} \to X_{B_p}, \Diamond \to X_{B_1}, \ldots, \Diamond \to X_{B_p}$. Now we define a procedure that builds, from $\mathcal{H}$ and the flat sequent $FS(\mathcal{H})$, a particular bi-colored graph $\mathcal{G}_{\mathcal{H}}$ that is associated to $\mathcal{H}$.

The **nodes** of $\mathcal{G}_{\mathcal{H}}$ are defined from the set of the nodes of the decomposition tree of all the formulae of $\mathcal{H}$ (set of the subformulae occurrences).
Moreover we introduce a new variable $X_F$ for every occurrence $F$ of subformula of $D$ in $\mathcal{H}$. It is the corresponding node of $F$. The nodes are signed as follows: we have $-$ at the root $D^-$ if $D$ is a hypothesis else we have $+$ and we propagate the signs as usual.[1]

[1] The connectors $\wedge$ and $\vee$ preserve the signs and $\to$ preserves the sign on the righthand side and inverses the sign on the lefthand side.

We can write $\mathcal{X}_F^+$ or $\mathcal{X}_F^-$ in order to emphasize the signs.
We also add the node denoted $V$ for all propositional variables of $\mathcal{H}$. Thus several occurrences of $V$ generate only one node $V$ and several nodes $\mathcal{X}_V^+$ or $\mathcal{X}_V^-$. Moreover we add two new nodes denoted $\Diamond$ et $\Box$.

The **arrows** of $\mathcal{G}_{\mathcal{H}}$ are defined as follows: we describe the green and red arrows between the nodes together with the boolean expressions indexing them.
First we start with the non-indexed arrows introduced independently of the internal structure of $\mathcal{H}$'s formulae. We add
- a red arrow $\mathcal{X}_D^- \Rightarrow \Diamond$ for any formula $D$ of the multi-set of conclusions of $\mathcal{H}$.
- a red arrow $\mathcal{X}_B^- \to \mathcal{X}_A^+$ for any formula $A$ and $B$ of $\mathcal{H}$ that belong to the same component.
- a green arrow $V \to \mathcal{X}_V^-$ for any negative occurrence of variable $V$ and a green arrow $\mathcal{X}_V^+ \to V$ for any positive occurrence of variable $V$.
- a green arrow $\Box \to V$ for any variable $V$, a green arrow $\Box \to \mathcal{X}_\perp^-$ for any negative occurrence of $\perp$ and a green arrow $\mathcal{X}_\perp^+ \to \Box$ for any positive occurrence of $\perp$.
Secondly we consider the introduction of arrows for the internal nodes. Let us start with the non-indexed arrows. We add
- two green arrows $\mathcal{X}_C^+ \to \mathcal{X}_A^+$ and $\mathcal{X}_C^+ \to \mathcal{X}_B^+$ for any positive subformula occurrence $C \equiv A \wedge B$.
- two green arrows $\mathcal{X}_A^- \to \mathcal{X}_C^-$ and $\mathcal{X}_B^- \to \mathcal{X}_C^-$ for any negative subformula occurrence $C \equiv A \vee B$.
We now complete with the indexed arrows, i.e., arrows indexed by boolean expressions of the form $x$ or $\overline{x}$ with $x$ propositional variable. Then we introduce a new boolean variable for any subformula occurrence. We add
- a new boolean variable $x$ and two green arrows $\mathcal{X}_A^- \to_x \mathcal{X}_C^-$ and $\mathcal{X}_B^- \to_{\overline{x}} \mathcal{X}_C^-$ for any negative subformula occurrence $C \equiv A \wedge B$.
- a new boolean variable $x$ and two green arrows $\mathcal{X}_C^+ \to_x \mathcal{X}_A^+$ and $\mathcal{X}_C^+ \to_{\overline{x}} \mathcal{X}_B^+$ for any positive subformula occurrence $C \equiv A \vee B$.
- a new boolean variable $x$, two green arrows $\mathcal{X}_B^- \to_x \mathcal{X}_C^-$ and $\Diamond \to_{\overline{x}} \mathcal{X}_C^-$ and two red arrows $\mathcal{X}_B^- \Rightarrow_x \mathcal{X}_A^+$ and $\mathcal{X}_B^- \Rightarrow_x \Diamond$ for any negative subformula occurrence $C \equiv A \to B$.
- a new boolean variable $x$ and two green arrows $\mathcal{X}_C^+ \to_x \mathcal{X}_B^+$ and $\mathcal{X}_A^- \to_{\overline{x}} \mathcal{X}_B^+$ for any positive subformula occurrence $C \equiv A \to B$.

