ALPHA-STABLE MATRIX FACTORIZATION
SUPPLEMENTARY DOCUMENT

Umut Şimşekli, Antoine Liutkus, Ali Taylan Cemgil

ABSTRACT

This document is a supplementary material for the paper “Alpha-Stable Matrix Factorization”. In this document, we describe the details of the numerical method for computing the α-stable probability density functions.

1. EVALUATING THE α-STABLE DENSITIES

In order to evaluate the stable densities, we have developed a numerical method, inspired from the numerical methods proposed in [1]. For evaluating the stable densities, we make use of the power series representation of the stable distribution [2, 3]:

$$p(x) = \frac{1}{\pi x} \sum_{k=1}^{\infty} V_k \tag{1}$$

where

$$V_k = \begin{cases} \frac{\Gamma(k\alpha+1)}{\Gamma(k+1)} (-1)^k (x^{-\alpha})^k \sin \frac{k\pi}{2}(\gamma - \alpha) & 0 < \alpha < 1 \\ \frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)} (-1)^k x^k \sin \frac{k\pi}{2\alpha}(\gamma - \alpha) & 1 < \alpha < 2 \end{cases}$$

where the scale parameter is set to $a = (1 + \mu^2 \tan^2(\pi\alpha/2))^{1/(2\alpha)}$. The main idea in our method is to locate the indices k where the terms V_k make the major contribution to the sum. In this section we describe the method for the case when $0 < \alpha < 1$. It is straightforward to extend the method for $1 < \alpha < 2$.

In order to locate the region of the contributive terms, we first determine the index of the most contributive term $k^\star$. We define an envelope V_k^{env} for V_k by discarding the terms $(-1)^k$ and $\sin(\cdot)$, where we have $V_k^{env} \geq |V_k(\cdot)|$ for all k. Then we approximate the gamma functions in the envelope term by using Stirling’s formula:

$$\log \Gamma(x+1) \approx (x + \frac{1}{2}) \log x - x + \frac{1}{2} \log 2\pi$$

For $0 < \alpha < 1$ we obtain:

$$\log V_k^{env} \approx (k\alpha - k) \log k + k\alpha \log \frac{\alpha}{x} + k - k\alpha$$

This equation allows us to find an approximate mode $k^\star$ by solving $(d \log V_k^{env})/(dk) = 0$ as if k is continuous. The lower and the upper bounds for k is found by evaluating the terms that are at either side of $k^\star$ until their contributions are negligible. For $0 < \alpha < 1$ we obtain the mode at:

$$k^\star = \exp(\frac{\alpha \log \frac{\alpha}{x}}{1 - \alpha}). \tag{2}$$

2. REFERENCES

[1] P. K. Dunn and G. S. Smyth, “Series evaluation of tweedie exponential dispersion model densities,” *Stats. & Comp.*, vol. 15, pp. 267–280, 2005.

[2] Harald Bergström, “On some expansions of stable distribution functions,” *Arkiv för Matematik*, vol. 2, no. 4, pp. 375–378, 1952.

[3] Ercan Engin Kuruoglu, *Signal processing in α-stable noise environments: a least lp-norm approach*, Ph.D. thesis, University of Cambridge, 1999.