Computational Complexity of the GPAC

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Joint work with Olivier Bournez and Daniel Graça

April 10, 2014

Outline

GPAC

General Purpose Analog Computer

- by Claude Shanon (1941)

GPAC

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- idealization of an analog computer: Differential Analyzer

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General Purpose Analog Computer

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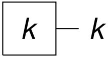
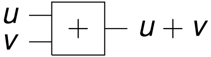
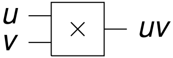
- idealization of an analog computer: Differential Analyzer
- circuit built from:



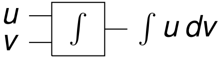
A constant unit



An adder unit



An multiplier unit



An integrator unit

GPAC: beyond the circuit approach

Theorem

y is generated by a GPAC iff it is a component of the solution $y = (y_1, \ldots, y_d)$ of the Polynomial Initial Value Problem (PIVP):

$$\begin{cases} y' &= p(y) \\ y(t_0) &= y_0 \end{cases}$$

where p is a vector of polynomials.

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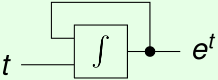
Remark

- continuous dynamical system
- the GPAC is just one reason to look at them[a]

[a] Ask question

GPAC: examples

Example (One variable, linear system)



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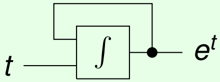

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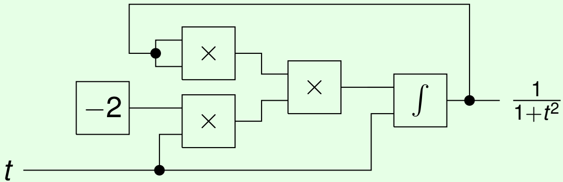

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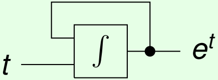

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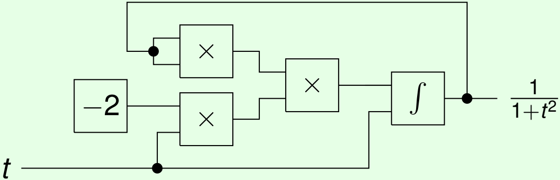

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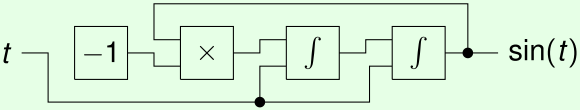

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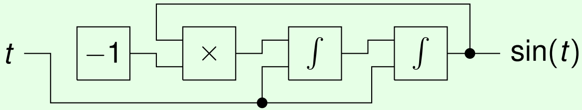

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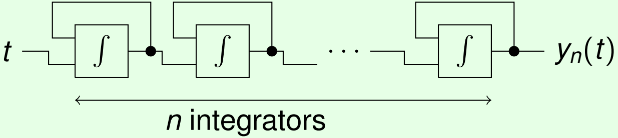

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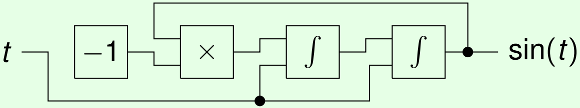

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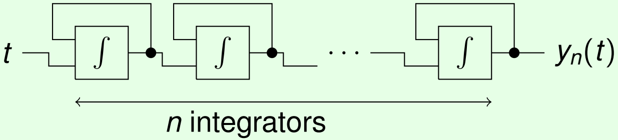

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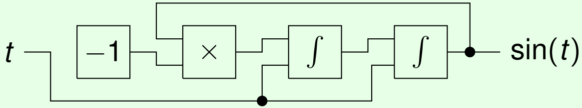

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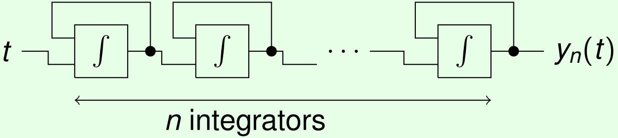

$$\begin{cases} y_1(t) = e^t \\ y_2(t) = e^{e^t} \\ \dots \\ y_n(t) = e^{e^{\cdot^{\cdot^{\cdot^t}}}} \end{cases}$$

Motivation

1. Study the computational power of such systems:

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Example (Counter-Example)

$$r = \sum_{n=0}^{\infty} d_n 2^{-n}$$

where

$$d_n = 1 \Leftrightarrow \text{the } n^{th} \text{ Turing Machine halts on input } n$$

Computable function

Definition (Computable Function)

A function $f : \mathbb{R} \to \mathbb{R}$ is computable if there exist a Turing Machine M s.t. for any $x \in \mathbb{R}$ and oracle $\mathcal{O}$ computing x, $M^{\mathcal{O}}$ computes $f(x)$.

