Labelled Tableaux for Proofs and Models in BI logics

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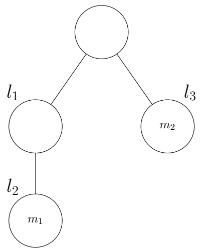
Separation Logic

- Introduced by Reynolds&O'Hearn 01 to model:
  - a **resource** logic
  - properties of the memory space (cells)
  - aggregation of cells into wider structures
- Combines:
  - classical logic connectives: $\wedge$, $\vee$, $\rightarrow \ldots$
  - multiplicative conjunction: $*$
- Defined via Kripke semantics extended by:

$$m \Vdash A * B \quad \text{iff} \quad \exists a, b \text{ s.t. } a, b \triangleright m \wedge a \Vdash A \wedge b \Vdash B$$

Separation models

- Decomposition $a, b \triangleright m$ interpreted in various structures:
  - stacks in pointer logic (Reynolds&O'Hearn&Yang 01), $a \uplus b \subseteq m$
  - but also $a \uplus b = m$ (Calcagno&Yang&O'Hearn 01)
  - trees in spatial logics (Calcagno&Cardelli&Gordon 02) $a \mid b \equiv m$
  - resource trees in BI-Loc (Biri&Galmiche07)



- Additive → can be Boolean (pointwise) or intuitionistic

Bunched Implication logic (BI)

- Introduced by Pym 99, 02
  - **intuitionistic** logic connectives: $\wedge$, $\vee$, $\rightarrow$ $\ldots$
  - multiplicative connectives of MILL: $*$, $-\!*$, I
  - sound and complete bunched sequent calculus, with cut elimination
- Kripke semantics (Pym&O'Hearn 99, Galmiche&Mery&Pym 02)
  - partially ordered partial commutative monoids $(\mathcal{M}, \circ, \leqslant)$
  - intuitionistic Kripke semantics for additives
  - relevant Kripke semantics for multiplicatives
  - sound and complete Kripke semantics for BI

BI Logic continued

- In BI, decomposition interpreted by $a \circ b \leqslant m$:
  - resource monoids (partial, ordered)
  - intuitionistic additives and relevant multiplicatives
- BI has proof systems:
  - cut-free bunched sequent calculus (Pym 99)
  - resource tableaux (Galmiche&Mery&Pym 05)
  - inverse method (Donnelly&Gibson et al. 04)
- Additives are intuitionistic in BI, mostly Boolean in Separation Logic

Boolean BI (BBI)

- Loosely defined by Pym as $\mathsf{BI} + \{\neg\neg A \to A\}$
  - no known pure sequent based proof system
  - Kripke semantics by relational monoids (Larchey&Galmiche 06)
  - faithfully embeds S4 and thus IL
  - Display Logic based cut-free proof-system (Brotherston 09)
- Other definition (logical core of Separation and Spatial logics)
  - additive implication → Kripke **interpreted pointwise**
  - based on (commutative) partial monoids $(\mathcal{M}, \circ)$
  - has a sound and complete (labelled tableaux) proof-system
  - still embeds S4 and IL and even BI (Larchey&Galmiche 09)

In this talk

- We focus on provability, not validity checking (specific model).
- Tools for propositional tautologies in (monoidal) BI and BBI
  - BI defined by partially ordered partial monoids
  - BBI defined by partial monoids
- Common methodology for BI/BBI
  - words and constraints based Kripke models
  - labels and contraints based tableaux calculi
- From properties of proof-search based models
  - representation of BI-models by BBI-models
  - embedding of BI into BBI

Words and constraints based models for BI/BBI

- **Resources as Words** of $L^\star$ = multisets of letters
- Constraints = (ordered) pairs of words: $m \mathrel{-\!\!\bullet\!\!-} n$ with $m, n \in L^\star$
- Partial monoidal order (PMO): $\sqsubseteq$ closed under $\langle \varepsilon, l, r, d, c, t \rangle$
- Partial monoidal equivalence (PME): $\sim$ closed under $\langle \varepsilon, s, d, c, t \rangle$

PMOs	PMEs	PMOs & PMEs	
$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{x \mathrel{-\!\!\bullet\!\!-} x}\,\langle l \rangle$	$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} x}\,\langle s \rangle$	$\dfrac{}{\varepsilon \mathrel{-\!\!\bullet\!\!-} \varepsilon}\,\langle \varepsilon \rangle$	$\dfrac{ky \mathrel{-\!\!\bullet\!\!-} ky \qquad x \mathrel{-\!\!\bullet\!\!-} y}{kx \mathrel{-\!\!\bullet\!\!-} ky}\,\langle c \rangle$
$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} y}\,\langle r \rangle$		$\dfrac{xy \mathrel{-\!\!\bullet\!\!-} xy}{x \mathrel{-\!\!\bullet\!\!-} x}\,\langle d \rangle$	$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y \qquad y \mathrel{-\!\!\bullet\!\!-} z}{x \mathrel{-\!\!\bullet\!\!-} z}\,\langle t \rangle$

