# Computability in Closure Spaces

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# Plan

# Generic Situation

In various contexts $\mathfrak{C}$ (group theory, logic, dynamics), the following situation appears:

- Structures of $\mathfrak{C}$ can be defined by the set of axioms they satisfy
- Someone proves: there exists a structure of $\mathfrak{C}$ that is not computable
- Someone proves: if we suppose that the structure has additional property $\mathfrak{P}$, then the structure is computable

# Context 1: Combinatorial Group theory

Finitely generated groups with two generators $a, b$:

- We are given two generators $a$ and $b$, and maybe some set of relations $\mathcal{R}$ between products of the generators.
- The group is the set of all possible products of $a, b, a^{-1}, b^{-1}$
- If we have the relations $x = y$ then the element $x$ and the element $y$ are identified.
- Principal question of comb. group theory: Given two elements $s$ and $t$ , do we have $s = t$ ?

# Examples

Suppose we have no relations.
Then

- $aba = abb^{-1}bab^{-1}b$
- $baa^{-1} = bb^{-1}b$
- But $baa \neq abb^{-1}$

# Examples

Suppose we have the relation $ab = ba$
Then

- $aba = abb^{-1}bab^{-1}b$ (same as before)
- $abb = bab = bba$
- But $baa \neq abb^{-1}$

# Context 1: Combinatorial Group Theory

**Definition**

A presentation of a group with generators $a, b$ is just a set of relations for the group.

**Theorem (Boone 57, Novikov 55)**

*There exists a group with a finite presentation for which the equality is not computable*

**Theorem (ess. Kuznetsov 58)**

*Any simple group with a finite presentation has a computable equality.*

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# Context 2: Subshifts

**Definition**

Subshifts over an alphabet $\{0, 1\}$ are sets of infinite words given by a set of forbidden patterns, i.e. finite words they cannot contain

# Example

No forbidden patterns: we can produce all infinite words:

- 01111111111111111...
- 10101010101000000...
- 1100100100001111110110... (bits of $\pi$)

# Example

Forbidden pattern: 01

- 000000000000000000 . . .
- 111111000000000000 . . .
- 11111111111111111 . . .

# Example

Forbidden patterns: $10^n1$ for all $n$

- $000000000000000000\ldots$
- $000000100000000000\ldots$

# Context 2: Symbolic Dynamics (Tilings)

**Definition**

A presentation of a subshift is a set of forbidden patterns that define it.

**Theorem (Robinson 71)**

*There exists subshifts with a finite presentation which is noncomputable*

**Theorem (Ballier-J. 08, Hochman 09)**

*Minimal subshifts with finite presentation are computable.*

# Generic Situation

In various contexts $\mathfrak{C}$ (group theory, dynamics, logic), the following situation appears:

- Structures of $\mathfrak{C}$ can be defined by the set of axioms they satisfy
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# Example: First order logic

Fix a vocabulary $V$

**Definition**

An axiomatisation of a theory $T$ is a set of axioms for $T$.

**Theorem (Robinson 1950)**

*There exists theories with finitely many axioms that are not computable.*

**Theorem (Folklore ?)**

*Complete theories with finitely many axioms are computable.*

# Generic Situation

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# Goal of the talk

**Is there a common framework ?**

- Universal Algebra (observed by Kuznetsov 55, see also Mal'tsev 61) captures group but does not capture logic and dynamics.
- Model Theory : not well adapted for finitely generated structures.
- Category Theory : not clear

# Goal of the talk

We use here a framework from formal logic called closure spaces

- Start from topology (Moore 1910)
- Large role in universal logic (Tarski 1930's)
- Then in universal algebra (*closure systems*)

We add some *computable* flavor to this framework

# Plan

# General Framework

In all our cases, structures are entirely defined by the properties they satisfy.

Almost by definition.

A structure can be identified with the set of properties it satisfies

We want to see a structure as a *closed* subset of properties, so we need a notion of closure.

# General Framework

- We want to introduce a *consequence operator* on properties
- $\mathcal{C}(\mathcal{X})$ will be the set of consequences of $\mathcal{X}$.