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$$f(x) = \lceil x \rceil$$

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Can we fix this ?

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y is **generated** by a GPAC iff it is a component of the solution $y = (y_1, \dots, y_d)$ of the ordinary differential equation (ODE):

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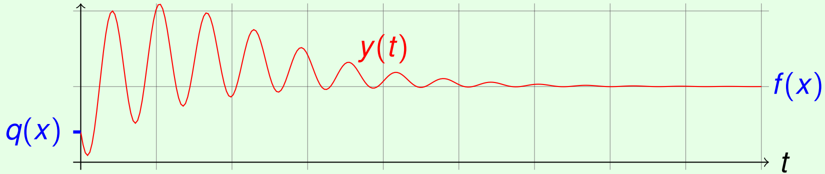

Computable Analysis = GPAC ? (again)

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- Any solution to a PIVP is computable + convergence
- Simulate a Turing machine with a GPAC[a]

□

[a]Details on blackboard

What about complexity ?

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- Computable Analysis: nice complexity theory (from Turing Machines)

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Conjecture (☹)

Computable Analysis = General Purpose Analog Computer, *at the complexity level*

Time Scaling

System	#1	#2
ODE	$\begin{cases} y'(t) = p(y(t)) \\ y(1) = y_0 \end{cases}$	$\begin{cases} z'(t) = u(t)p(z(t)) \\ u'(t) = u(t) \\ z(t_0) = y_0 \\ u(1) = 1 \end{cases}$

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Remark

Same curve, different speed: $u(t) = e^t$ and $z(t) = y(e^t)$

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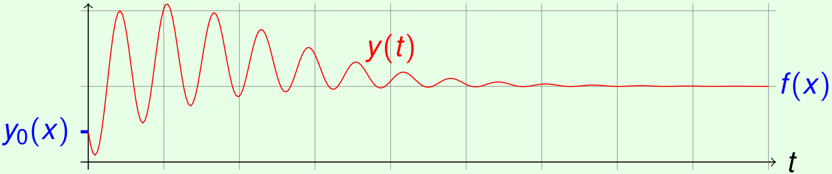

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Computed Function	$f(x) = \lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} z_1(t)$	

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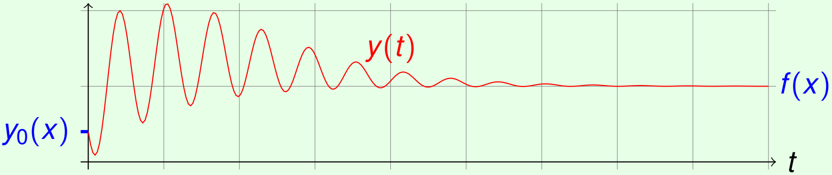

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Convergence	Eventually	Exponentially faster

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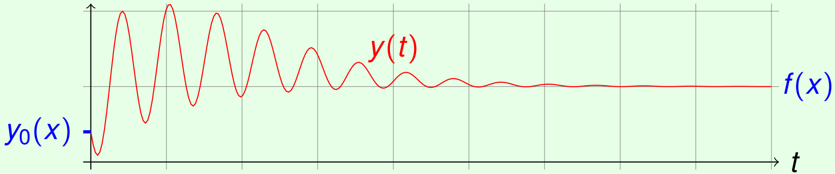

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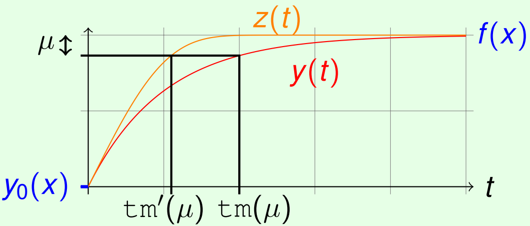

$$\|y_1(\mathtt{tm}(\mu)) - f(x)\| \leqslant \mu$$

Time Scaling

ODE	$y' = p(y)$	$\begin{cases} z' = up(z) \\ u' = u \end{cases}$
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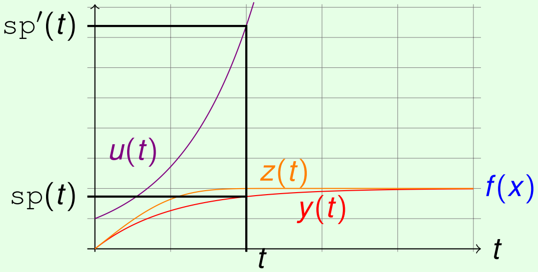
Bounding box for ODE at time t	$\mathtt{sp}(t)$	$\mathtt{sp}'(t) = \max(\mathtt{sp}(e^t), e^t)$