- $\langle s \rangle + \langle t \rangle$ implies $\langle l \rangle$ and $\langle r \rangle$, hence a PME is also a PMO
- Constraints solving: given $\mathcal{C}$, how to compute the closure $\sqsubseteq_{\mathcal{C}}$ / $\sim_{\mathcal{C}}$?

Constraints based Kripke models for BI/BBI

- $R \equiv \sqsubseteq$ for BI / $R \equiv \sim$ for BBI
- Usual (pointwise) Kripke interpretation for $\wedge$, $\vee$, $\perp$ and $\top$

BI/BBI	$m \Vdash_R \; \mathsf{I} \quad \text{iff} \quad \varepsilon\, R\, m$ $m \Vdash_R A * B \quad \text{iff} \quad \exists x, y\; xy\, R\, m \wedge x \Vdash_R A \wedge y \Vdash_R B$ $m \Vdash_R A \mathrel{-\!\!*} B \quad \text{iff} \quad \forall x, y\; (xm\, R\, y \wedge x \Vdash_R A) \Rightarrow y \Vdash_R B$
BI	$m \Vdash_{\sqsubseteq} A \rightarrow B \quad \text{iff} \quad \forall x\; (m \sqsubseteq x \wedge x \Vdash_{\sqsubseteq} A) \Rightarrow x \Vdash_{\sqsubseteq} B$
BBI	$m \Vdash_{\sim} A \rightarrow B \quad \text{iff} \quad m \Vdash_{\sim} A \Rightarrow m \Vdash_{\sim} B$ $m \Vdash_{\sim} \neg A \quad \text{iff} \quad m \nVdash_{\sim} A$

Complete constraints based Kripke semantics

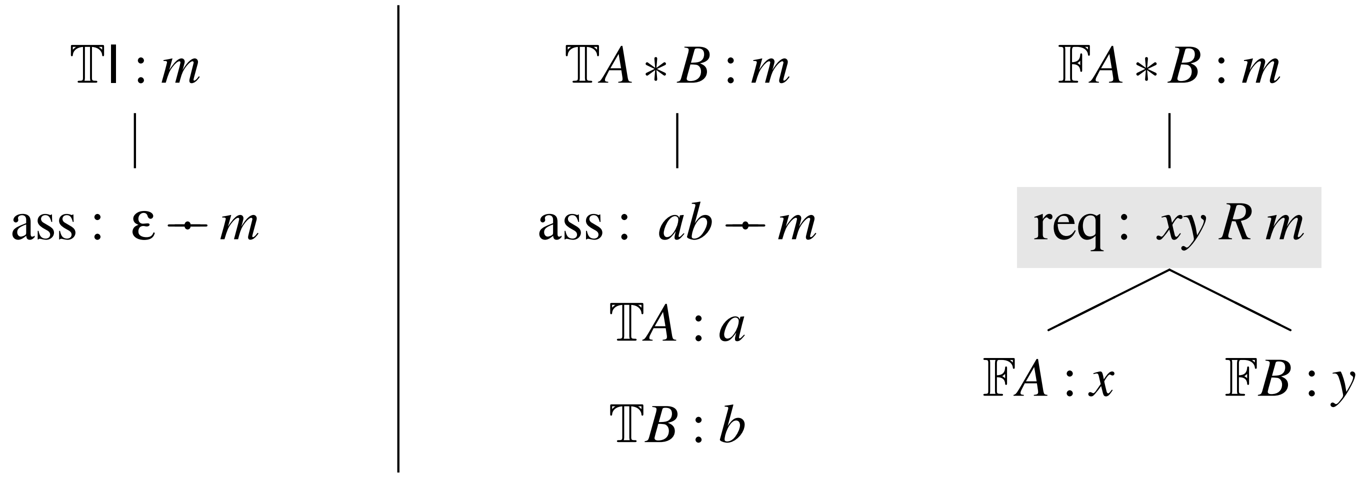
- Quotient monoids:
  - $L^\star/\sqsubseteq$ = partially ordered partial monoid
  - $L^\star/\sim$ = partial monoid
- These quotient maps $\sqsubseteq \mapsto L^\star/\sqsubseteq$ and $\sim \mapsto L^\star/\sim$ are full:
  - any partially ordered partial monoid is of the form $L^\star/\sqsubseteq$
  - any partial monoid is of the form $L^\star/\sim$
- Completeness theorem:
  - $\Vdash_\sqsubseteq$ sound and complete Kripke semantics for BI
  - $\Vdash_\sim$ sound and complete Kripke semantics for BBI

