The Braga method and the extraction of complex recursive algorithms

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Standard Recursion in Coq

- Structural recursion, rec. calls on sub-terms

$$\texttt{fact}\ \texttt{0} = \texttt{S}\ \texttt{0} \qquad \texttt{fact}\,(\texttt{S}\ n) = \texttt{S}\ n \times \texttt{fact}\ n$$

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- Well-founded recursion for $R : X \to X \to \mathtt{Prop}$

$$\begin{array}{l}\mathtt{Inductive}\ \mathtt{Acc}\ R\ x : \mathtt{Prop} := \\ \quad \mid \mathtt{Acc\_intro} : (\forall y,\ R\ y\ x \to \mathtt{Acc}\ R\ y) \to \mathtt{Acc}\ R\ x.\end{array}$$

$$\begin{array}{l}\mathtt{Fixpoint}\ f\ x\ (T_x : \mathtt{Acc}\ R\ x)\ \{\mathtt{struct}\ T_x\} := \\ \qquad\qquad \dots \quad f\ y\ T_y \quad \dots\end{array}$$

- Must define R before f, prove $\mathtt{Acc}\ R\ x$ and ensure $\boxed{T_y <_{\text{struct}} T_x}$

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- Must define R before f, prove $\mathtt{Acc}\ R\ x$ and ensure $\boxed{T_y <_{\mathrm{struct}} T_x}$
- Particular case, decreasing measure (using `lt_wf`):
  - $R\ x\ y$ is $m\ x < m\ y$ for some $m : X \to \mathtt{nat}$
- Too strong constraints for general recursion?

Complicated recursive schemes

- No obvious structural recursion, nor termination:

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$$\texttt{th}\ f\ x\ y = \texttt{if}\ x = y\ \texttt{then}\ 0\ \texttt{else}\ 1 + \texttt{th}\ f\ (f\ x)\ (f^2\ y)$$

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- Complicated termination proof ($succs : \mathcal{V} \to \mathbb{L}\,\mathcal{V}$):

$$\begin{array}{lcll}
\texttt{dfs}\ v\ [] & = & v & \\
\texttt{dfs}\ v\ (x :: l) & = & \texttt{dfs}\ v\ l & \text{if } x \in v \\
\texttt{dfs}\ v\ (x :: l) & = & \texttt{dfs}\ (x :: v)\ (succs\ x \mathbin{+\!\!+} l) & \text{if } x \notin v
\end{array}$$

More complicated recursive schemes

- Nesting/mutual recursion:

  McCarthy $\quad f_{91}\ x = \texttt{if}\ x > 100\ \texttt{then}\ x - 10\ \texttt{else}\ f_{91}^2(x+11)$

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Knuth 1991 $k_{91}\ x = \texttt{if}\ x > a\ \texttt{then}\ x - b\ \texttt{else}\ k_{91}^c(x + d)$

where $f^n\ x = \texttt{iter}_p\ f\ n\ x$

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- Nesting&hard termination: $\texttt{unif}\ (m \cdot n)\ (m' \cdot n')$ is

$$\begin{cases} \emptyset & \text{if } \texttt{unif}\ m\ m' = \emptyset \\ \emptyset & \text{if } \texttt{unif}\ m\ m' = \lfloor \rho \rfloor \text{ and } \texttt{unif}\ (\rho\ n)\ (\rho\ n') = \emptyset \\ \lfloor \sigma \circ \rho \rfloor & \text{if } \texttt{unif}\ m\ m' = \lfloor \rho \rfloor \text{ and } \texttt{unif}\ (\rho\ n)\ (\rho\ n') = \lfloor \sigma \rfloor \end{cases}$$

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  - auto. maps a Coq term to a program (OCaml)
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- Consider a fully specified term t:

$$t : \forall x : X, \mathbb{D}\ x \to \{y : Y \mid \mathbb{G}\ x\ y\}$$

$\mathbb{D} : X \to \texttt{Prop}$	Domain	Pre-condition
$\mathbb{G} : X \to Y \to \texttt{Prop}$	Specification	Post-condition

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  - are erased at extraction
  - $\textsc{extr}(t) : \textsc{extr}(X) \to \textsc{extr}(Y)$

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  - $\text{EXTR}(t) : \text{EXTR}(X) \to \text{EXTR}(Y)$
- What do $\mathbb{D}$ and $\mathbb{G}$ become?
  - it depends...
  - now: they are just erased
  - ideally (shortly ?): correctness of $\text{EXTR}(t)$

Certification by Extraction

- How to certify by extraction ?
  - From a given OCaml algo. $\varphi : \alpha \to \beta$
  - Get $\varphi = \mathrm{EXTR}(t_\varphi) : \mathrm{EXTR}(X_\alpha) \to \mathrm{EXTR}(X_\beta)$

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- $\mathbb{D}_\varphi\, x$ (domain) and $\mathbb{G}_\varphi\, x\, y$ (spec)
  - erased at extraction
  - but contain the statement of correctness
- Problem: how to define such a t_φ in Coq ?
  - no `let rec`, only restricted `Fixpoints` (`struct`)
  - How to control the CC ?

Some influencial references

- Non-constructive recursion:
  - *Termination of Nested and Mutually Recursive Algorithms* (Giesl 97)
  - *Partial and Nested Recursive Function Definitions in Higher-Order Logic* (Krauss 09)
  - *Partiality and Recursion in Interactive Theorem Provers - An Overview* (Bove&Krauss&Sozeau 15)

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  - the `Equations` package (2010 & 2019)
  - *The Braga method* (Types 2018 & WS 2021)

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  - the `Equations` package (2010 & 2019)
  - *The Braga method* (Types 2018 & WS 2021)
- Extraction related:
  - Extraction in Coq (P. Letousey's thesis 2004)
  - MetaCoq and Œuf (CPP'18)

The Braga method, an overview

- Techniques in Coq with standard tools:
  - implement spec while controlling CC
  - separate defs. from correctness proofs
  - non-terminating algo.
  - nested&mutual non-terminating algo
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  - but compatible with Braga
- Illustrated on examples:
  - ∞-loop, DFS, F91 (generalized)
  - control of CC and separation of LC from CC

F91 for the knowledgeable (10 loc)

- One nested recursive call

$$\texttt{f91}\ n = \texttt{if}\ n > 100\ \texttt{then}\ n{-}10\ \texttt{else}\ \texttt{f91}(\texttt{f91}(n{+}11))$$

- Provide you know the spec: $\boxed{\texttt{f91\_pred}\ n\ (\texttt{f91}\ n)}$

$$\texttt{f91\_pred}\ n\ o := \begin{cases} & n > 100 \wedge o = n - 10 \\ \vee & n \leq 100 \wedge o = 91 \end{cases}$$

- And the termination measure: $\boxed{M := \lambda\, n,\ 101 - n}$
- One can define:

$$\texttt{f91\_pwc}\ (n : \texttt{nat}) : \{o : \texttt{nat} \mid \texttt{f91\_pred}\ n\ o\}$$

- By (strong) induction on the measure $101 - n$
  - $\texttt{f91\_pred}\ (n + 11)\ x$ needed for term. of $\texttt{f91}\ x$

F91 for the clueless

- The clueless has no intuition on 10, 11, 100...
- Sees the generalization for $a : X \to \texttt{bool}$ and $b, c : X \to X$

$$f\ x = \texttt{if}\ a\,x\ \texttt{then}\ b\,x\ \texttt{else}\ f(f(c\,x))$$

- The spec is harder to guess...
- Might not terminate
  - eg if $a\,x = \texttt{false}$ for any x
- How to proceed in this case?
  - we need a mechanic procedure
- The Braga method is agnostic:
  - to post-conditions (specs)
  - to pre-conditions (termination)

The induction principle for `False`

- The empty proposition `False`:

  `Inductive False : Prop := .`

The induction principle for False

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  Inductive False : Prop := .

