From triangles to curves

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22nd European Workshop on Computational Geometry
March 2006 - $\Delta\varepsilon\lambda\varphi o\acute{\iota}$

Warning

- focus on practical methods
- non exhaustive, biased

mostly (not only)

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“Commercial”: ECG book coming out soon...

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- non exhaustive, biased

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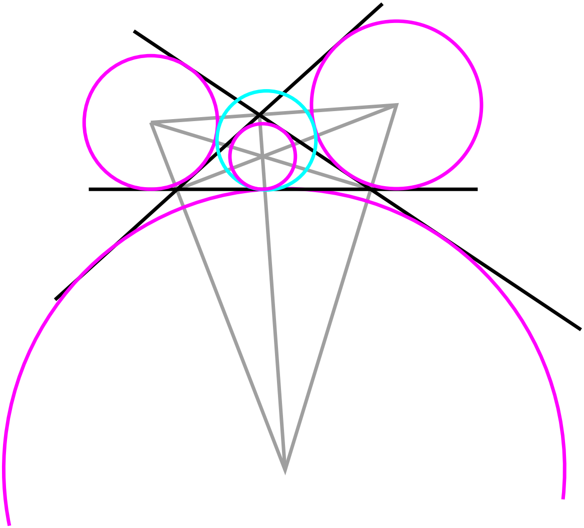
Advice to people having some knowledge of Computer Algebra:
you may leave the room

- non technical, superficial. . .

Circles are never far from triangles



Construction of curves from lines

Parabola: smooth connection between line segments

Du point Q on peut mener deux tangentes.

II. — *Tracer, parallèlement à une droite donnée* RR′, *une tangente à la parabole* (*fig.* 107).

Abaisser du foyer F une perpendiculaire FM sur RR′, puis élever une perpendiculaire QS au milieu de MF.

§ 93. **Raccord obtenu par un arc de parabole.** — Décrire une parabole, et mener deux tangentes QC et QC′ (*fig.* 108); tracer la corde des contacts CC′, soit ED une tangente quelconque. Porter CD en QD_1, et joindre D_1E ; constater que D_1E est parallèle à CC′.

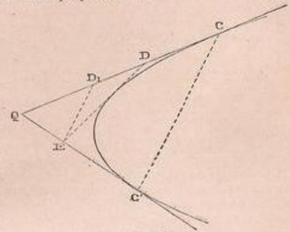


Fig. 108.

De cette constatation nous tirons la conclusion suivante :

$$\frac{QD_1}{QC} = \frac{QE}{QC'},$$

que nous pouvons énoncer ainsi :

Les points d'intersection D *et* E *partagent les tangentes* QC *et* QC′ *en segments proportionnels inversement placés par rapport au point* Q.

Proposons-nous de raccorder par un arc de parabole les deux directrices concourantes QC et QC′ (*fig.* 109) :

Partager les distances QC et QC′ en un même nombre de parties égales, cinq par exemple, et numéroter les points de division de

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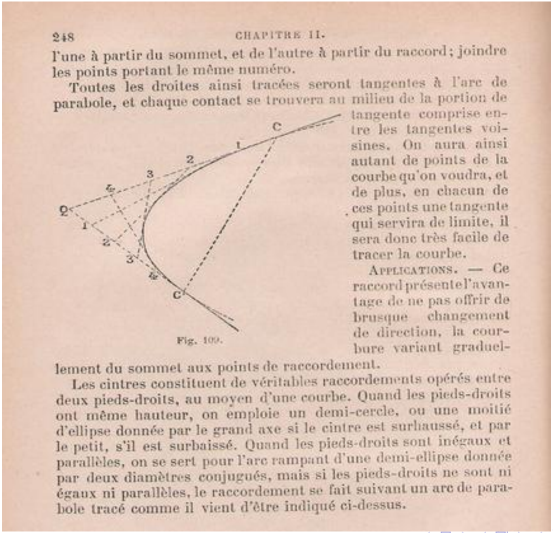


l'une à partir du sommet, et de l'autre à partir du raccord; joindre les points portant le même numéro.

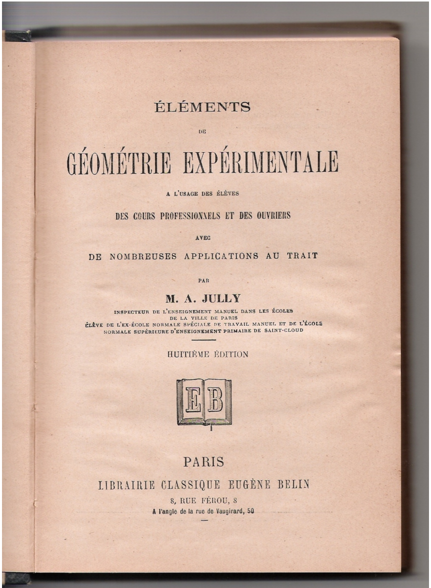
Toutes les droites ainsi tracées seront tangentes à l'arc de parabole, et chaque contact se trouvera au milieu de la portion de tangente comprise entre les tangentes voisines. On aura ainsi autant de points de la courbe qu'on voudra, et de plus, en chacun de ces points une tangente qui servira de limite, il sera donc très facile de tracer la courbe.