Labelled tableaux for BI and BBI

- Statements ($\mathbb{T}A : m$, $\mathbb{F}B : n$) and assertions (ass : $m \multimap n$)

- Requirements (req : $m\, R\, n$) with $R = \sqsubseteq$ or $\sim$ (side condition)

- Tableaux expansion rules for I and $*$:



$\mathbb{T}\mathsf{I} : m$

ass : $\varepsilon \multimap m$

$\mathbb{T}A * B : m$

ass : $ab \multimap m$

$\mathbb{T}A : a$

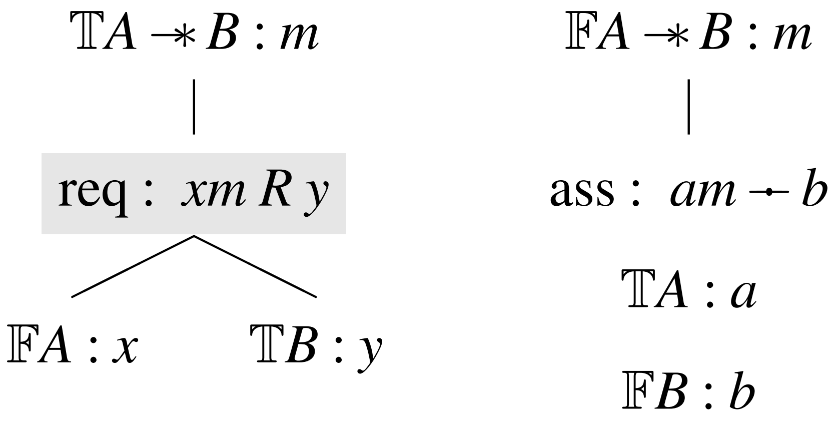
$\mathbb{T}B : b$

$\mathbb{F}A * B : m$

req : $xy\, R\, m$

$\mathbb{F}A : x$ $\quad$ $\mathbb{F}B : y$

- Tableaux expansion rules for $-\!\!*$:



$\mathbb{T}A -\!\!* B : m$

req : $xm\,R\,y$

$\mathbb{F}A : x$ $\quad$ $\mathbb{T}B : y$

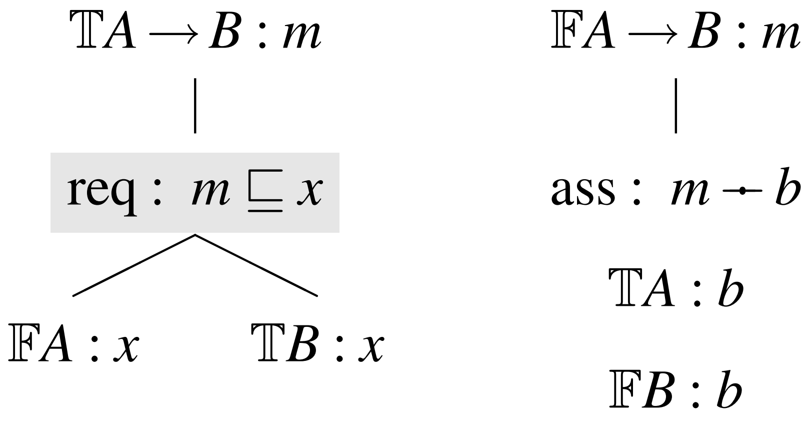
$\mathbb{F}A -\!\!* B : m$

ass : $am -\!\!\bullet\!\!- b$

$\mathbb{T}A : a$

$\mathbb{F}B : b$

- Tableaux expansion rules for $\longrightarrow$ (only BI):



