Static Analysis of Functional Programs

using Tree Automata

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Outline

1. Motivating example
2. Background on tree automata completion
3. What is missing for a decent static analysis of functional programs?

... Related work scattered in subsections
Motivating example

OCaml type checking

```ocaml
let rec append l1 l2 = match l1 with
| [] -> l2
| h::t -> h :: (append t l2 );;
# val append: 'a list -> 'a list -> 'a list = <fun>

let rec rev l = match l with
| [] -> []
| h::t -> append (rev t) [h];;
# val rev: 'a list -> 'a list = <fun>
```

Motivating example

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val append : 'a list -> 'a list -> 'a list = <fun>

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val rev : 'a list -> 'a list = <fun>
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Motivating example

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let rec rev l = match l with
  | [] -> []
  | h::t -> append (rev t) [h];;
# val rev: 'a list -> 'a list = <fun>
```

We would like to have... more than simple types

```ocaml
# val rev: 'a list -> empty list
```
Motivating example (II)

OCaml type checking

```ocaml
let rec append l1 l2 = match l1 with
  | [] -> l2
  | h::t -> h :: (append t l2 );;

let rec rev l = match l with
  | [] -> []
  | h::t -> append (rev t) [h];;
```

We would like to have...

val rev : list of As then Bs -> list of Bs then As

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Verification of OCaml using TA
Motivating example (II)

**OCaml type checking**

```ocaml
let rec append l1 l2 = match l1 with
  | []  -> l2
  | h :: t -> h :: (append t l2);

let rec rev l = match l with
  | []  -> []
  | h :: t -> append (rev t) [h];
```

We would like to have...

