

Implementing Delaunay triangulations of the Bolza surface

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33rd International Symposium on Computational Geometry
Brisbane, Australia

Outline

- 1 Introduction
- 2 The Bolza Surface
- 3 Background from [BTV, SoCG'16]
- 4 Data Structure
- 5 Incremental Insertion
- 6 Results
- 7 Future work

Outline

1 Introduction

2 The Bolza Surface

3 Background from [BTV, SoCG'16]

4 Data Structure

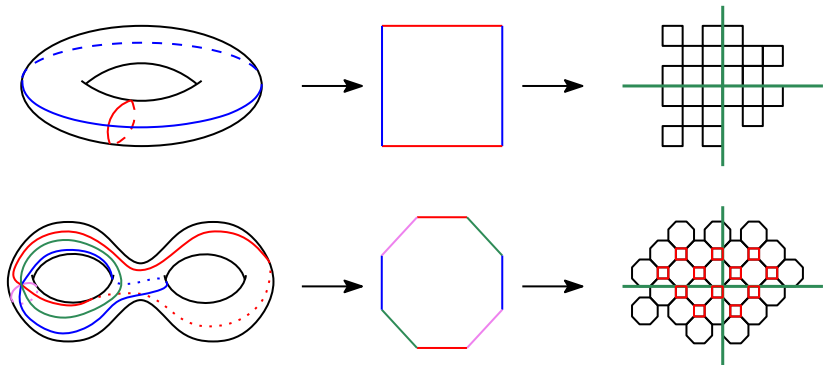
5 Incremental Insertion

6 Results

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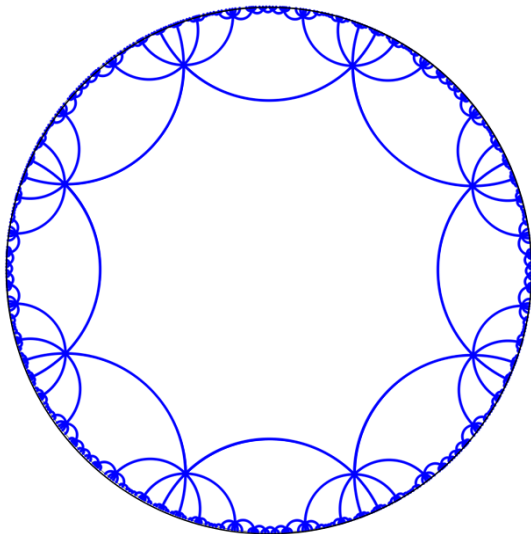
Motivation

Periodic triangulations in the Euclidean plane



Motivation

Periodic triangulations in the hyperbolic plane

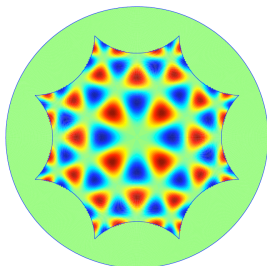


Motivation

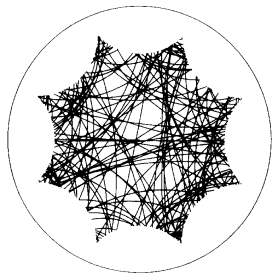
Applications



[Sausset, Tarjus, Viot]



[Chossat, Faye, Faugeras]

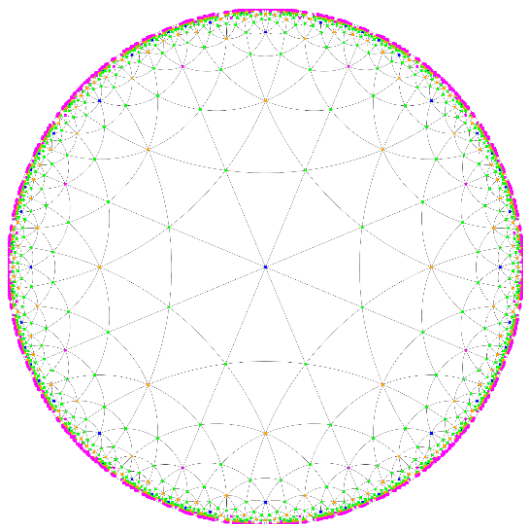


[Balazs, Voros]

Motivation

Beautiful groups

- Fuchsian groups
- finitely presented groups
- triangle groups
- ...



State of the art

Closed Euclidean manifolds

- Algorithms 2D [Mazón, Recio], 3D [Dolbilin, Huson], d D [Caroli, Teillaud, DCG'16]
- Software (square/cubic flat torus) 2D [Kruithof], 3D [Caroli, Teillaud]


 The logo for CGAL (Computational Geometry Algorithms Library) is displayed in a stylized, yellow, outlined font.

Closed hyperbolic manifolds

- Algorithms 2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG'16]
- Software (Bolza surface) [this paper]

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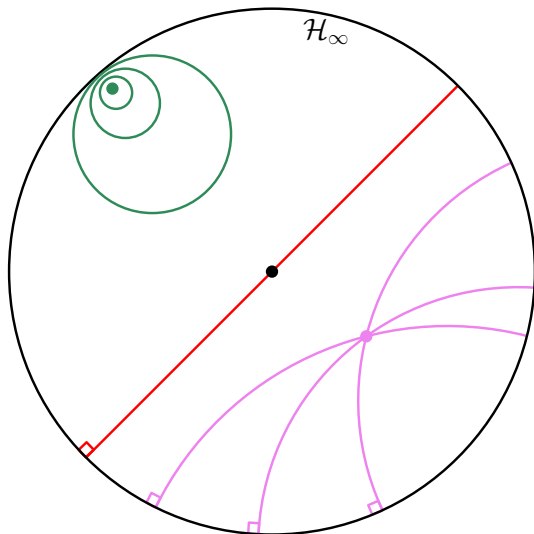
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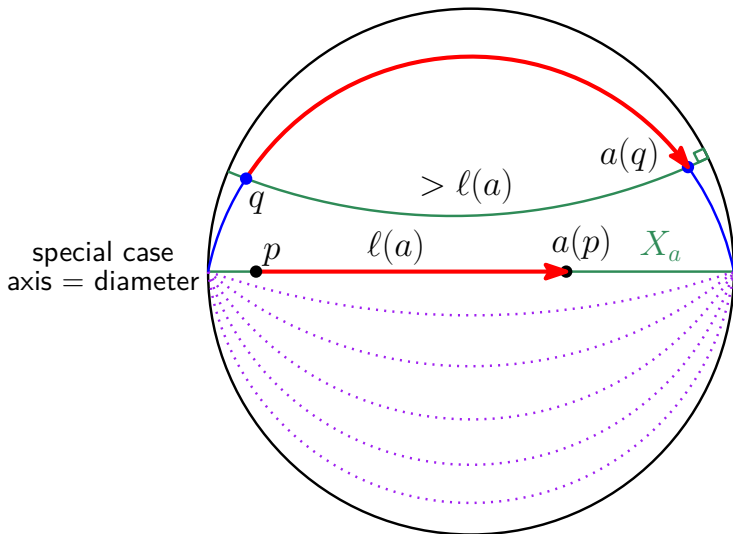
6 Results