- False_rect : $\forall X$: Type, False $\to X$

  Definition False_rect X (f : False) : X :=
  match f return X with end.

The induction principle for False

- The empty proposition False:

  $\texttt{Inductive False} : \texttt{Prop} := .$

- $\texttt{False\_rect} : \forall X : \texttt{Type}, \texttt{False} \to X$

  $$\begin{array}{l}\texttt{Definition False\_rect}\ X\ (f : \texttt{False}) : X := \\ \qquad\qquad \texttt{match}\ f\ \texttt{return}\ X\ \texttt{with end}.\end{array}$$

- extracts to

  $$\texttt{let false\_rect \_ = assert false}$$

- interpretation of partiality: exception/error

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  $$\texttt{let false\_rect \_ = assert false}$$

- interpretation of partiality: exception/error
- is there another proof of False_rect?

Eliminating False with an infinite loop

- Looping on False with dummy unit argument:
- $\texttt{loop} : \forall X : \texttt{Type}, \texttt{unit} \to \texttt{False} \to X$

$$\begin{array}{l}\texttt{fix loop } \{X\}\ (\_ : \texttt{unit})\ (\underline{f} : \texttt{False}) : X := \\ \quad \texttt{loop tt (match } f \texttt{ return False with end).}\end{array}$$

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- Alt. elim.: $\texttt{False\_loop} : \forall X : \texttt{Type}, \texttt{False} \to X$

  $\texttt{Definition False\_loop}\ X := \texttt{@loop}\ X\ \texttt{tt}$

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- extracts to

```
let false_loop _ =
  let rec loop _ = loop ()
  in loop ()
```

First take home idea

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- is **structurally smaller than any term** in X
  - when X is an inductive type

First take home idea

- Matching on `False`:

$$\boxed{\texttt{match}\ f : \texttt{False}\ \texttt{return}\ X\ \texttt{with end}}$$

- is a term of **any type** X
- is **structurally smaller than any term** in X
  - when X is an inductive type
- used extensively to rule out absurd cases
  - the `exfalso` tactic
  - the `discriminate` tactic
  - the `destruct` tactic on $H : \ldots \to \ldots \to \texttt{False}$
  - absurd cases for `inversion`

Depth First Search

$$
\begin{array}{l}
\texttt{let rec}\ x \in^?_{\mathcal{V}} v = \\
\quad \texttt{match}\ v\ \texttt{with} \\
\quad\quad |\ [] \quad\ \ \rightarrow \texttt{false} \\
\quad\quad |\ y :: w \rightarrow y = x\ \texttt{or}\ x \in^?_{\mathcal{V}} w
\end{array}
$$

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- For $=^?_{\mathcal{V}} : \forall x\, y : \mathcal{V}, \{b \mid x = y \iff b = \texttt{true}\}$
- $\texttt{succs} : \mathcal{V} \to \texttt{list } \mathcal{V}$ (directed graph structure)

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- For $=^?_{\mathcal{V}} : \forall x\, y : \mathcal{V}, \{b \mid x = y \iff b = \texttt{true}\}$
- $\texttt{succs} : \mathcal{V} \to \texttt{list } \mathcal{V}$ (directed graph structure)
- Specification is not obvious
  - When/why does it terminate?
  - What is the output?

From the algo. to its computational graph

- From the `dfs` algorithm only

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- Graph $\mathbb{G}_{\texttt{dfs}} : \texttt{list}\ \mathcal{V} \to \texttt{list}\ \mathcal{V} \to \texttt{list}\ \mathcal{V} \to \texttt{Prop}$

$$
\frac{}{\mathbb{G}_{\texttt{dfs}}\ v\ []\ v}
\qquad
\frac{x \in^?_{\mathcal{V}} v \quad \mathbb{G}_{\texttt{dfs}}\ v\ l\ o}{\mathbb{G}_{\texttt{dfs}}\ v\ (x :: l)\ o}
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$$
\frac{x \notin^?_{\mathcal{V}} v \quad \mathbb{G}_{\texttt{dfs}}\ (x :: v)\ (\texttt{succs}\ x \mathbin{+\!\!+} l)\ o}{\mathbb{G}_{\texttt{dfs}}\ v\ (x :: l)\ o}
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$$\frac{x \notin^?_{\mathcal{V}} v \quad \mathbb{G}_{\texttt{dfs}}\ (x :: v)\ (\texttt{succs}\ x +\!\!+\ l)\ o}{\mathbb{G}_{\texttt{dfs}}\ v\ (x :: l)\ o}$$

- functional: $\mathbb{G}_{\texttt{dfs}}\ v\ l\ o_1 \to \mathbb{G}_{\texttt{dfs}}\ v\ l\ o_2 \to o_1 = o_2$.

From the graph to the domain predicate

- Graph $\mathbb{G}_{\mathtt{dfs}}$

$$\frac{}{\mathbb{G}_{\mathtt{dfs}}\ v\ []\ v} \qquad \frac{x \in^?_{\mathcal{V}} v \quad \mathbb{G}_{\mathtt{dfs}}\ v\ l\ o}{\mathbb{G}_{\mathtt{dfs}}\ v\ (x :: l)\ o}$$

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- We will show: $\mathbb{D}_{\texttt{dfs}}\ v\ l \iff \exists o,\ \mathbb{G}_{\texttt{dfs}}\ v\ l\ o$

DFS packed with conformity to $\mathbb{G}_{\texttt{dfs}}$

- $\texttt{dfs\_pwc} : \forall v\, l,\ \mathbb{D}_{\texttt{dfs}}\ v\ l \to \{o \mid \mathbb{G}_{\texttt{dfs}}\ v\ l\ o\}$
- By structural induction on the domain predicate D

```
Fixpoint dfs_pwc v l (D : 𝔻dfs v l) : {o | 𝔾dfs v l o}.
Proof. refine(
  match l with
    | []     ⇒ λ D, exist _ v 𝒪?₁
    | x :: l ⇒ λ D,
    match x ∈?_𝒱 v as b return x ∈?_𝒱 v = b → _ with
      | true  ⇒ λ E,
            let (o, G_o) := dfs_pwc v l 𝒯?₂
            in  exist _ o 𝒪?₂
      | false ⇒ λ E,
            let (o, G_o) := dfs_pwc (x :: v) (succs x ++ l) 𝒯?₃
            in  exist _ o 𝒪?₃
    end eq_refl
  end D).
  (* Proof obligations *)
Qed.
```

- But not by pattern matching on D!