We illustrate this construction with the hypersequent $\mathcal{H}_3 = A \vdash B \mid A \to B \vdash B$. The bi-colored graph $\mathcal{G}_{\mathcal{H}_3}$ associated to $\mathcal{H}_3$ is given in Figure 3.
It is clear that the construction of the graph $\mathcal{G}_{\mathcal{H}}$ for a given hypersequent $\mathcal{H}$ is in linear time because we add at most four arrows for each subformula instance. Now we have to define a characterization of the validity of $\mathcal{H}$ from the associated bi-colored graph.

### 5.3 A procedure for countermodel search

We have proved, in the previous section, that deciding if a hypersequent $\mathcal{H}$ is valid or has a countermodel in $\mathsf{LC}_n$ can be reduced to deciding if the flat sequent associated to $\mathcal{H}$ (see Theorem 5.1 and 5.2) is valid or has a countermodel in $\mathsf{LC}_n$. Then we focus

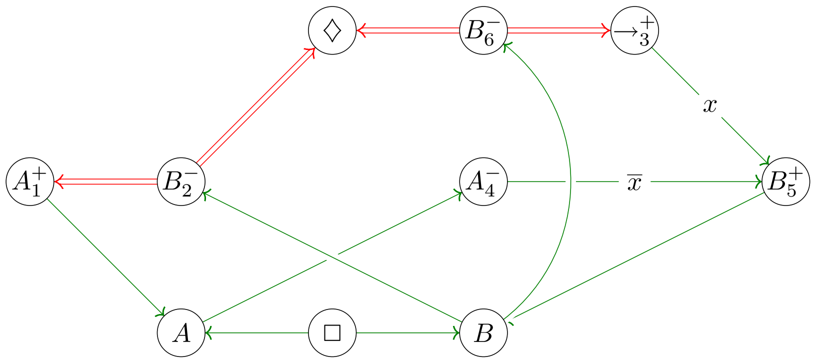



**Fig. 3.** A Bi-colored Graph of a Hypersequent

now on the graph $\mathcal{G}_{\mathcal{H}}$ and analyze validity of $\mathcal{H}$ from it.
First any flat sequent can be reduced or transformed into a set of implicational sequents, i.e., sequents of the form $\mathcal{S} = X_1 \rightarrow Y_1, \ldots, X_k \rightarrow Y_k \vdash A_1 \rightarrow B_1, \ldots, A_l \rightarrow B_l$. Secondly we can associate a bi-colored graph $\mathcal{G}_{\mathcal{S}}$ to such a sequent $\mathcal{S}$ as follows: the set of nodes is the set of the variables of $\mathcal{S}$, namely $\{X_i\} \cup \{Y_i\} \cup \{A_i\} \cup \{B_i\}$ and the set of arrows is $\{X_1 \rightarrow Y_1, \ldots, X_k \rightarrow Y_k\} \cup \{B_1 \Rightarrow A_1, \ldots, B_l \Rightarrow A_l\}$. Then an implicational sequent $\mathcal{S}$ has a countermodel in $\mathsf{LC}_n$ (resp. in $\mathsf{LC}$) if and only if its associated bi-colored graph does not contain a $(n+1)$-alternating chain (resp. a $\Rightarrow$-cycle) [16]. Thus we now study if we can relate the search of particular chains in an instance of the associated graph $\mathcal{G}_{\mathcal{H}}$ to the existence of countermodels.

**Theorem 5.3** **($n = \infty$).** *Let $\mathcal{H}$ be a hypersequent and $\mathcal{G}_{\mathcal{H}}$ its bi-colored graph, $\mathcal{H}$ has a countermodel in* $\mathsf{LC}$ *if and only if there exists an instance of $\mathcal{G}_{\mathcal{H}}$ that does not contain a $\Rightarrow$-cycle.*

*Proof.* Let $\mathcal{H}$ be a hypersequent and $\mathcal{S}$ be the associated flat sequent. An interpretation $[\![\cdot]\!]$ is a countermodel of $\mathcal{S}$ if and only if at least it is a countermodel of one of the implicational sequents issued of the transformation of $\mathcal{S}$. The bi-colored graphs associated to these implicational sequents exactly correspond to the instances of $\mathcal{G}_{\mathcal{H}}$. By the above-mentioned results $\mathcal{S}$ has a countermodel if and only if one of the instances of $\mathcal{G}_{\mathcal{H}}$ does not contain a $\Rightarrow$-cycle. Thus, $\mathcal{H}$ has a countermodel if and only if one of the instances of $\mathcal{G}_{\mathcal{H}}$ does not contain a $\Rightarrow$-cycle.

