Characterizing NC^k

from words to trees and back to words

Guillaume Bonfante, Reinhard Kahle, Jean-Yves Marion
and Isabel Oitavem

Implicit Computational Complexity and applications:
Resource control, security, real-number computation

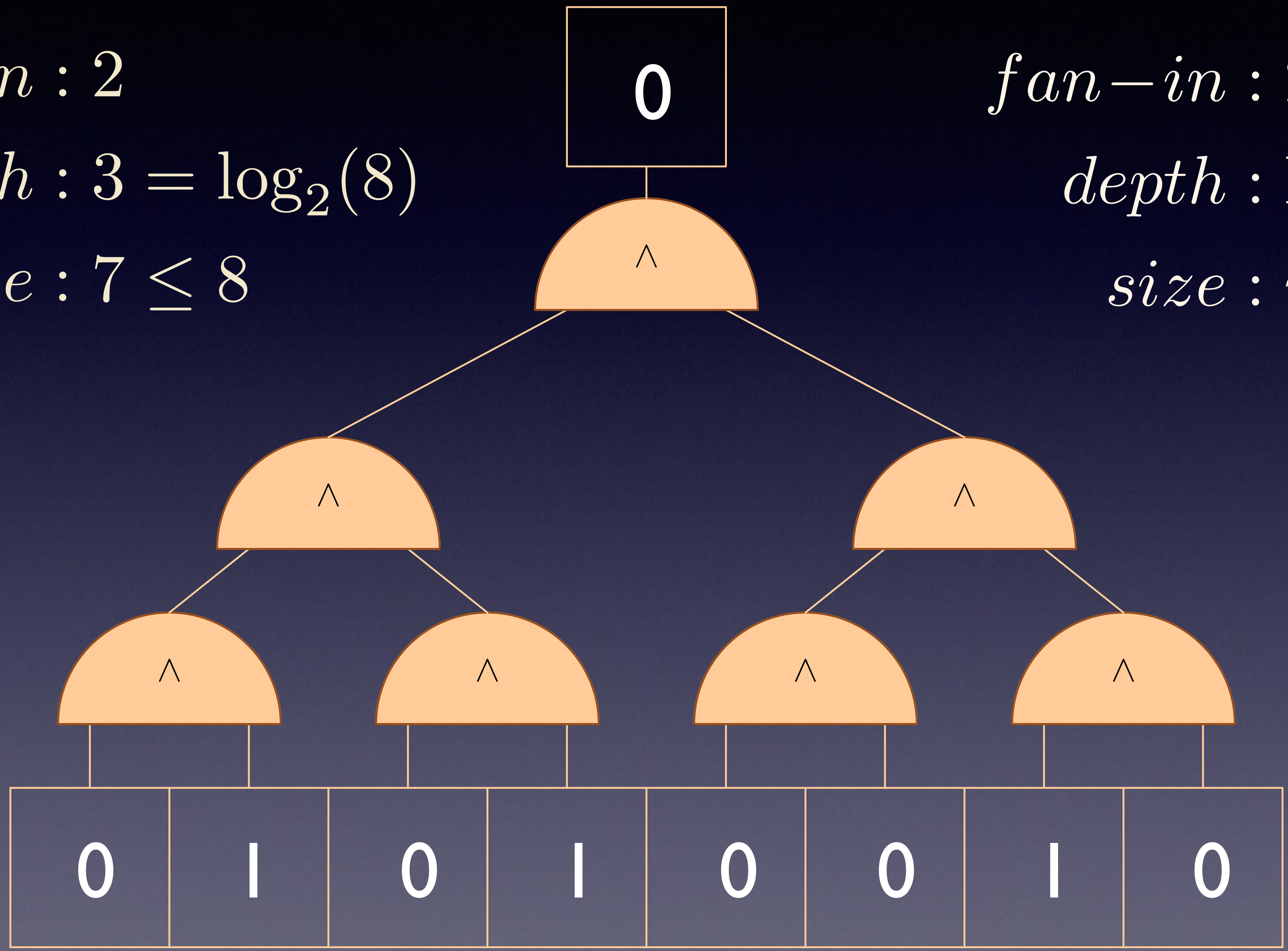
Shonan Village - 2013

Is there a 0?

$fan\text{-}in : 2$

$depth : 3 = \log_2(8)$

$size : 7 \leq 8$



$fan\text{-}in : 2$

$depth : \log_2(n)$

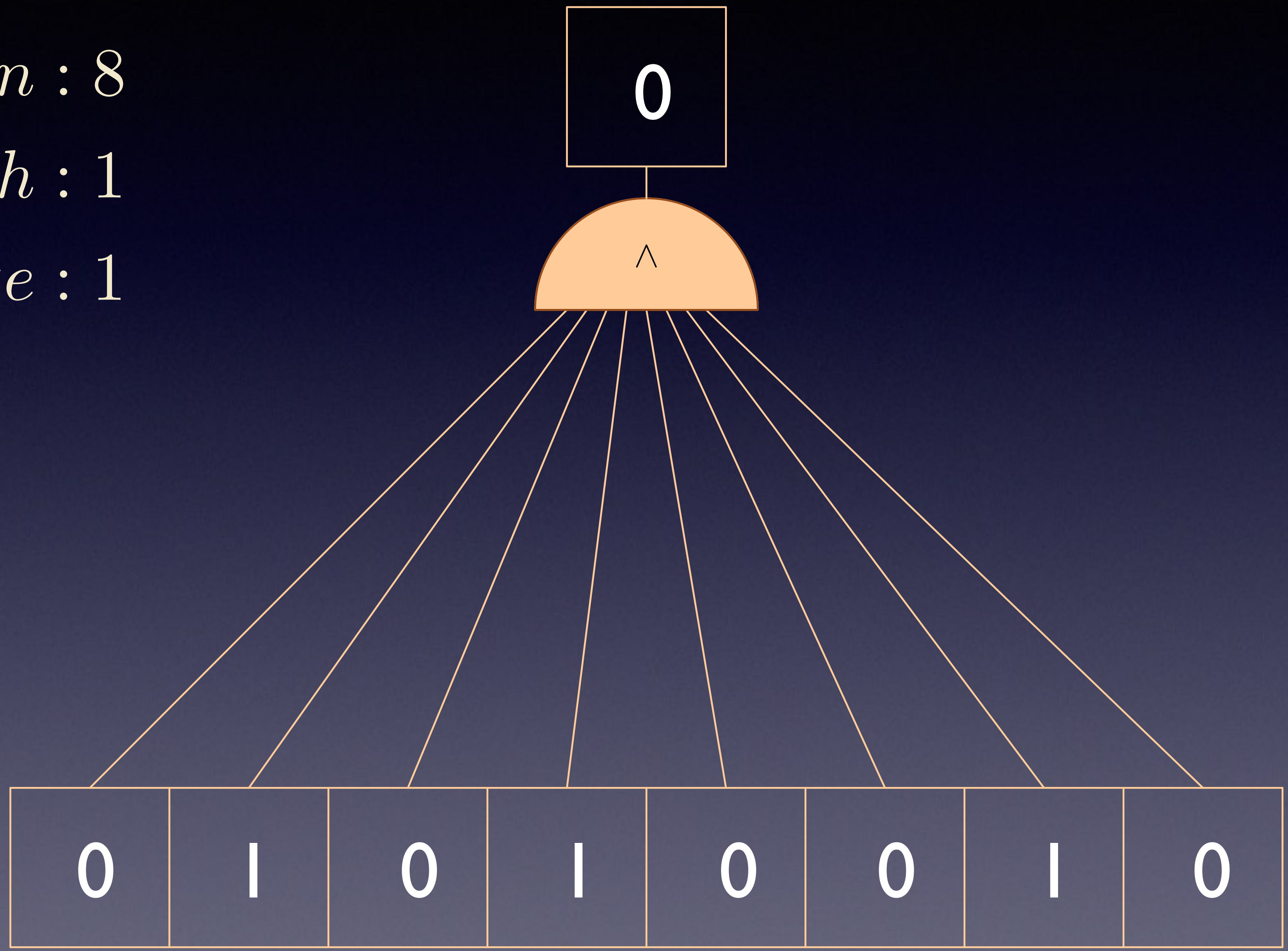
$size : n - 1$

Is there a 0?

