Symbolic dynamics as a categorical notion

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Plan

Situation

Main open problem of symbolic dynamics:

Decide if two subshifts of finite type are conjugate.

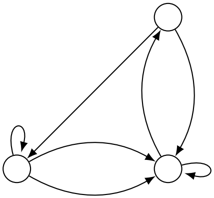
Subshifts of finite type (SFT) can be defined in various ways. Here we focus on the graph approach.

SFTs

Given a finite graph G, the subshift of finite type X_G associated to G is the set of all biinfinite paths on G.

We may think either of G as a graph, or equivalently as a matrix with nonnegative coefficients.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$



Conjugacy

- We say that two SFTs are conjugate if the dynamical systems they represent are conjugate.
- If we write the biinfinite paths as words over some infinite alphabet, then the conjugacy is a cellular automaton.

Main problem of symbolic dynamics: decide conjugacy.

Conjugacy

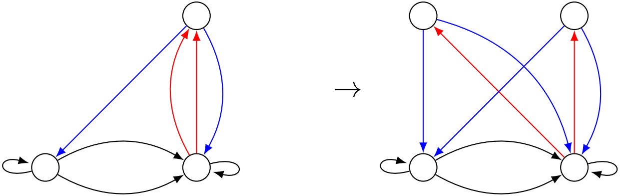
In terms of matrices:

M is *Strong Shift Equivalent* to *N*, if $M \sim N$ where $\sim$ is the smallest equivalence relation s.t. $RS \sim SR$ for all nonsquare integral nonnegative matrices R, S

In terms of graph:

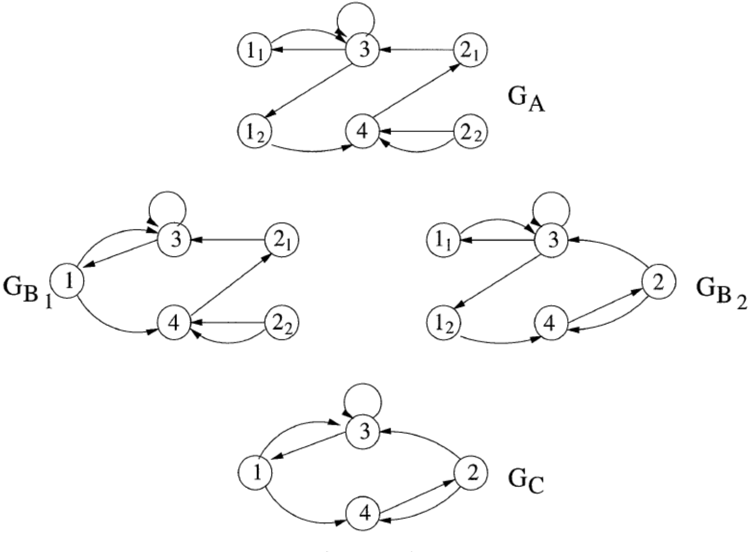
G is conjugate to *G'* if *G* can be obtained from *G'* by a series of incoming/outgoing splits and amalgamations.

Incoming split: transform one vertex u into two vertices u_1, u_2, split the inputs and share the outputs.



Examples

All pictures from Kitchen's book (Symbolic Dynamics):



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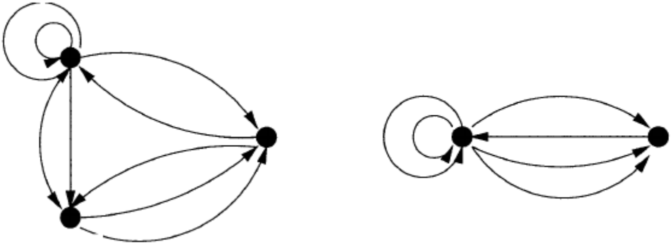