```ocaml
val rev : list of As then Bs -> list of Bs then As
```
# Background: Term Rewriting

## Sets of symbols and variables

- **Set of ranked symbols**
  
  \[ \mathcal{F} = \{ \text{app} : 2, \text{cons} : 2, \text{nil} : 0, a : 0 \} \]

- **Set of variables**
  
  \[ \mathcal{X} = \{ x, y, z, \ldots \} \]
## Background: Term Rewriting

### Sets of symbols and variables
- **Set of ranked symbols**
  \[ F = \{ \text{app} : 2, \text{cons} : 2, \text{nil} : 0, a : 0 \} \]
- **Set of variables**
  \[ X = \{ x, y, z, \ldots \} \]

### Sets of terms
- **Ground terms**
  \[ T(F) = \{ a, \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, \text{cons}(a, \text{nil})), \ldots \} \]
- **Terms**
  \[ T(F, X) = \{ x, \text{cons}(x, y), \text{app}(\text{nil}, a), \ldots \} \]
Background: Term Rewriting

### Sets of symbols and variables
- Set of ranked symbols \( \mathcal{F} = \{\text{app : 2, cons : 2, nil : 0, a : 0}\} \)
- Set of variables \( \mathcal{X} = \{x, y, z, \ldots\} \)

### Sets of terms
- Ground terms \( \mathcal{T}(\mathcal{F}) = \{a, \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, \text{cons}(a, \text{nil})), \ldots\} \)
- Terms \( \mathcal{T}(\mathcal{F}, \mathcal{X}) = \{x, \text{cons}(x, y), \text{app}(\text{nil}, a), \ldots\} \)

### Term Rewriting Systems (TRS)
Set of rewrite rules \( l \rightarrow r \) with \( l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \) and \( \text{Var}(r) \subseteq \text{Var}(l) \) e.g.
\[
\mathcal{R} = \left\{ \begin{array}{c}
\text{app}(\text{nil}, x) \rightarrow x \\
\text{app}(\text{cons}(x, y), z) \rightarrow \text{cons}(x, \text{app}(y, z))
\end{array} \right\}
\]
## Background: Term Rewriting

### Sets of symbols and variables

- **Set of ranked symbols**
  \[ \mathcal{F} = \{\text{app} : 2, \text{cons} : 2, \text{nil} : 0, a : 0\} \]

- **Set of variables**
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### Sets of terms

- **Ground terms**
  \[ \mathcal{T}(\mathcal{F}) = \{a, \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, \text{cons}(a, \text{nil})), \ldots\} \]

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Term Rewriting (II)

Rewriting term $app(cons(a, nil), cons(b, nil))$ using

$$\mathcal{R} = \left\{ \begin{array}{l}
app(nil, x) \rightarrow x \\
app(cons(x, y), z) \rightarrow cons(x, app(y, z))
\end{array} \right\}$$
Term Rewriting (II)

- Rewriting term \( \text{app}(\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})) \) using

\[
\mathcal{R} = \begin{cases} 
\text{app}(\text{nil}, x) \rightarrow x \\
\text{app}(\text{cons}(x, y), z) \rightarrow \text{cons}(x, \text{app}(y, z)) 
\end{cases}
\]

\[
\text{app}(\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})) \rightarrow_{\mathcal{R}} \text{cons}(a, \text{app}(\text{nil}, \text{cons}(b, \text{nil})))
\]
Term Rewriting (II)

- Rewriting term \( \text{app} (\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})) \) using

\[
R = \left\{ \begin{array}{l}
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\text{app}(\text{cons}(x, y), z) \rightarrow \text{cons}(x, \text{app}(y, z)) \\
\end{array} \right\}
\]

\[
\text{app}(\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})) \rightarrow_R \text{cons}(a, \text{app}(\text{nil}, \text{cons}(b, \text{nil}))) \\
\rightarrow_R \text{cons}(a, \text{cons}(b, \text{nil}))
\]
Term Rewriting (II)

- Rewriting term $app(cons(a, nil), cons(b, nil))$ using

$$\mathcal{R} = \left\{ \begin{array}{l}
app(nil, x) \rightarrow x \\
app(cons(x, y), z) \rightarrow cons(x, app(y, z))
\end{array} \right\}$$

$$app(cons(a, nil), cons(b, nil)) \rightarrow_{\mathcal{R}} cons(a, app(nil, cons(b, nil)))$$
$$\quad \rightarrow_{\mathcal{R}} cons(a, cons(b, nil))$$

- Set of reachable terms: $\mathcal{R}^*(\mathcal{L}) = \{ u \mid s \in \mathcal{L} \land s \rightarrow_{\mathcal{R}^*} u \}$
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\[
\text{app}(\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})) \quad \rightarrow_{\mathcal{R}} \quad \text{cons}(a, \text{app}(\text{nil}, \text{cons}(b, \text{nil})))
\]

\[
\rightarrow_{\mathcal{R}} \quad \text{cons}(a, \text{cons}(b, \text{nil}))
\]

- Set of reachable terms: \( \mathcal{R}^*(\mathcal{L}) = \{ u \mid s \in \mathcal{L} \land s \rightarrow_{\mathcal{R}^*} u \} \)

\[
\mathcal{R}^*(\{ \text{app}(\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})) \}) =
\]

\[
\{ \text{app}(\text{cons}(a, \text{nil}), \text{cons}(b, \text{nil})), \\
\text{cons}(a, \text{app}(\text{nil}, \text{cons}(b, \text{nil}))), \\
\text{cons}(a, \text{cons}(b, \text{nil})) \}
\]
Equational abstraction [Meseguer & al. 03] [Takai 04]

\[ \mathcal{R} = \begin{cases} 
(1) \ f(x, y) \rightarrow f(g(x), y) \\
(2) \ f(x, y) \rightarrow f(x, h(y)) 
\end{cases} \]

prove that \( f(a, b) \not\rightarrow_{\mathcal{R}^*} f(a, h(g(b))) \)?
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\]