$\mathbb{T}A \longrightarrow B : m$

req : $m \sqsubseteq x$

$\mathbb{F}A : x$ $\quad$ $\mathbb{T}B : x$

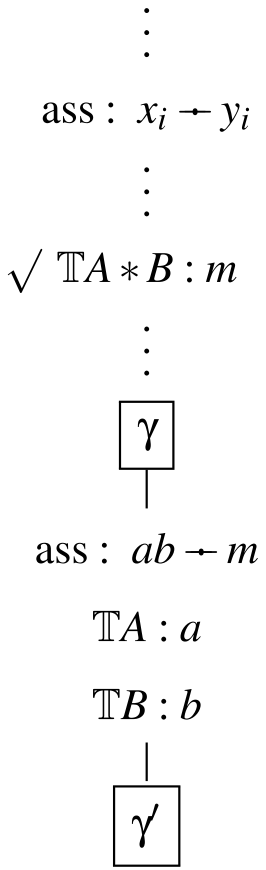
$\mathbb{F}A \longrightarrow B : m$

ass : $m -\!\!\bullet\!\!- b$

$\mathbb{T}A : b$

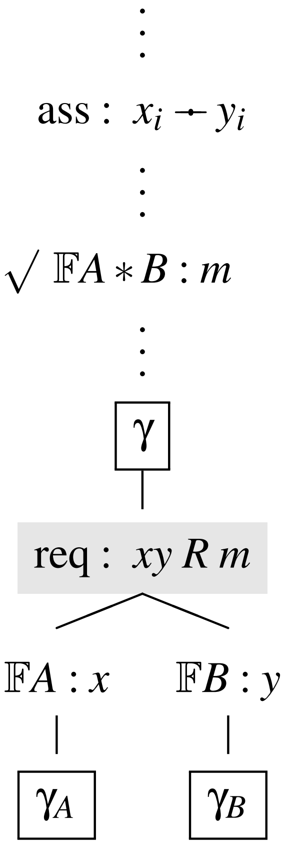
$\mathbb{F}B : b$

Assertions and proof-search



- $\mathcal{C} = \{\ldots, x_i \multimap y_i, \ldots\}$ from γ
- $\mathsf{A}_\gamma = \mathsf{A}_\mathcal{C} = \{c \in L \mid c$ occurs in $\mathcal{C}\}$
- $\sqsubseteq_\gamma = \sqsubseteq_\mathcal{C}$ / $\sim_\gamma = \sim_\mathcal{C}$
- branch expansion
  - $a \neq b$ new $(a, b \notin \mathsf{A}_\gamma)$
  - $\mathcal{C}' = \mathcal{C} \cup \{ab \multimap m\}$
  - $\sqsubseteq_{\gamma'} = \sqsubseteq_\gamma + \{ab \multimap m\}$
  - $\sim_{\gamma'} = \sim_\gamma + \{ab \multimap m\}$