$fan\text{-}in : 8$

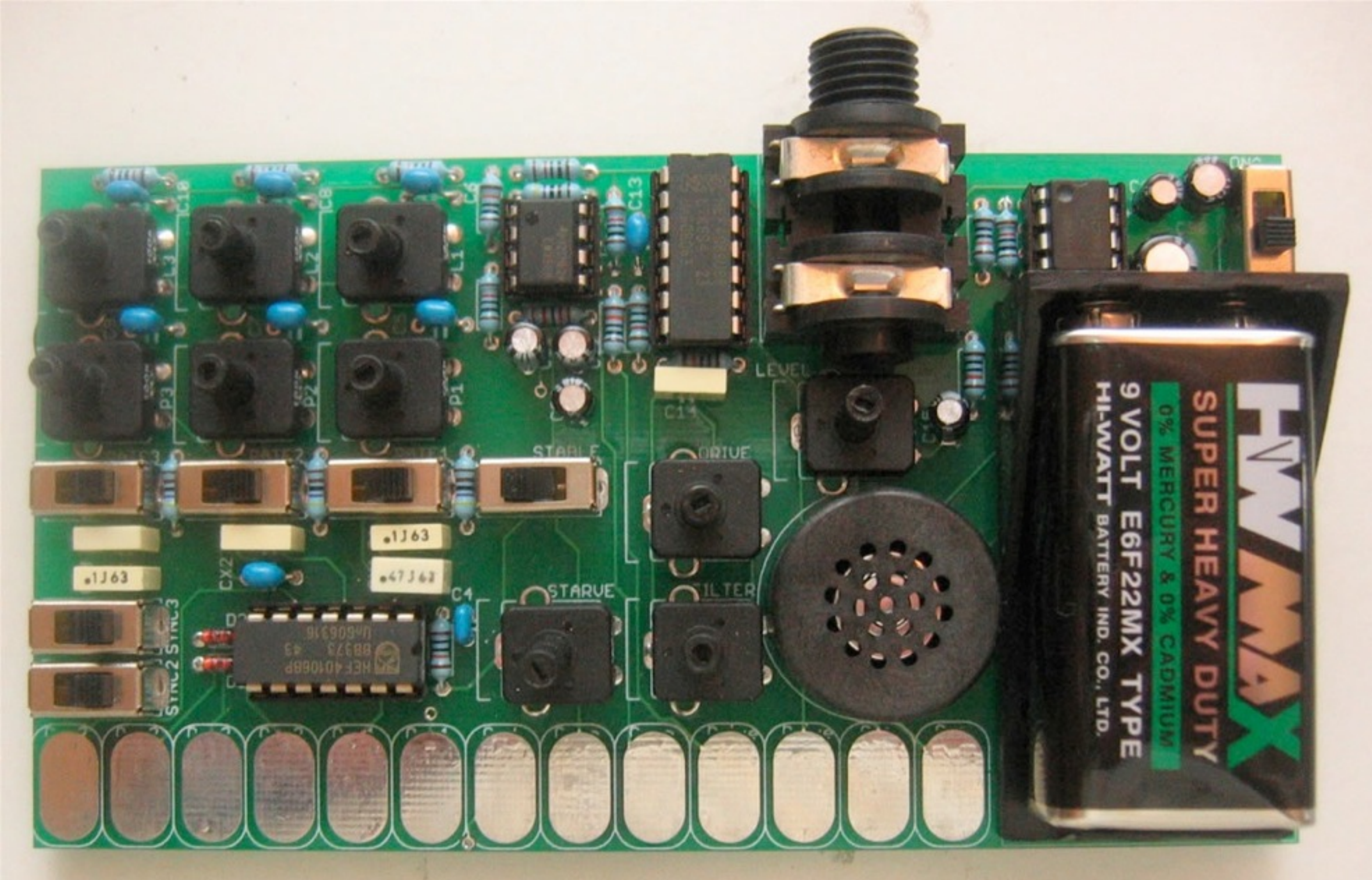
$depth : 1$

$size : 1$



Circuits NC^k

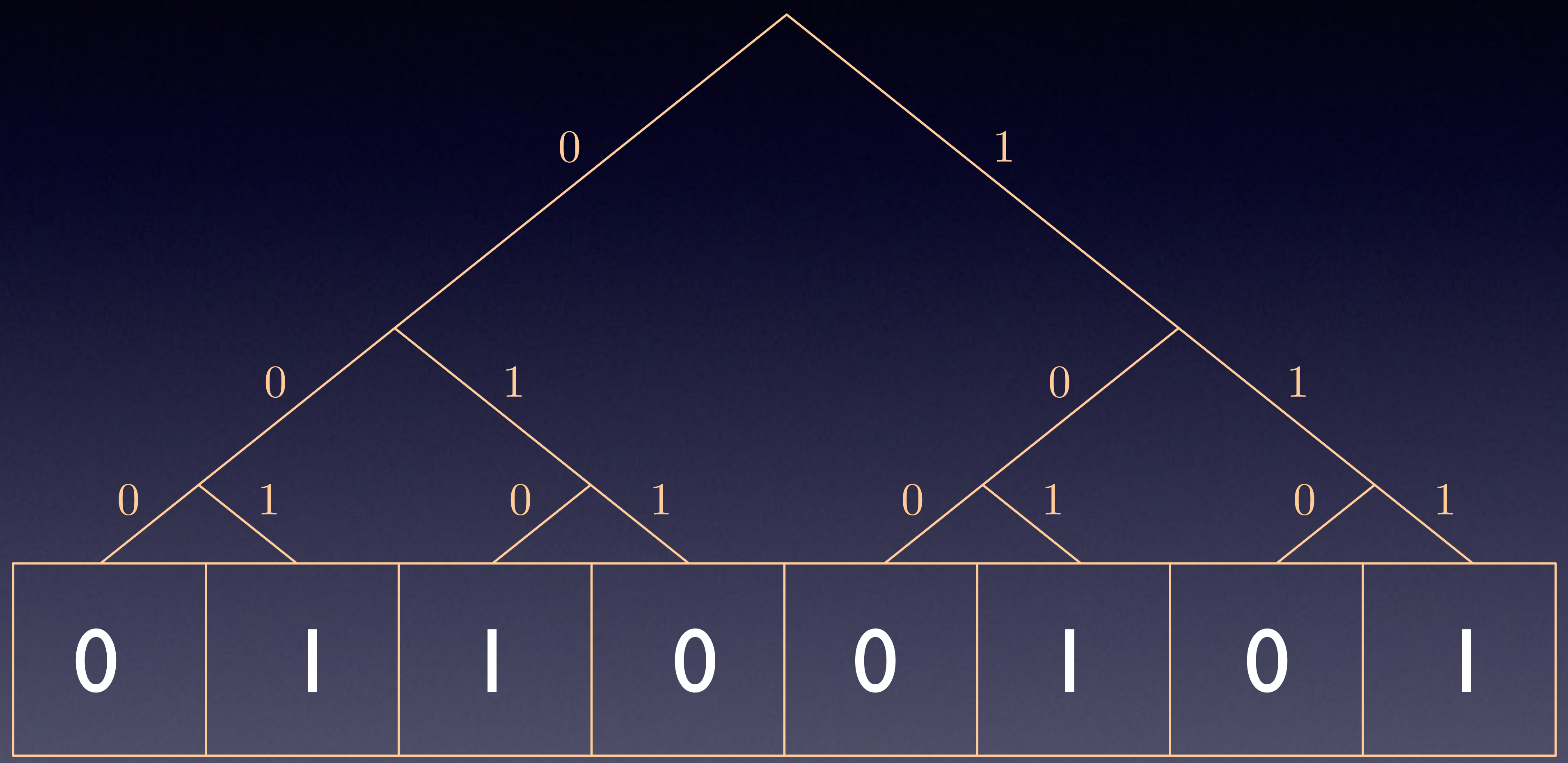
- For $k \geq 1$, NC^k is the class of uniform boolean circuits such that:
  - constant fan-in,
  - polynomial size (w.r.t. the size of inputs)
  - depth is bounded by $O(\log^k(n))$
  - LOGSPACE-Uniform