Figure 2.1.5

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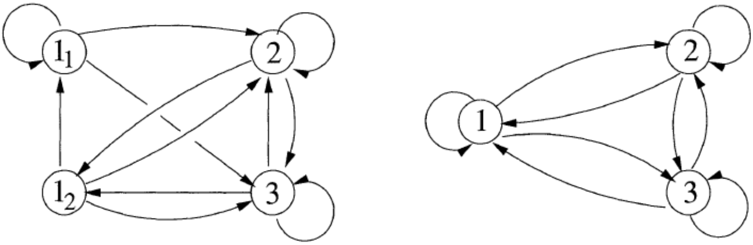


Figure 2.1.6

History

I will use “Strong Shift equivalence” (SSE) instead of conjugacy

- Williams 1973: SSE is introduced
- Williams 1973: SSE is decidable for one-sided SFTs (only incoming splits/amalgamations)
- Franks 1984: Flow equivalence (a variant of SSE) is decidable
- Kim-Roush 1988: Shift equivalence (a variant of SSE) is decidable
- Kim-Roush 1992: Shift equivalence is not the same as SSE
- Folklore: SSE is decidable for matrices in $\mathbb{Z}$ rather than in $\mathbb{Z}_+$ (graphs with negative edges)

Conclusion: while SSE is not known to be decidable, there are a lot of variants that are.

This talk

- SSE is complicated because the split/amalgamation stuff is complicated
- We will introduce a simplified version of the split/amalgamation
- The equations we obtain will remind us of category theory, and we will use category theory to obtain some results

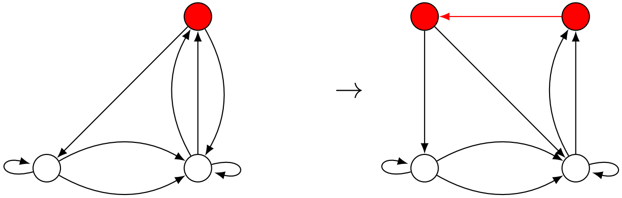
Plan

Flow equivalence (Parry-Sullivan 1975)

We will first focus on *flow equivalence*, a variant of SSE.

Flow equivalence is just SSE with a looser notion of time

i.e. we can now stretch a vertex:



Plan

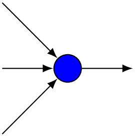
- We will reformulate flow equivalence with simpler equations
- Then we will go back to the original problem

Goal: get rid of the split/amalgamations equations.

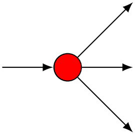
Idea

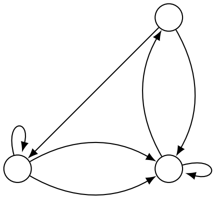
Represent the graph in a new formalism with two kinds of vertices:

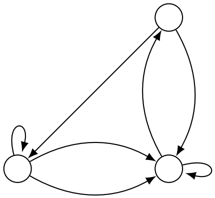
- Vertices that collect incoming edges

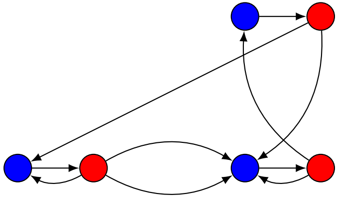


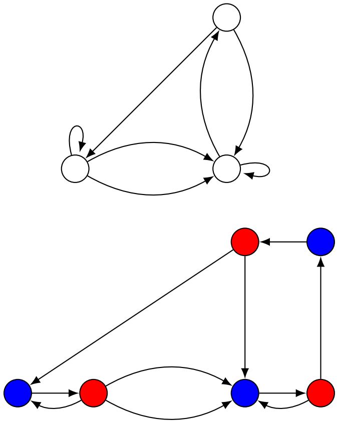
- Vertices that distribute outgoing edges:











Rules

How does flow equivalence translate into rules for red-blue graphs ?