# Group Theory

- $\text{“}aa^{-1}b = bb^{-1}b\text{''} \in \mathcal{C}(\emptyset)$
- $\text{“}aab = baa\text{''} \in \mathcal{C}(\text{“}ab = ba\text{''})$
- $\text{“}a = b\text{''} \in \mathcal{C}(\text{“}ab = ba\text{''}, \text{“}abb = baa\text{''})$

# Symbolic Dynamics

- "0 is forbidden" $\in \mathcal{C}$("01 is forbidden", "00 is forbidden")
- "10 is forbidden" $\in \mathcal{C}$("00 is forbidden", "101 is forbidden")

# First-Order Logic

On vocabulary $V = \{P(\cdot), a, b\}$.

- $``\forall x, P(x) \vee \exists x \neg P(x)'' \in \mathcal{C}(\emptyset)$
- $``\exists x, P(x)'' \in \mathcal{C}(``\forall x, P(x)'')$
- $``P(a) \vee P(b)'' \in \mathcal{C}(``P(a)'')$

# The closure operator

($I$ is the set of all properties/axioms)
The closure operator satisfies all the following rules:

- For all $R \subseteq I$, $R \subseteq C(R)$
- For all $R \subseteq I$, $C(C(R)) = C(R)$
- For all $A \subseteq B$, $C(A) \subseteq C(B)$

And is finitary:

- Every consequence of $R$ is a consequence of a finite subset of $R$.

$$C(R) = \bigcup_{R' \subseteq R, R' \text{finite}} C(R')$$

The closure operator form what is known as a *Tarski* space.

# Remark

In all three cases, the structures can be identified as closed subsets of properties

- A group with two generators $a, b$ can be identified to a normal subgroup of $F_{a,b}$.
- A subshift can be identified with the set of finite words that do not appear in it.
- A theory is (by definition) a set of axioms closed under consequences.

# Closure operators

Where do the closure operators come from ?

Are there other ways to see why these operators exist ?

# First method

- We start from a collection of rules of the form:

$$p_1, p_2 \dots p_n \vdash p_0$$

  Every structure that has properties $p_1 \dots p_n$ should have property $p_0$
- $\mathcal{C}(\mathcal{S})$ is the smallest superset of $\mathcal{S}$ that is closed under all rules.

Finitary by construction.

# Example: Subshifts

- $w$ forbidden $\vdash$ $w0$ forbidden .
- $w$ forbidden $\vdash$ $w1$ forbidden .
- $w$ forbidden $\vdash$ $0w$ forbidden .
- $w$ forbidden $\vdash$ $1w$ forbidden .
- $w0$ forbidden , $w1$ forbidden $\vdash$ $w$ forbidden .

# Example: Groups

- $\vdash x = x$
- $\vdash aa^{-1} = 1$
- $\vdash a^{-1}a = 1$
- $x = y, x' = y' \vdash xx' = yy'$
- $x = y \vdash x^{-1} = y^{-1}$

# Other examples

- Closed subsets $X$ of $[0, 1]$.
  - Properties of the form: $]p, q[$ does not intersect $X$
    - If interval $I$ does not intersect $X$, then no smaller interval intersect $X$
    - If $I$ and $J$ overlap and do not intersect $X$, then $I \cup J$ does not intersect $X$.
- Connected subsets $X$ of a computable graph $G$.
  - Properties of the form: $u$ belongs to $X$
  - If $u \in X$ and $R(u, v)$ then $v \in X$.

# Closure operators

Closure operators might also come from:

- Galois connections
- Topology

# Closure systems

Our objects (f.g. groups, subshifts and theories) are exactly the closed subsets of a closure system.

- Groups: closure system on the free group $F_{a,b}$
- Subshifts: closure system on the set of finite words over $\{0, 1\}$
- Theories: closed system on the set of formulas

# Uses

What do we get from closure spaces ?

# Order

- Subsets of properties are naturally ordered by subset inclusion.
- Closed subsets correspond to structures
- This gives an order on closed subsets.