LEVEL
DRIVE
STABLE
STARVE
FILTER
SYNC2
SYNC3
CX2
C4
C13
C14
.1J63
.47J63
HEF40106BP
SUPER HEAVY DUTY
0% MERCURY & 0% CADMIUM
9 VOLT E6F22MX TYPE
HI-WATT BATTERY IND. CO., LTD.

Implicit computational complexity

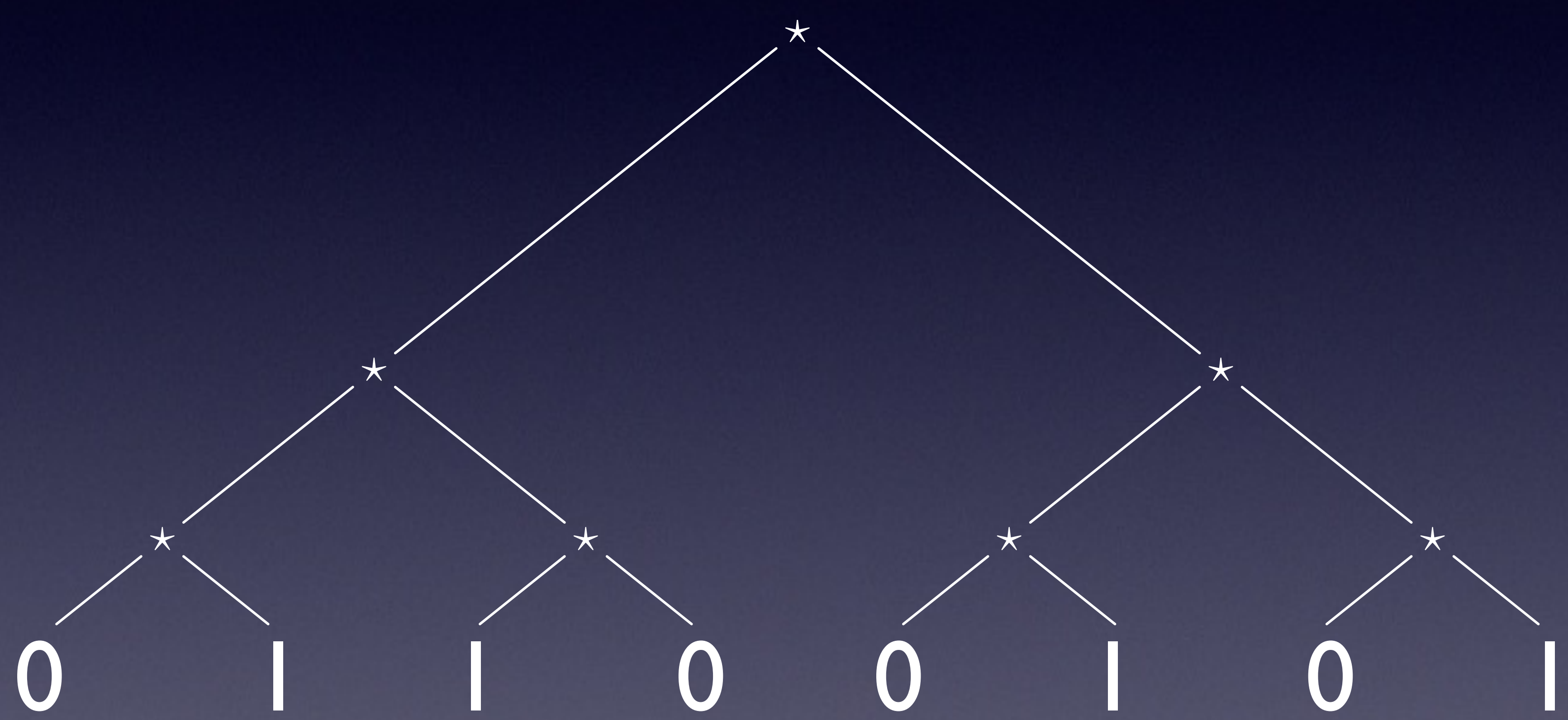
- A characterization of the classes NC^k
  - machine independent
  - without a priori bounds (cf. Clote, Cobham)
  - Infinite structures (cf Cook, Buss)
  - Logical approaches (cf. Mogbil)
  - Recursion Theory (Leivant, Bloch, Oitavem)

Computing on trees



Computing on trees

0	1	1	0	0	1	0	1



$$((0 \star 1) \star (1 \star 0)) \star ((0 \star 1) \star (0 \star 1))$$

Basic functions

$$\begin{aligned}
\mathsf{d}_0(c) = \mathsf{d}_1(c) &= c, & c \in \{0,1\} \\
\mathsf{d}_0(t_0 \star t_1) &= t_0, \\
\mathsf{d}_1(t_0 \star t_1) &= t_1, \\
\mathsf{cond}(c, x_0, x_1, x_\star) &= x_c, & c \in \{0,1\} \\
\mathsf{cond}(t_0 \star t_1, x_0, x_1, x_\star) &= x_\star
\end{aligned}$$

A characterization of NC by Leivant

Ramified Schematic Recurrence

$$f(c, \vec{u}; \vec{x}) = g_c(\vec{u}; \vec{x})$$

$$f(t_0 \star t_1, \vec{u}; \vec{x}) = g_\star(\vec{u}; f(t_0, \vec{u}; h_1(\vec{x})), \ldots, f(t_0, \vec{u}; h_d(\vec{x})),$$
$$f(t_1, \vec{u}; h'_1(\vec{x})), \ldots, f(t_1, \vec{u}; h'_{d'}(\vec{x})))$$

Theorem (Leivant):

RSR characterize NC-computable functions

Mutual In-Place Recursion (MIP)