**Theorem 5.4** **($n < \infty$).** *Let $\mathcal{H}$ be a hypersequent and $\mathcal{G}_{\mathcal{H}}$ its associated bi-colored graph, $\mathcal{H}$ has a countermodel in* $\mathsf{LC}_n$ *if and only if there exists an instance of $\mathcal{G}_{\mathcal{H}}$ that does not contain a $(n+1)$-alternating chain.*

*Proof.* For $\mathsf{LC}_n$ with $n \neq \infty$, the proof is similar by replacing the notion of $\Rightarrow$-cycle by the one of $(n+1)$-alternating chain.

Let us come back to our example with the hypersequent $\mathcal{H}_3 = A \vdash B \mid A \rightarrow B \vdash B$. If we consider its associated graph $\mathcal{G}_{\mathcal{H}_3}$ (see previous subsection) we observe that it has two instances ($x = 0$ and $x = 1$). The first one ($x = 0$) contains the following $\Rightarrow$-cycle: $B_2^- \Rightarrow A_1^+ \rightarrow A \rightarrow A_4^- \rightarrow B_5^+ \rightarrow B \rightarrow B_2^-$. The second one ($x = 1$) contains the following $\Rightarrow$-cycle: $B_6^- \Rightarrow \rightarrow_3^+ \rightarrow B_5^+ \rightarrow B \rightarrow B_6^-$. Then we deduce that $\mathcal{H}_3$ does not contain countermodels in $\mathsf{LC}$. An example with countermodel generation is given in appendix B.

These results on mono-conclusioned hypersequents can be easily extended to (multi-conclusioned) hypersequents.
Let $\mathcal{H}$=$A_1^1,\ldots,A_{n_1}^1 \vdash B_1^1,\ldots,B_{m_1}^1 \mid \ldots \mid A_1^p,\ldots,A_{n_p}^p \vdash B_1^p,\ldots,B_{m_p}^p$ be a given hypersequent, we build the flat sequent $FS(\mathcal{H}) = \delta^+(A_1^1),\ldots,\delta^+(A_{n_p}^p),\delta^-(B_1^1),\ldots,\delta^-(B_{m_p}^p)$, $(\mathcal{X}_{\vdash_1}) \rightarrow \mathcal{X}_{A_1^1},\ldots,(\mathcal{X}_{\vdash_1}) \rightarrow \mathcal{X}_{A_{n_1}^1},\ldots,(\mathcal{X}_{\vdash_p}) \rightarrow \mathcal{X}_{A_{n_p}^p} \vdash (\mathcal{X}_{\vdash_1}) \rightarrow \mathcal{X}_{B_1^1},\ldots,(\mathcal{X}_{\vdash_1}) \rightarrow \mathcal{X}_{B_{m_1}^1},\ldots,$ $(\mathcal{X}_{\vdash_p}) \rightarrow \mathcal{X}_{B_1^p},\ldots,(\mathcal{X}_{\vdash_p}) \rightarrow \mathcal{X}_{B_{m_p}^p},\diamondsuit \rightarrow \mathcal{X}_{B_1^1},\ldots,\diamondsuit \rightarrow \mathcal{X}_{B_{m_p}^p}$. Then we can prove theorems similar to Theorem 5.1 and Theorem 5.2 and then directly use the procedure defined for bi-colored graph construction. In addition our new procedure can be applied to a specific case of hypersequents that are sequents. Therefore we also provide a new procedure for deciding provability of $\mathsf{LC}$ sequents through bi-colored graphs with generation of countermodels.