prove that \( f(a, b) \not\rightarrow_{\mathcal{R}}^* f(a, h(g(b))) \)?

using \( E = \{ g(g(x)) = g(x), h(h(x)) = h(x) \} \)

\[
\begin{aligned}
f(a, b) & \rightarrow f(g(a), b) \\
f(g(a), b) & \rightarrow f(g(g(a)), b) \\
f(a, h(b)) & \rightarrow \ldots \\
f(a, h(b)) & \rightarrow \ldots
\end{aligned}
\]
Equational abstraction [Meseguer & al. 03] [Takai 04]

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(1) f(x, y) \rightarrow f(g(x), y) \\
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prove that \( f(a, b) \not\rightarrow_{\mathcal{R}^*} f(a, h(g(b))) \)?

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(1) & f(x, y) \rightarrow f(g(x), y) \\
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prove that \( f(a, b) \not\rightarrow_{\mathcal{R}^*} f(a, h(g(b))) \)?

using \( E = \{ g(g(x)) = g(x), h(h(x)) = h(x) \} \)

\[
f(a, b) \not\rightarrow_{\mathcal{R}/E} f(a, h(g(b)))
\]
Equational abstraction [Meseguer & al. 03] [Takai 04]

\[ \mathcal{R} = \begin{cases} 
(1) f(x, y) \rightarrow f(g(x), y) \\
(2) f(x, y) \rightarrow f(x, h(y)) \end{cases} \]

prove that \( f(a, b) \not\rightarrow^{\mathcal{R}^*} f(a, h(g(b))) \)?

using \( E = \{ g(g(x)) = g(x), h(h(x)) = h(x) \} \)

\[
\begin{array}{c}
\text{1} \\
\text{f(g^+(a),b)} \\
\text{f(a,b)} \\
\text{2} \\
\text{f(g^+(a),h^+(b))} \\
\text{1} \\
\text{2} \\
\text{f(a,h^+b)} \\
\text{2} \\
\end{array}
\]

\[
f(a, b) \not\rightarrow^{\mathcal{R}/E^*} f(a, h(g(b))) \implies f(a, b) \not\rightarrow^{\mathcal{R}^*} f(a, h(g(b)))
\]
Background: Tree Automata

Recognized language \( \mathcal{L}(A, q) \)

\[
A = \langle \mathcal{F}, Q, Q_f, \Delta \rangle \\
\text{with } Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \} \\
Q_f = \{ q_f \} \text{ and}
\]

\[
\Delta = \begin{cases} 
    a \to q_a \\
    b \to q_b \\
    \text{nil} \to q_{la} \\
    \text{nil} \to q_{lb} \\
    \text{cons}(q_a, q_{la}) \to q_{la} \\
    \text{cons}(q_b, q_{lb}) \to q_{lb} \\
    \text{app}(q_{la}, q_{lb}) \to q_f 
\end{cases}
\]
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Recognized language $L(\mathcal{A}, q) = \{ s \in T(\mathcal{F}) \mid s \to^{*}_\Delta q \}$

$$\mathcal{A} = \langle \mathcal{F}, Q, Q_f, \Delta \rangle$$
with $Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \}$

$Q_f = \{ q_f \}$ and

$$\Delta = \begin{cases} 
    a & \to q_a \\
    b & \to q_b \\
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    \text{cons}(q_a, q_{la}) & \to q_{la} \\
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\end{cases}$$
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Recognized language $L(\mathcal{A}, q) = \{ s \in T(\mathcal{F}) \mid s \xrightarrow{\Delta}^* q \}$

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\end{cases}$$
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$$
\mathcal{A} = \langle \mathcal{F}, Q, Q_f, \Delta \rangle \\
\text{with } Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \} \\
Q_f = \{ q_f \} \text{ and } \\
\Delta = \\
a \rightarrow q_a \\
b \rightarrow q_b \\
nil \rightarrow q_{la} \\
nil \rightarrow q_{lb} \\
cons(q_a, q_{la}) \rightarrow q_{la} \\
cons(q_b, q_{lb}) \rightarrow q_{lb} \\
app(q_{la}, q_{lb}) \rightarrow q_f \\
$$

`cons`: a  
`cons`: nil  
`cons`: qa  
  `cons`: a  
  `nil`  
  qa  
  `nil`  

`nil`: nil  
`nil`: nil  

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Background: Tree Automata

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\[ \mathcal{A} = \langle \mathcal{F}, Q, Q_f, \Delta \rangle \]

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\[ \Delta = \begin{cases} 
  a \rightarrow q_a \\
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\end{cases} \]
Background: Tree Automata

Recognized language $\mathcal{L}(\mathcal{A}, q) = \{ s \in \mathcal{T}(\mathcal{F}) \mid s \xrightarrow{\Delta}^* q \}$

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\mathcal{A} = \langle \mathcal{F}, Q, Q_f, \Delta \rangle \\
\text{with } Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \} \\
Q_f = \{ q_f \} \text{ and } \\
\Delta = \\
\begin{cases} 
    a \rightarrow q_a \\
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    \text{nil} \rightarrow q_{lb} \\
    \text{cons}(q_a, q_{la}) \rightarrow q_{la} \\
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    \text{app}(q_{la}, q_{lb}) \rightarrow q_f 
\end{cases}
$$
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$A = \langle F, Q, Q_f, \Delta \rangle$
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$$
\Delta = \begin{cases} 
    a \rightarrow q_a \\
    b \rightarrow q_b \\
    nil \rightarrow q_{la} \\
    nil \rightarrow q_{lb} \\
    \text{cons}(q_a, q_{la}) \rightarrow q_{la} \\
    \text{cons}(q_b, q_{lb}) \rightarrow q_{lb} \\
    \text{app}(q_{la}, q_{lb}) \rightarrow q_f 
\end{cases}
$$

\[ 
\begin{align*}
    a &\xrightarrow{\Delta} \text{cons} \\