Proof obligations: postconditions $\mathcal{O}^?_{1,2,3}$

- Postcondition e.g. $\mathcal{O}^?_2$

  $[\mathcal{O}^?_2] :\ \ldots, E : x \in^?_{\mathcal{V}} v = \texttt{true}, G_o : \mathbb{G}_{\texttt{dfs}}\ v\ l\ o \vdash \mathbb{G}_{\texttt{dfs}}\ v\ (x{::}l)\ o$

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- is trivial to handle
- second constructor of the graph $\mathbb{G}_{\mathtt{dfs}}$:

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- same holds for $\mathcal{O}^?_1$ and $\mathcal{O}^?_3$
- Termination certificates $\mathcal{T}^?_2$ and $\mathcal{T}^?_3$:
  - much more complicated to handle

Proof obligations: termination certificates

- Termination certificates $\mathcal{T}_2^?$ and $\mathcal{T}_3^?$

$$[\mathcal{T}_2^?] : \ldots, D : \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l), E : x \in_{\mathcal{V}}^? v = \texttt{true} \vdash \mathbb{D}_{\texttt{dfs}}\ v\ l$$

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- We need to provide a term of type:

  $$\pi_{\mathbb{D}_{\texttt{dfs}}}\_2 : \forall v\, x\, l,\ \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l) \to x \in_{\mathcal{V}}^? v = \texttt{true} \to \mathbb{D}_{\texttt{dfs}}\ v\ l$$

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- The projection $\pi_{\mathbb{D}_{\texttt{dfs}}}\_2$ inverts the constructor $\mathbb{D}^2_{\texttt{dfs}}$:

$$\frac{x \in_{\mathcal{V}}^? v \qquad \mathbb{D}_{\texttt{dfs}}\ v\ l}{\mathbb{D}_{\texttt{dfs}}\ v\ (x :: l)}\ \langle \mathbb{D}^2_{\texttt{dfs}} \rangle \qquad \frac{x \in_{\mathcal{V}}^? v \qquad \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l)}{\mathbb{D}_{\texttt{dfs}}\ v\ l}\ \langle \pi_{\mathbb{D}_{\texttt{dfs}}}\_2 \rangle$$

- `Fixpoint` guard cond.: $\pi_{\mathbb{D}_{\texttt{dfs}}}\_2\ v\ x\ l\ D\ E <_{\mathrm{struct}} D$
  - `inversion` tactic works but unreadable
  - Small inversions give a human checkable term

Small Inversions (J.F. Monin)

- Termination certificate $\mathcal{T}_2^?$ by dep. pattern matching
- Generic code which explicits structural decrease

$$
\begin{array}{l}
\texttt{Let shape } (b : \texttt{bool})\ v\ l := \\
\quad \texttt{match } l \texttt{ with} \\
\qquad \mid [\,] \quad \Rightarrow \texttt{False} \\
\qquad \mid x :: l \Rightarrow x \in_{\mathcal{V}}^{?} v = b \\
\quad \texttt{end.}
\end{array}
$$

$$
\begin{array}{l}
\texttt{Let p\_tl } \{b\ v\ l\} : \texttt{shape}\, b\, v\, l \to \texttt{list } \mathcal{V} := \\
\quad \texttt{match } l \texttt{ with} \\
\qquad \mid [\,] \quad \Rightarrow \lambda\, s, \texttt{match } s : \texttt{False with end} \\
\qquad \mid \_ :: l \Rightarrow \lambda\, \_, l \\
\quad \texttt{end.}
\end{array}
$$

$$
\begin{array}{l}
\texttt{Let } \pi_{\mathbb{D}_{\texttt{dfs}}}\texttt{\_2\_gen } \{v\ l\}\ (D_{vl} : \mathbb{D}_{\texttt{dfs}}\ v\ l) : \forall s : \texttt{shape}\,\texttt{true}\, v\, l,\ \mathbb{D}_{\texttt{dfs}}\ v\ (\texttt{p\_tl } s) := \\
\quad \texttt{match } D_{vl} \texttt{ in } \mathbb{D}_{\texttt{dfs}}\ v'\ l' \texttt{ with} \\
\quad \texttt{return } \forall s : \texttt{shape}\,\texttt{true}\, v'\, l', \mathbb{D}_{\texttt{dfs}}\ v'\ (\texttt{p\_tl } s) \\
\quad \texttt{with} \\
\qquad \mid \mathbb{D}^1_{\texttt{dfs}}\ v \qquad\qquad \Rightarrow \lambda\, s, \texttt{match } s : \texttt{False with end} \\
\qquad \mid \mathbb{D}^2_{\texttt{dfs}}\ v\ x\ l\ \_\ D \Rightarrow \lambda\, \_, D \\
\qquad \mid \mathbb{D}^3_{\texttt{dfs}}\ v\ x\ l\ H\ \_ \Rightarrow \lambda\, s, \texttt{match not\_mem\_true } H\ s : \texttt{False with end} \\
\quad \texttt{end.}
\end{array}
$$

$$\texttt{Let } \pi_{\mathbb{D}_{\texttt{dfs}}}\texttt{\_2 } v\ x\ l : \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l) \to x \in_{\mathcal{V}}^{?} v = \texttt{true} \to \mathbb{D}_{\texttt{dfs}}\ v\ l := \pi_{\mathbb{D}_{\texttt{dfs}}}\texttt{\_2\_gen}.$$

The partial DFS algorithm

- $\mathtt{dfs\_pwc} : \forall v\, l, \mathbb{D}_{\mathtt{dfs}}\ v\ l \to \{o \mid \mathbb{G}_{\mathtt{dfs}}\ v\ l\ o\}$

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- We define $\texttt{dfs}\ v\ l\ D := \pi_1(\texttt{dfs\_pwc}\ v\ l\ D)$

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- We define $\texttt{dfs}\ v\ l\ D := \pi_1(\texttt{dfs\_pwc}\ v\ l\ D)$
- get conformity via π_2 (low-level):

$$\boxed{\texttt{dfs\_spec} : \forall v\, l\, D,\ \mathbb{G}_{\texttt{dfs}}\ v\ l\ (\texttt{dfs}\ v\ l\ D)}$$

- Then fixpoint eqs and proof irrelevance

$$\begin{array}{l}
\texttt{dfs_pirr}\ \ : \texttt{dfs}\ v\ l\ D_1 = \texttt{dfs}\ v\ l\ D_2. \\
\texttt{dfs_fix_1} : \texttt{dfs}\ v\ []\ (\mathbb{D}^1_{\texttt{dfs}}\ v) = v. \\
\texttt{dfs_fix_2} : \texttt{dfs}\ v\ (x :: l)\ (\mathbb{D}^2_{\texttt{dfs}}\ v\ x\ l\ H\, D) = \texttt{dfs}\ v\ l\ D. \\
\texttt{dfs_fix_3} : \texttt{dfs}\ v\ (x :: l)\ (\mathbb{D}^3_{\texttt{dfs}}\ v\ x\ l\ H\, D) = \texttt{dfs}\ (x :: v)\ (\texttt{succs}\, x +\!\!+\, l)\ D.
\end{array}$$

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$$\begin{array}{ll} \texttt{dfs\_pirr} & : \texttt{dfs}\ v\ l\ D_1 = \texttt{dfs}\ v\ l\ D_2. \\ \texttt{dfs\_fix\_1} & : \texttt{dfs}\ v\ []\ (\mathbb{D}^1_{\texttt{dfs}}\ v) = v. \\ \texttt{dfs\_fix\_2} & : \texttt{dfs}\ v\ (x :: l)\ (\mathbb{D}^2_{\texttt{dfs}}\ v\ x\ l\ H\ D) = \texttt{dfs}\ v\ l\ D. \\ \texttt{dfs\_fix\_3} & : \texttt{dfs}\ v\ (x :: l)\ (\mathbb{D}^3_{\texttt{dfs}}\ v\ x\ l\ H\ D) = \texttt{dfs}\ (x :: v)\ (\texttt{succs}\, x +\!\!+\, l)\ D. \end{array}$$