## 6 Conclusion and perspectives

In this paper we propose new characterizations of validity in $\mathsf{LC}$ and $\mathsf{LC}_n$ based on the construction, from a hypersequent, of a specific bi-colored graph on which the search of particular chains corresponds to countermodel search. It leads to new decision procedures for hypersequents in Gödel-Dummett logics that is well adapted to countermodel generation. These results present an alternative approach to works on proof-search with analytic calculi. Thus we aim at developing it for other logics including substructural or intermediate logics [18].
Recent works have studied the relationships between parallel dialogue games and hypersequents for some intermediate logics including $\mathsf{LC}$ [10]. We also aim at relating bi-colored graphs and such games in such a way that we could generate directly winning strategies from bi-colored graphs associated to sequents or hypersequents. From preliminary results for graphs associated to implicational sequents we expect to study how to deal with general sequents or hypersequents.

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## A Proof of Theorem 5.2

To prove the result we need the three following propositions, proved in [16].

**Proposition A.1.** *Let $A$ be a formula without $\bot$, $\alpha$ be a value in $\overline{[0,n)}$ and $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ be an interpretation such that, for any variable $X$ of $A$, $[\![X]\!] \geqslant \alpha$. We have $[\![A]\!] \geqslant \alpha$ and if $\alpha < \infty$ then $[\![A]\!]_{-\alpha} = [\![A]\!] - \alpha$.*

**Proposition A.2.** *Let $D$ be a formula, we have the following properties:*
*1. For any subformula $K$ of $D$, the formulae of $\delta^+(K)$ and $\delta^-(K)$ are flat and do not contain the constant $\bot$;*
*2. For any variable $V$ of $D$, the atomic implication $\Box \to V$ is in $\delta^+(K)$ and in $\delta^-(K)$;*
*3. The size of $\delta^+(K)$ and $\delta^-(K)$ is linear in the size of $D$.*

**Proposition A.3.** *Let $D$ be a formula, for any subformula $K$ of $D$, the two sequents $\delta^+(K), X_K \vdash K_\Box$ and $\delta^-(K), K_\Box \vdash X_K$ are valid in $\mathsf{LC}_n$.*

**Theorem 5.2**
*Let $\mathcal{H}$=$A^1_1,\dots,A^1_{n_1} \vdash B_1 \mid \dots \mid A^p_1,\dots,A^p_{n_p} \vdash B_p$, if $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ is a countermodel of the sequent $FS(\mathcal{H})$ in $\mathsf{LC}_n$ then $[\![\Box]\!] < \infty$ and for $\alpha = [\![\Box]\!]$, the translated interpretation $[\![\cdot]\!]_{-\alpha}$ is a countermodel of the hypersequent $\mathcal{H}$ in $\mathsf{LC}_n$.*

*Proof.* Let $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ be a countermodel of the sequent $\mathcal{S}$. The following properties are satisfied:

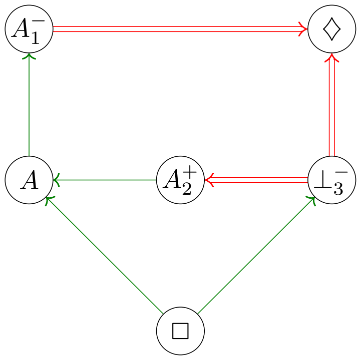
1. $\forall i, j \in [1,p]$ and $\forall k \in [1,n_i]$, $[\![\delta^+(A^i_k)]\!] > [\![\Diamond \to X_{B_j}]\!]$;
2. $\forall i, j \in [1,p]$, $\forall k \in [1,n_i]$ and $\forall l \in [1,n_j]$, $[\![\delta^+(A^i_k)]\!] > [\![X_{A^j_l} \to X_{B_j}]\!]$;
3. $\forall i, j \in [1,p]$, $[\![\delta^-(B_i)]\!] > [\![\Diamond \to X_{B_j}]\!]$;
4. $\forall i, j \in [1,p]$ and $\forall k \in [1,n_j]$, $[\![\delta^-(B_i)]\!] > [\![X_{A^j_k} \to X_{B_j}]\!]$.