Requirements and proof-search



$\vdots$

ass : $x_i \mathrel{-\!\!\bullet\!\!-} y_i$

$\vdots$

$\surd\ \mathbb{F}A * B : m$

$\vdots$

$\boxed{\gamma}$

req : $xy\,R\,m$

$\mathbb{F}A : x$ $\mathbb{F}B : y$

$\boxed{\gamma_A}$ $\boxed{\gamma_B}$

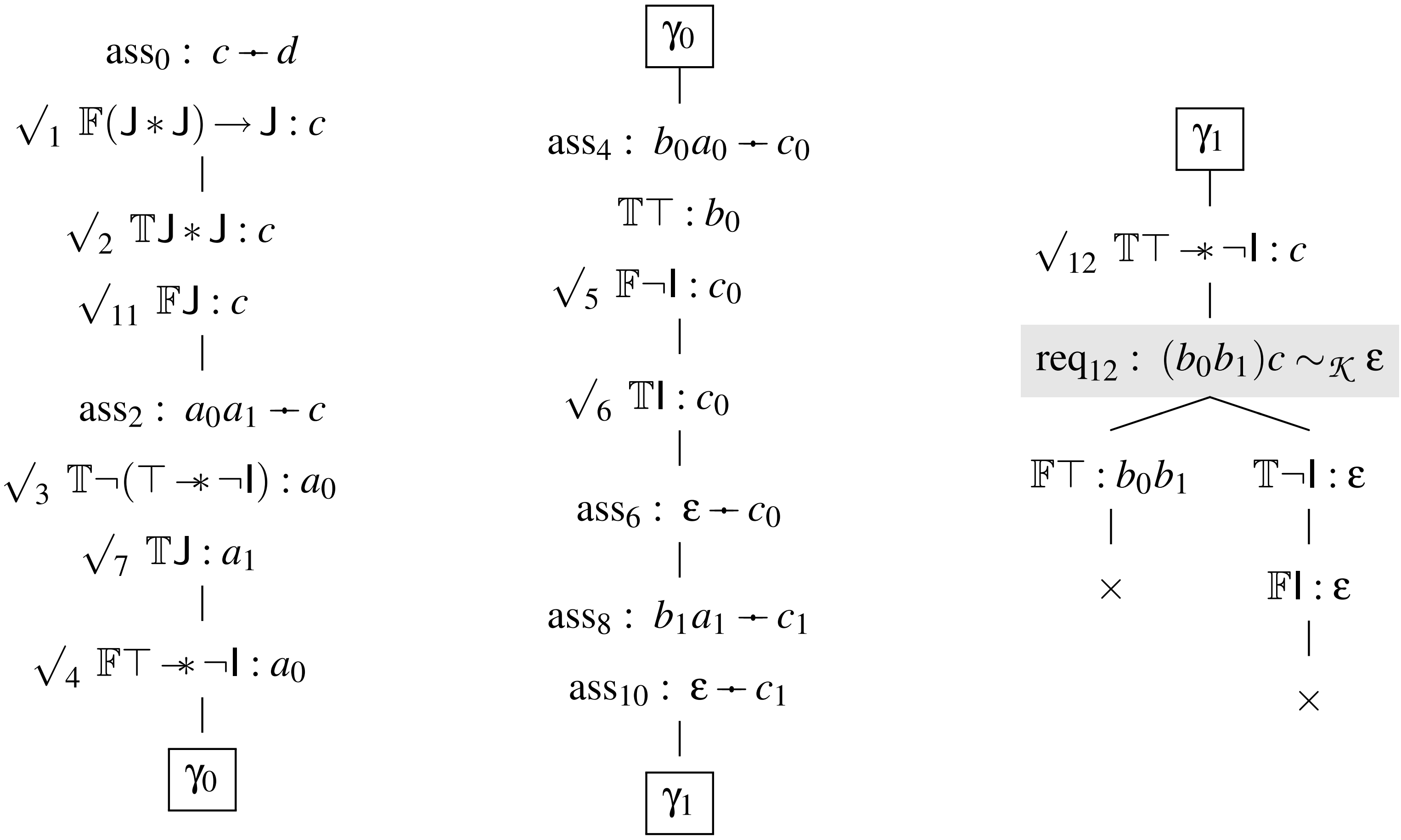
- $\mathcal{C} = \{\ldots, x_i \mathrel{-\!\!\bullet\!\!-} y_i, \ldots\}$ from γ
- $\mathsf{A}_\gamma = \mathsf{A}_\mathcal{C} = \{c \in L \mid c$ occurs in $\mathcal{C}\}$
- $\sqsubseteq_\gamma = \sqsubseteq_\mathcal{C}$ / $\sim_\gamma = \sim_\mathcal{C}$
- branch expansion
  - x, y s.t. $xy \sqsubseteq_\gamma m$ / $xy \sim_\gamma m$
  - $\mathcal{C}_A = \mathcal{C}_B = \mathcal{C}$
  - $\sqsubseteq_{\gamma_A} = \sqsubseteq_{\gamma_B} = \sqsubseteq_\gamma$
  - $\sim_{\gamma_A} = \sim_{\gamma_B} = \sim_\gamma$

Closure condition for proof-search

$$\begin{array}{c}\vdots\\ \text{ass}:\ x_i \mathrel{-\!\!\bullet\!\!-} y_i\\ \mathbb{T}X : m\\ \vdots\\ \mathbb{F}X : n\\ \vdots\\ \boxed{\gamma}\\ |\\ \times\end{array}$$

- $\mathcal{C} = \{\ldots, x_i \mathrel{-\!\!\bullet\!\!-} y_i, \ldots\}$ from γ
- $\mathsf{A}_\gamma = \mathsf{A}_\mathcal{C} = \{c \in L \mid c \text{ occurs in } \mathcal{C}\}$
- $\sqsubseteq_\gamma = \sqsubseteq_\mathcal{C}$ / $\sim_\gamma = \sim_\mathcal{C}$
- branch closure
  - $m \sqsubseteq_\gamma n$ / $m \sim_\gamma n$

BBI proof of $(\mathsf{J} * \mathsf{J}) \to \mathsf{J}$ with $\mathsf{J} = \neg(\top \mathrel{-\!\!*} \neg \mathsf{I})$