Rules

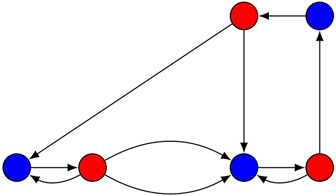
We want to think of the blue vertex as gathering incoming edges:

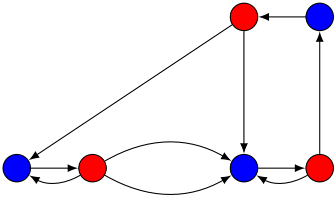
- Gathering one incoming edge is the same as doing nothing
- Gathering three incoming edges is the same as gathering the first two, then gathering the result with the third

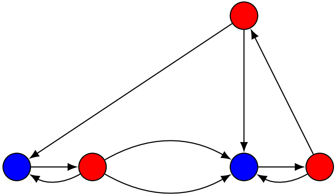
We only need blue vertices of incoming degree 2

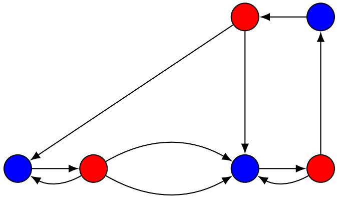
(Technically we also need vertices of incoming degree 0)

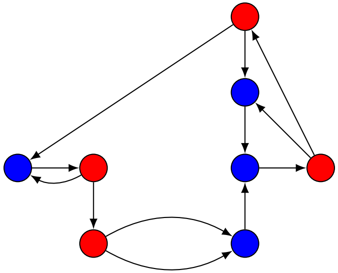
The same is true for red vertices

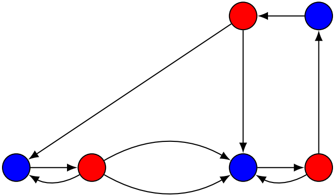


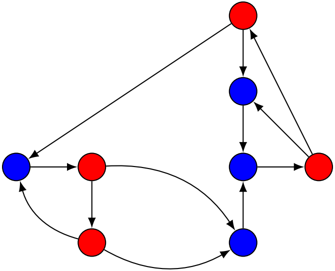


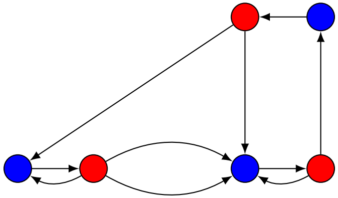


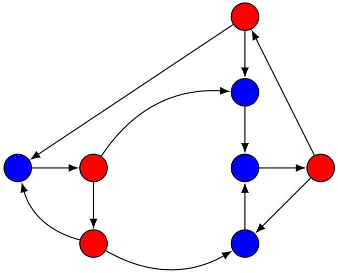




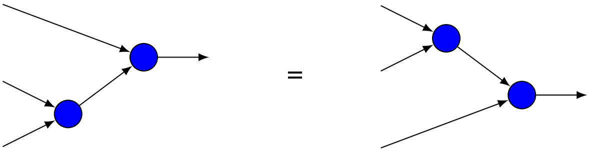








Two rules



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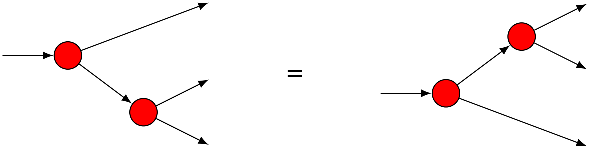
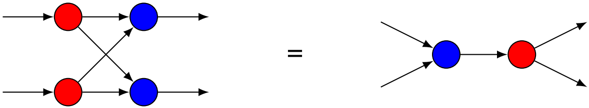


Axioms

What axioms do we need to take into account amalgamations/split ?

Axioms

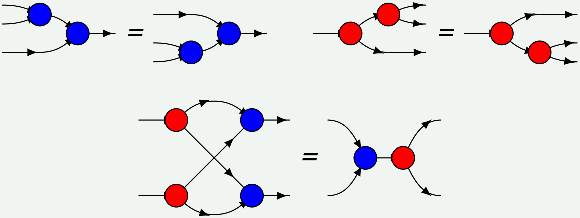
We only need ONE additional axiom:



Theorem 1

Theorem

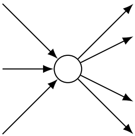
Flow equivalence, when expressed on bicolored graphs is entirely given by the following equations;



(plus other axioms for degenerate graphs, i.e. graphs with sources and sinks)

Idea of the proof

As an example, how to do the following split?