- $\mathbb{D}_{\texttt{dfs}}$ has a dependent recursion principle

$$\begin{array}{l} \texttt{Theorem}\ \mathbb{D}_{\texttt{dfs}}\texttt{\_rect}\ (P : \forall v\ l,\ \mathbb{D}_{\texttt{dfs}}\ v\ l \to \boxed{\texttt{Type}}) : \\ \quad\ \ \big(\forall v\ l\ D_1\ D_2,\ P\ v\ l\ D_1 \to P\ v\ l\ D_2\big) \\ \ \to \big(\forall v,\ P\ \_\ \_\ (\mathbb{D}^1_{\texttt{dfs}}\ v)\big) \\ \ \to \big(\forall v\ x\ l\ H D,\ P\ \_\ \_\ D \to P\ \_\ \_\ (\mathbb{D}^2_{\texttt{dfs}}\ v\ x\ l\ H\ D)\big) \\ \ \to \big(\forall v\ x\ l\ H D,\ P\ \_\ \_\ D \to P\ \_\ \_\ (\mathbb{D}^3_{\texttt{dfs}}\ v\ x\ l\ H\ D)\big) \\ \ \to \big(\forall v\ l\ D, P\ v\ l\ D\big). \end{array}$$

Simulated Inductive-Recursive Scheme

- Thus we simulate the IR-scheme (Dybjer 2000)

$$
\begin{array}{l}
\texttt{Inductive}\ \mathbb{D}_{\texttt{dfs}} : \texttt{list}\ \mathcal{V} \to \texttt{list}\ \mathcal{V} \to \boxed{\texttt{Prop}} := \\
\quad \mid \mathbb{D}^1_{\texttt{dfs}} : \forall v, \qquad\qquad\qquad\quad \mathbb{D}_{\texttt{dfs}}\ v\ [] \\
\quad \mid \mathbb{D}^2_{\texttt{dfs}} : \forall v\,x\,l,\ x \in^?_{\mathcal{V}} v \to \mathbb{D}_{\texttt{dfs}}\ v\ l \\
\qquad\qquad\qquad\qquad\qquad\quad \to \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l) \\
\quad \mid \mathbb{D}^3_{\texttt{dfs}} : \forall v\,x\,l,\ x \notin^?_{\mathcal{V}} v \to \mathbb{D}_{\texttt{dfs}}\ (x :: v)\ (\texttt{succs}\ x +\!\!+ l) \\
\qquad\qquad\qquad\qquad\qquad\quad \to \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l) \\
\\
\texttt{with Fixpoint dfs}\ v\ l\ (D : \mathbb{D}_{\texttt{dfs}}\ v\ l) : \texttt{list}\ \mathcal{V} := \\
\quad \texttt{match}\ D\ \texttt{with} \\
\quad \mid \mathbb{D}^1_{\texttt{dfs}}\ v \qquad\qquad \Rightarrow v \\
\quad \mid \mathbb{D}^2_{\texttt{dfs}}\ v\ x\ l\ \_\ D \Rightarrow \texttt{dfs}\ v\ l\ D \\
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\qquad\qquad\qquad\qquad\qquad\quad \to \mathbb{D}_{\texttt{dfs}}\ v\ (x :: l) \\
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- Degenerate here because no nesting

High-level partial correctness for DFS

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$\mathtt{Definition}\ \ \mathtt{dfs\_invariant}_t\ (v\ l\ i : \mathtt{list}\,\mathcal{V}) :=$
$v \subseteq i \wedge l \subseteq i \wedge (\forall x,\ x \in^?_{\mathcal{V}} i \to x \in^?_{\mathcal{V}} v \vee \mathtt{succs}\, x \subseteq i).$

$\mathtt{Theorem}\ \ \mathtt{dfs\_invariant}\ v\ l\ (D : \mathbb{D}_{\mathtt{dfs}}\, v\, l) :$

$$\wedge \begin{cases} \mathtt{dfs\_invariant}_t\ v\ l\ (\mathtt{dfs}\ v\ l\ D) \\ \forall i,\ \mathtt{dfs\_invariant}_t\ v\ l\ i \to \mathtt{dfs}\ v\ l\ D \subseteq i. \end{cases}$$

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  - when there is an invariant
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$$\texttt{Theorem}\ \mathbb{D}_{\texttt{dfs}}\texttt{\_domain}\ v\ l :$$
$$\mathbb{D}_{\texttt{dfs}}\, v\, l \iff \exists i, \texttt{dfs\_invariant}_t\ v\ l\ i.$$

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- No high-level knowledge of φ needed
  - Termination is not needed for partial correctness
  - Partial correctness could be use for termination
- Beware with nested algos. (see later)
  - Projecting the graph a bit more complicated

Back to F91 (generalized)

- $a : X \to \texttt{bool}$ and $b, c : X \to X$

$$f\ x = \texttt{if}\ a\,x\ \texttt{then}\ b\,x\ \texttt{else}\ f(f(c\,x))$$

- The graph $\mathbb{G}_f : X \to X \to \texttt{Prop}$ of f

$$\frac{a\,x = \texttt{true}}{\mathbb{G}_f\ x\ (b\,x)} \qquad \frac{a\,x = \texttt{false} \quad \mathbb{G}_f\ (c\,x)\ y \quad \mathbb{G}_f\ y\ o}{\mathbb{G}_f\ x\ o}$$

- The domain $\mathbb{D}_f : X \to \texttt{Prop}$ of f

$$\frac{a\,x = \texttt{true}}{\mathbb{D}_f\ x} \qquad \frac{\begin{array}{c} a\,x = \texttt{false} \\ \mathbb{D}_f(c\,x) \\ \forall y, \mathbb{G}_f\ (c\,x)\ y \to \mathbb{D}_f\ y \end{array}}{\mathbb{D}_f\ x}$$

Simulated IR Scheme for F91 (gen.)

- We simulate the following IR-scheme

$$
\begin{array}{l}
\texttt{Inductive}\ \mathbb{D}_f : X \to \texttt{Prop} := \\
\quad \mid \mathbb{D}_f^0 : \forall x,\ a\,x = \texttt{true} \to \mathbb{D}_f\, x \\
\quad \mid \mathbb{D}_f^1 : \forall x,\ a\,x = \texttt{false} \\
\qquad\quad \to \forall dc : \mathbb{D}_f(c\,x),\ \mathbb{D}_f(f\,(c\,x)\,dc) \to \mathbb{D}_f\, x \\
\texttt{with Fixpoint}\ f\ x\ (d : \mathbb{D}_f\, x) : X := \\
\ \ \texttt{match}\ d\ \texttt{with} \\
\quad \mid \mathbb{D}_f^0\ x\ e \Rightarrow b\,x \\
\quad \mid \mathbb{D}_f^1\ x\ e\ (dc : \mathbb{D}_f(c\,x))\ (df : \mathbb{D}_f(f\,(c\,x)\,dc)) \\
\qquad\qquad \Rightarrow f\,(f\,(c\,x)\,dc)\,df \\
\ \ \texttt{end}
\end{array}
$$