$\mathrm{ass}_0 : \ c \mathrel{-\!\!\bullet} d$

$\surd_1 \ \mathbb{F}(\mathsf{J} * \mathsf{J}) \to \mathsf{J} : c$

$\surd_2 \ \mathbb{T}\mathsf{J} * \mathsf{J} : c$

$\surd_{11} \ \mathbb{F}\mathsf{J} : c$

$\mathrm{ass}_2 : \ a_0 a_1 \mathrel{-\!\!\bullet} c$

$\surd_3 \ \mathbb{T}\neg(\top \mathrel{-\!\!*} \neg \mathsf{I}) : a_0$

$\surd_7 \ \mathbb{T}\mathsf{J} : a_1$

$\surd_4 \ \mathbb{F}\top \mathrel{-\!\!*} \neg \mathsf{I} : a_0$

$\boxed{\gamma_0}$

$\boxed{\gamma_0}$

$\mathrm{ass}_4 : \ b_0 a_0 \mathrel{-\!\!\bullet} c_0$

$\mathbb{T}\top : b_0$

$\surd_5 \ \mathbb{F}\neg \mathsf{I} : c_0$

$\surd_6 \ \mathbb{T}\mathsf{I} : c_0$

$\mathrm{ass}_6 : \ \varepsilon \mathrel{-\!\!\bullet} c_0$

$\mathrm{ass}_8 : \ b_1 a_1 \mathrel{-\!\!\bullet} c_1$

$\mathrm{ass}_{10} : \ \varepsilon \mathrel{-\!\!\bullet} c_1$

$\boxed{\gamma_1}$

$\boxed{\gamma_1}$

$\surd_{12} \ \mathbb{T}\top \mathrel{-\!\!*} \neg \mathsf{I} : c$

$\mathrm{req}_{12} : \ (b_0 b_1) c \sim_{\mathcal{K}} \varepsilon$

$\mathbb{F}\top : b_0 b_1$

$\times$

$\mathbb{T}\neg \mathsf{I} : \varepsilon$

$\mathbb{F}\mathsf{I} : \varepsilon$

$\times$

- with $\mathcal{K} = \{c \mathrel{-\!\!\bullet} d, a_0 a_1 \mathrel{-\!\!\bullet} c, b_0 a_0 \mathrel{-\!\!\bullet} c_0, \varepsilon \mathrel{-\!\!\bullet} c_0, b_1 a_1 \mathrel{-\!\!\bullet} c_1, \varepsilon \mathrel{-\!\!\bullet} c_1\}$

Checking the requirement

- $\mathcal{K} = \{c \mathrel{-\!\!\bullet\!\!-} d, a_0 a_1 \mathrel{-\!\!\bullet\!\!-} c, b_0 a_0 \mathrel{-\!\!\bullet\!\!-} c_0, \varepsilon \mathrel{-\!\!\bullet\!\!-} c_0, b_1 a_1 \mathrel{-\!\!\bullet\!\!-} c_1, \varepsilon \mathrel{-\!\!\bullet\!\!-} c_1\}$
- We check the requirement $b_0 b_1 c \sim_{\mathcal{K}} \varepsilon$ by solving $\mathcal{K}$
- $\{c, d, a_0, a_1, b_0, b_1, c_0, c_1\}^{\star} / \sim_{\mathcal{K}}$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ with:

$$\begin{array}{cc} c_0 = c_1 = \varepsilon = (0,0) & a_0 = -b_0 = (1,0) \\ c = d = (1,1) & a_1 = -b_1 = (0,1) \end{array}$$

- $b_0 b_1 c \sim_{\mathcal{K}} \varepsilon$ because $(-1,0) + (0,-1) + (1,1) = (0,0)$
- Remark: the solution of the (finite) set $\mathcal{K}$ is infinite

Tableaux completeness and counter-models

- Labels and constraints based methods:
  - calculi with constraints: $\mathbb{T}A : m, \mathbb{F}B : n, m \mathrel{-\!\!\bullet\!\!-} n$
  - sound/complete proof-search method for tautologies of BI/BBI
  - counter-models from open and saturated proof-search branch
- Why study the counter-models generated by proof-search:
  - implement/optimize proof assistants
  - extract complete sub-classes of counter-models
  - expressivity properties of BI and BBI
  - model theoretic and logical links between BI and BBI

PMO extensions in BI-tableaux (i)

- a and b are new letters ($a \not\sqsubseteq a$ and $b \not\sqsubseteq b$)
- m defined in $\sqsubseteq$ ($m \sqsubseteq m$)
- Four types of extensions

$$\begin{array}{ll}
\sqsubseteq' = \sqsubseteq + \{ab \mathrel{-\!\bullet\!-} m\} \text{ (rule } \mathbb{T}*\text{)} & \sqsubseteq' = \sqsubseteq + \{am \mathrel{-\!\bullet\!-} b\} \text{ (rule } \mathbb{F}{-\!*}\text{)} \\
\sqsubseteq' = \sqsubseteq + \{m \mathrel{-\!\bullet\!-} b\} \quad \text{(rule } \mathbb{F}{\rightarrow}\text{)} & \sqsubseteq' = \sqsubseteq + \{\varepsilon \mathrel{-\!\bullet\!-} m\} \quad \text{(rule } \mathbb{T}\mathsf{I}\text{)}
\end{array}$$