7 Future work

Poincaré model of the hyperbolic plane \mathbb{H}^2

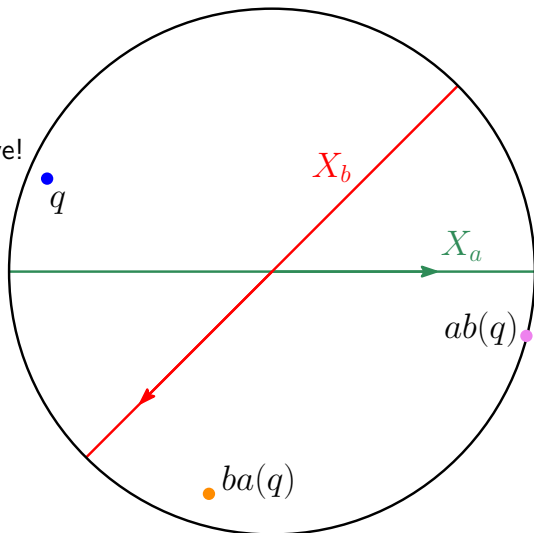


Hyperbolic translations



Hyperbolic translations

non-commutative!

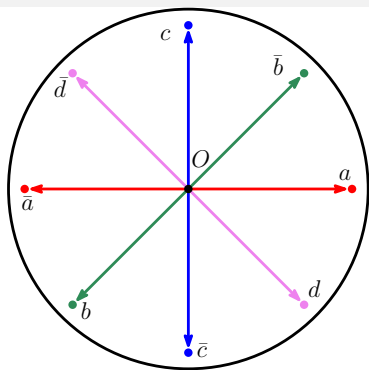


Bolza surface

What is it?

- Closed, compact, orientable surface of genus 2.
- Constant negative curvature \rightarrow locally hyperbolic metric.
- The most symmetric of all genus-2 surfaces.

Bolza surface



Fuchsian group \mathcal{G} with finite presentation

$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

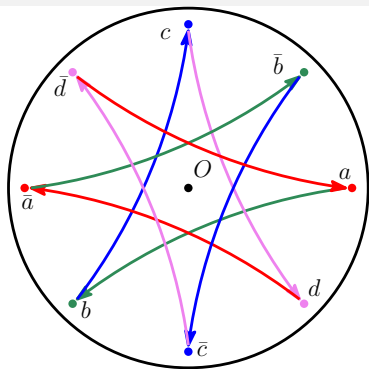
\mathcal{G} contains only translations (and $\mathbb{1}$)

Bolza surface

$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \rightarrow \mathcal{M}$

Bolza surface



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Bolza surface

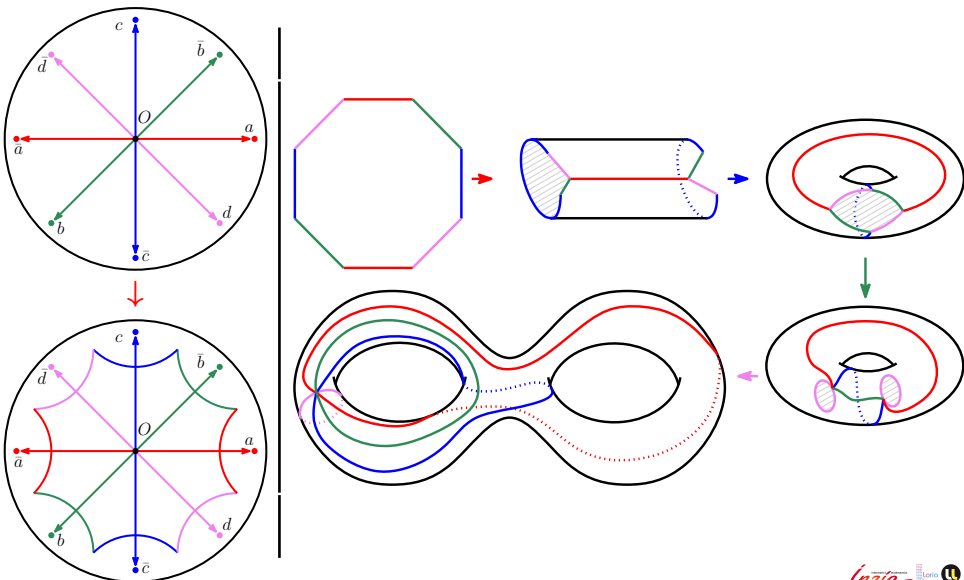
$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \rightarrow \mathcal{M}$