    &\xrightarrow{\Delta} q_a \text{ nil} \\
    &\xrightarrow{\Delta} q_a \\
    &\xrightarrow{\Delta} q_{la} \\
    \ldots &\xrightarrow{\Delta} q_{la} \\
\end{align*}
\]
Recognized language: $\mathcal{L}(A, q) = \{ s \mid s \rightarrow^* q \}$

$A = \langle \mathcal{F}, Q, Q_f, \Delta \rangle$
with $Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \}$
$Q_f = \{ q_f \}$ and

$\Delta = \left\{ \begin{array}{l}
a \rightarrow q_a \\
b \rightarrow q_b \\
nil \rightarrow q_{la} \\
nil \rightarrow q_{lb} \\
\text{cons}(q_a, q_{la}) \rightarrow q_{la} \\
\text{cons}(q_b, q_{lb}) \rightarrow q_{lb} \\
\text{app}(q_{la}, q_{lb}) \rightarrow q_f \end{array} \right.$

$L(A, q_{la}) = \{ \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, \ldots) \}$
Recognized language: $L(A, q) = \{ s \mid s \xrightarrow{\Delta}^* q\}$

$A = \langle \mathcal{F}, Q, Q_f, \Delta \rangle$

with $Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \}$

$Q_f = \{ q_f \}$ and

$\Delta =$

\[
\begin{align*}
    a & \rightarrow q_a \\
    b & \rightarrow q_b \\
    \text{nil} & \rightarrow q_{la} \\
    \text{nil} & \rightarrow q_{lb} \\
    \text{cons}(q_a, q_{la}) & \rightarrow q_{la} \\
    \text{cons}(q_b, q_{lb}) & \rightarrow q_{lb} \\
    \text{app}(q_{la}, q_{lb}) & \rightarrow q_f
\end{align*}
\]

$L(A, q_{la}) = \{ \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, ...) \}$

$L(A, q_{lb}) = \{ \text{nil}, \text{cons}(b, \text{nil}), \text{cons}(b, ...) \}$
Recognized language: \( \mathcal{L}(\mathcal{A}, q) = \{ s \mid s \rightarrow^*_\Delta q \} \)

\[ \begin{align*}
\mathcal{A} & = \langle \mathcal{F}, Q, Q_f, \Delta \rangle \\
\text{with } Q & = \{ q_a, q_b, q_{la}, q_{lb}, q_f \} \\
Q_f & = \{ q_f \} \text{ and}
\end{align*} \]

\[ \Delta = \begin{cases} 
    a \rightarrow q_a \\
    b \rightarrow q_b \\
    \text{nil} \rightarrow q_{la} \\
    \text{nil} \rightarrow q_{lb} \\
    \text{cons}(q_a, q_{la}) \rightarrow q_{la} \\
    \text{cons}(q_b, q_{lb}) \rightarrow q_{lb} \\
    \text{app}(q_{la}, q_{lb}) \rightarrow q_f 
\end{cases} \]

\[ \begin{align*}
\mathcal{L}(\mathcal{A}, q_{la}) & = \{ \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, \ldots) \} \\
\mathcal{L}(\mathcal{A}, q_{lb}) & = \{ \text{nil}, \text{cons}(b, \text{nil}), \text{cons}(b, \ldots) \} \\
\mathcal{L}(\mathcal{A}, q_f) & = \{ \text{app}(la, lb) \mid la \in \mathcal{L}(\mathcal{A}, q_{la}) \land lb \in \mathcal{L}(\mathcal{A}, q_{lb}) \} 
\end{align*} \]
Tree Automata (II)

Recognized language: \( \mathcal{L}(\mathcal{A}, q) = \{ s \mid s \rightarrow^*_\Delta q \} \)

\[ \mathcal{A} = \langle \mathcal{F}, Q, Q_f, \Delta \rangle \]
with \( Q = \{ q_a, q_b, q_{la}, q_{lb}, q_f \} \)
\( Q_f = \{ q_f \} \) and

\[ \Delta = \begin{cases} 
  a \rightarrow q_a \\
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  \text{app}(q_{la}, q_{lb}) \rightarrow q_f
\end{cases} \]

\[ \mathcal{L}(\mathcal{A}, q_{la}) = \{ \text{nil}, \text{cons}(a, \text{nil}), \text{cons}(a, \ldots) \} \]

\[ \mathcal{L}(\mathcal{A}, q_{lb}) = \{ \text{nil}, \text{cons}(b, \text{nil}), \text{cons}(b, \ldots) \} \]

\[ \mathcal{L}(\mathcal{A}, q_f) = \{ \text{app}(la, lb) \mid la \in \mathcal{L}(\mathcal{A}, q_{la}) \land \ lb \in \mathcal{L}(\mathcal{A}, q_{lb}) \} \]

\[ \mathcal{L}(\mathcal{A}) = \{ s \in \mathcal{T}(\mathcal{F}) \mid s \rightarrow^*_\Delta q \land q \in Q_f \} \]
Tree Automata Completion to approximate $R^*(L)$

Tree automata completion semi-algorithm (particular ARTMC)

- **Input:** a TRS $R$, a tree automaton $A$ and approximation equations $E$
- **Output:** an automaton $A_{R,E}^*$

Theorem 1 (Upper bound)
Given a left-linear TRS $R$, a tree automaton $A$ and a set of equations $E$, if completion terminates on $A_{R,E}^*$, then $L(A_{R,E}^* E) \supseteq R^* (L(A))$.

Theorem 2 (Lower bound)
Given a left-linear TRS $R$, a tree automaton $A$ and a set of equations $E$, if $A$ is $R/E$-coherent then $L(A_{R,E}^*) \subseteq R^* E (L(A))$. 

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Tree Automata Completion to approximate $\mathcal{R}^*(\mathcal{L})$

Tree automata completion semi-algorithm (particular ARTMC)

- **Input:** a TRS $\mathcal{R}$, a tree automaton $\mathcal{A}$ and approximation equations $E$
- **Output:** an automaton $\mathcal{A}_{\mathcal{R},E}^*$

$$\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A}_{\mathcal{R},E}^*) \subseteq \mathcal{R}_E^*(\mathcal{L}(\mathcal{A}))$$  \[with \ V. \ Rusu, \ 2010\]

**Theorem 1 (Upper bound)**

*Given a left-linear TRS $\mathcal{R}$, a tree automaton $\mathcal{A}$ and a set of equations $E$, if completion terminates on $\mathcal{A}_{\mathcal{R},E}^*$ then $\mathcal{L}(\mathcal{A}_{\mathcal{R},E}^*) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$.*