- Basic PMO = (finite or infinite) **sequence** of such extensions
- Extensions can be solved:

$$\begin{aligned}
\sqsubseteq + \{ab \mathrel{-\!\bullet\!-} m\} = \sqsubseteq & \cup \{ax \mathrel{-\!\bullet\!-} ay \mid x \sqsubseteq y \text{ and } mx \sqsubseteq my\} \\
& \cup \{bx \mathrel{-\!\bullet\!-} by \mid x \sqsubseteq y \text{ and } mx \sqsubseteq my\} \\
& \cup \{abx \mathrel{-\!\bullet\!-} y \mid mx \sqsubseteq y\}
\end{aligned}$$

PMO extensions in BI-tableaux (ii)

- Properties of basic PMO $\sqsubseteq_{\mathcal{C}}$ (by induction on $\mathcal{C}$):
  - ε-minimality: if $m \sqsubseteq_{\mathcal{C}} \varepsilon$ then $m = \varepsilon$
  - **no square**: if $mm \sqsubseteq_{\mathcal{C}} mm$ then $m = \varepsilon$
  - regularity: if $kx \sqsubseteq_{\mathcal{C}} ky$ then $x \sqsubseteq_{\mathcal{C}} y$

$\Rightarrow$ **finiteness**: $\{m \in L^{\star} \mid m \sqsubseteq_{\mathcal{C}} m\}$ is finite ($\mathcal{C}$ finite sequence)

- Solving constraints in $\mathcal{C}$: (finite) resource graph (Mery 04)
- Complete sub-class for BI:
  - these properties hold for infinite sequences of basic extensions
  - regular monoids where ε is minimal and without square
- Application: no BI-formula F such that $m \Vdash_{\sqsubseteq} F$ iff $mm \sqsubseteq mm$

PME extensions in BBI-tableaux (i)

- a and b are new letters, m defined in $\sim$ (i.e. $m \sim m$)
- Three types of extensions

$$\sim' \;=\; \sim + \{ab \mathrel{-\!\!\bullet\!\!-} m\} \qquad \text{(rule } \mathbb{T}*\text{)}$$

$$\sim' \;=\; \sim + \{am \mathrel{-\!\!\bullet\!\!-} b\} \qquad \text{(rule } \mathbb{F}{-\!\!*}\text{)}$$

$$\sim' \;=\; \sim + \{\varepsilon \mathrel{-\!\!\bullet\!\!-} m\} \qquad \text{(rule } \mathbb{T}\mathsf{I}\text{)}$$

- Basic PME = (finite or infinite) sequence of such extensions
- Extensions $ab \mathrel{-\!\!\bullet\!\!-} m$ (and $am \mathrel{-\!\!\bullet\!\!-} b$) solved when $\boxed{mm \nsim mm}$:

$$\begin{aligned}\sim + \{ab \mathrel{-\!\!\bullet\!\!-} m\} = \sim &\cup \{ax \mathrel{-\!\!\bullet\!\!-} ay, bx \mathrel{-\!\!\bullet\!\!-} by \mid x \sim y \text{ and } mx \sim my\}\\ &\cup \{abx \mathrel{-\!\!\bullet\!\!-} aby \mid mx \sim my\}\\ &\cup \{abx \mathrel{-\!\!\bullet\!\!-} y, y \mathrel{-\!\!\bullet\!\!-} abx \mid mx \sim y\}\end{aligned}$$

PME extensions in BBI-tableaux (ii)

- Problems with the $\sim + \{\varepsilon \multimap m\}$ extension:
  - does not preserve regularity
  - introduce squares (if $\varepsilon \sim m$ then $mm \sim mm$)
  - ε-minimality irrelevant

$\Rightarrow$ Invertible letters produce $\boxed{\text{infinite models}}$ (not as in BI)

- No simple solution for $\sim + \{ab \multimap m\}$ when $mm \sim mm$
- Automated constraint solving for basic PME not detailed here
- Not the same as the word problem in Thue systems (partiality)