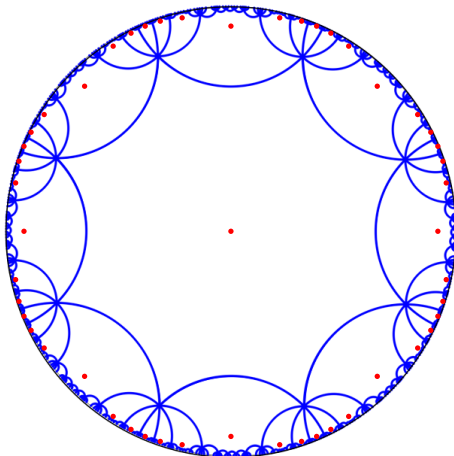
$$\mathcal{A} = [a, \bar{b}, c, \bar{d}, \bar{a}, b, \bar{c}, d] = [g_0, g_1, \dots, g_7]$$

$$g_k = \begin{bmatrix} \alpha & \beta_k \\ \bar{\beta}_k & \bar{\alpha} \end{bmatrix}, \quad g_k(z) = \frac{\alpha z + \beta_k}{\bar{\beta}_k z + \bar{\alpha}}, \quad \alpha = 1 + \sqrt{2}, \quad \beta_k = e^{ik\pi/4} \sqrt{2\alpha}$$

Bolza surface

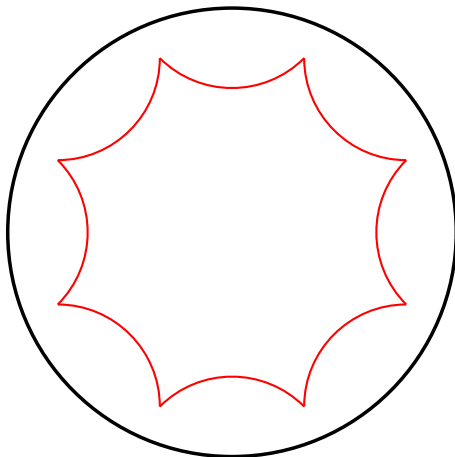


Hyperbolic octagon



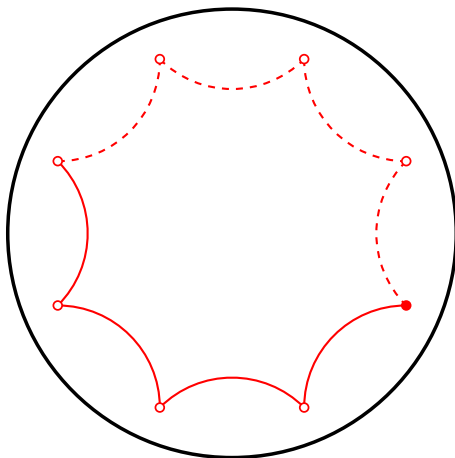
Voronoi diagram of $\mathcal{G}O$

Hyperbolic octagon



Fundamental domain $\mathcal{D}_O =$ Dirichlet region of O

Hyperbolic octagon

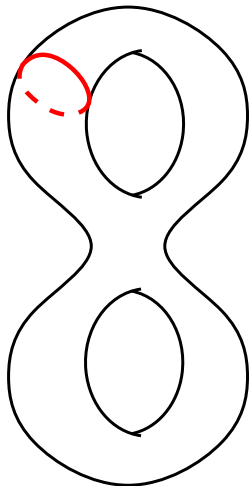


“Original” domain \mathcal{D} : contains exactly one point of each orbit

Outline

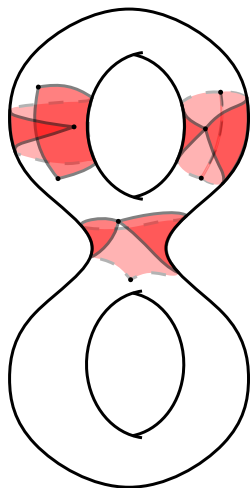
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Criterion



$\text{Systole } \text{sys}(\mathcal{M}) =$
minimum length of a
non-contractible loop on \mathcal{M}

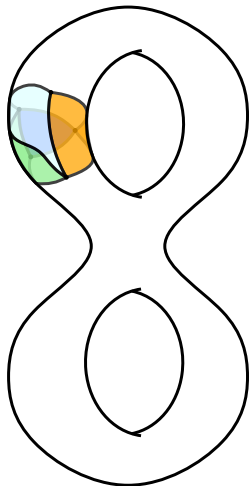
Criterion



Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on \mathcal{M}

$$\pi_{\mathcal{M}}(DT_{\mathbb{H}}(\mathcal{G}S))$$

Criterion



Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on \mathcal{M}

S set of points in \mathbb{H}^2
 $\delta_S =$ diameter of largest disks in \mathbb{H}^2 not containing any point of \mathcal{G}_S

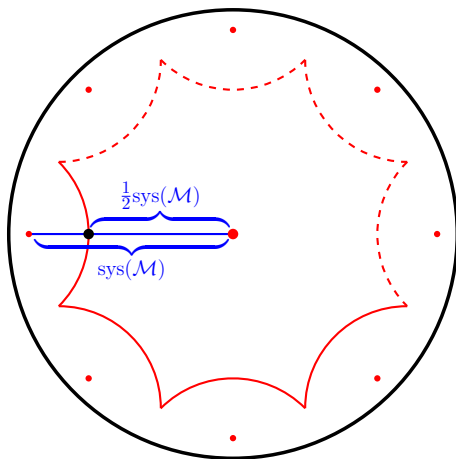
$$\delta_S < \frac{1}{2} \text{sys}(\mathcal{M})$$

$\Rightarrow \pi_{\mathcal{M}}(DT_{\mathbb{H}}(\mathcal{G}_S)) = DT_{\mathcal{M}}(S)$
 is a simplicial complex

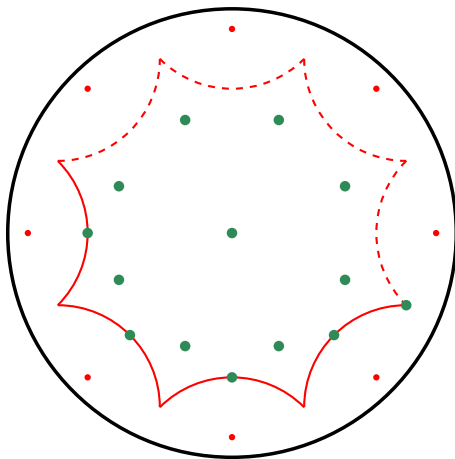
\Rightarrow The usual incremental algorithm can be used