Fig. 109.

APPLICATIONS. — Ce raccord présente l'avantage de ne pas offrir de brusque changement de direction, la courbure variant graduellement du sommet aux points de raccordement.

Les cintres constituent de véritables raccordements opérés entre deux pieds-droits, au moyen d'une courbe. Quand les pieds-droits ont même hauteur, on emploie un demi-cercle, ou une moitié d'ellipse donnée par le grand axe si le cintre est surhaussé, et par le petit, s'il est surbaissé. Quand les pieds-droits sont inégaux et parallèles, on se sert pour l'arc rampant d'une demi-ellipse donnée par deux diamètres conjugués, mais si les pieds-droits ne sont ni égaux ni parallèles, le raccordement se fait suivant un arc de parabole tracé comme il vient d'être indiqué ci-dessus.

Construction of curves from lines



ÉLÉMENTS

DE

GÉOMÉTRIE EXPÉRIMENTALE

A L'USAGE DES ÉLÈVES

DES COURS PROFESSIONNELS ET DES OUVRIERS

AVEC

DE NOMBREUSES APPLICATIONS AU TRAIT

PAR

M. A. JULLY

INSPECTEUR DE L'ENSEIGNEMENT MANUEL DANS LES ÉCOLES
DE LA VILLE DE PARIS
ÉLÈVE DE L'EX-ÉCOLE NORMALE SPÉCIALE DE TRAVAIL MANUEL ET DE L'ÉCOLE
NORMALE SUPÉRIEURE D'ENSEIGNEMENT PRIMAIRE DE SAINT-CLOUD

HUITIÈME ÉDITION

EB

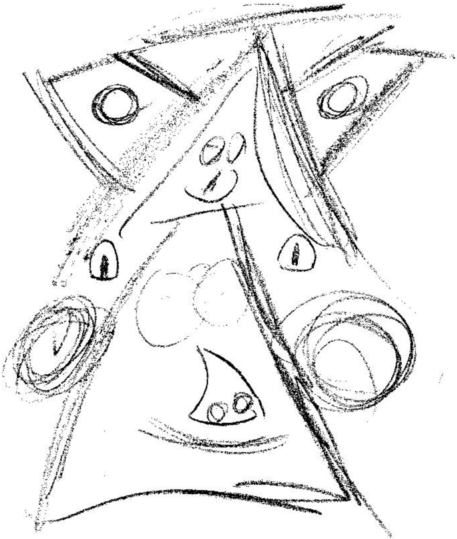
PARIS

LIBRAIRIE CLASSIQUE EUGÈNE BELIN

8, RUE FÉROU, 8

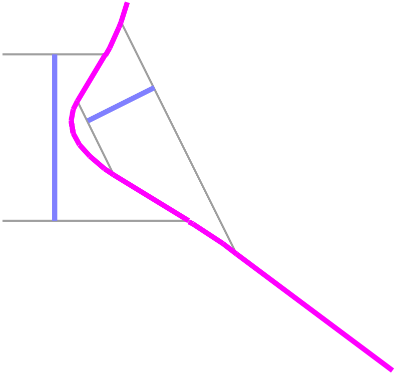
A l'angle de la rue de Vaugirard, 50

Triangles and curves



[Florence, 1997]
Triangular period

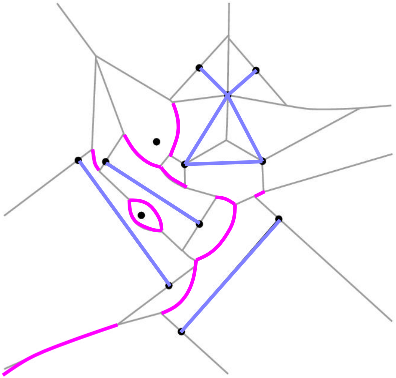
Curves already appear for linear input



Bisecting curve

2D line segments
arcs of parabolas

Curves already appear for linear input



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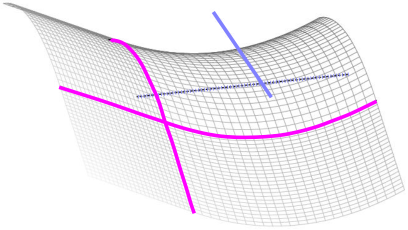
Voronoi diagram

2D line segments
arcs of parabolas

Curves already appear for linear input



Voronoi diagram

3D line segments
patches of quadric surfaces

More generally:

manipulations of algebraic curves and surfaces

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manipulations of algebraic curves and surfaces

Only considered here
Exact Geometric Computation [Yap][. . .]

Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand

Why we should not be afraid of Computer Algebra

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- not so hard to understand
- people are nice

Why we should not be afraid of Computer Algebra

trying to convice myself...

- useful
- interesting
- not so hard to understand (?)
- some people are nice

One tool: Resultant

Resultant of a system of polynomial equations

= necessary and sufficient condition
such that it has a root.

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Resultant of a system of polynomial equations

= necessary and sufficient condition
such that it has a root.

How to compute the resultant? hard problem

Sylvester resultant

Univariate case

$$\begin{cases} P &= a_0 x^m + \cdots + a_m \\ Q &= b_0 x^n + \cdots + b_n \end{cases}$$

$a_0 \neq 0, b_0 \neq 0, m > n$,
coefficients in a field $\mathbb{K}$ (algebraically closed).

Sylvester resultant

$$\left\{\begin{array}{lcl} P & = & a_0x^m + \cdots + a_m \\ Q & = & b_0x^n + \cdots + b_n \end{array}\right.$$

$$\text{Sylvester resultant} = \begin{vmatrix}
a_0 & & & & & & b_0 & & & \\
a_1 & a_0 & & & & & b_1 & b_0 & & \\
 & a_1 & \ddots & & & & & b_1 & \ddots & \\
 & & \ddots & \ddots & & & & & \ddots & b_0 \\
\vdots & & & \ddots & \ddots & & \vdots & & & b_1 \\
 & \vdots & & & \ddots & a_0 & & \vdots & & \\
\vdots & & \vdots & & & a_1 & \vdots & & \vdots & \\
a_m & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & a_m & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \ddots & & \vdots & & b_n & & \vdots & \\
 & & & \ddots & & \vdots & & b_n & & \vdots \\
 & & & & \ddots & & & & \ddots & \\
 & & & & & a_m & & & & b_n
\end{vmatrix}$$

Sylvester resultant

$$\left\{ \begin{array}{lcl} P & = & a_0 x^m + \cdots + a_m \\ Q & = & b_0 x^n + \cdots + b_n \end{array} \right.$$

Sylvester resultant =

$$\left| \begin{array}{cccccccccc}
a_0 & & & & & & b_0 & & & \\
a_1 & a_0 & & & & & b_1 & b_0 & & \\
 & a_1 & \ddots & & & & & b_1 & \ddots & \\
 & & \ddots & \ddots & & & & & \ddots & b_0 \\
\vdots & & & \ddots & \ddots & & \vdots & & & b_1 \\
 & \vdots & & & \ddots & a_0 & & \vdots & & \\
\vdots & & \vdots & & & a_1 & \vdots & & \vdots & \\
a_m & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & a_m & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \ddots & & \vdots & & b_n & & \vdots & \\
 & & & \ddots & & \vdots & & b_n & & \vdots \\
 & & & & \ddots & & & & \ddots & \\
 & & & & & a_m & & & & b_n
\end{array} \right|$$

$= 0$ *iff*

P and Q have a

common root in $\mathbb{K}$.

Demystifying Resultant - I

$$\begin{cases} ax + by - c &= 0 \\ dx + ey - f &= 0 \end{cases}$$

seen as: x unknown, y parameter

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$$\text{Sylvester Resultant} = \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix}$$

Demystifying Resultant - I

$$\begin{cases} ax + by - c &= 0 \\ dx + ey - f &= 0 \end{cases}$$

seen as: x unknown, y parameter

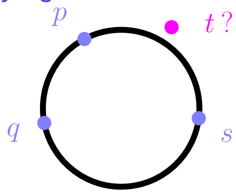
$$\begin{aligned} \text{Sylvester Resultant} &= \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix} \\ &= a(ey - f) - d(by - c) \end{aligned}$$

Boils down to eliminate x

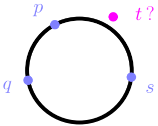
Demystifying Resultant - II

p, q, s three points in the plane,
t a fourth point.

Is t lying on the circle $\mathcal{C}_{pqs}$?