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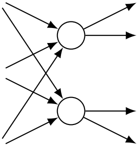
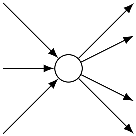


Proof



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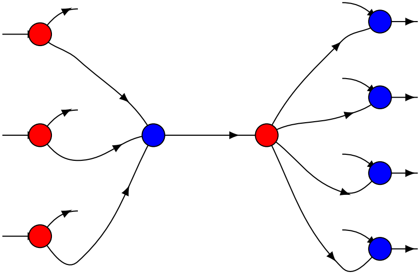

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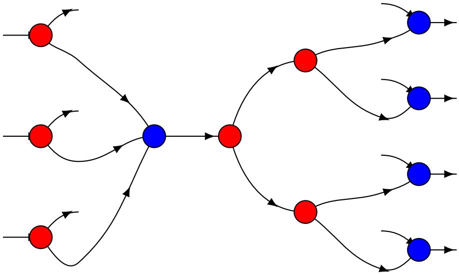

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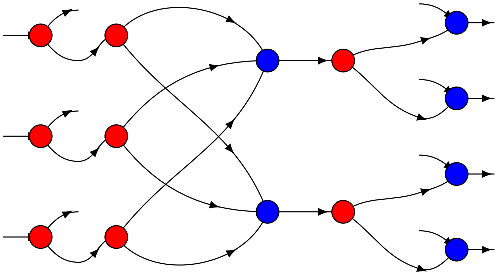

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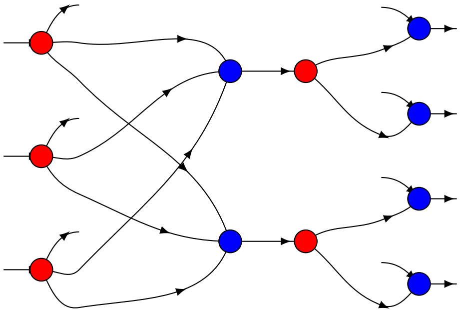

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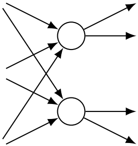

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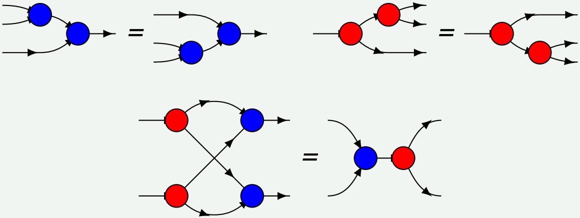


(plus other axioms for degenerate graphs, i.e. graphs with sources and sinks)

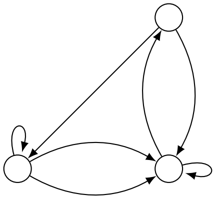
Strong Shift Equivalence

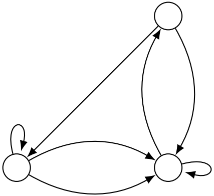
How to go back to strong shift equivalence (conjugacy) ?

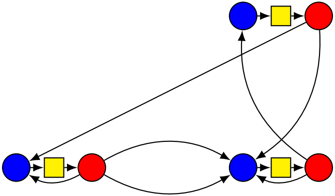
Flow equivalence is just SSE with a looser notion of time

SSE is just flow equivalence with a stronger notion of time.

(formal statement uses results from Boyle and Wagoner)
We will add a new vertex that represents one unit of time







Theorem 2

Theorem

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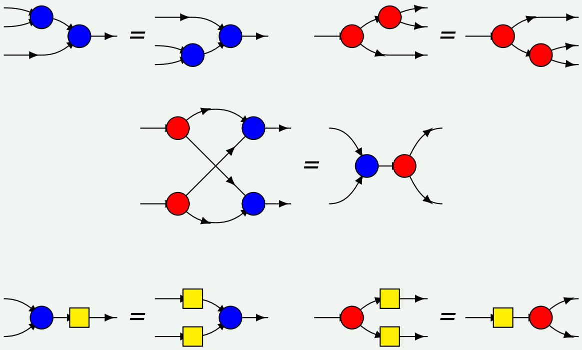
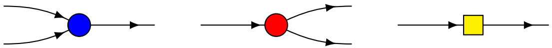


(plus other axioms for degenerate graphs)

Plan

Categories

Idea: Do not see these boxes as nodes in a graph, but as operators :



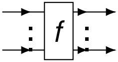
Typically, the blue node takes two inputs, and converts them to one output, similarly for the others.
What do we need to represent graphs ?