Definition

$(f_i)_{i \in I}$ is defined by MIP if for all i:

$$f_i(t_0 \star t_1, \vec{u}) = f_j(t_0, \sigma_{i,0}(t_0 \star t_1, \vec{u})) \star f_k(t_1, \sigma_{i,1}(t_0 \star t_1, \vec{u}))$$

$$f_i(c, \vec{u}) = g_{i,c}(\vec{u})$$

$g_{i,c}(\vec{u})$ range in $\{0, 1\}$

$\sigma_{i,b}$ is a destructor , $b = 0, 1$

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Computing the palyndrome

$$
\begin{aligned}
f_0(t_0 \star t_1) &= f_1(t_0, t_1) \star f_1(t_1, t_0) \\
f_0(c) &= 1 \\
f_1(t_0 \star t_1, u) &= f_1(t_0, \mathsf{d}_1(u)) \star f_1(t_1, \mathsf{d}_0(u)) \\
f_1(c, c') &= c == c'
\end{aligned}
$$

with

$$
\begin{aligned}
\mathsf{d}_0(t_0 \star t_1) &= t_0 \\
\mathsf{d}_1(t_0 \star t_1) &= t_1
\end{aligned}
$$

Computing the palyndrome

$$f_0(((0 \star 1) \star (1 \star 0)) \star ((0 \star 1) \star (1 \star 0)))$$

$$= ((1 \star 1) \star (1 \star 1)) \star ((1 \star 1) \star (1 \star 1))$$

$$\begin{aligned} f(t_0 \star t_1, u) &= f(t_0, \mathsf{d}_0(u)) \star f(t_1, \mathsf{d}_1(u)), \\ f(c, u) &= \wedge(c, \wedge(\mathsf{d}_0(u), \mathsf{d}_1(u))); \end{aligned}$$

$$\begin{aligned} \mathsf{AND}(t_0 \star t_1) &= f(t_0, t_0 \star t_1) \star f(t_1, t_0 \star t_1), \\ \mathsf{AND}(c) &= c \end{aligned}$$

Computing the palyndrome

Claim:

$$\text{AND}^{\log_2(n)}(t) = (((((b \star \cdots) \cdots)$$

with $b = 0 \text{ iff } t \text{ contains a } 0$

Thus

$$\text{AND}^{\log_2(n)}(f_0(t)) = (((((b \star \cdots) \cdots)$$

with $b = 1 \text{ iff } t \text{ is a palyndrome}$

Time iteration

$$f(t_1' \star t_1'', t_2, \ldots, t_k, s, \vec{u}) = h(f(t_1', t_2, \ldots, t_k, s, \vec{u}), \vec{u})$$

$$f(c_1, t_2' \star t_2'', t_3, \ldots, t_k, s, \vec{u}) = f(s, t_2', t_3 \ldots, t_k, s, \vec{u})$$

$$\vdots$$

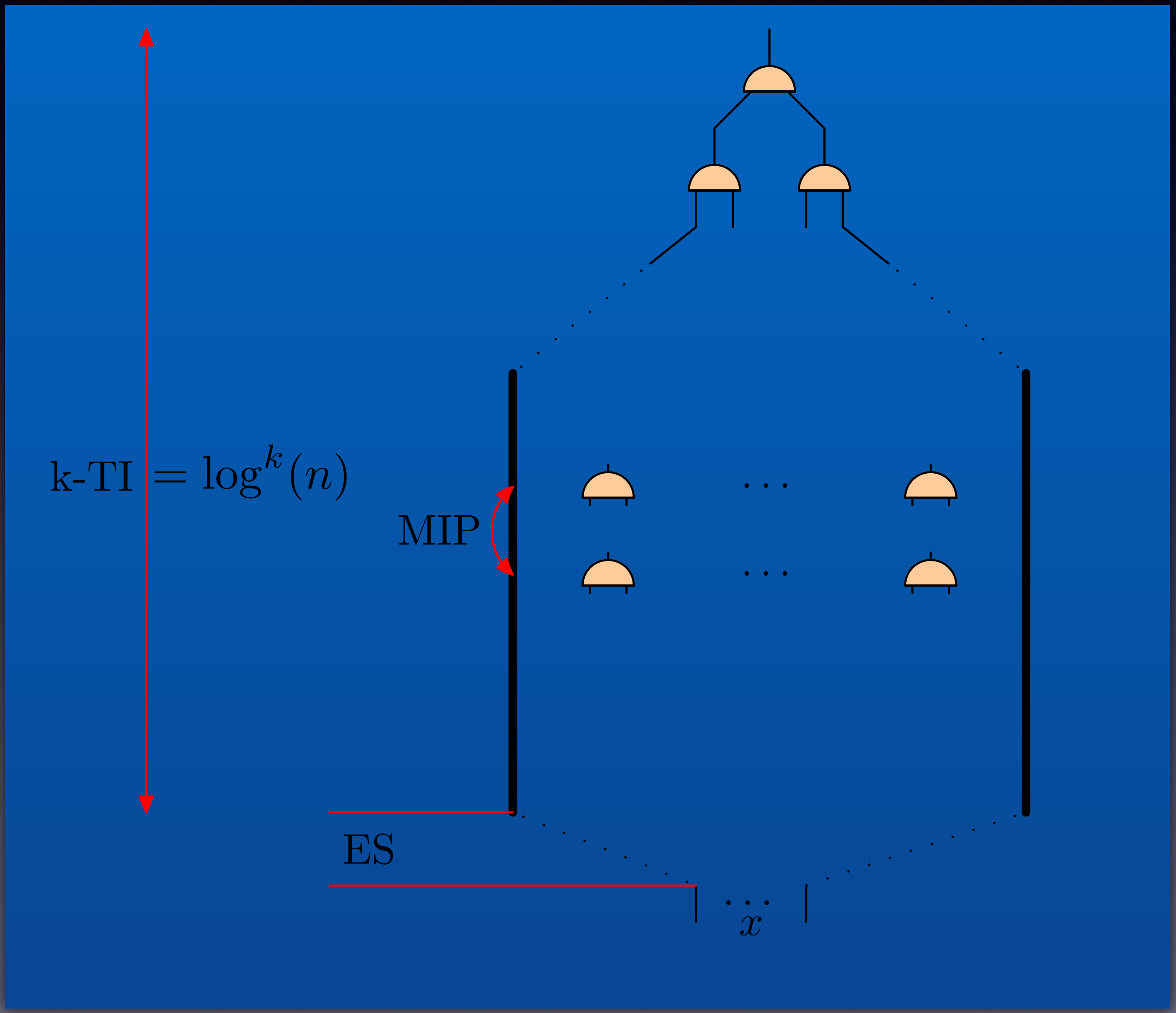
$$f(c_1, \ldots, c_{i-1}, t_i' \star t_i'', t_{i+1}, \ldots, t_k, s, \vec{u}) = f(c_1, \ldots, c_{i-2}, s, t_i', t_{i+1}, \ldots, t_k, s, \vec{u})$$