By property 4, we have $\forall i \in [1,p]$, $[\![\delta^-(B_i)]\!] > [\![\Diamond \to X_{B_i}]\!]$ and we deduce that for any $i \in [1,p]$, $[\![\Diamond \to X_{B_i}]\!] < \infty$ and $[\![X_{B_i}]\!] = [\![\Diamond]\!] \twoheadrightarrow [\![X_{B_i}]\!] < \infty$. Then, for any $i \in [1,p]$, $[\![X_{B_i}]\!] < [\![\delta^-(B_i)]\!]$. By Proposition A.3, $\forall i \in [1,p]$, $\delta^-(B_i), (B_i)_\Box \vdash X_{B_i}$ is a valid sequent and then $\forall i \in [1,p]$, $[\![\delta^-(B_i)]\!] \wedge [\![(B_i)_\Box]\!] \leqslant [\![X_{B_i}]\!] < [\![\delta^-(B_i)]\!]$. Therefore $\forall i \in [1,p]$, $[\![\delta^-(B_i)]\!] \wedge [\![(B_i)_\Box]\!] < [\![\delta^-(B_i)]\!]$ and we obtain $[\![(B_i)_\Box]\!] < [\![\delta^-(B_i)]\!]$ for any $i \in [1,p]$.
- We now prove that, for any variable $V$ of $\mathcal{H}$, $[\![V]\!] \geqslant [\![\Box]\!]$. First, if $\mathcal{H}$ does not contain variables, i.e., all its atoms are occurrences of $\perp$, then the previous property is trivially verified. Else, let $B_i$ be one of its conclusion formulae and $V_0$ be a variable of $(B_i)_\Box$ that realizes the minimal value $\gamma$ of the non-empty set $\{[\![\Box \to V]\!], V$ variable of $B_{i\Box}\}$. Thus $\gamma = [\![\Box \to V_0]\!]$ and for any variable $V$ of $(B_i)_\Box$, $[\![\Box \to V]\!] > \gamma$. If $V_0$ is in the variable set of $B_i$ then, by Proposition A.2, $\Box \to V_0 \in \delta^-(B_i)$ and then $[\![\delta^-(B_i)]\!] \leqslant [\![\Box \to V_0]\!] = \gamma$. Else $V_0 = \Box$ and therefore $[\![\delta^-(B_i)]\!] \leqslant [\![\Box \to V_0]\!] = [\![\Box \to \Box]\!] = \infty$. In both cases $[\![\delta^-(B_i)]\!] \leqslant [\![\Box \to V_0]\!] = \gamma$.
- We now prove that $[\![\Box]\!] \leqslant \gamma$. Let us suppose $[\![\Box]\!] > \gamma = [\![\Box \to V_0]\!]$ and let $V$ be a variable de $(B_i)_\Box$: either $V = \Box$ and $[\![V]\!] = [\![\Box]\!] > \gamma$ or $V$ is a variable of $B_i$ and $[\![\Box \to V]\!] \geqslant \gamma$, and then $[\![V]\!] \geqslant \gamma$. By Proposition A.1, as $(B_i)_\Box$ does not contain $\perp$, $[\![(B_i)_\Box]\!] \geqslant \gamma$. We deduce $\gamma \leqslant [\![(B_i)_\Box]\!] < [\![\delta^-(B_i)]\!] \leqslant [\![\Box \to V_0]\!] = \gamma$, that is contradictory and then $\Box \leqslant \gamma$. For any variable $V$ of $(B_i)_\Box$, $[\![\Box]\!] \leqslant \gamma \leqslant [\![\Box \to V]\!] = [\![\Box]\!] \twoheadrightarrow [\![V]\!]$. Then , for any variable $V$ of $(B_i)_\Box$, $[\![\Box]\!] \leqslant [\![V]\!]$ and by Proposition A.1 $[\![(B_i)_\Box]\!] \geqslant [\![\Box]\!]$. Thus $\forall i \in [1,p]$ $[\![(B_i)_\Box]\!] \geqslant [\![\Box]\!]$ because we have no hypothesis on $B_i$.
- We now prove that, for any variable $V$ of $\mathcal{H}$, $[\![V]\!] \geqslant [\![\Box]\!]$. We suppose that there exists a variable $V_1$ such that $[\![V_1]\!] < [\![\Box]\!]$. By Proposition A.2, $\Box \to V_1$ belongs to the multiset of hypotheses of $\mathcal{S}$. Then $\forall i \in [1,p]$ $[\![\Box \to V_1]\!] = [\![V_1]\!] > X_{B_i}$. We have $\forall i \in [1,p]$ $[\![\delta^-(B_i)]\!] > [\![X_{B_i}]\!]$ and by Proposition A.3, $\delta^-(B_i), (B_i)_\Box \vdash X_{B_i}$ is a valid sequent and then $\forall i \in [1,p]$ $[\![(B_i)_\Box]\!] \leqslant [\![X_{B_i}]\!]$. Thus we have $\forall i \in [1,p]$ $[\![\Box]\!] \leqslant [\![(B_i)_\Box]\!] \leqslant [\![X_{B_i}]\!] < [\![V_1]\!]$ that is contradictory.
By property 3, $\forall i, j \in [1,p]$ $\forall k \in [1,n_i]$ $\forall l \in [1,n_j]$ $[\![\delta^+(A^i_k)]\!] > [\![X_{A^j_l} \to X_{B_j}]\!]$ and we deduce that $\forall j \in [1,p]$ $\forall l \in [1,n_j]$ $[\![X_{A^j_l} \to X_{B_j}]\!] < \infty$. Then $\forall i \in [1,p]$ $\forall j \in [1,n_i]$ $[\![X_{A^i_j}]\!] > [\![X_{B_i}]\!]$ and finally $\forall i \in [1,p]$ $\forall j \in [1,n_i]$ $[\![\delta^+(A^i_j)]\!] > [\![X_{B_i}]\!]$ and then $[\![\delta^+(A^i_j), X_{A^i_j}]\!] > [\![X_{B_i}]\!]$. By Proposition A.3, $\forall i \in [1,p]$ $\forall j \in [1,n_i]$ $\delta^+(A^i_j), (A^i_j)_\Box \vdash X_{A^i_j}$ is a valid sequent. Thus $\forall i \in [1,p]$ $\forall j \in [1,n_i]$ $[\![(A^i_j)_\Box]\!] > [\![X_{B_i}]\!]$. Moreover $\forall i \in [1,p]$ $\forall j \in [1,n_i]$