- But restricted to proof-irrelevant predicates (PIRR)
  - f itself is PIRR: $f\,x\,d_1 = f\,x\,d_2$

Third take home ideas

- Domain $\mathbb{D}$ for nested schemes
  - use $\mathbb{G}$ to characterize (nested) output values
  - and define $\mathbb{D}$ after&using $\mathbb{G}$

Third take home ideas

- Domain $\mathbb{D}$ for nested schemes
  - use $\mathbb{G}$ to characterize (nested) output values
  - and define $\mathbb{D}$ after&using $\mathbb{G}$
- Correctness of nested schemes
  - can be studied independently of termination
  - hence, can be used to establish termination

Conclusion

- The Braga method separates tasks
  - definition of the function in Coq
  - prove its partial correctness (IR or graph ind.)
  - prove (partial) termination (from correctness)

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  - erases the Logical Contents (LC)
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  - give access to partial algorithms
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Conclusion

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  - definition of the function in Coq
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  - give access to partial algorithms
  - incl. nested and mutual
- Perspectives
  - better integrate with existing tools
  - more examples, e.g. Knuth k_{91}, μ-rec. algos.
  - partial functions as guarded total functions

Larry Paulson's normalization (1985)

$$
\begin{array}{l}
\texttt{let rec nm } e = \texttt{match } e \texttt{ with} \\
\quad \begin{array}{ll}
\mid \alpha & \to \alpha \\
\mid \omega(\alpha, y, z) & \to \omega(\alpha, \texttt{nm } y, \texttt{nm } z) \\
\mid \omega(\omega(a, b, c), y, z) & \to \texttt{nm}(\omega(a, \texttt{nm}(\omega(b, y, z)), \\
 & \qquad\qquad\qquad \texttt{nm}(\omega(c, y, z))))
\end{array}
\end{array}
$$

- Expressions in Ω : $b, x, y ::= \alpha \mid \omega\ b\ x\ y$
  - α is atomic expression
  - $\omega\ b\ x\ y$ denotes "$\texttt{if}\ b\ \texttt{then}\ x\ \texttt{else}\ y$"

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- Expressions in $\Omega : b, x, y ::= \alpha \mid \omega\ b\ x\ y$
  - α is atomic expression
  - $\omega\ b\ x\ y$ denotes "$\texttt{if } b \texttt{ then } x \texttt{ else } y$"
- Interest of this algorithm:
  - recurring example (Giesl 97, B&C 05...)
  - has nested recursion but still compact
  - idealized but meaningful

Inductive capture of $\mathbb{D}_{\mathtt{nm}} : \Omega \to \mathtt{Prop}$

- Using the computational graph $\mathbb{G}_{\mathtt{nm}} : \Omega \to \Omega \to \mathtt{Prop}$

$$\frac{}{\mathbb{G}_{\mathtt{nm}}\ \alpha\ \alpha} \qquad \frac{\mathbb{G}_{\mathtt{nm}}\ y\ n_y \qquad \mathbb{G}_{\mathtt{nm}}\ z\ n_z}{\mathbb{G}_{\mathtt{nm}}\ (\omega\ \alpha\ y\ z)\ (\omega\ \alpha\ n_y\ n_z)}$$

$$\frac{\mathbb{G}_{\mathtt{nm}}\ (\omega\ b\ y\ z)\ n_b \qquad \mathbb{G}_{\mathtt{nm}}\ (\omega\ c\ y\ z)\ n_c \qquad \mathbb{G}_{\mathtt{nm}}\ (\omega\ a\ n_b\ n_c)\ n_a}{\mathbb{G}_{\mathtt{nm}}\ (\omega\ (\omega\ a\ b\ c)\ y\ z)\ n_a}$$

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- Define $\mathbb{D}_{\mathtt{nm}} \simeq \lambda\, e, \exists n, \mathbb{G}_{\mathtt{nm}}\, e\, n$ inductively by:

$$\frac{}{\mathbb{D}_{\mathtt{nm}}\ \alpha} \qquad \frac{\mathbb{D}_{\mathtt{nm}}\ y \qquad \mathbb{D}_{\mathtt{nm}}\ z}{\mathbb{D}_{\mathtt{nm}}\ (\omega\ \alpha\ y\ z)}$$

$$\frac{\begin{array}{c}\mathbb{D}_{\mathtt{nm}}\ (\omega\ b\ y\ z) \qquad \mathbb{D}_{\mathtt{nm}}\ (\omega\ c\ y\ z) \\ \forall n_b\, n_c, \mathbb{G}_{\mathtt{nm}}\ (\omega\ b\ y\ z)\ n_b \to \mathbb{G}_{\mathtt{nm}}\ (\omega\ c\ y\ z)\ n_c \to \mathbb{D}_{\mathtt{nm}}\ (\omega\ a\ n_b\ n_c)\end{array}}{\mathbb{D}_{\mathtt{nm}}\ (\omega\ (\omega\ a\ b\ c)\ y\ z)}$$

Inductive capture of $\mathbb{D}_{\mathtt{nm}} : \Omega \to \mathtt{Prop}$

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$$\frac{}{\mathbb{G}_{\mathtt{nm}}\ \alpha\ \alpha} \qquad \frac{\mathbb{G}_{\mathtt{nm}}\ y\ n_y \qquad \mathbb{G}_{\mathtt{nm}}\ z\ n_z}{\mathbb{G}_{\mathtt{nm}}\ (\omega\ \alpha\ y\ z)\ (\omega\ \alpha\ n_y\ n_z)}$$

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- The rules for $\mathbb{D}_{\mathtt{nm}}$ use $\mathbb{G}_{\mathtt{nm}}$ **for nested calls**

Def. $\texttt{nm\_pwc} : \forall e,\ \mathbb{D}_{\texttt{nm}}\ e \to \{n \mid \mathbb{G}_{\texttt{nm}}\ e\ n\}$

$$
\begin{array}{l}
\texttt{Fixpoint nm\_pwc}\ e\ (\underline{D} : \mathbb{D}_{\texttt{nm}}\ e) : \{n \mid \mathbb{G}_{\texttt{nm}}\ e\ n\}.\\
\texttt{refine(}\\
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\qquad \mid \omega\ \alpha\ y\ z \qquad\quad\ \Rightarrow \lambda D,\ \texttt{let}\ (n_y, C_y) := \texttt{nm\_pwc}\ y\ \mathcal{T}_y^?\ \texttt{in}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \texttt{let}\ (n_z, C_z) := \texttt{nm\_pwc}\ z\ \mathcal{T}_z^?\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \texttt{in}\ \ \texttt{exist}\ \_\ (\omega\ \alpha\ n_y\ n_z)\ \mathcal{O}_1^?\\
\qquad \mid \omega\ (\omega\ a\ b\ c)\ y\ z \Rightarrow \lambda D,\ \texttt{let}\ (n_b, C_b) := \texttt{nm\_pwc}\ (\omega\ b\ y\ z)\ \mathcal{T}_b^?\ \texttt{in}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \texttt{let}\ (n_c, C_c) := \texttt{nm\_pwc}\ (\omega\ c\ y\ z)\ \mathcal{T}_c^?\ \texttt{in}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \texttt{let}\ (n_a, C_a) := \texttt{nm\_pwc}\ (\omega\ a\ n_b\ n_c)\ \mathcal{T}_a^?\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \texttt{in}\ \ \texttt{exist}\ \_\ n_a\ \mathcal{O}_2^?\\
\quad \texttt{end}\ D\texttt{);}\ \texttt{simpl in}\ *.\\
\texttt{Proof.}\ \text{ of certificates } \mathcal{T}_y^?, \mathcal{T}_z^?, \mathcal{T}_b^?, \mathcal{T}_c^?, \mathcal{T}_a^? \text{ and post-conditions } \mathcal{O}_0^?, \mathcal{O}_1^?, \mathcal{O}_2^?\ \ \texttt{Qed.}
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$$