$$\vdots$$

$$f(c_1, \ldots, c_k, s, \vec{u}) = g(s, \vec{u})$$

MIP+TI^k = NC^k

Theorem (BKMO): MIP+TI^k = NC^k



Rational Bitwise Equations

$$f(w_0, \ldots, w_k) = w$$

$$|w_0| = |w|$$

$$w[p] = h(\phi_0(p), w_{e_1}[\phi_1(p)], \ldots, w_{e_m}[\phi_m(p)])$$

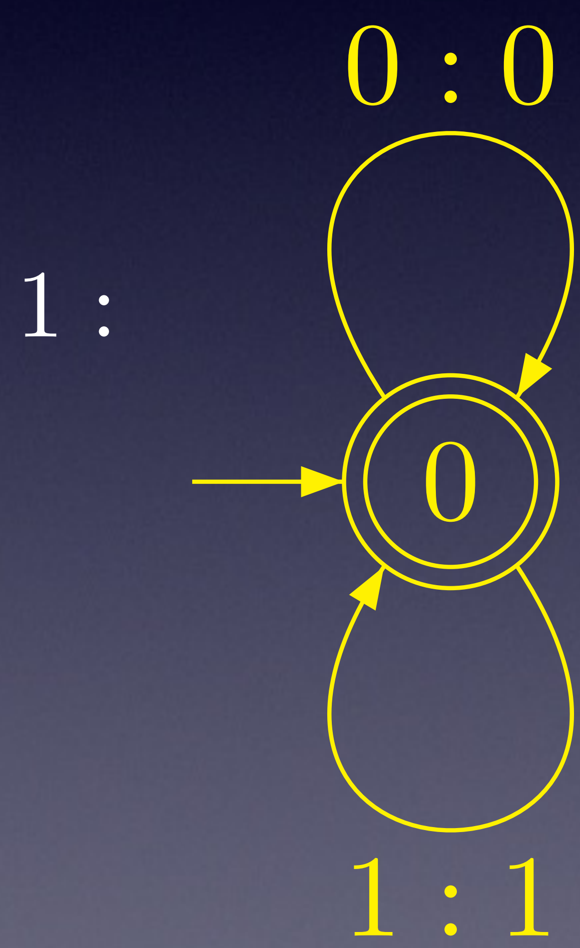
$\phi_0, \ldots, \phi_m$ some functional transducers on $\{0,1\}^*$

