A sequent calculus with labels for *PAL*

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Abstract

We present a sequent calculus with labels for *PAL*.

Keywords: Public announcement logic; proof theory; sequent calculus.

1 Introduction

The main motivation to formalize the dynamics of knowledge in a group of agents is to characterize how agents' knowledges evolve by adding new information. From that perspective, public announcement logic (*PAL*) is a dynamic epistemic logic *par excellence*. Besides the ordinary epistemic constructs K_a, formulas $K_a\phi$ being read "agent a knows that ϕ", *PAL* associates with each formula ϕ the modal constructs $[\phi]$, formulas $[\phi]\psi$ being read "if ϕ, then ψ after the announcement of ϕ".

PAL is a convenient language to describe knowledge and announcement. Although it does not define a normal modal logic as it is not closed under the inference rule of uniform substitution, validity can be completely axiomatized by means of so-called "reduction axioms" ($[\phi]p \leftrightarrow (\phi \to p)$, $[\phi][\psi]\chi \leftrightarrow [\phi \wedge [\phi]\psi]\chi$, etc) [4,10]. Validity can also be decided. As an alternative to Lutz' decision procedure in polynomial space [6], a tableaux-based decision procedure in polynomial space has been proposed in [2].

A proof-theoretical analysis of *PAL* has been proposed in [7] in terms of a sequent calculus following the labelled approach of [9]. In this sequent calculus, formulas are labelled expressions of the form $x(\phi_1, \ldots, \phi_n) : \phi$ being read "in

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the model restricted by the sequence $(\phi_1, \ldots, \phi_n)$, ϕ holds at state x". Unfortunately, the sequent calculus for PAL proposed in [7] is not complete as it cannot prove the valid formula $[p \wedge p]q \leftrightarrow [p]q$. In this paper, putting right the defects in [7], we present a sequent calculus with labels for PAL.

2 Syntax

Formulas are inductively defined as follows:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \psi) \mid K_a\phi \mid [\phi]\psi$$

where p ranges over a countably infinite set of propositional variables and a ranges over a countably infinite set of agents. The other Boolean constructs for formulas ($\vee$, $\rightarrow$, $\bot$, $\top$) are defined as usual. The modal constructs $\hat{K}_a\cdot$ and $\langle\cdot\rangle\cdot$ for formulas are defined as follows: $\hat{K}_a\phi ::= \neg K_a \neg\phi$ and $\langle\phi\rangle\psi ::= \neg[\phi]\neg\psi$. We will follow the standard rules for omission of the parentheses.

The size of formula ϕ, in symbols $\sharp(\phi)$, is defined as follows:

$$\begin{aligned}
\sharp(p) &= 1 \\
\sharp(\neg\phi) &= \sharp(\phi) + 1 \\
\sharp(\phi \wedge \psi) &= \sharp(\phi) + \sharp(\psi) + 1 \\
\sharp(K_a\phi) &= \sharp(\phi) + 2 \\
\sharp([\phi]\psi) &= \sharp(\phi) + \sharp(\psi) + 2
\end{aligned}$$

The size of sequence $(\phi_1, \ldots, \phi_n)$ of formulas, in symbols $\sharp(\phi_1, \ldots, \phi_n)$, is defined as follows:

$$\sharp(\phi_1, \ldots, \phi_n) = \sharp(\phi_1) + \ldots + \sharp(\phi_n) + n$$