[Bowyer]

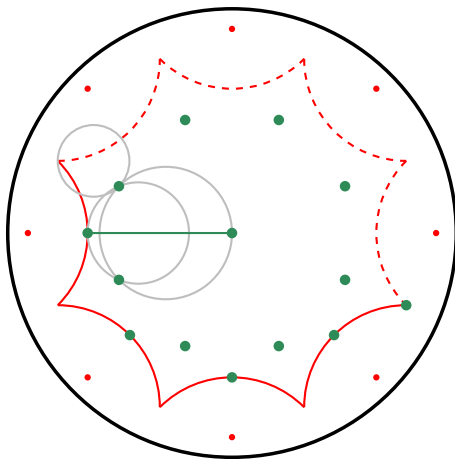
Systole on the octagon



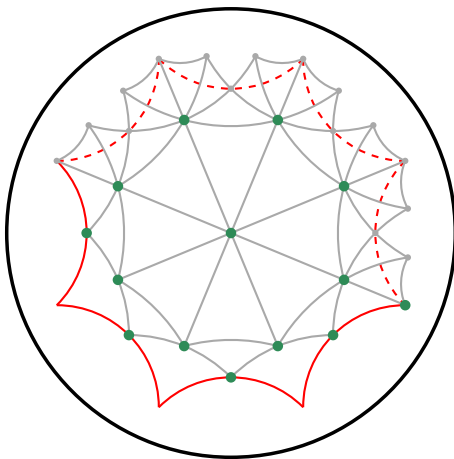
Set of dummy points



Set of dummy points vs. criterion



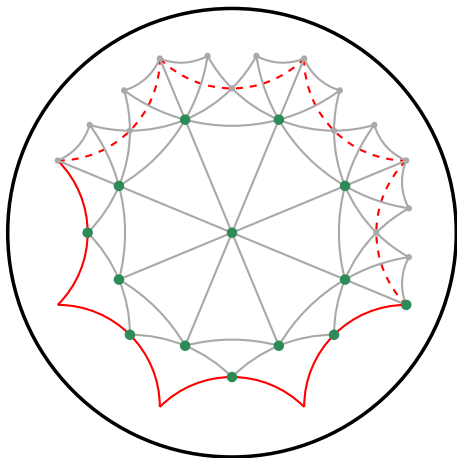
Delaunay triangulation of the dummy points



Delaunay triangulation of the Bolza surface

Algorithm:

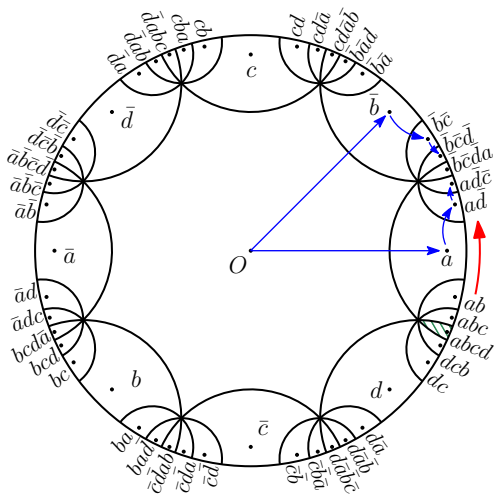
- 1 initialize with dummy points
- 2 insert points in S
- 3 remove dummy points



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Notation



$g(O)$, $g \in \mathcal{G}$, denoted as g

$\mathcal{D}_g = g(\mathcal{D}_O)$, $g \in \mathcal{G}$

$\mathcal{N} = \{g \in \mathcal{G} \mid \mathcal{D}_g \cap \mathcal{D}_O \neq \emptyset\}$

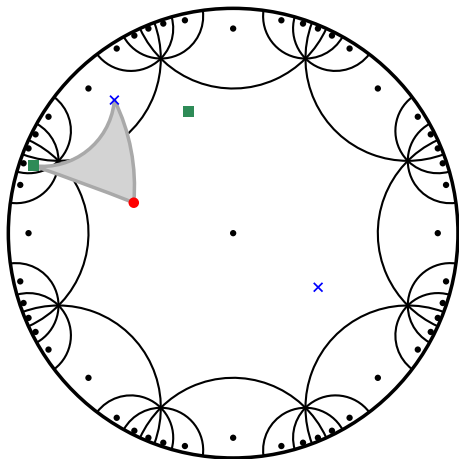
$\mathcal{D}_{\mathcal{N}} = \bigcup_{g \in \mathcal{N}} \mathcal{D}_g$

Property of $DT_{\mathbb{H}}(\mathcal{G}S)$

$S \subset \mathcal{D}$ input point set
 s.t. criterion $\delta_S < \frac{1}{2} \text{sys}(\mathcal{M})$ holds

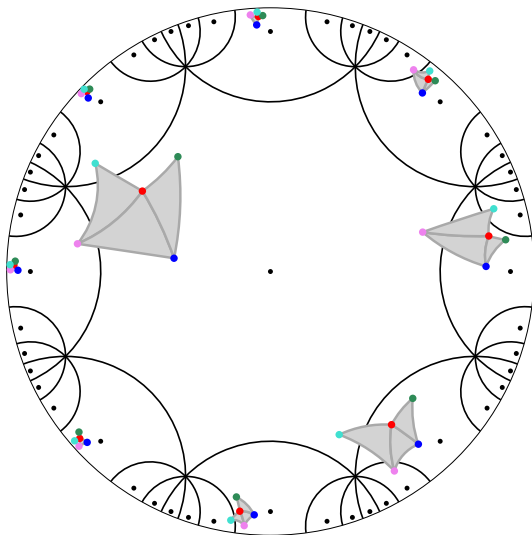
σ face of $DT_{\mathbb{H}}(\mathcal{G}S)$ with at least one
 vertex in \mathcal{D}