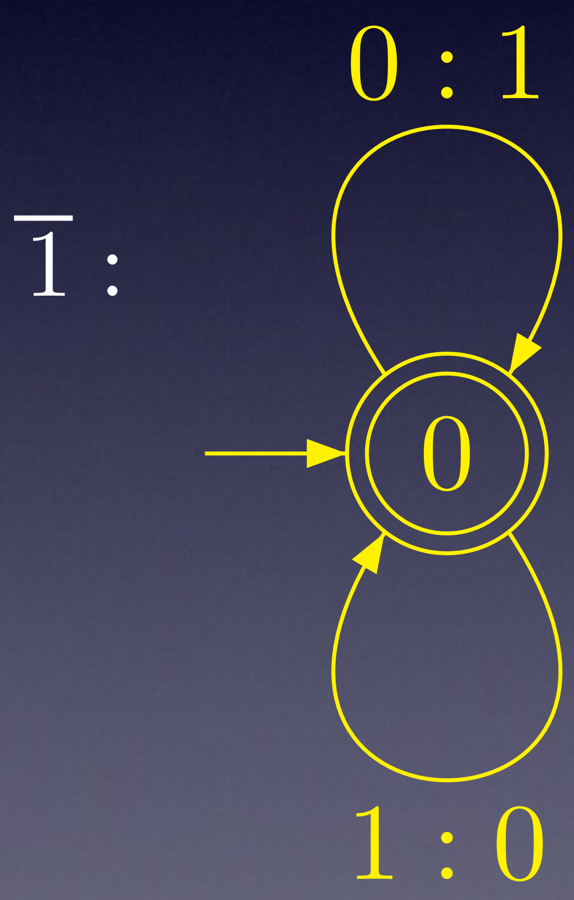
h is a finite mapping in $\{0,1\}$

On the palyndrome

$$f(w_0) = w$$

$$w[p] = \text{XNOR}(w_0[1(p)], w_0[\overline{1}(p)])$$



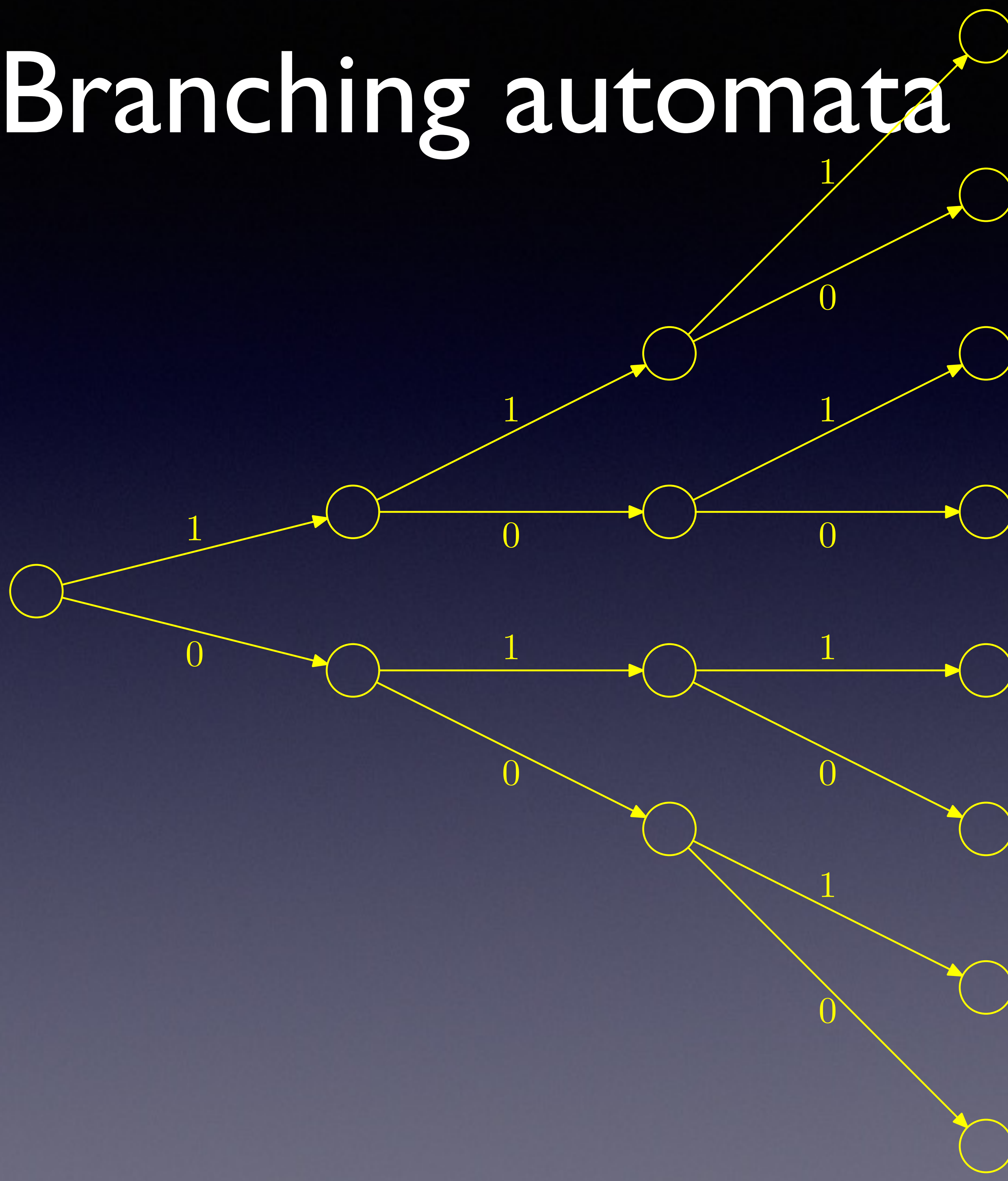


MIP = RBE

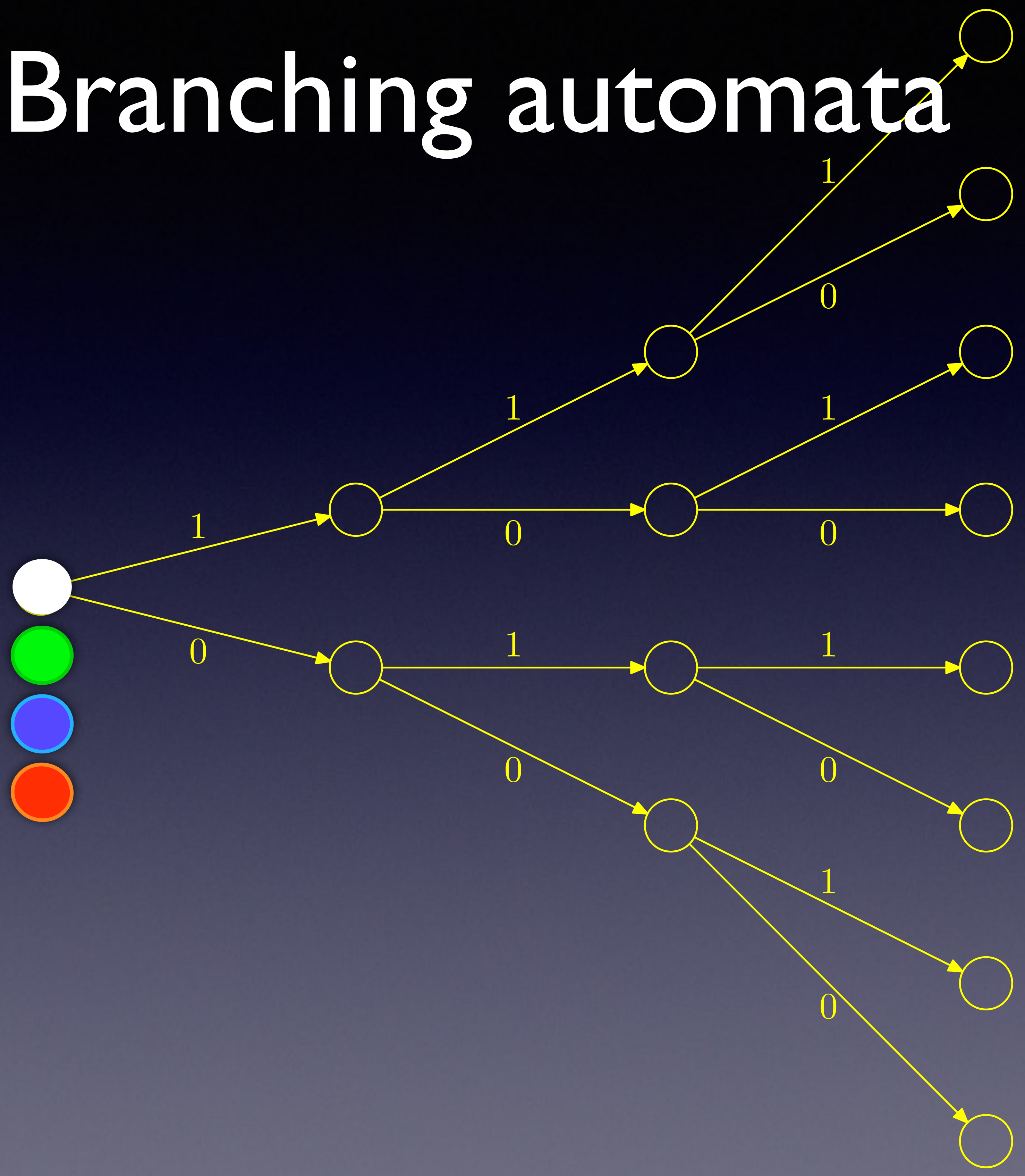
Theorem : MIP = RBE

A proof?