Demystifying Resultant - II



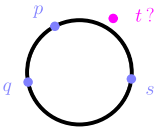
t on $\mathcal{C}_{pqs}$

$\mathcal{C}_{pqs}$ center (x_c, y_c) radius r

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

$$\textit{iff} \quad \left\{ \begin{array}{lcl} 2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) & = & 0 \\ 2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) & = & 0 \\ 2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) & = & 0 \\ 2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) & = & 0 \end{array} \right.$$

Demystifying Resultant - II



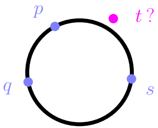
t on $\mathcal{C}_{pqs}$

$$
\text{iff} \left\{ \begin{array}{rcl}
2x_pX + 2y_pY + R - (x_p^2 + y_p^2)Z & = & 0 \\
2x_qX + 2y_qY + R - (x_q^2 + y_q^2)Z & = & 0 \\
2x_sX + 2y_sY + R - (x_s^2 + y_s^2)Z & = & 0 \\
2x_tX + 2y_tY + R - (x_t^2 + y_t^2)Z & = & 0
\end{array} \right.
$$

has a non-trivial solution (X, Y, R, Z) and

$$
\begin{array}{rcl}
X/Z & = & x_c \\
Y/Z & = & y_c \\
R/Z & = & r^2 - x_c^2 - y_c^2.
\end{array}
$$

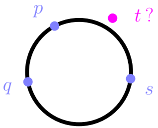
Demystifying Resultant - II



$$
\text{iff} \begin{cases}
2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) & = & 0 \\
2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) & = & 0 \\
2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) & = & 0 \\
2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) & = & 0
\end{cases}
$$

$$
\text{iff} \begin{vmatrix}
x_p & y_p & 1 & x_p^2 + y_p^2 \\
x_q & y_q & 1 & x_q^2 + y_q^2 \\
x_s & y_s & 1 & x_s^2 + y_s^2 \\
x_t & y_t & 1 & x_t^2 + y_t^2
\end{vmatrix} = 0
$$

Demystifying Resultant - II



$$\text{iff}\quad \left\{\begin{array}{lcl} 2x_px_c + 2y_py_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) & = & 0 \\ 2x_qx_c + 2y_qy_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) & = & 0 \\ 2x_sx_c + 2y_sy_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) & = & 0 \\ 2x_tx_c + 2y_ty_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) & = & 0 \end{array}\right.$$

$$\text{iff}\quad \begin{vmatrix} x_p & y_p & 1 & x_p^2 + y_p^2 \\ x_q & y_q & 1 & x_q^2 + y_q^2 \\ x_s & y_s & 1 & x_s^2 + y_s^2 \\ x_t & y_t & 1 & x_t^2 + y_t^2 \end{vmatrix} = 0$$

= resultant of the system

Allows to eliminate x_c, y_c, r^2

Resultant

- Resultant often used in simple cases without noticing

Resultant

- Resultant often used in simple cases without noticing
- Linear algebra helps solve non-linear problems

Digression on algebraic degree

One measure of efficiency and precision of a predicate:

algebraic degree

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has minimal degree $\Longrightarrow$ optimal predicate?

Digression on algebraic degree

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Resultant has minimal degree $\Longrightarrow$ optimal predicate?

No:

- methods often return a multiple of the resultant
  $\longrightarrow$ resultant hard to compute

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has minimal degree $\Longrightarrow$ optimal predicate?

No:

- methods often return a multiple of the resultant
  $\longrightarrow$ resultant hard to compute
- the resultant may be factored
  $\longrightarrow$ predicate can have a lower degree

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has minimal degree $\Longrightarrow$ optimal predicate?

No:

- methods often return a multiple of the resultant
  $\longrightarrow$ resultant hard to compute
- the resultant may be factored
  $\longrightarrow$ predicate can have a lower degree
- a factor may be $P^2 + Q^2$
  $\longrightarrow$ the degree does not mean so much

Digression on algebraic degree

- filtering techniques used for efficiency
  $\longrightarrow$ maybe not such an interesting measure ?

Digression on algebraic degree

- Degree of a predicate
  $\longrightarrow$ not trivial
- Degree of an algorithm
  $\longrightarrow$ depends on the algebraic expressions of predicates
- Degree of a geometric problem
  $\longrightarrow$?

Digression $\mapsto$ thread

Another tool: Sturm sequences

$\mathcal{P} = P_0, P_1, \ldots, P_d \in \mathbb{R}[X]$

$\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$

$Var(\mathcal{P}; \alpha)$ = number of sign variations in the sequence $P_0(\alpha), P_1(\alpha), \ldots, P_d(\alpha)$

$$Var(\mathcal{P}; \alpha, \beta) = Var(\mathcal{P}; \alpha) - Var(\mathcal{P}; \beta)$$

Sturm sequences

$P, Q \in \mathbb{K}[X]$ signed remainder sequence of P and Q =
sequence $\mathcal{S}(P, Q) : P_0, P_1, \ldots, P_k$

$$\begin{array}{rcl}
P_0 & = & P \\
P_1 & = & Q \\
P_2 & = & -Rem(P_0, P_1) \\
 & \vdots & \\
P_k & = & -Rem(P_{k-2}, P_{k-1}) \\
P_{k+1} & = & -Rem(P_{k-1}, P_k) = 0
\end{array}$$

where
$Rem(A, B) =$ remainder of the Euclidean division of A by B

Sturm sequences

Sturm sequence of P =
sequence $\mathcal{S}(P, P')$ of signed reminders of P and P'