→ σ is contained in $\mathcal{D}_{\mathcal{N}}$



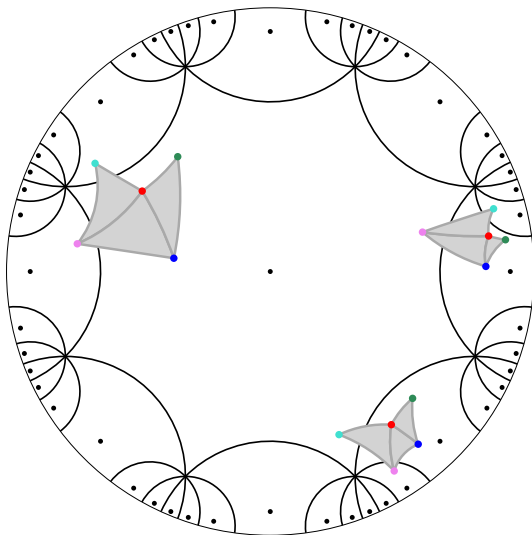
Canonical representative of a face

Each face of $DT_{\mathcal{M}}(S)$ has infinitely many pre-images in $DT_{\mathbb{H}}(\mathcal{G}S)$



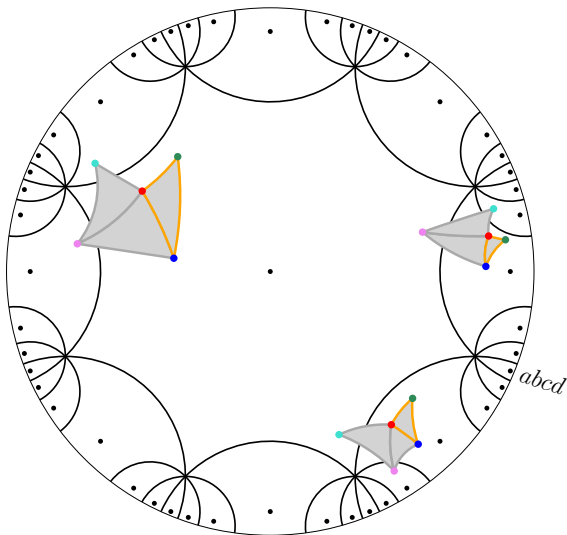
Canonical representative of a face

at least one pre-image with at least one vertex in \mathcal{D}



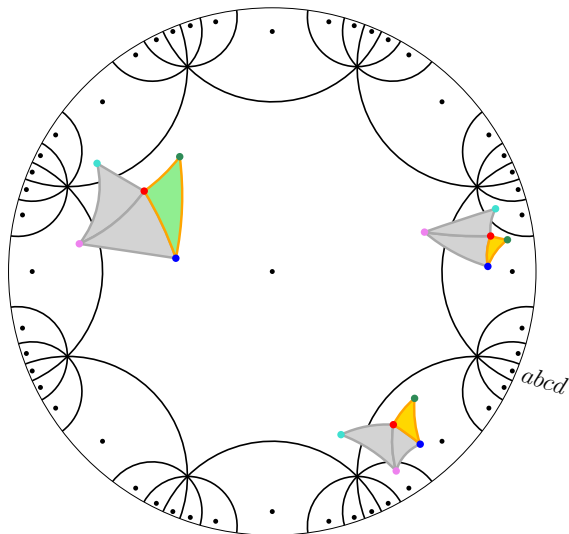
Canonical representative of a face

Case: face with 3 vertices in \mathcal{D}



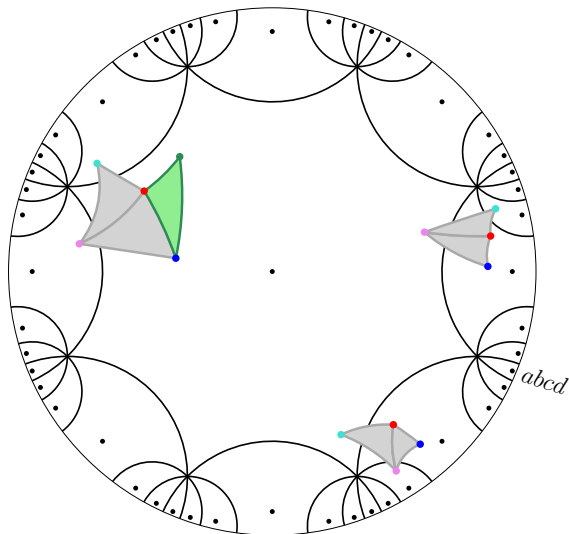
Canonical representative of a face