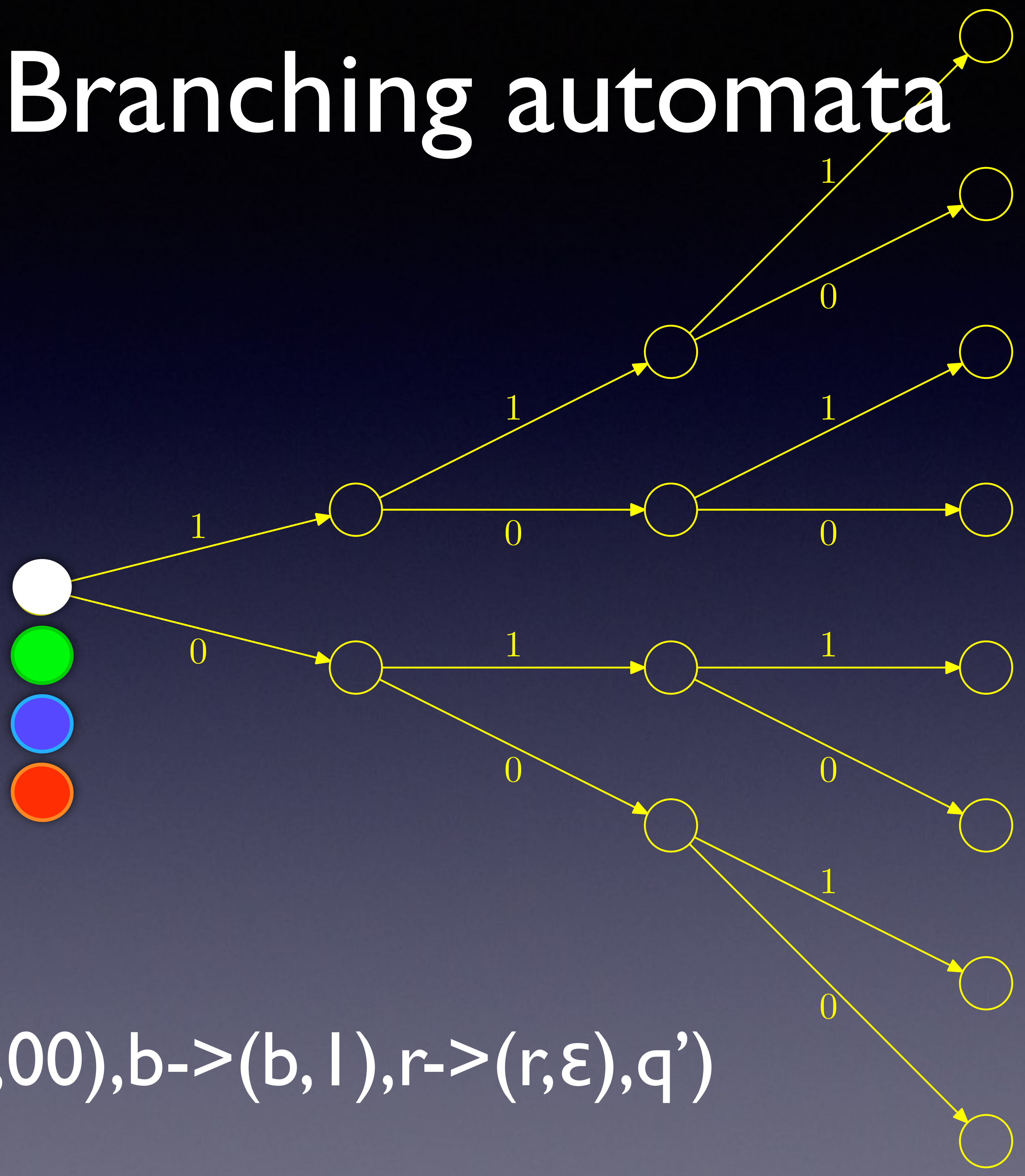
Branching automata



Branching automata

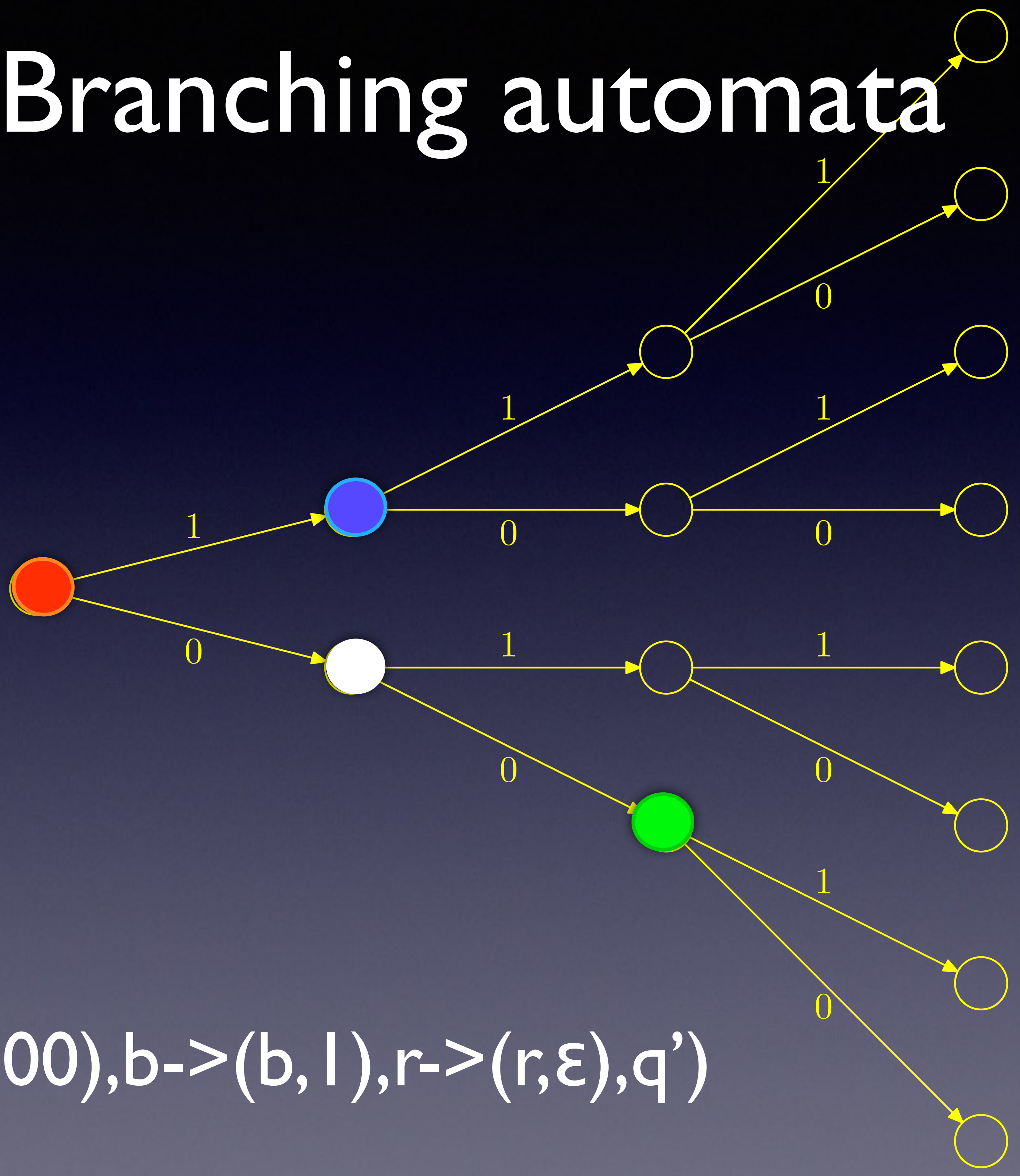


Branching automata



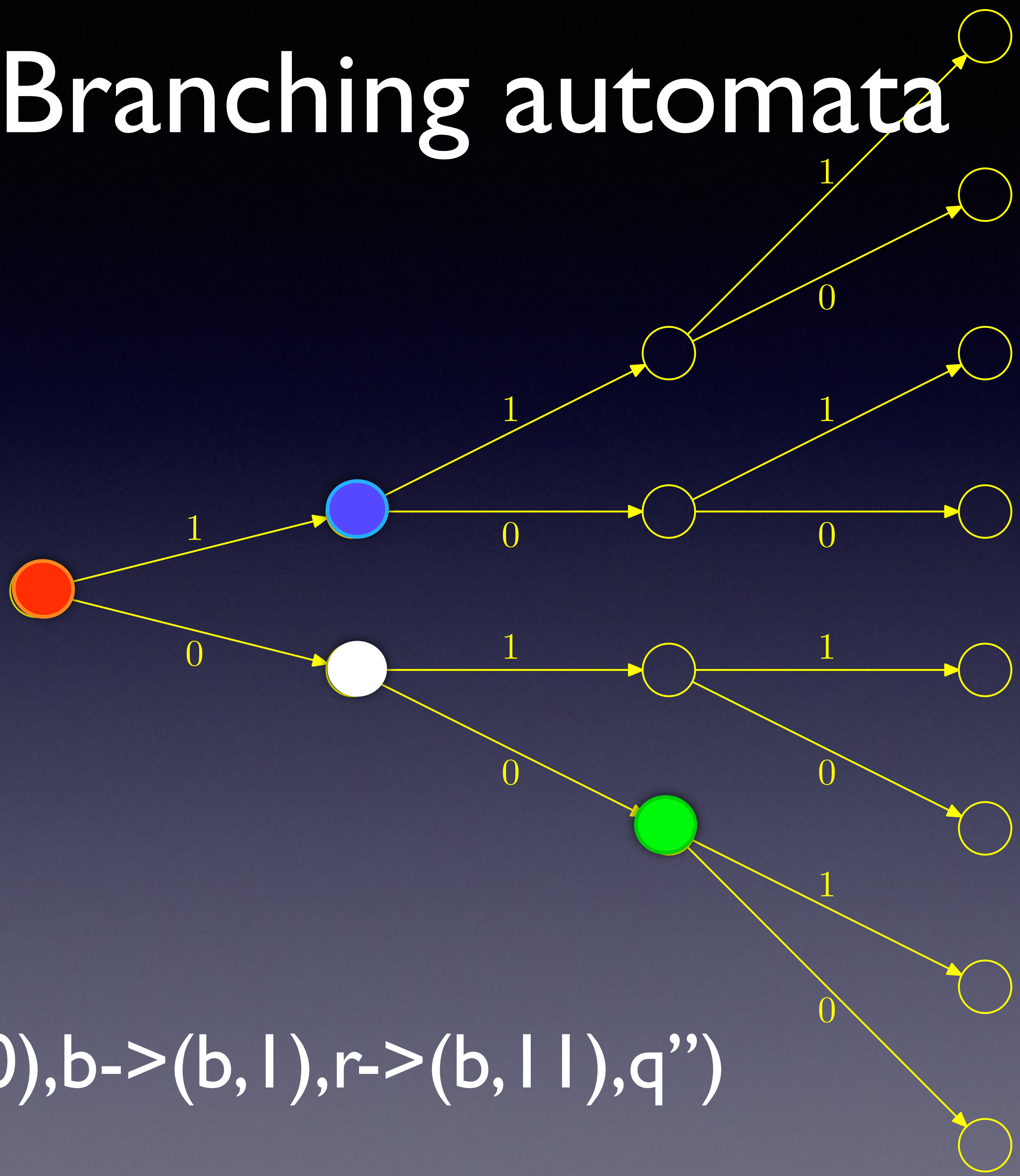
(q0,0,g->(r,00),b->(b,1),r->(r,ε),q')