$$Var(\mathcal{S}(P, P'); \alpha, \beta)$$

is the number of roots of P in the interval $[\alpha, \beta]$

Sturm sequences for dummies

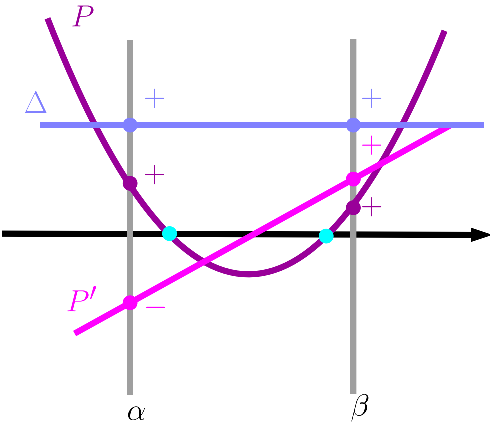
Sturm sequences ~~for dummies~~ by a dummy

$$P = aX^2 + bX + c$$

$$P' = 2aX + b$$
$$P = P'.(\frac{X}{2} + \frac{b}{4a}) - (\frac{b^2}{4a} - c)$$

Sturm sequences ~~for dummies~~ by a dummy

$$P = aX^2 + bX + c$$



$$P' = 2aX + b$$
$$P = P'.(\frac{X}{2} + \frac{b}{4a}) - (\frac{b^2}{4a} - c)$$

$$P_0 = P, P_1 = P', P_2 = \Delta$$

if $\Delta > 0$

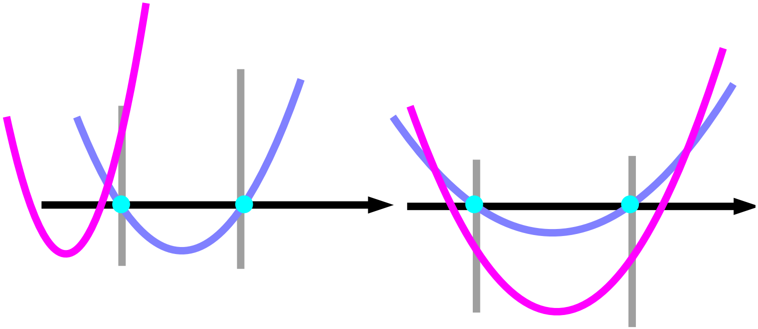
$$\alpha = -\infty, \beta = +\infty$$

$$Var(\mathcal{P}; \alpha) = 2$$
$$Var(\mathcal{P}; \beta) = 0$$

2 roots

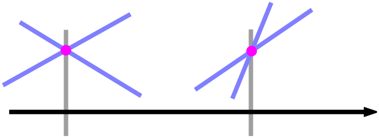
Sturm sequences

Sequence $\mathcal{S}(P, P'Q)$ of signed reminders of P and $P'Q$ counts the number of roots of P at which Q is positive



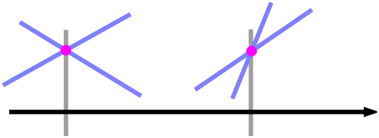
Sturm sequences allow to compare roots of P and Q

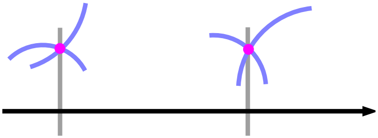
Comparing intersection points



signs of
polynomial expressions

Comparing intersection points

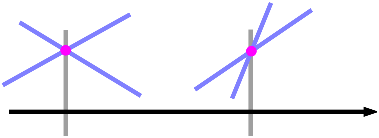




signs of
polynomial expressions

comparison of
algebraic numbers

Comparing intersection points



signs of
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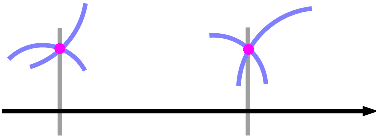


comparison of
algebraic numbers

Sturm sequences $\longrightarrow$
signs of
polynomial expressions

Practical efficiency

Arithmetic filters for sign computations:

Approximate evaluation $P^a(x)$
$+$ Error ε

$|P^a(x)| > \varepsilon$
?

y

n

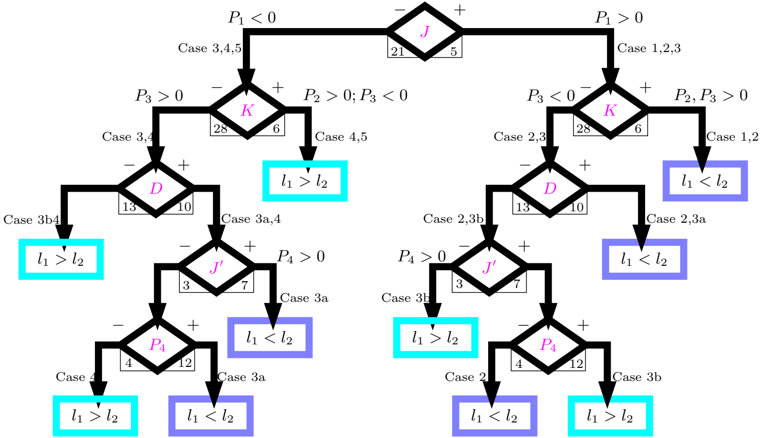
$\text{sign}\ (P(x)) = \text{sign}\ (P^a(x))$