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\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \texttt{in}\ \ \texttt{exist}\ \_\ (\omega\ \alpha\ n_y\ n_z)\ \mathcal{O}_1^? \\
\qquad \mid \omega\ (\omega\ a\ b\ c)\ y\ z \Rightarrow \lambda D,\ \texttt{let}\ (n_b, C_b) := \texttt{nm\_pwc}\ (\omega\ b\ y\ z)\ \mathcal{T}_b^?\ \texttt{in} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \texttt{let}\ (n_c, C_c) := \texttt{nm\_pwc}\ (\omega\ c\ y\ z)\ \mathcal{T}_c^?\ \texttt{in} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \texttt{let}\ (n_a, C_a) := \texttt{nm\_pwc}\ (\omega\ a\ n_b\ n_c)\ \mathcal{T}_a^? \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \texttt{in}\ \ \texttt{exist}\ \_\ n_a\ \mathcal{O}_2^? \\
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\end{array}
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- use of dependent pattern matching

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\qquad \begin{array}{lll}
\mid \alpha & \Rightarrow \lambda D, & \texttt{exist \_}\ \alpha\ \mathcal{O}_0^?\\
\mid \omega\ \alpha\ y\ z & \Rightarrow \lambda D, & \texttt{let}\ (n_y, C_y) := \texttt{nm\_pwc}\ y\ \mathcal{T}_y^?\ \texttt{in}\\
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 & & \texttt{in}\ \ \texttt{exist \_}\ (\omega\ \alpha\ n_y\ n_z)\ \mathcal{O}_1^?\\
\mid \omega\ (\omega\ a\ b\ c)\ y\ z & \Rightarrow \lambda D, & \texttt{let}\ (n_b, C_b) := \texttt{nm\_pwc}\ (\omega\ b\ y\ z)\ \mathcal{T}_b^?\ \texttt{in}\\
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 & & \texttt{in}\ \ \texttt{exist \_}\ n_a\ \mathcal{O}_2^?
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$$

- use of dependent pattern matching
- LC (i.e. proof obligations) separated from CC

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\quad\quad \mid \omega\ \alpha\ y\ z \quad \Rightarrow \lambda D,\ \mathtt{let}\ (n_y, C_y) := \mathtt{nm\_pwc}\ y\ \mathcal{T}_y^?\ \mathtt{in}\\
\qquad\qquad\qquad\qquad \mathtt{let}\ (n_z, C_z) := \mathtt{nm\_pwc}\ z\ \mathcal{T}_z^?\\
\qquad\qquad\qquad\qquad \mathtt{in}\ \ \mathtt{exist}\ \_\ (\omega\ \alpha\ n_y\ n_z)\ \mathcal{O}_1^?\\
\quad\quad \mid \omega\ (\omega\ a\ b\ c)\ y\ z \Rightarrow \lambda D,\ \mathtt{let}\ (n_b, C_b) := \mathtt{nm\_pwc}\ (\omega\ b\ y\ z)\ \mathcal{T}_b^?\ \mathtt{in}\\
\qquad\qquad\qquad\qquad \mathtt{let}\ (n_c, C_c) := \mathtt{nm\_pwc}\ (\omega\ c\ y\ z)\ \mathcal{T}_c^?\ \mathtt{in}\\
\qquad\qquad\qquad\qquad \mathtt{let}\ (n_a, C_a) := \mathtt{nm\_pwc}\ (\omega\ a\ n_b\ n_c)\ \mathcal{T}_a^?\\
\qquad\qquad\qquad\qquad \mathtt{in}\ \ \mathtt{exist}\ \_\ n_a\ \mathcal{O}_2^?\\
\quad \mathtt{end}\ D);\ \mathtt{simpl}\ \mathtt{in}\ *.\\
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\end{array}
$$

- use of dependent pattern matching
- LC (i.e. proof obligations) separated from CC
- LC divided: termination certificates, post-conditions

Proof obligations (Logical Contents)

- Post-conditions by the constructors of $\mathbb{G}_{\mathtt{nm}}$

$$
\begin{array}{lll}
\mathcal{O}_0^? & /\!\!/ & \ldots \vdash \mathbb{G}_{\mathtt{nm}}\ \alpha\ \alpha \\
\mathcal{O}_1^? & /\!\!/ & \ldots, C_y : \mathbb{G}_{\mathtt{nm}}\ y\ n_y, C_z : \mathbb{G}_{\mathtt{nm}}\ z\ n_z \vdash \mathbb{G}_{\mathtt{nm}}\ (\omega\ \alpha\ y\ z)\ (\omega\ \alpha\ n_y\ n_z) \\
\mathcal{O}_2^? & /\!\!/ & \ldots, C_b : \mathbb{G}_{\mathtt{nm}}\ (\omega\ b\ y\ z)\ n_b, C_c : \mathbb{G}_{\mathtt{nm}}\ (\omega\ c\ y\ z)\ n_c, \ldots \\
 & & \quad \ldots C_a : \mathbb{G}_{\mathtt{nm}}\ (\omega\ a\ n_b\ n_c)\ n_a \vdash \mathbb{G}_{\mathtt{nm}}\ (\omega\ (\omega\ a\ b\ c)\ y\ z)\ n_a
\end{array}
$$

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- Post-conditions by the constructors of $\mathbb{G}_{\mathrm{nm}}$

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\mathcal{O}_2^? & /\!/ & \ldots, C_b : \mathbb{G}_{\mathrm{nm}}\ (\omega\ b\ y\ z)\ n_b, C_c : \mathbb{G}_{\mathrm{nm}}\ (\omega\ c\ y\ z)\ n_c, \ldots \\
 & & \quad \ldots C_a : \mathbb{G}_{\mathrm{nm}}\ (\omega\ a\ n_b\ n_c)\ n_a \vdash \mathbb{G}_{\mathrm{nm}}\ (\omega\ (\omega\ a\ b\ c)\ y\ z)\ n_a
\end{array}$$

- Termination certificates

$$\begin{array}{lll}
\mathcal{T}_y^? & /\!/ & \ldots, D : \mathbb{D}_{\mathrm{nm}}\ (\omega\ \alpha\ y\ z) \vdash \mathbb{D}_{\mathrm{nm}}\ y \\
\mathcal{T}_b^? & /\!/ & \ldots, D : \mathbb{D}_{\mathrm{nm}}\ (\omega\ (\omega\ a\ b\ c)\ y\ z) \vdash \mathbb{D}_{\mathrm{nm}}\ (\omega\ b\ y\ z) \\
\mathcal{T}_a^? & /\!/ & \ldots, D : \mathbb{D}_{\mathrm{nm}}\ (\omega\ (\omega\ a\ b\ c)\ y\ z), H_b : \mathbb{G}_{\mathrm{nm}}\ (\omega\ b\ y\ z)\ n_b, \ldots \\
 & & \quad \ldots H_c : \mathbb{G}_{\mathrm{nm}}\ (\omega\ c\ y\ z)\ n_c \vdash \mathbb{D}_{\mathrm{nm}}\ (\omega\ a\ n_b\ n_c)
\end{array}$$