3 Semantics

A model is a 3-tuple $\mathcal{M} = (W, R, V)$ where W is a nonempty set of states, R is a function from the set of all agents into the set of all binary relations between states and V is a valuation on W, i.e. a function from the set of all propositional variables into the set of all sets of states. In a model $\mathcal{M} = (W, R, V)$, we define the property "formula ϕ is true at state x", in symbols $\mathcal{M}, x \models \phi$, as follows:

$$\begin{array}{lll}
\mathcal{M}, x \models p & \text{iff} & x \in V(p) \\
\mathcal{M}, x \models \neg\phi & \text{iff} & \mathcal{M}, x \not\models \phi \\
\mathcal{M}, x \models \phi \wedge \psi & \text{iff} & \mathcal{M}, x \models \phi \text{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models K_a\phi & \text{iff} & \text{for all } y \in W, \text{ if xR(a)y then } \mathcal{M}, y \models \phi \\
\mathcal{M}, x \models [\phi]\psi & \text{iff} & \text{if } \mathcal{M}, x \models \phi \text{ then } \mathcal{M}^\phi, x \models \psi
\end{array}$$

where $\mathcal{M}^\phi$ is the restriction of $\mathcal{M}$ to those states z such that $\mathcal{M}, z \models \phi$.

In a model $\mathcal{M} = (W, R, V)$, we define the property "formula ϕ is true at state x with respect to a sequence $(\phi_1, \ldots, \phi_n)$ of formulas", in symbols $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \phi$, as follows ($\epsilon$ for the empty sequence, φ for any se-

quence):

$\mathcal{M}, x, \epsilon \Vdash p$	iff	$x \in V(p)$
$\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash p$	iff	$\mathcal{M}, x, (\varphi) \Vdash \phi_{n+1}$ and $\mathcal{M}, x, (\varphi) \Vdash p$
$\mathcal{M}, x, (\varphi) \Vdash \neg\phi$	iff	$\mathcal{M}, x, (\varphi) \nVdash \phi$
$\mathcal{M}, x, (\varphi) \Vdash \phi \wedge \psi$	iff	$\mathcal{M}, x, (\varphi) \Vdash \phi$ and $\mathcal{M}, x, (\varphi) \Vdash \psi$
$\mathcal{M}, x, \epsilon \Vdash K_a\phi$	iff	for all $y \in W$, if $xR(a)y$ then $\mathcal{M}, y, \epsilon \Vdash \phi$
$\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash K_a\phi$	iff	for all $y \in W$, if $xR(a)y$ and $\mathcal{M}, y, (\varphi) \Vdash \phi_{n+1}$ then $\mathcal{M}, y, (\varphi, \phi_{n+1}) \Vdash \phi$
$\mathcal{M}, x, (\varphi) \Vdash [\phi]\psi$	iff	if $\mathcal{M}, x, (\varphi) \Vdash \phi$ then $\mathcal{M}, x, (\varphi, \phi) \Vdash \psi$

Remark 3.1

The above definition of $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \phi$ is correct decreasing on $\sharp(\phi_1, \ldots, \phi_n) + \sharp(\phi)$, seeing that in particular: $\sharp(\phi_1, \ldots, \phi_n) + \sharp(\phi) < \sharp(\phi_1, \ldots, \phi_n) + \sharp([\phi]\psi)$ and $\sharp(\phi_1, \ldots, \phi_n, \phi) + \sharp(\psi) < \sharp(\phi_1, \ldots, \phi_n) + \sharp([\phi]\psi)$.

The main difference from the proposition of [7] lies in the semantics of K_a above wich is distinguished whether the sequence of announcements is empty or not, thereby not introducing occurences of a restricted relation $xR^{\varphi}(a)y$.

Proposition 3.2 *Let $(\phi_1, \ldots, \phi_n)$ be a sequence of formulas and ϕ be a formula. For all models $\mathcal{M} = (W, R, V)$ and for all $x \in W$, the following conditions are equivalent:*

- $\mathcal{M}, x \models [\phi_1] \ldots [\phi_n]\phi$;
- *if* $\mathcal{M}, x, \epsilon \Vdash \phi_1, \ldots, \mathcal{M}, x, (\phi_1, \ldots, \phi_{n-1}) \Vdash \phi_n$ *then* $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \phi$.

Proof. By induction on $\sharp(\phi_1, \ldots, \phi_n) + \sharp\phi$. □

Validity is defined as usual: formula ϕ is valid iff for all models $\mathcal{M} = (W, R, V)$ and for all $x \in W$, $\mathcal{M}, x \models \phi$.