Exact computation

Exact geometric computation $\neq$ Exact arithmetics

Practical efficiency

Comparison of algebraic numbers of degree 2:

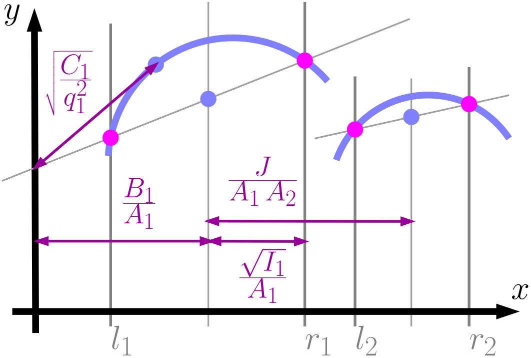


polynomial expressions pre-computed

static Sturm sequences

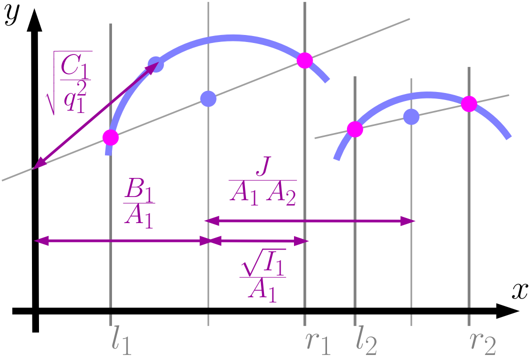
Algebra is not just “computations”

it has a meaning...!



$K = 0 \iff l_1, r_1, l_2, r_2$ harmonic division

Algebra is not just “computations”
it has a meaning...!



$K = 0 \iff l_1, r_1, l_2, r_2$ harmonic division

- Geometric interpretation in more complicated cases...?

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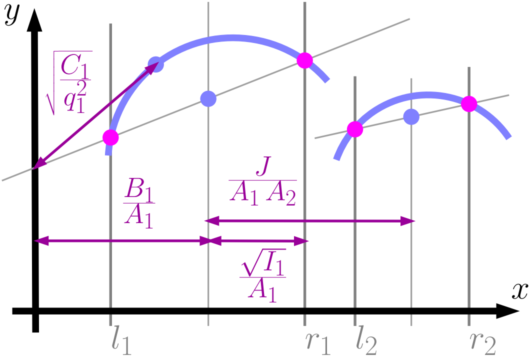


$K = 0 \iff l_1, r_1, l_2, r_2$ harmonic division

- Geometric interpretation in more complicated cases...?
- Optimal degree...?

Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Release 3.2 soon

CGAL

Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Release 3.2 soon

Exclusive news: Out before Microsoft new OS!

Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Release 3.2 soon

- new: 2D Circular Kernel
  manipulations of circular arcs

CGAL

Computational Geometry Algorithms Library

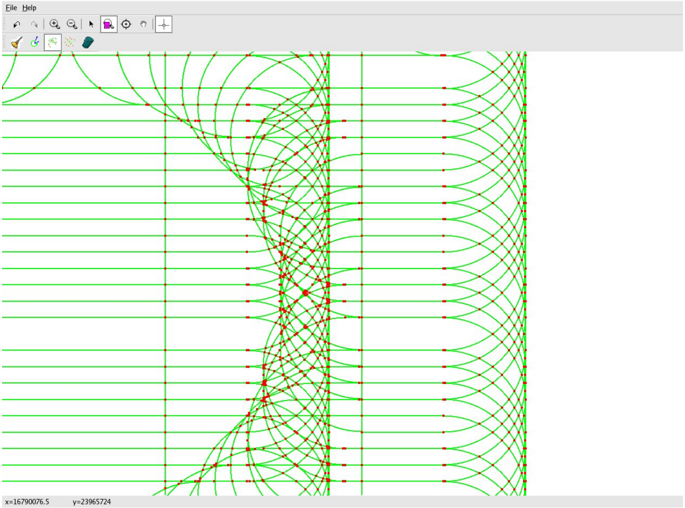
Open Source Project

www.cgal.org

Release 3.2 soon

- new: 2D Circular Kernel
  manipulations of circular arcs
- Arrangement package redesigned
- ...