- A way to compose these operators sequentially
- A way to compose these operators in parallel

What we need is a symmetric monoidal category.

Categories

A *prop* is the data, for each pair (n, m) of a set $P[m, n]$.
Think of elements of $P[m, n]$ as boxes with m inputs and n outputs. We write $f : m \to n$.



We also need :

- A composition $P[n, p] \times P[m, n] \to P[m, p]$ satisfying the obvious axioms.
- An identity element:
- A tensor product : $P[m_1, n_1] \times P[m_2, n_2] \to P[m_1 + m_2, n_1 + n_2]$ satisfying the obvious axioms
- A swap element:

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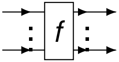

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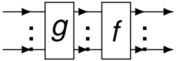
  $f \circ g$: 

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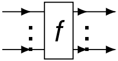

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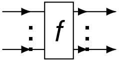

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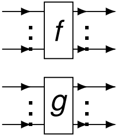
$f \otimes g$ 

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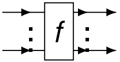

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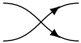
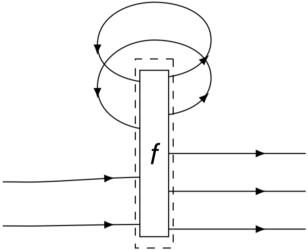
σ: 

Categories

A *traced prop* is a prop that contains an operator:
$[n+1, m+1] \to [n, m]$, called the trace satisfying obvious axioms



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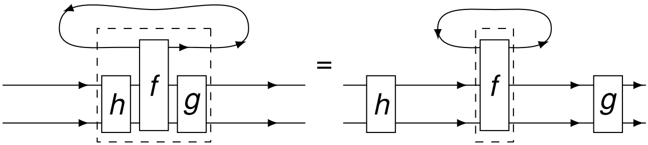

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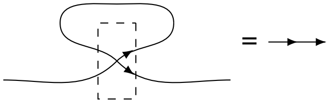

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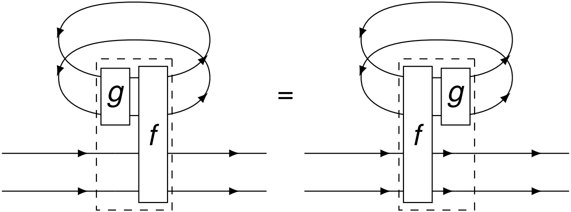

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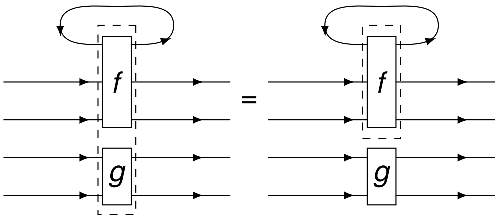


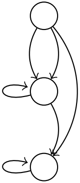
Main idea

- Find a traced prop which contains a *bigebra*, that is
  - An element $2 \to 1$ to represent the blue node
  - An element $1 \to 2$ to represent the red node
  - An arrow $1 \to 1$ to represent the square
- Suppose these three things satisfy the axioms we gave previously
- Then one can “interpret” graphs/matrice/SFTs in this category in such a way that SFTs that are conjugate corresponds to the same element of the prop.
- This gives a way to obtain *invariants*

Main idea

- Start from a graph/matrix

$$M = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



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- Start from a graph/matrix

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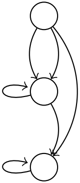
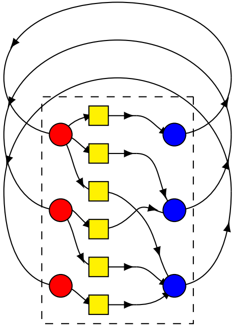
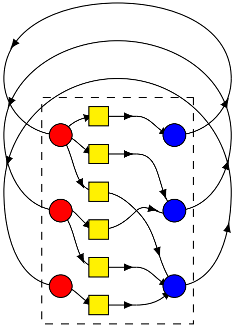