- beware of $\boxed{\text{structural decrease}}$ in term. certificates
  - by the `inversion` tactic
  - or "small inversion" (human readable)

Simulated IR scheme

- $\texttt{nm}\ e\ D := \pi_1(\texttt{nm\_pwc}\ e\ D)$ and $\pi_2 : \mathbb{G}_{\texttt{nm}}\ e\ (\texttt{nm}\ e\ D)$

Simulated IR scheme

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$$
\begin{array}{l}
\mathtt{Inductive}\ \ \mathbb{D}_{\mathtt{nm}} : \Omega \to \boxed{\mathtt{Prop}} := \\
\quad \mid \mathbb{D}^1_{\mathtt{nm}} \qquad\qquad\qquad\qquad : \ \mathbb{D}_{\mathtt{nm}}\ \alpha \\
\quad \mid \mathbb{D}^2_{\mathtt{nm}}\ y\ z \qquad\qquad\qquad : \ \mathbb{D}_{\mathtt{nm}}\ y \to \mathbb{D}_{\mathtt{nm}}\ z \to \mathbb{D}_{\mathtt{nm}}(\omega\ \alpha\ y\ z) \\
\quad \mid \mathbb{D}^3_{\mathtt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c \quad : \ \mathbb{D}_{\mathtt{nm}}(\omega\ a\ (\mathtt{nm}\ (\omega\ b\ y\ z)\ D_b)\ (\mathtt{nm}\ (\omega\ c\ y\ z)\ D_c)) \\
\qquad\qquad\qquad\qquad\qquad\quad \to \mathbb{D}_{\mathtt{nm}}(\omega\ (\omega\ a\ b\ c)\ y\ z) \\
\mathtt{with\ Fixpoint}\ \ \mathtt{nm}\ e\ (D_e : \mathbb{D}_{\mathtt{nm}}\ e) : \Omega := \\
\ \mathtt{match}\ D_e\ \mathtt{with} \\
\quad \mid \mathbb{D}^1_{\mathtt{nm}} \qquad\qquad\qquad\qquad \Rightarrow \alpha \\
\quad \mid \mathbb{D}^2_{\mathtt{nm}}\ y\ z\ D_y\ D_z \qquad\quad \Rightarrow \omega\ \alpha\ (\mathtt{nm}\ y\ D_y)\ (\mathtt{nm}\ z\ D_z) \\
\quad \mid \mathbb{D}^3_{\mathtt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c\ D_a \Rightarrow \mathtt{nm}\ \big(\omega\ a\ (\mathtt{nm}\ (\omega\ b\ y\ z)\ D_b) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathtt{nm}\ (\omega\ c\ y\ z)\ D_c)\big)\ D_a \\
\ \mathtt{end.}
\end{array}
$$

Simulated IR scheme

- $\mathtt{nm}\ e\ D := \pi_1(\mathtt{nm\_pwc}\ e\ D)$ and $\pi_2 : \mathbb{G}_{\mathtt{nm}}\ e\ (\mathtt{nm}\ e\ D)$

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathtt{Inductive}\ \ \mathbb{D}_{\mathtt{nm}} : \Omega \to \boxed{\mathtt{Prop}} :=} \\
\quad \mid \mathbb{D}^1_{\mathtt{nm}} & : & \mathbb{D}_{\mathtt{nm}}\ \alpha \\
\quad \mid \mathbb{D}^2_{\mathtt{nm}}\ y\ z & : & \mathbb{D}_{\mathtt{nm}}\ y \to \mathbb{D}_{\mathtt{nm}}\ z \to \mathbb{D}_{\mathtt{nm}}(\omega\ \alpha\ y\ z) \\
\quad \mid \mathbb{D}^3_{\mathtt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c & : & \mathbb{D}_{\mathtt{nm}}(\omega\ a\ (\mathtt{nm}\ (\omega\ b\ y\ z)\ D_b)\ (\mathtt{nm}\ (\omega\ c\ y\ z)\ D_c)) \\
 & \to & \mathbb{D}_{\mathtt{nm}}(\omega\ (\omega\ a\ b\ c)\ y\ z) \\
\multicolumn{3}{l}{\mathtt{with\ Fixpoint}\ \ \mathtt{nm}\ e\ (D_e : \mathbb{D}_{\mathtt{nm}}\ e) : \Omega :=} \\
\multicolumn{3}{l}{\ \mathtt{match}\ D_e\ \mathtt{with}} \\
\quad \mid \mathbb{D}^1_{\mathtt{nm}} & \Rightarrow & \alpha \\
\quad \mid \mathbb{D}^2_{\mathtt{nm}}\ y\ z\ D_y\ D_z & \Rightarrow & \omega\ \alpha\ (\mathtt{nm}\ y\ D_y)\ (\mathtt{nm}\ z\ D_z) \\
\quad \mid \mathbb{D}^3_{\mathtt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c\ D_a & \Rightarrow & \mathtt{nm}\ (\omega\ a\ (\mathtt{nm}\ (\omega\ b\ y\ z)\ D_b) \\
 & & \qquad\qquad (\mathtt{nm}\ (\omega\ c\ y\ z)\ D_c))\ D_a \\
\multicolumn{3}{l}{\ \mathtt{end.}}
\end{array}
$$

- The domain $\mathbb{D}_{\mathtt{nm}} : \Omega \to \mathtt{Prop}$ is **non-informative**

Simulated IR scheme

- $\mathtt{nm}\ e\ D := \pi_1(\mathtt{nm\_pwc}\ e\ D)$ and $\pi_2 : \mathbb{G}_{\mathtt{nm}}\ e\ (\mathtt{nm}\ e\ D)$

$$
\begin{array}{ll}
\texttt{Inductive}\ \mathbb{D}_{\mathtt{nm}} : \Omega \to \boxed{\texttt{Prop}} := & \\
\mid \mathbb{D}^1_{\mathtt{nm}} & : \mathbb{D}_{\mathtt{nm}}\ \alpha \\
\mid \mathbb{D}^2_{\mathtt{nm}}\ y\ z & : \mathbb{D}_{\mathtt{nm}}\ y \to \mathbb{D}_{\mathtt{nm}}\ z \to \mathbb{D}_{\mathtt{nm}}(\omega\ \alpha\ y\ z) \\
\mid \mathbb{D}^3_{\mathtt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c & : \mathbb{D}_{\mathtt{nm}}(\omega\ a\ (\mathtt{nm}\ (\omega\ b\ y\ z)\ D_b)\ (\mathtt{nm}\ (\omega\ c\ y\ z)\ D_c)) \\
& \to \mathbb{D}_{\mathtt{nm}}(\omega\ (\omega\ a\ b\ c)\ y\ z) \\
\texttt{with Fixpoint}\ \mathtt{nm}\ e\ (D_e : \mathbb{D}_{\mathtt{nm}}\ e) : \Omega := & \\
\quad \texttt{match}\ D_e\ \texttt{with} & \\
\mid \mathbb{D}^1_{\mathtt{nm}} & \Rightarrow \alpha \\
\mid \mathbb{D}^2_{\mathtt{nm}}\ y\ z\ D_y\ D_z & \Rightarrow \omega\ \alpha\ (\mathtt{nm}\ y\ D_y)\ (\mathtt{nm}\ z\ D_z) \\
\mid \mathbb{D}^3_{\mathtt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c\ D_a & \Rightarrow \mathtt{nm}\ (\omega\ a\ (\mathtt{nm}\ (\omega\ b\ y\ z)\ D_b) \\
& \qquad (\mathtt{nm}\ (\omega\ c\ y\ z)\ D_c))\ D_a \\
\quad \texttt{end.} &
\end{array}
$$