VLSI - CAD



Intersection of two quadrics Q_S and Q_T

Levin's pencil method

- find a “good” quadric in the pencil $Q_{R(\lambda)=\lambda S-T}$
  λ root of degree 3 pol.
- Diagonalize $R(\lambda)$.
  Eigenvalues = roots of degree 2 pol. $\in \mathbb{Q}(\lambda)$.
  Normalize eigenvectors.
- Plug the parameterization of $Q_R(\lambda)$ in Q_T.
  Degree 2 in one of the parameters. Solve

“good” = simple ruled

$$\begin{vmatrix} \begin{vmatrix} x & x & x \\ x & x & x \\ x & x & x \end{vmatrix} & \vdots \\ \cdots & . \end{vmatrix}$$

principal subdeterminant =0

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Improvement

- work in $\mathbb{P}^3$
- Relax the constraint on $Q_{R(\lambda)}$
  Rational, ruled.
- Apply Gauss reduction of the quadratic form: $P^T RP$ diagonal.
  Rational transformation.
- Plug the parameterization in Q_T.
  Degree 2 in one of the parameters. Solve

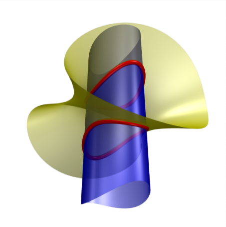
Intersection of quadrics

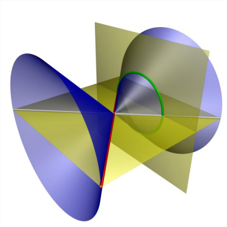
Levin's pencil method

$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\ }}}}}$$

New parameterization
- rational when it exists, involves $\sqrt{\text{pol.}}$ otherwise.
- quasi-optimal in $\sqrt{\ }$.

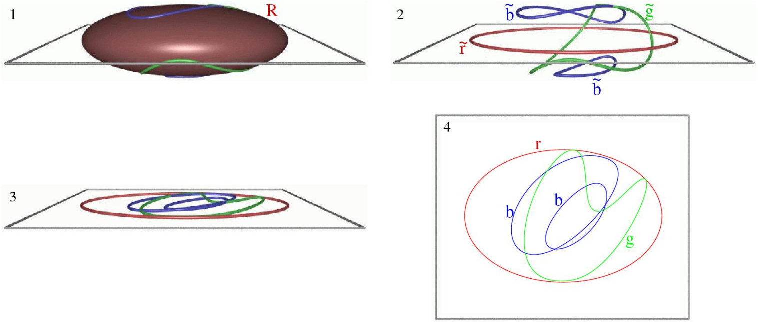
Implemented





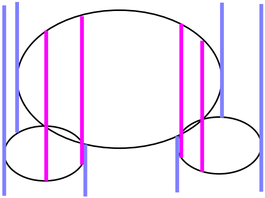
Arrangement of quadrics

Projection approach



Planar arrangement of curves of degree 4
a curve can have 6 singular points
Sort out (upper, lower) $\longrightarrow$ arrangement on each quadric

Surfacic approach



Volumic approach:
vertical decomposition

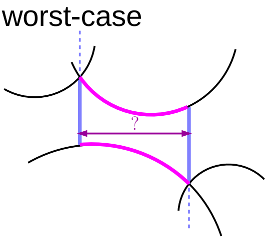
Sweeping plane:
Trapezoidal map of evolving conics

Events:

- new quadric
- features in the map intersect

Arrangement of quadrics

Sweeping approach



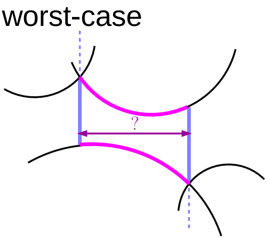
Events:

- new quadric
- features in the map intersect

x solution of

$$\exists \; y, z_1, z_2 \quad \text{s.t.} \left\{ \begin{array}{rcl} Q_i(x,y,z_1) & = & 0 \\ Q_j(x,y,z_1) & = & 0 \end{array} \right. \text{ and } \left\{ \begin{array}{rcl} Q_k(x,y,z_2) & = & 0 \\ Q_l(x,y,z_2) & = & 0 \end{array} \right.$$

x in an extension field of degree 16



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x in an extension field of degree 16

Comparison of events:
difference of events in an extension field of degree 256...

- Optimal degree...?

Apollonius diagram

Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i)$, $p_i \in \mathbb{R}^2, r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$

Apollonius diagram

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$$\begin{aligned} C_i \subset \mathbb{R}^3 : \quad & x_3 = \|x - p_i\| - r_i \\ \iff \quad & (x_3 + r_i)^2 = (x - p_i)^2 \quad x_3 + r_i > 0 \quad \text{half-cone} \end{aligned}$$

Apollonius diagram

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Apollonius diagram =

lower envelope of the half-cones.

Bisector of σ_i and σ_j =

projection of a plane conic section $C_i \cap C_j$.