$\mathfrak{M} \leq \mathfrak{N}$ if $\mathfrak{N}$ satisfies more relations than $\mathfrak{M}$

# Examples

- $G \leq H$ iff $H$ is a quotient of $G$.
  - The largest group is therefore the trivial group with one element.
- $S \leq T$ iff $T \subseteq S$ (as sets of infinite words)
  - The largest subshift is therefore the empty subshift
- $T \leq T'$ iff $T \subseteq T'$ for theories.

# Plan

# Computability

What typically are the computable properties of the closure system ?

# The framework

In typical applications:

- The set of all properties $I$ is computable
- The set of all rules

  $$p_1, p_2 \dots p_n \vdash p_0$$

  can be computably enumerated

What does it mean for the closure operator ?

# Key ingredient - Enumeration reducibility

Most results involve *enumeration reducibility*

**Definition (Informal)**

$A \leq_e B$ if there is a program that can list the elements of $A$ from any list of all elements of $B$

(Note: $B$ is usually infinite)

**Definition (Informal)**

$A \leq_e^f B$ if the program is called $f$.

$\leq_e$ is a preorder, smallest degree is exactly the computably enumerated sets.

# First Theorem

**Theorem**

*For all $X$, $C(X) \leq_e X$*

There is some program that can list $C(X)$ from any list of $X$.

The program does not depend on $X$.

# First Theorem

What does it mean ?

# What does it mean ?

Enumeration reducibility speaks about “positive information”

$A \leq_e B$ : Some program can give me a list of the elements in $A$ given any list of all elements in $B$

As $B$ is (usually) infinite, at any moment my program knows only that some elements are in $B$, but will never know that some elements are not in $B$.

To produce the list of elements in $A$, the program cannot use negative information about $B$, i.e. information about which elements are not in $B$.

# What does it mean ?

$C(X) \leq_e X$

To enumerate the consequences of $X$, we only need positive information about $X$.

# Plan

# Presentations

## Definition

If $X = C(R)$, we usually say that $R$ is a presentation of $X$.
$X$ is finitely presented if $X = C(R)$ for $R$ finite.

Finitely presented groups = finitely presented groups.
Finitely presented subshifts = subshifts of finite type.
Finitely presented theories = finitely axiomatizable theories.

# First theorem

**Proposition (Classical Theorem 1)**

*If $X \subseteq I$ is finitely presented, then the set of consequences of $X$ is computably enumerable*

Indeed, $X \leq_e R$ for $R$ finite, therefore $X$ is enumerable given the enumeration of a finite set, i.e. enumerable.
encompasses exactly what happens in the three situations

# Craig's theorem

## Definition

Say that the set of properties is redundant if for any finite set $S$ of properties there exists a disjoint set $T$ with $\mathcal{C}(S) = \mathcal{C}(T)$

All examples have redundant properties.

## Proposition (Craig's theorem)

*Suppose the set of properties is redundant.*
*If $X = \mathcal{C}(S)$ for a computably enumerable set $S$, then $X = \mathcal{C}(T)$ for a computable $T$.*

encompasses exactly what happens in the three situations

# Maximal elements

## Definition (Maximal points)

A closed subset $X$ is maximal if any closed set $Y$ s.t. $X \subseteq Y$ satisfies $Y = I$ or $Y = X$.

($I$ is the set of all properties).
The only structure “above” $X$ is the structure with all properties, i.e. the “trivial” structure.
Alternatively, for any $e \notin X$, $\mathcal{C}(X \cup \{e\}) = I$, the whole set.

# Examples

In first-order logic

- *T* is maximal iff adding any axiom to *T* would lead to the theory consisting of all axioms i.e. the inconsistent theory
- *T* is therefore a *complete* theory.