**Theorem 2 (Lower bound)**

*Given a left-linear TRS $\mathcal{R}$, a tree automaton $\mathcal{A}$ and a set of equations $E$, if $\mathcal{A}$ is $R/E$-coherent then $\mathcal{L}(\mathcal{A}_{\mathcal{R},E}^i) \subseteq \mathcal{R}_E^*(\mathcal{L}(\mathcal{A}))$.***
Timbuk provides

- Tree automata completion
- Equational approximations
- Coq checker for completion results
- Beta: CEGAR, Abstract Domains (e.g. integer intervals)

Used for Cryptographic Protocol, Java and JavaScript verification

Demo:

- demo_basic.txt
- demo_reverseBug.txt
What is missing for static analysis of functional languages?

1. Define equations guaranteeing termination of completion

2. Deal with higher order functions

3. Take evaluation strategies into account
   - call by value (e.g. Ocaml) ≈ innermost rewrite strategy
   - call by need (e.g. Haskell) ≈ outermost rewrite strategy + sharing
   - order in pattern matching ≈ priority rewrite strategy

4. Deal with built-in types (int, float, char, strings, ...)

5. Have a modular analysis

6. Have a user friendly way to display/define language annotations ...
What is missing for static analysis of functional languages?

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Equations guaranteeing termination of completion

Intuition: finite set of $E$-equivalence classes $\Rightarrow$ completion terminates

$\mathcal{T}(\mathcal{F})/\equiv_E$ is

$\begin{align*}
  S &\quad u & t \\
  &\quad s \\
  &\quad v & w & k \\
  &\quad \ldots
\end{align*}$
Equations guaranteeing termination of completion

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$\mathcal{T}(\mathcal{F})/\equiv_E$ is finite if $\mathcal{T}(\mathcal{F})/\equiv_E$ is finite.

Example 3 (Equations for the `append` function)

Let $F = \{\text{app}: 2, \text{cons}: 2, \text{nil}: 0, \text{a}: 0\}$. With $E = \{\text{cons}(x, \text{cons}(y, z)) = \text{cons}(x, z)\}$, but $T(F)/\equiv_E$ is not finite!
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$\mathcal{T}(\mathcal{F})/\equiv_E$ is finite set of states $\Rightarrow$ terminating completion

\[ T(F)/\equiv_E \]

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Equations guaranteeing termination of completion (II)

With \( E = \{ \text{cons}(x, \text{cons}(y, z)) = \text{cons}(x, z) \} \), \( \mathcal{T}(\mathcal{F})/\equiv_E \) is not finite!

Infinitely many classes of ill-typed terms

\[
\begin{array}{ccc}
\text{cons} & \text{cons} & \text{cons} \\
\text{a} & \text{a} & \text{a} \\
\end{array}
\]

are all in different classes!

Ill-typed terms incompatible with \( \text{cons} : \alpha \to \alpha \text{ list} \to \alpha \text{ list} \)
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We restrict to well-typed terms $\mathcal{T}(\mathcal{F})^S$
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With $E = \{\text{cons}(x, \text{cons}(y, z)) = \text{cons}(x, z)\}$, $\mathcal{T}(\mathcal{F}) \equiv_E$ is not finite!

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With \( E = \{ cons(x, cons(y, z)) = cons(x, z) \} \), \( T(F) / \equiv_E \) is not finite!

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We restrict to well-typed terms \( T(F)^S \)

With \( E = \{ cons(x, cons(y, z)) = cons(x, z) \} \), \( T(F)^S / \equiv_E \) is not finite!

Infinitely many classes of partially evaluated terms

\[
\begin{array}{c|c|c}
\text{app} & \text{app} & \text{are all in different classes!} \\
\text{nil} & \text{nil} & \ldots \\
\end{array}
\]

Partially evaluated terms
Equations guaranteeing termination of completion (III)

Proposed solution: use $\mathcal{F} = \mathcal{C} \cup \mathcal{D}$ and $E = E^c_C \cup E_R$

- **Defined** and **Constructor** e.g. $\mathcal{D} = \{\text{app}\}$ and $\mathcal{C} = \{a, \text{cons}, \text{nil}\}$
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- **Defined** and **Constructor** e.g. $\mathcal{D} = \{\text{app}\}$ and $\mathcal{C} = \{a, \text{cons}, \text{nil}\}$
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- Define $E_\mathcal{R} = \{l = r \mid l \rightarrow r \in \mathcal{R}\}$ and $E = E^c_\mathcal{C} \cup E_\mathcal{R}$
- Theorem: If $\mathcal{R}$ is sufficiently complete then $\mathcal{T}(\mathcal{F})^S / \equiv_E$ is finite

$\mathcal{R}$ Sufficiently complete:

$\forall s \in \mathcal{T}(\mathcal{F})^S. \exists t \in \mathcal{T}(\mathcal{C})^S.s \rightarrow_{\mathcal{R}}^* t$
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Demo: demo_reverse.txt
A word about Higher-Order functions

Static analysis of higher-order functional programs

- use higher-order formalisms: e.g. HORS [L. Ong, 2006], PMRS [L. Ong & S. Ramsay, 2011]
A word about Higher-Order functions

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- Use first-order formalisms (e.g. tree automata and TRS) with an encoding of higher-order into first-order e.g. [N. Jones, 1987]

Example 4 (Encoding of H.O. functions into TRS)

Use an explicit function application operator '@'.