- The domain $\mathbb{D}_{\mathtt{nm}} : \Omega \to \texttt{Prop}$ is **non-informative**
- $\mathtt{nm} : \forall e, \mathbb{D}_{\mathtt{nm}}\ e \to \Omega$ is **proof-irrelevant**, i.e. $\mathtt{nm}\ x\ D_1 = \mathtt{nm}\ x\ D_2$

Simulated IR scheme

- $\texttt{nm}\ e\ D := \pi_1(\texttt{nm\_pwc}\ e\ D)$ and $\pi_2 : \mathbb{G}_{\texttt{nm}}\ e\ (\texttt{nm}\ e\ D)$

$$
\begin{array}{l}
\texttt{Inductive}\ \ \mathbb{D}_{\texttt{nm}} : \Omega \to \boxed{\texttt{Prop}} := \\
\quad \mid \mathbb{D}^1_{\texttt{nm}} \hspace{12em} : \mathbb{D}_{\texttt{nm}}\ \alpha \\
\quad \mid \mathbb{D}^2_{\texttt{nm}}\ y\ z \hspace{10em} : \mathbb{D}_{\texttt{nm}}\ y \to \mathbb{D}_{\texttt{nm}}\ z \to \mathbb{D}_{\texttt{nm}}(\omega\ \alpha\ y\ z) \\
\quad \mid \mathbb{D}^3_{\texttt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c \hspace{5em} : \mathbb{D}_{\texttt{nm}}(\omega\ a\ (\texttt{nm}\ (\omega\ b\ y\ z)\ D_b)\ (\texttt{nm}\ (\omega\ c\ y\ z)\ D_c)) \\
\hspace{16em} \to \mathbb{D}_{\texttt{nm}}(\omega\ (\omega\ a\ b\ c)\ y\ z) \\
\texttt{with Fixpoint}\ \ \texttt{nm}\ e\ (D_e : \mathbb{D}_{\texttt{nm}}\ e) : \Omega := \\
\ \texttt{match}\ D_e\ \texttt{with} \\
\quad \mid \mathbb{D}^1_{\texttt{nm}} \hspace{12em} \Rightarrow \alpha \\
\quad \mid \mathbb{D}^2_{\texttt{nm}}\ y\ z\ D_y\ D_z \hspace{7em} \Rightarrow \omega\ \alpha\ (\texttt{nm}\ y\ D_y)\ (\texttt{nm}\ z\ D_z) \\
\quad \mid \mathbb{D}^3_{\texttt{nm}}\ a\ b\ c\ y\ z\ D_b\ D_c\ D_a \Rightarrow \texttt{nm}\ (\omega\ a\ (\texttt{nm}\ (\omega\ b\ y\ z)\ D_b) \\
\hspace{22em} (\texttt{nm}\ (\omega\ c\ y\ z)\ D_c))\ D_a \\
\ \texttt{end.}
\end{array}
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- $\texttt{nm} : \forall e, \mathbb{D}_{\texttt{nm}}\ e \to \Omega$ is **proof-irrelevant**, i.e. $\texttt{nm}\ x\ D_1 = \texttt{nm}\ x\ D_2$
- Constructors, dep. elim. scheme and fixpoint equations *retrieved*

Extraction unaltered by $\mathbb{D}_{\texttt{nm}}$ in $\texttt{Prop}$

- In $\texttt{nm}\ e\ (D : \mathbb{D}_{\texttt{nm}}\ e)$ extract. erases $D : \mathbb{D}_{\texttt{nm}}\ e : \texttt{Prop}$

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- Hence Extraction nm gives the intended term:

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\texttt{let rec nm}\ e = \texttt{match}\ e\ \texttt{with} \\
\quad \mid \alpha \qquad\qquad\quad \to \alpha \\
\quad \mid \omega(x,y,z) \quad\ \to \texttt{match}\ x\ \texttt{with} \\
\qquad \mid \alpha \qquad\qquad \to \omega(\alpha, \texttt{nm}\ y, \texttt{nm}\ z) \\
\qquad \mid \omega(a,b,c) \to \texttt{nm}(\omega(a, \texttt{nm}(\omega(b,y,z)), \texttt{nm}(\omega(c,y,z))))
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- The proof term $D : \mathbb{D}_{\mathtt{nm}}\ e$
  - has **no impact** on extracted algorithm
  - great complexity does not matter
  - use high-level tool (lex. prod, WQOs)

Termination postponed after definition

- Proving termination of $\texttt{nm}$ at e is a term $D : \mathbb{D}_{\texttt{nm}}\ e$
  - a "meaningful" characterization of $\mathbb{D}_{\texttt{nm}}\ e$
  - for partial fun.: $P : \Omega \to \texttt{Prop}$ and $P \subseteq \mathbb{D}_{\texttt{nm}}$
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- $\texttt{nm\_dec} : \forall\, e\ (D : \mathbb{D}_{\texttt{nm}}\ e),\ |\texttt{nm}\ e\ D| \leqslant |e|$
  - some "size" $|\cdot| : \Omega \to \texttt{nat}$ is preserved (Giesl 97)

$$|\alpha| = 1 \qquad |\omega\ x\ y\ z| = |x| \cdot (1 + |y| + |z|)$$

Totality of $\mathbb{D}_{\mathtt{nm}}$ / Termination of $\mathtt{nm}$

$$\mathbb{D}_{\mathtt{nm}}\mathtt{\_total} : \forall e,\ \mathbb{D}_{\mathtt{nm}}\ e$$

- By induction on the size $|e|$

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  - we use $\mathtt{nm\_dec} : \forall\, e\ (D : \mathbb{D}_{\mathtt{nm}}\ e),\ |\mathtt{nm}\ e\ D| \leqslant |e|$
  - and $|\omega\ x\ y\ z| \leqslant |\omega\ x'\ y'\ z'|$ (monotonic)
    - i.e. when $|x| \leqslant |x'|$, $|y| \leqslant |y'|$, $|z| \leqslant |z'|$
  - and $|\omega\ u\ y\ z| < |\omega\ v\ y\ z|$ when $|u| < |v|$
  - and $|y| < |\omega\ x\ y\ z|$ and $|z| < |\omega\ x\ y\ z|$
  - & $|\omega\ a\ (\omega\ b\ y\ z)\ (\omega\ c\ y\ z)| < |\omega\ (\omega\ a\ b\ c)\ y\ z|$
- Partial correction / termination indep. of definition

$$\boxed{\mathtt{paulson\_nm} : \forall e : \Omega,\ \{n_e : \Omega \mid e \simeq_\Omega n_e \wedge \mathtt{normal}\ e\}}$$