- Convert it into a red/blue graph:



Main idea

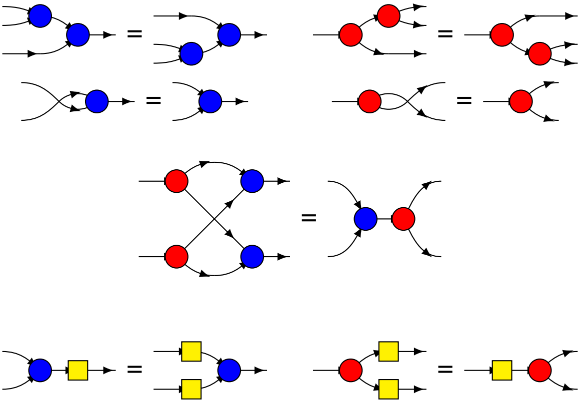
- Convert it into a red/blue graph: 

- Interpret the nodes as operators in some category:

54

The equations

These equations are incredibly common, and appear in many parts of math:



Plan

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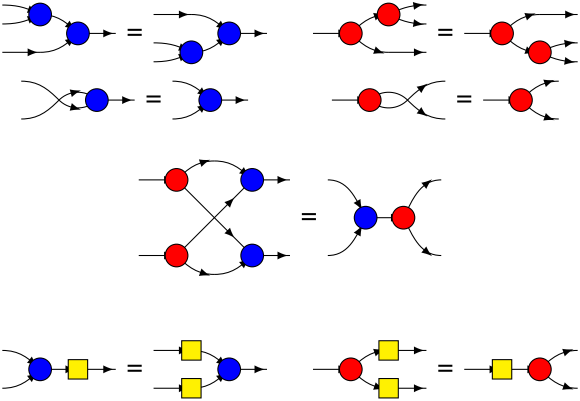

Plan

1. Introduction
2. The simplification
3. Categories
4. Examples
   - Monoids
   - Bialgebras and Hopf Algebras
   - Groups
5. Conclusion

Monoids

Let $\mathcal{M}$ be a commutative monoid. Inputs and outputs are elements of $\mathcal{M}$:

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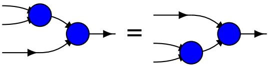

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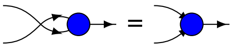

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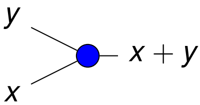

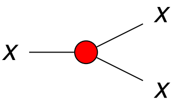

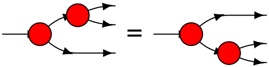

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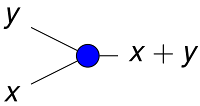

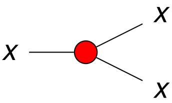

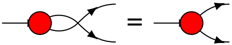

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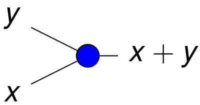

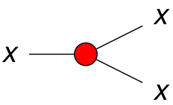

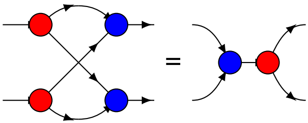

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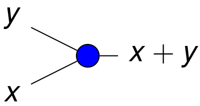

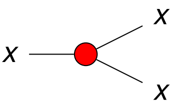

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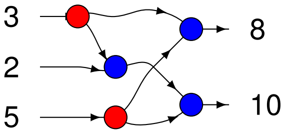

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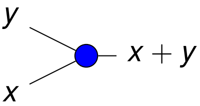

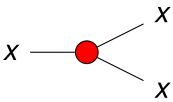

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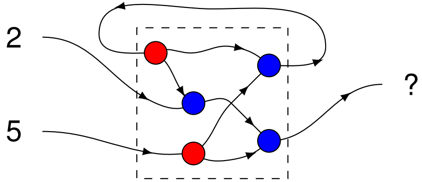