Representing basic PMOs by basic PMEs

- Let $\sqsubseteq \,=\, \sqsubseteq_{\mathcal{C}}$ be a basic PMO over L with $\mathcal{C} = \{x_0 \mathrel{-\!\bullet\!-} y_0, \ldots\}$
- $(K, \sim)$ is a representation of $(L, \sqsubseteq)$ if
  - $\sim$ is PME over $L \cup K \cup \ldots$
  - $\boxed{x \sqsubseteq y \text{ iff } \exists \delta \in K^\star, \delta x \sim y}$ (for any $x, y \in L^\star$)
- Result: every basic PMO can be represented by a basic PME:
  - $\sqsubseteq' \,=\, \sqsubseteq + \{ab \mathrel{-\!\bullet\!-} m\} \quad \rightsquigarrow \quad \sim' \,=\, \sim + \{\delta c \mathrel{-\!\bullet\!-} m, ab \mathrel{-\!\bullet\!-} c\}$
  - $\sqsubseteq' \,=\, \sqsubseteq + \{am \mathrel{-\!\bullet\!-} b\} \quad \rightsquigarrow \quad \sim' \,=\, \sim + \{cm \mathrel{-\!\bullet\!-} b, \delta a \mathrel{-\!\bullet\!-} c\}$
  - δ, c are new, $\delta \in K$ and $c \notin L \cup K$
  - this representation is compatible with limits (by compactness)

Validity in BI/BBI and PMO/PME representations

- Let K (resp. L) be a new variable for K (resp. L)
- $F \mapsto F^\circ$ is a (linear) map from BI to BBI:

$$X^\circ = \mathsf{K} * X \quad \mathsf{I}^\circ = \mathsf{K} * \mathsf{I} \quad \bot^\circ = \bot \quad \top^\circ = \top$$

$$(A \oplus B)^\circ = A^\circ \oplus B^\circ \text{ for } \oplus \in \{\wedge, \vee\}$$

$$(A \to B)^\circ = \mathsf{K} \mathrel{-\!\!*} \big((\mathsf{L} \wedge A^\circ) \to B^\circ\big)$$

$$(A * B)^\circ = \mathsf{K} * \big((\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ)\big)$$

$$(A \mathrel{-\!\!*} B)^\circ = \big(\mathsf{K} * (\mathsf{L} \wedge A^\circ)\big) \mathrel{-\!\!*} (\mathsf{L} \to B^\circ)$$

- Result: if $(K, \sim)$ represents $(L, \sqsubseteq)$, then for any $F \in \mathsf{BI}$ and $m \in \mathcal{L}^{\sqsubseteq}$

$$\boxed{m \Vdash_{\sqsubseteq} F \quad \text{iff} \quad m \Vdash_{\sim} F^\circ}$$

- Relates (in)validity but not provability

Faithfully embedding BI into BBI

- Let $\mathsf{H} = (\mathsf{L} \wedge \mathsf{K}) \wedge \big((\top \mathrel{-\!\!*} (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L})) \wedge (\top \mathrel{-\!\!*} (\mathsf{K} * \mathsf{K} \rightarrow \mathsf{K}))\big)$
- $G \mapsto (\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ is faithful:
  - if G is invalid in BI then it has a basic counter-model $(L, \sqsubseteq)$: $\varepsilon \nVdash_{\sqsubseteq} G$
  - let $(K, \sim)$ be a representation of $(L, \sqsubseteq)$
  - then $\varepsilon \nVdash_{\sim} (\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ ($\sim$ is a BBI-counter-model)
- $G \mapsto (\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ is sound:
  - step-by-step transformation of BI-tableaux in BBI-tableaux
  - BI-expansions mapped into BBI-expansions
  - closure of BBI-branches with $\mathsf{I} \wedge \mathsf{H}$
- $\boxed{G \mapsto (\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ \text{ is a faithful embedding BI into BBI}}$ (MSCS 09)

Some remarks about the embedding

- Obtained by the study of counter-model generated by proof-search
  - labelled tableaux well-suited for this task
  - common framework for BI and BBI
- Not expected (counter-intuitive):
  - IL faithfully embeds CL (double negation, Gödel)
  - Boolean BI faithfully embeds (intuitionistic) BI
  - the embedding in the reverse direction
  - BBI into BI (BI decidable, BBI not decidable ?)

Conclusion and perspectives

- Achievements:
  - complete tableaux with constraints method for BBI
  - properties of proof-search generated BBI constraints
  - expressivity properties for BI and BBI, embedding
  - algorithmic solution to BBI constraints solving (to come)
- Perspectives:
  - implement constraint solving for proof-search in BBI
  - towards undecidability of BBI (Display Logic)
  - provide intuitive understanding of invertible resources