Branching automata



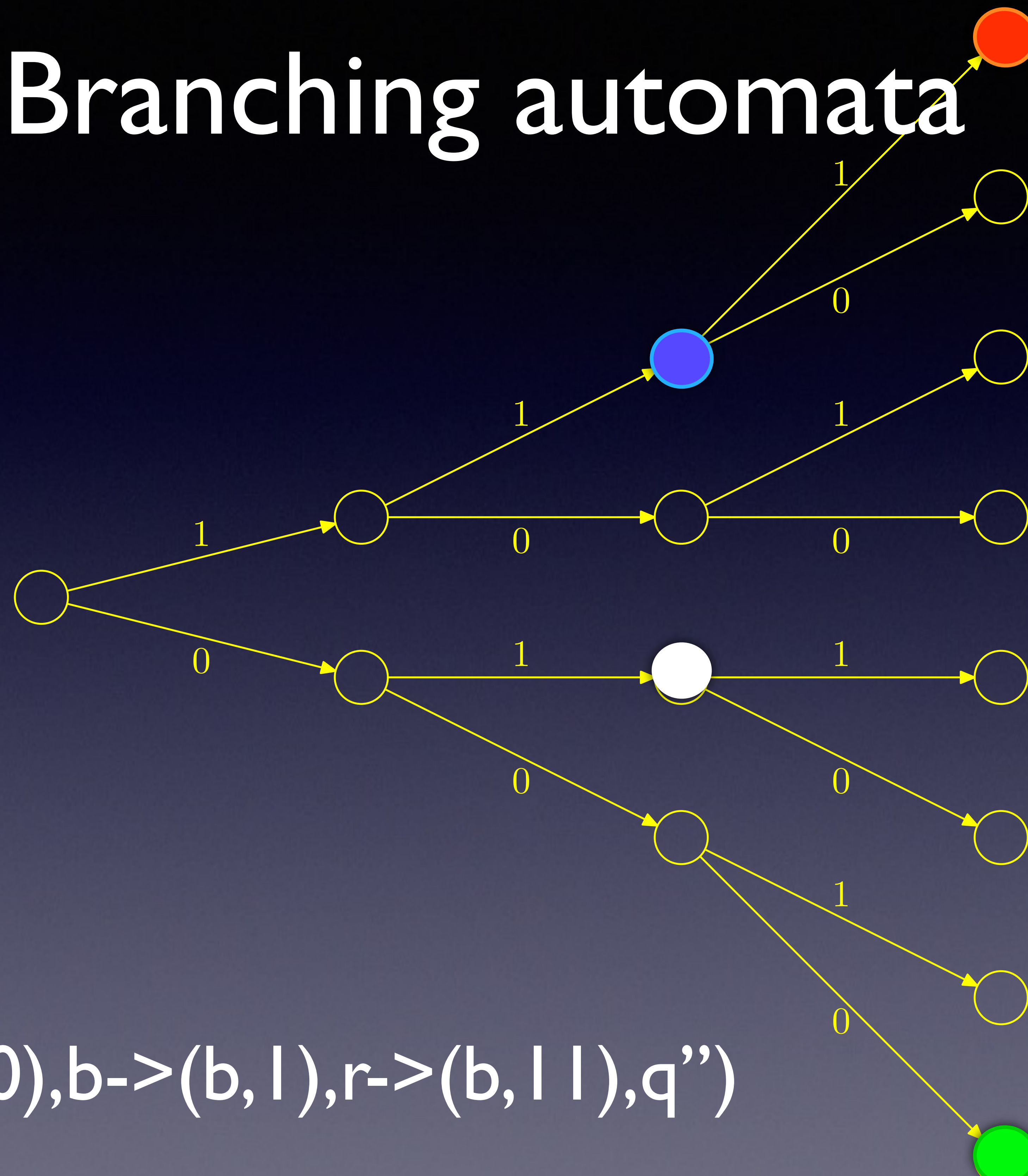
(q0,0,g->(r,00),b->(b,1),r->(r,ε),q')

Branching automata



(q’,1,g->(g,0),b->(b,1),r->(b,11),q”)

Branching automata



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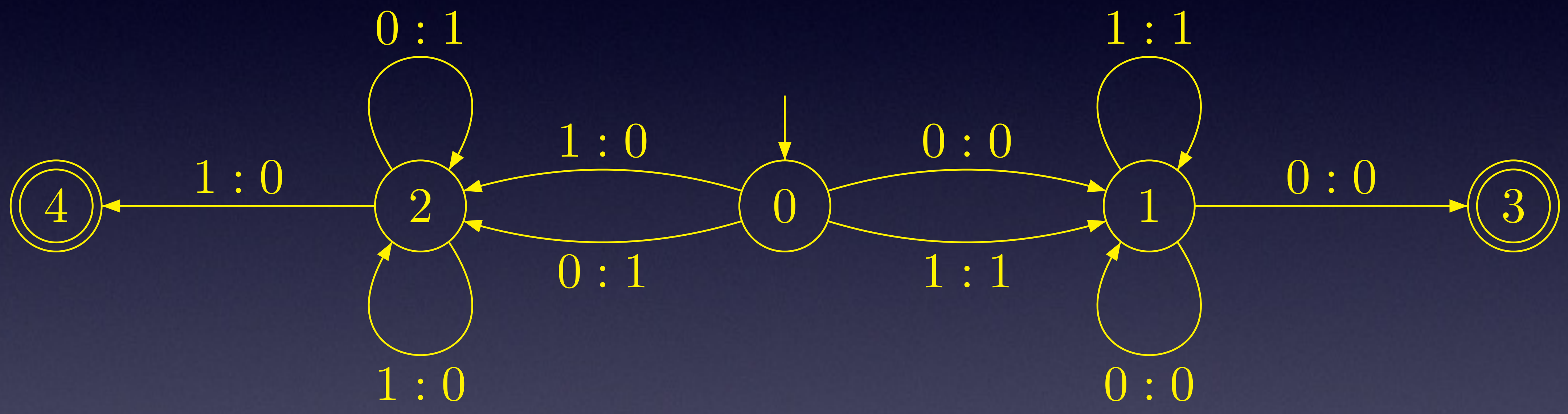
A proof?



Functional transducers

MIP == RBE iff FT == BA

Theorem (Elgot and Mezei, 1965):
Rational functions are the composition of a sequential and a co-sequential function

Exercice:
compute the finite transducer above with a BA

Conclusion

- A small step to NC^k
- A longer way to NC^0
- An even longer way to AC^k