```
let rec map f l1 =
  match l1 with |
  | [] > [] |
  | h :: t > (@ (f h) :: (map f t)) ; ;
```

Becomes

```
@(@ (map), f), nil) → nil
@(@ (map), f), cons (h, t)) → cons (@ (f, h), @ (@ (map), f), t))
```
A word about Higher-Order functions

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@(@((map, f), nil)) → nil
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A word about Higher-Order functions (II)

<table>
<thead>
<tr>
<th>Is the @$-$encoding enough?</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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</table>

```ocaml
goto f = fun x t -> if x then f t else t
goto nz i = match i with |
| 0 -> false |
| S ( x ) -> true |
goto rec filter p l = match l with |
| [] -> [] |
| h :: t -> if 2 ( p h ) then h :: ( filter p t ) else filter p l |
```
A word about Higher-Order functions (II)

Is the @-encoding enough?

- Authors of H.O. formalisms claim that the @-encoding is too imprecise.
- On H.O. examples of [L. Ong & S. Ramsay, 2011], we obtained similar results with the @-encoding, TRSs, and tree automata completion.
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Example 5 (filter nz on any nat list, results in a list without 0)

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```

Successful on some examples but needs to be investigated further!
A word about evaluation strategies

Example 6 (Terminating with call-by-need but not for call-by-value)