Monoids

Solution: monoids with multiplicities:

- Input of size *n*: an element of $\mathcal{M}^n \to \mathbb{N}_\infty$.
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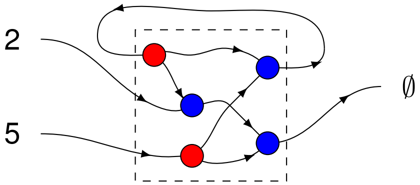

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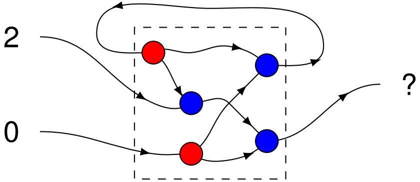

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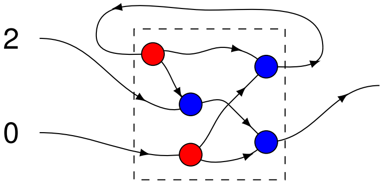
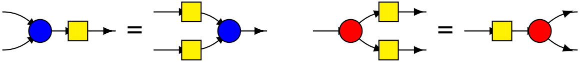


all elements of $\mathbb{R}$

Monoids

What about the square ?



It's just a morphism for the monoid (which will automatically work with the copy)

Theorem

Theorem

Let R be a matrix and $\mathcal{M}$ be a monoid, and h an homomorphism. When interpreting the diagram in the previous category, R represents the number of solutions of the equation $x = h(Rx)$ in the monoid $\mathcal{M}$.

Theorem

For all commutative monoids $\mathcal{M}$ and all homomorphisms h of $\mathcal{M}$, the number of solutions of the equation $x = h(Rx)$ in $\mathcal{M}$ is an invariant of conjugacy.

Plan

Bialgebras

Bialgebras and Hopf Algebras are well studied in representation theory and combinatorics.

- Input of size n: an element of $V^{\otimes n}$ where V is a vector space over some field $\mathbb{K}$
- If V is a vector space with basis e_i, $V \otimes V$ is a vector space with basis $e_i \otimes e_j$
- Boxes are linear maps

Monoid ring

Monoid ring : $\mathbb{K}[\mathcal{M}]$, vector space with basis e_x, $x \in \mathcal{M}$

- Multiplication: $e_x \otimes e_y \to e_{x+y}$
- By the multiplication:
  $3(e_2 \otimes e_3) - 2(e_1 \otimes e_4) + 3(e_1 \otimes e_5) \to e_5 + 3e_6$
- Comultiplication $e_x \to e_x \otimes e_x$
- By the comultiplication: $e_5 + 3e_6 \to e_5 \otimes e_5 + 3e_6 \otimes e_6$

Exactly the same example as before, presented differently.

The binomial bialgebra

The binomial bialgebra: $V = \mathbb{K}[X]$, basis $(X^n)_{n \geq 0}$

- Multiplication: $X^n \otimes X^m \to X^{n+m}$
- By the multiplication:
  $3(X^2 \otimes X^3) - 2(X^1 \otimes X^4) + 3(X^1 \otimes X^5) \to X^5 + 3X^6$
- Comultiplication $X^n \to \sum_k \binom{n}{k} X^k \otimes X^{n-k}$
- By the comultiplication: $X^2 \to 1 \otimes X^2 + 2X \otimes X + X^2 \otimes 1$
- Homomorphism: $X^n \to (\lambda X)^n$ for some $\lambda \in \mathbb{K}$

The binomial bialgebra

The canonical example $V = \mathbb{K}[X]$ does not have a trace, we need to tweak it:

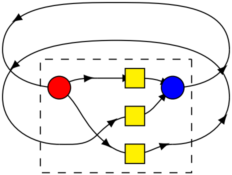
- Coefficients in the complete semiring $\mathbb{R}_\infty$ rather than in $\mathbb{R}$
- We allow infinite sums: $V = \mathbb{R}_\infty[[X]]$

Trace: sum over all n of the coefficient of X^n of the output if the input is X^n