Apollonius diagram

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X_i projection of x onto C_i

$$\begin{array}{llcll} x \in A(\sigma_i) & \mathit{iff} & \|x - p_i\| - r_i & < & \|x - p_j\| - r_j \quad (\forall j) \\ & \mathit{iff} & pow(X_i, \Sigma_i) & < & pow(X_i, \Sigma_j) \end{array}$$

Apollonius diagram
Additively weighted Voronoi diagram

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$$\delta_i(x) = \|x - p_i\| - r_i$$

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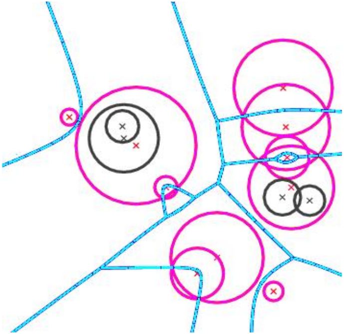
Apollonius diagram =
lower envelope of the half-cones.
Bisector of σ_i and σ_j =
projection of a plane conic section $C_i \cap C_j$.

Σ_i sphere $\subset \mathbb{R}^3$, center (p_i, r_i) radius $\sqrt{2} r_i$
$A(\sigma_i)$ = projection of the intersection of
the half-cone C_i with the power region of Σ_i

Same in $\mathbb{R}^d$



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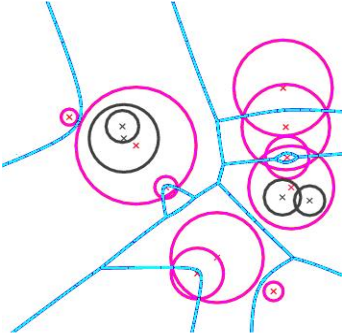
Tricky predicates
Degree 16



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Tricky predicates
Degree 16

- Implementation degree 20:
  degree 16 requires $\sim$ 100 times as many arithmetic operations...
- Optimal degree...?

Challenges

- theoretical: questions on degree...

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- Robust (Exact?) computation on higher degree
  curves and surfaces

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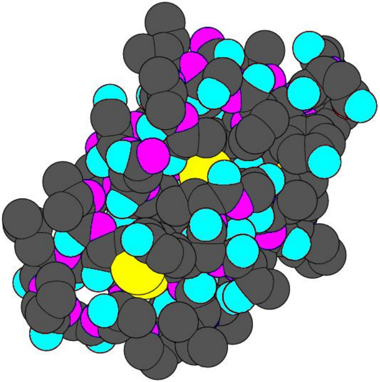
- theoretical: questions on degree...
- Robust (Exact?) computation on higher degree
  curves and surfaces
- Improvement of practical efficiency for low degree curves
  CAD-VLSI (circular arcs):
  $\sim$ 10 times slower than industrial non-robust code
  good start!

Challenges

- Applications to Structural biology
  Manipulations of a large number of spheres
  (low degree surfaces...)



$\varepsilon\upsilon\chi\alpha\rho\iota\sigma\tau\acute{\omega}$

Material taken from:

- Greece
  - National University of Athens
  - University of Crete
- Germany
  - Max-Planck Institut für Informatik
  - Universität des Saarlandes
- Israel
  - Tel-Aviv University
- France
  - Loria
  - INRIA Sophia Antipolis

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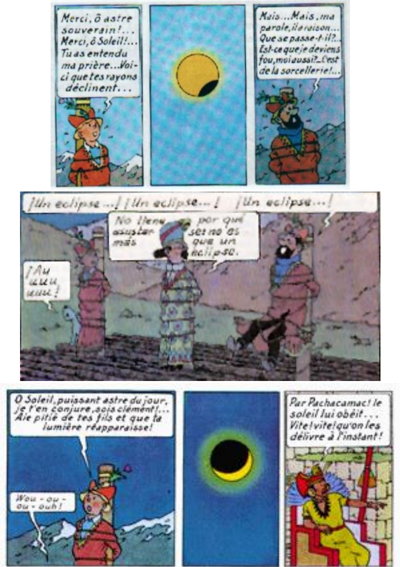

Merci, ô astre souverain!... Merci, ô Soleil!... Tu as entendu ma prière... Voici que tes rayons déclinent...
Mais... Mais, ma parole, il a raison... Que se passe-t-il?... Est-ce que je deviens fou, moi aussi?... C'est de la sorcellerie!...
¡Un eclipse...! ¡Un eclipse...! ¡Un eclipse...!
No tiene por qué asustar se; no es más que un eclipse.
¡Au uuu uuu!
O Soleil, puissant astre du jour, je t'en conjure, sois clément!... Aie pitié de tes fils et que ta lumière réapparaisse!
Wou - ou - ou - ouh!
Par Pachacamac! le soleil lui obéit... Vite! vite! qu'on les délivre à l'instant!