Case: face with 3 vertices in \mathcal{D}



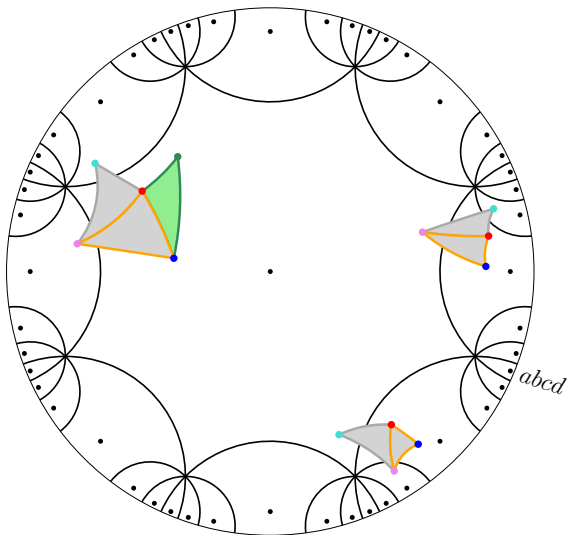
Canonical representative of a face

Case: face with 3 vertices in \mathcal{D}



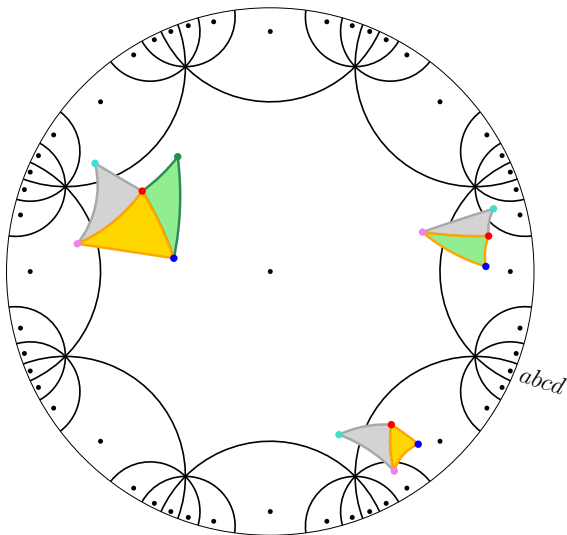
Canonical representative of a face

Case: face with 2 vertices in \mathcal{D}



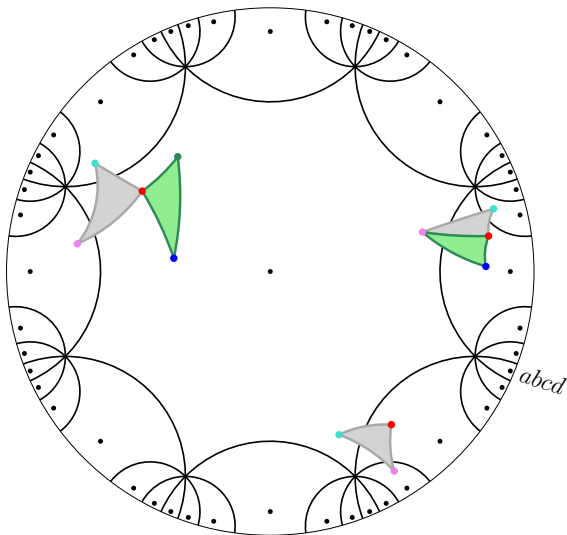
Canonical representative of a face

Case: face with 2 vertices in \mathcal{D}



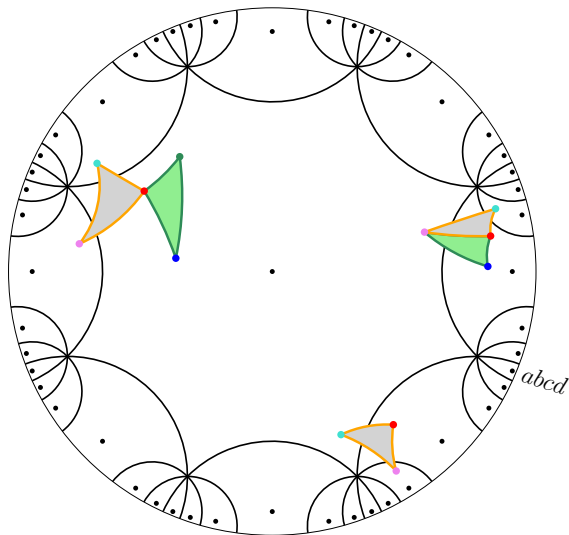
Canonical representative of a face

Case: face with 2 vertices in \mathcal{D}



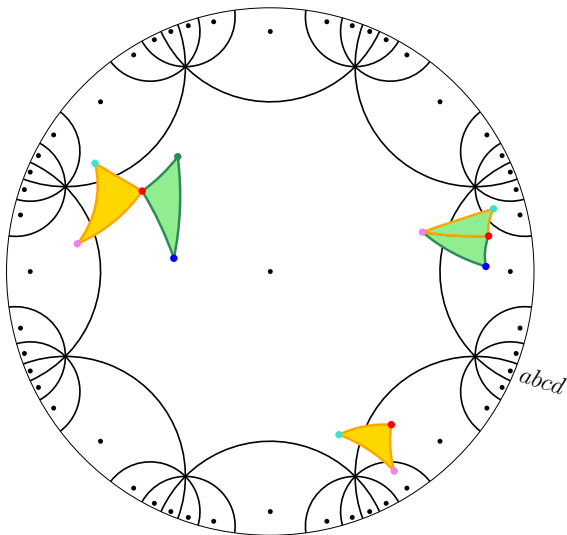
Canonical representative of a face