In groups:

- *G* is maximal iff adding any equality to *G* would lead to the trivial group
- *T* is therefore a *simple* group

For subshifts, maximal elements are usually called *minimal* subshifts. (due to duality)

# Main Theorem

**Theorem (Generalization of the classical theorem )**

*If $X$ is maximal and finitely presented, then $X$ is computable.*

encompasses exactly what happens in the three situations

# Corollary

**Theorem (Ballier-J. 08, Hochman 09)**

*Minimal subshifts with finite presentation are computable.*

**Theorem (ess. Kuznetsov 58)**

*Any simple group with a finite presentation has a computable equality.*

**Theorem (Folklore ?)**

*Complete theories with finitely many axioms are computable.*

# Main Theorem

**Theorem (Classical Theorem, modified)**

*If $X$ is maximal, then $\overline{X} \leq_e X$*

$\overline{X} = I \setminus X$ is the complement of $X$

In a maximal point, one can enumerate non-consequences from consequences.

Trivial for logic. Shows that $\overline{X} \leq_e X$ is an important relation to study in computable algebra.

# Converse

**Theorem**

*If $\overline{X} \leq_e X$, then there exists a (computable) closure system on $I$ for which $X$ is maximal*

Hence $\leq_e$ captures exactly what we want.

# Conclusion - Maximality

Many rigidity conditions on a structure $X$ can be expressed in terms of the natural order on the lattice of all structures.

The signification of these conditions is that they imply that negative information on $X$ can be obtained from positive information on $X$.

# Plan

# Conclusion

What do we gain from this ?

- Unified proofs
- Results that are valid also for nonrecursive structures
- Show that results from different areas are instances of a general theorem

# Where to go

Additional theorems we would like to unify. In all three examples:

**Boone-Novikov 55-57, Robinson 50, Robinson 71**

There exists finitely presented structures that are undecidable

**Higman 61, Kleene 52, Hochman 09**

A structure has a computable presentation iff it has a finite presentation over a larger vocabulary/alphabet/set of generators.

**folklore**

Finitely presented structures that are residually finite/with finite model property/with dense set of periodic points are computable

# Classical Theorem 2

**Theorem (Boone 57, Novikov 55)**

*There exists f.p. groups with undecidable word problems.*

**Theorem (Robinson 71)**

*There exists subshifts of finite type with an nonrecursive language*

**Theorem (Robinson 1950)**

*There exists finitely axiomatisable theories that are not recursive*

# Classical Theorem 3

**Theorem (Higman 61)**

*A group is recursively presented iff it embeds into a finitely presented group*

**Theorem (Kleene 52)**

*A theory with identity is recursively axiomatisable iff it is finitely axiomatisable using additional predicates*

**Theorem (Hochman 2009)**

*A subshift of dimension $d$ is effectively closed iff it is of "finite type" in dimension $d+2$*

# Classical Theorem 4

**Theorem (Higman-Thompson 80)**

*A group has a recursive word problem iff it embeds into a finitely presented simple group*

**Theorem (J.-Vanier 2019)**

*A subshift of dimension $d$ has a computable word problem iff it is a subaction of a minimal subshift of finite type.*

# Plan

# Galois connection

We start from a relation $R \subseteq I \times J$ ($I$ will be a set of attributes, $J$ a set of objects)

This gives us two mappings:

$$F(X) = \{b \in J, \forall a \in X, (a, b) \in R\}$$

$$G(Y) = \{a \in I, \forall b \in Y, (a, b) \in R\}$$

Our structures can be seen as sets of the form $F(X)$ or dually of the form $G(Y)$.

# Subshifts

- $I$ are finite words
- $J$ are infinite words
- $xRu$ if the finite word $x$ does not appear in $u$
- $F(X)$ the set of infinite words that contain no element of $X$
  - Exactly our subshifts
- $G(Y)$ the set of finite words that appear in no element of $Y$.

# First-order logic

- $I$ are first-order formulas over the alphabet $V$
- $J$ are first-order structures over the vocabulary $V$
- $\phi R \mathfrak{M}$ if $\phi$ is true on $\mathfrak{M}$
- $F(X)$ the set of models that satisfy all axioms of $\phi$.
- $G(Y)$ the set of formulas true in all models in $Y$
  - These are precisely theories

# Galois connection

The Galois connection gives us a closure system on $I$

$$C(X) = G(F(X))$$

NOT finitary in general. For first order logic, this is essentially the compactness property.

Our objects (f.g. groups, subshifts and theories) are exactly the closed subsets of this closure system.