```ml
let rec sumList(x, y) = (x + y) :: sumList(x + y, y + 1);;
let rec nth i (x :: l) = if i <= 0 then x else nth (i - 1) l; ;
let sum x = nth x (sumList(0, 0));;
```

(sum 4) = 10 with call by need and diverges with call-by-value

Completion covers all reachable terms (for all strategies)

\( R^* (\text{((sum 4)}) \subseteq L(A^*, E)) \) contains 10 (and intermediate computations)

Call-by-value ↔ innermost strategy for TRSs

Adapted tree automata completion for innermost strategy [with Y. Salmon]

\( R^* \text{in} ((\text{sum} x)) \subseteq L(A^* \text{in}, E) \) contains no normal form (no result)
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Call-by-value ↔ innermost strategy for TRSs
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\]

\(\text{(sum 4)} = 10\) with call by need and diverges with call-by-value

Completion covers all reachable terms (for all strategies)

\[\mathcal{R}^*((\text{sum 4})) \subseteq \mathcal{L}(A_{\mathcal{R}, E}^*)\] contains 10 (and intermediate computations)

Call-by-value \(\leftrightarrow\) innermost strategy for TRSs

Adapted tree automata completion for innermost strategy [with Y. Salmon]
A word about evaluation strategies

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\text{let rec sumList}(x, y) = (x + y) :: \text{sumList}(x + y, y + 1);
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\]
\[
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Adapted tree automata completion for innermost strategy [with Y. Salmon]

\[
\mathcal{R}_{in}^*((\text{sum } x)) \subseteq \mathcal{L}(A_{R_{in},E}^*) \text{ contains no normal form (no result)}
\]
A word about built-in types

Recall this example:

Example 7 (filter \texttt{nz} on any \texttt{nat list}, results in a list without 0)

\begin{verbatim}
let if2 c t e = match c with
  | true -> t
  | false -> e;

let nz i = match i with
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  | S(x) -> true;;
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Programs usually use machine integers instead of Peano numbers
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**Lattice Tree Automata completion [with Legay, Le Gall, Murat, 2013]**

LTA completion permits to seamlessly plug abstract domains in ARTMC
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\end{align*}
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\[
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\end{align*}
\]

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LTA completion permits to seamlessly plug abstract domains in ARTMC

e.g. integer lists with no zero:

\[
\begin{align*}
\text{cons}(q_i, q_l) & \rightarrow q_l \\
\text{nil} & \rightarrow q_l \\
[-\infty; -1] & \rightarrow q_i \\
[1; +\infty] & \rightarrow q_i \\
\end{align*}
\]
What about the presentation of the results/annotations?

A simple automaton for the $A$ then $B$ lists

Automaton $A_0$

States $q_A$, $q_B$, $q_{nil}$, $q_{lB}$, $q_{lAB}$

Final States $q_{lAB}$

Transitions

- $A \rightarrow q_A$
- $B \rightarrow q_B$
- $nil \rightarrow q_{nil}$
- $\text{cons}(q_B, q_{nil}) \rightarrow q_{lB}$
- $\text{cons}(q_B, q_{lB}) \rightarrow q_{lB}$
- $\text{cons}(q_A, q_{lB}) \rightarrow q_{lAB}$
- $\text{cons}(q_A, q_{lAB}) \rightarrow q_{lAB}$

Any suggestion for a short textual/graphical format is welcome!
What about the presentation of the results/annotations?

Contracts [D. Xu, 2009]

contract rev = {l | ab l} -> {l | ba l};;

where ab and ba are user defined functions discriminating the \( \langle A \text{ then } B \rangle \) lists etc. Contracts can be dynamically or statically checked.

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Liquid Types (and variants) [N. Vazou, P. Rondon, R. Jhala, 2013]

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Liquid types are statically checked.
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\text{contract } \text{rev} = \{ l \mid \text{ab } l \} \rightarrow \{ l \mid \text{ba } l \};;
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where \( \text{ab} \) and \( \text{ba} \) are user defined functions discriminating the \( \ll A \text{ then } B \ll \) lists etc. Contracts can be dynamically or statically checked.

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Two remarks and one question

+ Those techniques prove stronger properties (e.g. quicksort sorts)
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Liquid Types (and variants) [N. Vazou, P. Rondon, R. Jhala, 2013]

\[
\text{rev} :: [a]\langle h \text{ } v \rightarrow h \leq v \rangle \rightarrow [a]\langle h \text{ } v \rightarrow h \geq v \rangle
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+ Those techniques prove stronger properties (e.g. quicksort sorts)
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### Contracts [D. Xu, 2009]