Case: face with 1 vertex in \mathcal{D}

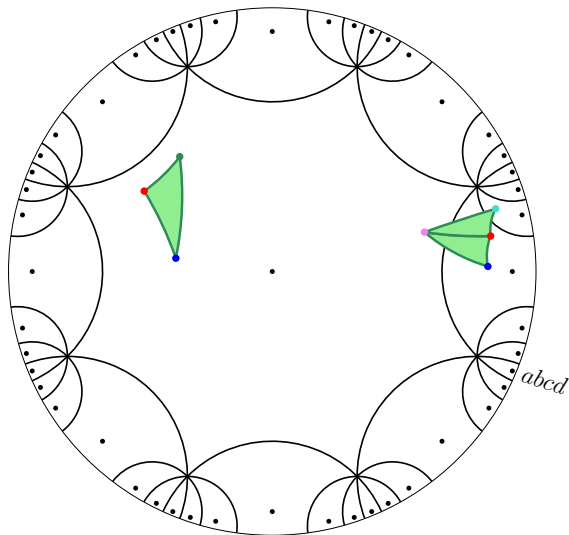


Canonical representative of a face

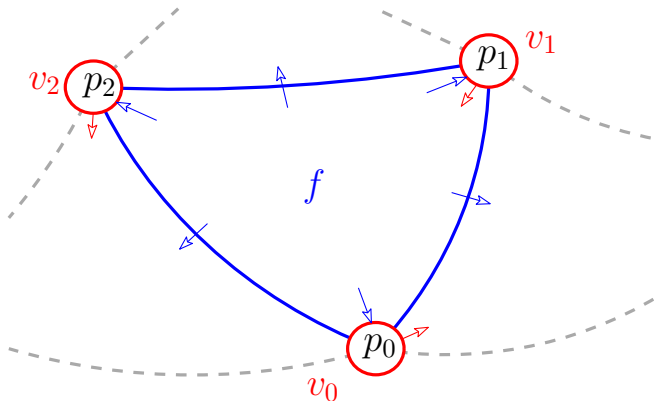
Case: face with 1 vertex in \mathcal{D}

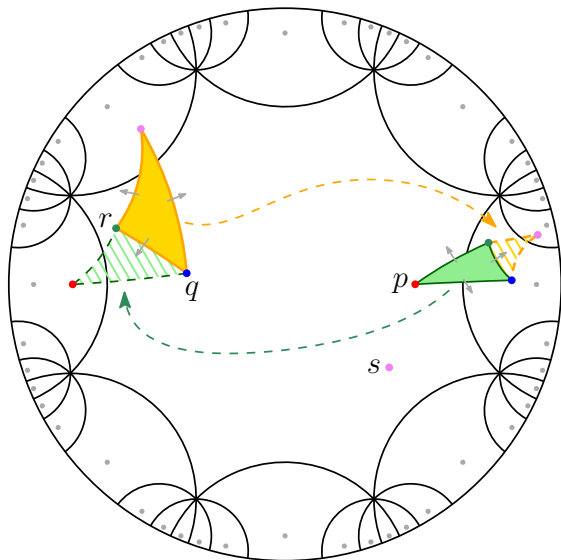


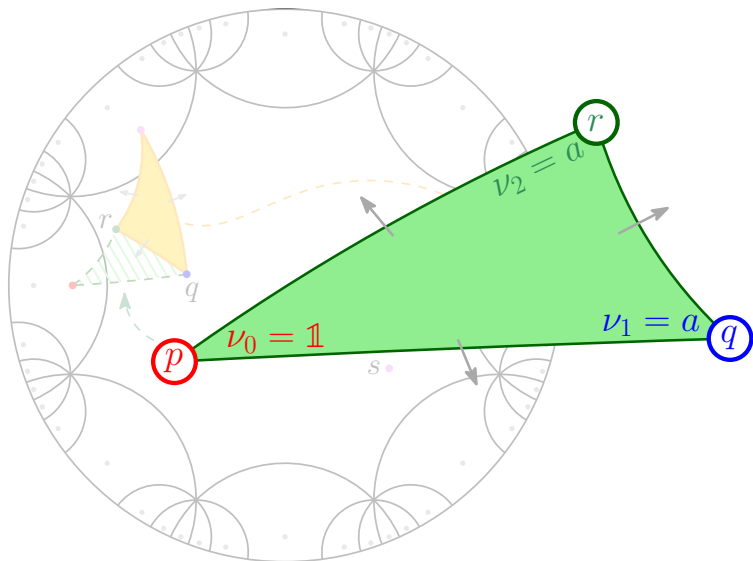
Canonical representative of a face



CGAL Triangulations



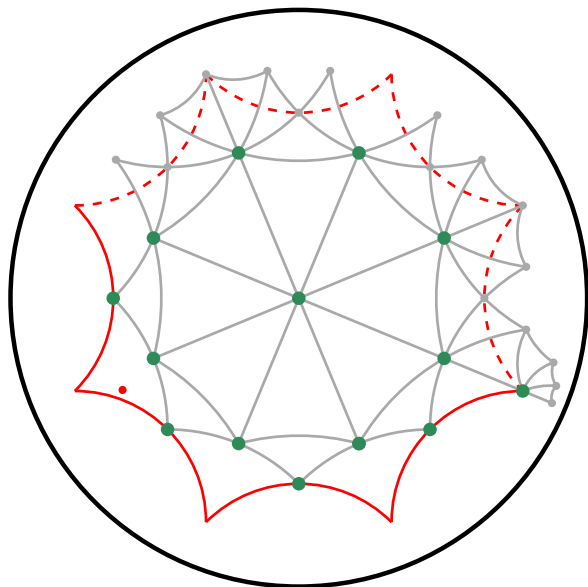
Face of $DT_M(S)$ 

Face of $DT_M(S)$ 

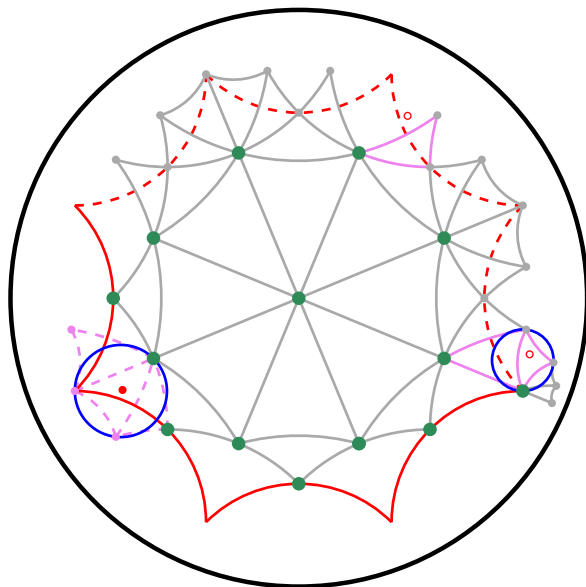
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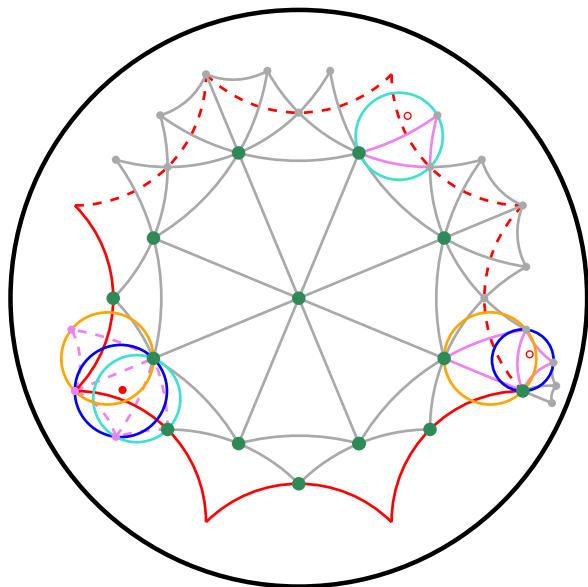
Point Location