Example



$$\mathtt{sp}(t) = \sup_{\xi\in[1,t]} \|y(\xi)\|$$

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Remark

- $\mathtt{tm}(\mu)$ and $\mathtt{sp}(t)$ depend on the convergence rate
- $\mathtt{sp}(\mathtt{tm}(\mu))$ seems not

Proper Measures

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- time scaling invariant
- property of the curve

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- Bounding Box at precision $\mu \Rightarrow$ Ok but geometric interpretation ?
- Length of the curve until precision $\mu \Rightarrow$ Much more intuitive

Definition (Polytime GPAC-Computable Function)

f is **polytime** computable by a GPAC iff for all $x \in \mathbb{R}$ the solution $y = (y_1, \dots, y_d)$ of the ordinary differential equation (ODE):

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where p,q are vectors of polynomials

satisfies $\left\| f(x) - y_1(\ell^{-1}(\mathtt{len}(x,\mu)) \right\| \leqslant e^{-\mu}$ where

- $\mathtt{len}$ is a polynomial [polytime]
- $\ell(t)$ is the length of the curve y from t_0 to t.
- $\ell^{-1}(l)$ is the time to reach a length l on the curve y

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Remark

- implies $f(x) = \lim_{t\to\infty} y_1(t)$
- length of a curve: $\ell(t) = \int_{t_0}^{t} \|p(y(u))\|\, du$
- $y_1(\ell^{-1}(l)) =$ position after travelling a length l on the curve y

Computable Analysis = GPAC ?

Theorem (Almost ☺)

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Remark (Polytime computable in CA)

f polytime computable:

- polynomial modulus of continuity mc: $\|x-y\| \leqslant 2^{-\mathrm{mc}(\mu)} \Rightarrow \|f(x)-f(y)\| \leqslant 2^{-\mu}$
- polynomial time computable over $\mathbb{Q}$

Conclusion

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- Equivalence with Computable Analysis for polynomial time

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Not mentioned in this talk:

- The GPAC as a language recogniser

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- The GPAC as a language recogniser
- Equivalence with *P* and *NP*

Future Work

- Notion of reduction ?

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- Space complexity ?

Questions ?

- Do you have any questions ?

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- decide $\approx \{Yes, No\} \approx \{0, 1\} \subseteq$ real numbers

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Remark

- words $\approx$ integers $\subseteq$ real numbers
- decide $\approx \{Yes, No\} \approx \{0, 1\} \subseteq$ real numbers
- language recogniser: special case of real function ?
  $f : \mathbb{N} \subseteq \mathbb{R} \rightarrow \{0, 1\} \subseteq \mathbb{R}$

GPAC as Language Recogniser

- GPAC as computable real function $\to$ Computable Analysis
- GPAC as language recogniser $\to$ classical computability ?

Remark

- words $\approx$ integers $\subseteq$ real numbers
- decide $\approx \{Yes, No\} \approx \{0,1\} \subseteq$ real numbers
- language recogniser: special case of real function ?
  $f : \mathbb{N} \subseteq \mathbb{R} \to \{0,1\} \subseteq \mathbb{R}$
- Yes but there is more !

Definition (GPAC-Recognisable Language)

$\mathcal{L} \subseteq \mathbb{N}$ GPAC-recognisable if for any $x \in \mathbb{N}$, the solution y to

$$\begin{cases} y' = p(y) \\ y(t_0) = q(x) \end{cases} \qquad \text{where } p, q \text{ are vectors of polynomials}$$

satisfies for $t \geqslant t_1(x)$:

- if $x \in \mathcal{L}$ then $y_1(t) \geqslant 1$ (accept)
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Theorem

The GPAC-recognisable languages are exactly the recursive languages.

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What about complexity ?

Definition (Polytime GPAC-Recognisable Language)

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The class of polytime GPAC-recognisable languages is exactly P.

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Remark (Why $\log(x)$?)

Classical complexity measure: length of word $\approx$ log of value

Definition (Non-deterministic Polytime GPAC-Recognisable Language)

$\mathcal{L} \subseteq \mathbb{N}$ non-deterministic poyltime GPAC-recognisable if for any $x \in \mathbb{N}$, the solution y to

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satisfies for $t \geqslant t_1(x)$:

- if $x \in \mathcal{L}$ then $y_1(t) \geqslant 1$ for at least one digital controller u
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Remark (Digital Controller)

Digital Controller $\approx u : \mathbb{R} \to \{0, 1\}$

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The class of non-deterministic polytime GPAC-recognisable languages is exactly *NP*.