4 Sequent calculus

Now, we present our sequent calculus with labels for *PAL*. It consists of the inference rules presented in Figure 1. Our sequents are pairs of finite sets of expressions either of the form $x(\phi_1, \ldots, \phi_n) : \phi$ read "state x satisfies ϕ with respect to the sequence $(\phi_1, \ldots, \phi_n)$", or of the form $xR(a)y$ read "state x is related to state y by means of a". The sequent $\Gamma \vdash \Delta$ means that the conjunction of the expressions in Γ implies the disjunction of the expressions in Δ. Provability is defined as usual: formula ϕ is provable iff the sequent $\vdash x(\epsilon) : \phi$ is derivable from these inference rules.

Proposition 4.1 *Let ϕ be a formula. The following conditions are equivalent:*

- *ϕ is valid;*
- *ϕ is provable.*

Proof. ($\Rightarrow$) Suppose ϕ is valid. By the completeness of the Hilbert-style axiomatization *HPAL* of Figure 2 considered in [4,10], there exists a proof of ϕ from the axioms and the inference rules of *HPAL*. The reader may easily verify

Fig. 1. Inference rules for PAL.

$$\frac{}{x(\epsilon) : p,\ \Gamma \vdash \Delta,\ x(\epsilon) : p}\text{ax}$$

$$\frac{x(\varphi) : \phi,\ x(\varphi) : p,\ \Gamma \vdash \Delta}{x(\varphi, \phi) : p,\ \Gamma \vdash \Delta}\text{Lp} \qquad \frac{\Gamma \vdash \Delta,\ x(\varphi) : \phi \quad \Gamma \vdash \Delta,\ x(\varphi) : p}{\Gamma \vdash \Delta,\ x(\varphi, \phi) : p}\text{Rp}$$

$$\frac{\Gamma \vdash \Delta,\ x(\varphi) : \phi}{x(\varphi) : \neg\phi,\ \Gamma \vdash \Delta}\text{L}\neg \qquad \frac{x(\varphi) : \phi,\ \Gamma \vdash \Delta}{\Gamma \vdash \Delta,\ x(\varphi) : \neg\phi}\text{R}\neg$$

$$\frac{x(\varphi) : \phi,\ x(\varphi) : \psi,\ \Gamma \vdash \Delta}{x(\varphi) : \phi \wedge \psi,\ \Gamma \vdash \Delta}\text{L}\wedge \qquad \frac{\Gamma \vdash \Delta,\ x(\varphi) : \phi \quad \Gamma \vdash \Delta,\ x(\varphi) : \psi}{\Gamma \vdash \Delta,\ x(\varphi) : \phi \wedge \psi}\text{R}\wedge$$

$$\frac{x(\epsilon) : K_a\phi,\ xR(a)y,\ y(\epsilon) : \phi,\ \Gamma \vdash \Delta}{x(\epsilon) : K_a\phi,\ xR(a)y,\ \Gamma \vdash \Delta}\text{L}K \qquad \frac{xR(a)y,\ \Gamma \vdash \Delta,\ y(\epsilon) : \phi}{\Gamma \vdash \Delta,\ x(\epsilon) : K_a\phi}\text{R}K$$

$$\frac{x(\varphi, \phi) : K_a\psi,\ xR(a)y,\ y(\varphi) : \phi,\ y(\varphi, \phi) : \psi,\ \Gamma \vdash \Delta}{x(\varphi, \phi) : K_a\psi,\ xR(a)y,\ y(\varphi) : \phi,\ \Gamma \vdash \Delta}\text{L}K^{\varphi}$$

$$\frac{xR(a)y,\ y(\varphi) : \phi,\ \Gamma \vdash \Delta,\ y(\varphi, \phi) : \psi}{\Gamma \vdash \Delta,\ x(\varphi, \phi)K_a\psi}\text{R}K^{\varphi}$$

$$\frac{\Gamma \vdash \Delta,\ x(\varphi) : \phi \quad x(\varphi, \phi) : \psi,\ \Gamma \vdash \Delta}{x(\varphi) : [\phi]\psi,\ \Gamma \vdash \Delta}\text{L}[] \qquad \frac{x(\varphi) : \phi,\ \Gamma \vdash \Delta,\ x(\varphi, \phi) : \psi}{\Gamma \vdash \Delta,\ x(\varphi) : [\phi]\psi}\text{R}[]$$

Fig. 2. Hilbert axiomatization for PAL.