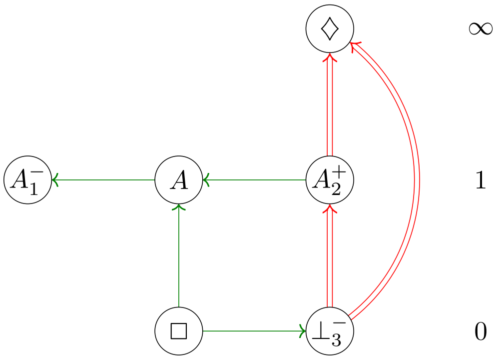
$[\![(A^i_j)_\square]\!] > [\![(B_i)_\square]\!]$ because $\forall k \in [1,p]$, $[\![B_{k\square}]\!] \leqslant [\![\mathcal{X}_{B_k}]\!]$.
We have proved that $\forall i \in [1,p]$ $[\![(B_i)_\square]\!] < [\![\delta^-(B_i)]\!]$. But $\forall i \in [1,p]$ $[\![(B_i)_\square]\!] \geqslant [\![\square]\!]$ and then $[\![\square]\!] < \infty$. Let $\alpha = [\![\square]\!]$, as $[\![\square]\!]_{-\alpha} = [\![\square]\!] - \alpha = 0 = [\![\bot]\!]_{-\alpha}$. We obtain $[\![D]\!]_{-\alpha} = [\![D_\square]\!]_{-\alpha}$, for any formula $D$ of $\mathcal{H}$, and by Proposition A.1, $[\![D_\square]\!]_{-\alpha} = [\![D_\square]\!] - \alpha$. We have $\forall i \in [1,p]$ $\forall j \in [1,n_i]$ $[\![A^i_{j\square}]\!] > [\![(B_i)_\square]\!]$ and $\alpha < \infty$. Thus $\forall i \in [1,p]$ $\forall j \in [1,n_i]$ $[\![(A^i_j)_\square]\!] - \alpha > [\![(B_i)_\square]\!] - \alpha$ and then $[\![A^i_j]\!]_{-\alpha} > [\![B_i]\!]_{-\alpha}$. Then $[\![\cdot]\!]_{-\alpha}$ is a countermodel.

# B An example with countermodel generation

The bi-colored graph of the hypersequent $\mathcal{H}_4 \equiv \vdash A \mid A \vdash \bot$ is




This graph has only one instance that does not contain $\Rightarrow$-cycle. Then we deduce that $\mathcal{H}_4$ has a countermodel. In order to extract it we modify the graph as follows: the red arrows always go up and the green arrows never go down.




Then $[\![\cdot]\!] : \mathsf{Var} \to \overline{[0,n)}$ such that $[\![A]\!] = 1$ and $n > 2$, is a countermodel of $\mathcal{H}_4$ in $\mathsf{LC}_n$.