```ocaml
contact rev = {l | ab l} -> {l | ba l};;
```

where `ab` and `ba` are user defined functions discriminating the «A then B lists» etc. **Contracts can be dynamically or statically checked.**

### Liquid Types (and variants) [N. Vazou, P. Rondon, R. Jhala, 2013]

```ocaml
rev :: [a]<{\ h \ v \ -> \ h \ <= \ v}> \rightarrow \ [a]<{\ h \ v \ -> \ h \ >= \ v}
```

**Liquid types are statically checked.**

### Two remarks and one question

+ Those techniques prove stronger properties (e.g. quicksort sorts)
+ (Co)-Domains annotations are given by the user (we infer them)
  - Can we define user friendly ”language annotations” close to types?
Conclusion

1. Define equations guaranteeing termination of completion ✓ ❨ systemctl ❩
2. Deal with higher order functions ❨ systemctl ❩
3. Take evaluation strategies into account
   - call by value (e.g. Ocaml) ≈ innermost rewrite strategy ✓
   - call by need (e.g. Haskell) ≈ outermost rewrite strategy + sharing
   - order in pattern matching ≈ priority rewrite strategy
4. Deal with built-in types ✓
5. Modularity of the analysis
6. User friendly way to display/define language annotations . . .
Further research

- Find a translation from OCaml to TRS s.t.
  - Typing is preserved
  - Higher-order functions can be encoded
  - OCaml pattern matching exhaustivity $\Rightarrow$ TRS sufficient completeness

Example 8 (sumList is not sufficiently complete)

```ocaml
let rec sumList(x, y) = (x + y) :: sumList(x + y, y + 1);

let rec nth i (x :: l) =
  if i < 0
  then x
  else nth(i - 1) l;

let sum x = nth x(sumList(0, 0));
```
Further research

- Find a translation from OCaml to TRS s.t.
  - Typing is preserved
  - Higher-order functions can be encoded
  - OCaml pattern matching exhaustivity $\Rightarrow$ TRS sufficient completeness

- Find other criteria guaranteeing finiteness of $\mathcal{T}(\mathcal{F})/\equiv_E$ or $\mathcal{T}(\mathcal{C})/\equiv_E$
Further research

- Find a translation from OCaml to TRS s.t.
  - Typing is preserved
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- Find other criteria guaranteeing finiteness of $T(F)/=E$ or $T(C)/=E$

  e.g. Discard the "sufficient completeness" requirement

Example 8 (sumList is not sufficiently complete)

```ocaml
let rec sumList(x, y) = (x+y) :: sumList(x+y, y+1);
let rec nth i (x :: l) = if i <= 0 then x else nth (i-1) l;
let sum x = nth x (sumList(0,0));
```
Completion algorithm

Tree automata completion principle

1. complete $\mathcal{A}$ with new transitions into $\mathcal{A}^1_R, \mathcal{A}^2_R, \ldots$
Completion algorithm

Tree automata completion principle

1. Complete $A$ with new transitions into $A_1^R, A_2^R, \ldots$

   $\forall l \rightarrow r \in R, \forall q \in Q, \forall \sigma : X \rightarrow Q$:

   $$l\sigma \xrightarrow{R} r\sigma$$

   $$A_i^R \star \\
   q$$
Completion algorithm

Tree automata completion principle

1. Complete $A$ with new transitions into $A_1^R, A_2^R, \ldots$

   $\forall l \rightarrow r \in R, \forall q \in Q, \forall \sigma : \mathcal{X} \rightarrow Q$:

   $$l\sigma \xrightarrow{\mathcal{R}} r\sigma$$

   $$A_i^R \xrightarrow{*} A_{i+1}^R$$

   $$q \xleftarrow{A_{i+1}} q'$$
Completion algorithm

Tree automata completion principle

1. complete $A$ with new transitions into $A_{R}^{1}, A_{R}^{2}, \ldots$
   \[ \forall l \rightarrow r \in R, \forall q \in Q, \forall \sigma : X \mapsto Q: \]

   \[
   \begin{array}{c}
   l\sigma \\
   \overset{R}{\longrightarrow} \\
   r\sigma
   \end{array}
   \begin{array}{c}
   A_{R}^{i}
   \end{array}
   \begin{array}{c}
   \ast
   \end{array}
   \begin{array}{c}
   \ast
   \end{array}
   \begin{array}{c}
   A_{R}^{i+1}
   \end{array}
   \begin{array}{c}
   q
   \end{array}
   \begin{array}{c}
   \overset{A_{i+1}}{\leftarrow}
   \end{array}
   \begin{array}{c}
   q'
   \end{array}
   \]

2. use approximation equations of $E$ to (possibly) converge on $A_{R,E}^{*}$

Genet & Salmon (IRISA)
Completion algorithm (II)

Definition 9 (Set $E_C^e$ of contracting equations)
The set of well-sorted equations $E_C^e$ is contracting if its equations are of the form $u = u|_p$ with $u$ linear and $p \neq \Lambda$ and if the set of normal forms of $T(C)^S$ w.r.t. the TRS $\overrightarrow{E_C^e} = \{u \rightarrow v \mid u = v \in E_C^e\}$ is finite.

R/E-coherence
Languages recognized by states of $A$ ($\epsilon$-free) are $E$-equivalent terms.