Example

The golden shift:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$



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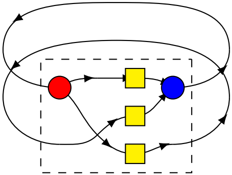

We look without the traces.
If we start from $X^n \otimes X^m$, the output is

$$\lambda^{n+m} \sum_k \binom{n}{k} X^{m+k} \otimes X^{n-k}$$

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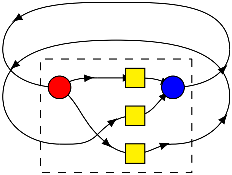

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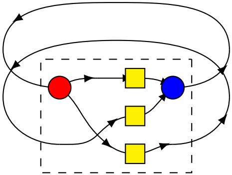


The coefficient of $X^n \otimes X^m$ in this sum is $\binom{n}{n-m}\lambda^{n+m}$
The value of the graph is therefore

$$\sum_{n,m} \binom{n}{n-m} \lambda^{n+m} = \frac{1}{1-\lambda^2-\lambda}$$

Zeta function

Theorem

Let M be a nonnegative matrix.
The result of the computation is $\zeta_M(\lambda)$, with $\zeta_M(t) = \frac{1}{\det(I - tM)}$.
Therefore ζ_M is an invariant of conjugacy.

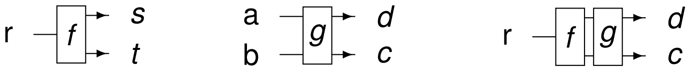
Consequence of McMahon master's theorem.

Plan

Cospans

New, weird category:

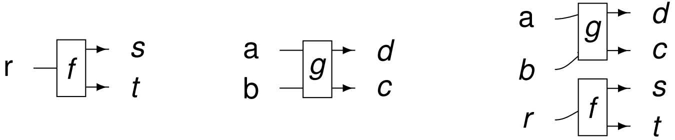
- A box with *n* inputs and *m* outputs is a *commutative group*, with at least $n+m$ generators, and a finite presentation.
- Inputs and outputs are to be understood as generators that can still be plugged in into other generators
- Composition is the new group obtained by identifying input and output generators that are plugged together (pushout)



$$\left\langle \begin{matrix} r,s, \\ t,u \end{matrix} \;\middle|\; u=2t+r \right\rangle \quad \left\langle \begin{matrix} a,b, \\ c,d \end{matrix} \;\middle|\; \begin{matrix} a-b=c+d \\ c-3d=a \end{matrix} \right\rangle \quad \left\langle \begin{matrix} a,b, \\ c,d, \\ r,s, \\ t,u \end{matrix} \;\middle|\; \begin{matrix} a-b=c+d \\ u=2t+r \\ c-3d=a \\ s=a \\ t=b \end{matrix} \right\rangle$$

Groups

- Tensor product is the new group obtained by putting the two groups side by side (sum of the group)



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Groups

- Trace consists in equating input and output
- We have to look at groups upto isomorphism of the internal generators.

What is the red and blue node ?

- Red node: group $\mathbb{Z} = \langle x, y, z | x = y = z \rangle$: all generators are equal
- Blue node: group $\mathbb{Z}^2 = \langle x, y, z | x + y = z \rangle$: output generator is equal to the sum of the input generators.

Note: the square is the trivial homomorphism

Theorem

Theorem

Starting from a matrix M (or a graph G), this construction associates to M the abelian group

$$G = \langle x | x = Mx \rangle$$

This is the Bowen-Franks group

Theorem

We can do the same with things other than groups: if we look at $\mathbb{Z}[t]$-modules instead of groups, we can have a nontrivial interpretation of the square, and obtain:

Theorem

Starting from a matrix M (or a graph G), this construction associates to M the $\mathbb{Z}[t]$ module:

$$G = \langle x | x = tMx \rangle$$

This is the dimension group (Krieger).

Plan

Conclusion

A systematic way to obtain invariants for symbolic dynamics by looking at algebraic structures in some categories.
We recover the classical invariants, which proves the method works:

- The Zeta function
- The Bowen-Franks group
- The Dimension group

Now: test other categories, to obtain new invariants!