- classical propositional tautologies

$$K \begin{cases} K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi) \\ K_a\phi \to \phi \\ K_a\phi \to K_aK_a\phi \\ \neg K_a\phi \to K_a\neg K_a\phi \end{cases} \qquad [\cdot]\cdot \begin{cases} [\phi]p & \leftrightarrow (\phi \to p) \\ [\phi]\neg\psi & \leftrightarrow (\phi \to \neg[\phi]\psi) \\ [\phi](\psi \wedge \chi) & \leftrightarrow ([\phi]\psi \wedge [\phi]\chi) \\ [\phi]K_a\psi & \leftrightarrow (\phi \to K_a[\phi]\psi) \\ [\phi][\psi]\chi & \leftrightarrow ([\phi \wedge [\phi]\psi]\chi) \end{cases}$$

$$\frac{\phi \quad \phi \to \psi}{\psi}\text{mp} \qquad \frac{\phi}{K_a\phi}\text{nec}$$

that these axioms and these inference rules are, respectively, provable and derivable in our sequent calculus. More precisely, if ψ is an axiom in $HPAL$, then the sequent $\vdash x(\epsilon) : \psi$ is derivable in our sequent calculus and if $\dfrac{\psi_1 \ \ldots \ \psi_n}{\psi}$ is an inference rule in $HPAL$, then the inference rule $\dfrac{\vdash x(\epsilon) : \psi_1 \ \ldots \ \vdash x(\epsilon) : \psi_n}{\vdash x(\epsilon) : \psi}$ is derivable in our sequent calculus. As a result, ϕ is provable in our sequent

calculus.

($\Leftarrow$) Let $\mathcal{M} = (W, R, V)$ be a model and $f : Var \mapsto W$. Sequents are pairs of finite sets of expressions either of the form $x(\phi_1, \ldots, \phi_n) : \phi$, or of the form $xR(a)y$. We define the property "$\mathcal{M}$ and f satisfy the expression exp", in symbols $\mathcal{M}, f \Vdash exp$, as follows:

$$\begin{array}{lll} \mathcal{M}, f \Vdash x(\phi_1, \ldots, \phi_n) : \phi & \text{iff} & \mathcal{M}, f(x), (\phi_1, \ldots, \phi_n) \Vdash \phi \\ \mathcal{M}, f \Vdash xR(a)y & \text{iff} & f(x)R(a)f(y) \end{array}$$

We will say that a sequent $\Gamma \vdash \Delta$ is valid iff for all models $\mathcal{M} = (W, R, V)$ and for all $f : Var \mapsto W$, if $\mathcal{M}$ and f satisfy every expression in Γ, then $\mathcal{M}$ and f satisfy some expression in Δ. The reader may easily verify that for all inference rules in Figure 1, if all sequents above the inference rule are valid, then the sequent below the inference rule is valid. Hence, if ϕ is provable in our sequent calculus with labels, then ϕ is valid. $\square$

5 Conclusion

We have developped a labelled sequent calculus for *PAL* that is sound and complete. Furthermore this calculus can be used for proof-search and to obtain decidability results. Indeed reading the rules bottom-up we can see that almost all the rules break down the main formula into subformulas: in the case of Lp and Rp the sequence of announcements decrease, and the LK^{φ} rule needs to be triggered using a strategy (e.g. when no other rule is possible and on every $xR(a)y$).

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