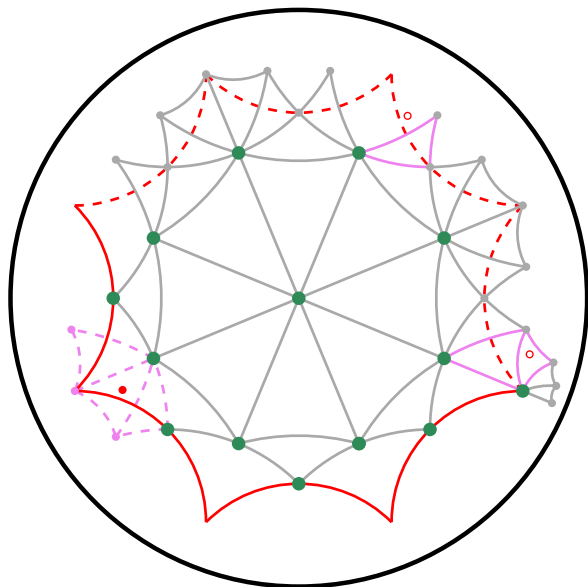
Point Location



Point Location

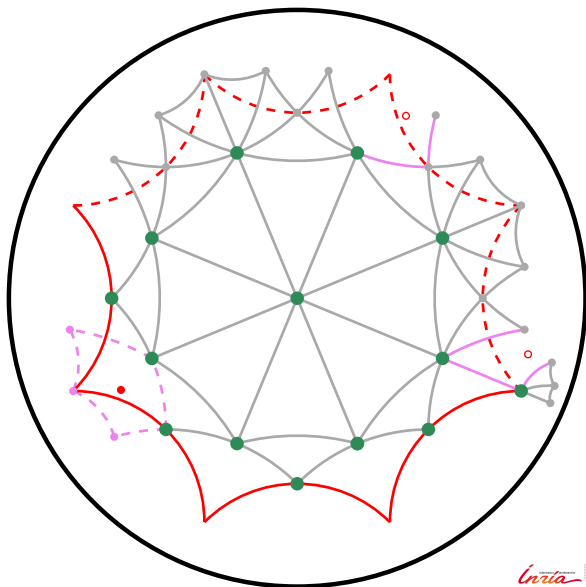


Point Location



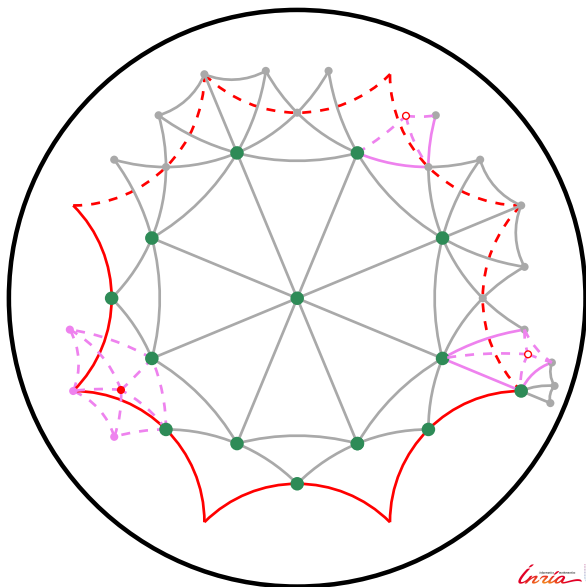
Point Insertion

“hole” = topological disk



Point Insertion

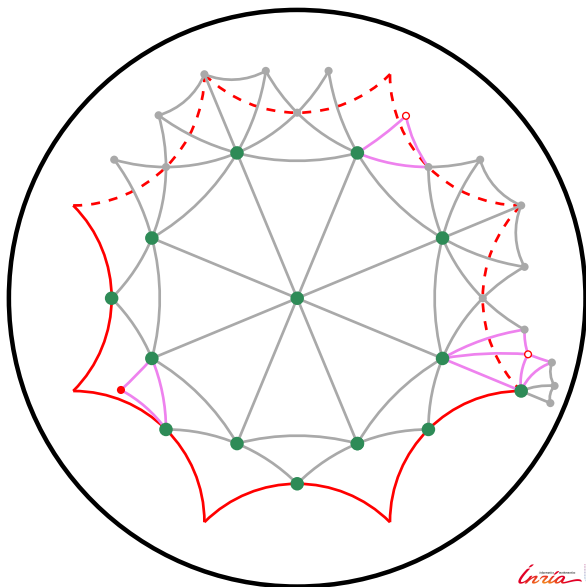
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Point Insertion

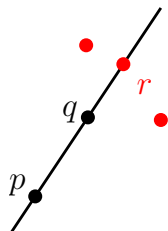
Computations
on translations

Dehn's algorithm
(slightly modified)

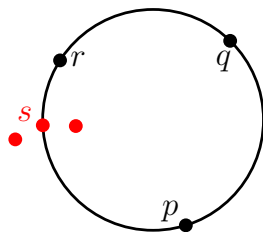


Predicates

$$\text{Orientation}(p, q, r) = \text{sign} \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$



$$\text{InCircle}(p, q, r, s) = \text{sign} \begin{vmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{vmatrix}$$



Predicates

Suppose that the points in S are **rational**.

Input of the predicates can be images of these points under $\nu \in \mathcal{N}$.

$$g_k(z) = \frac{\alpha z + e^{ik\pi/4} \sqrt{2\alpha}}{e^{-ik\pi/4} \sqrt{2\alpha} z + \alpha}, \quad \alpha = 1 + \sqrt{2}, \quad k = 0, 1, \dots, 7$$

- the *Orientation* predicate has algebraic degree at most 20
- the *InCircle* predicate has algebraic degree at most 72

Point coordinates represented with **CORE: :Expr**

→ (filtered) exact evaluation of predicates



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Experiments

Fully dynamic implementation



1 million random points

-  Euclidean DT (double) ~ 1 sec.
-  Euclidean DT (CORE::Expr) ~ 13 sec.
- Hyperbolic periodic DT (CORE::Expr) ~ 34 sec.

Experiments

Fully dynamic implementation

1 million random points

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Predicates

- 0.76% calls to predicates involving translations in \mathcal{N}
- responsible for 36% of total time spent in predicates

Dummy points can be removed after insertion of 17–72 random points.

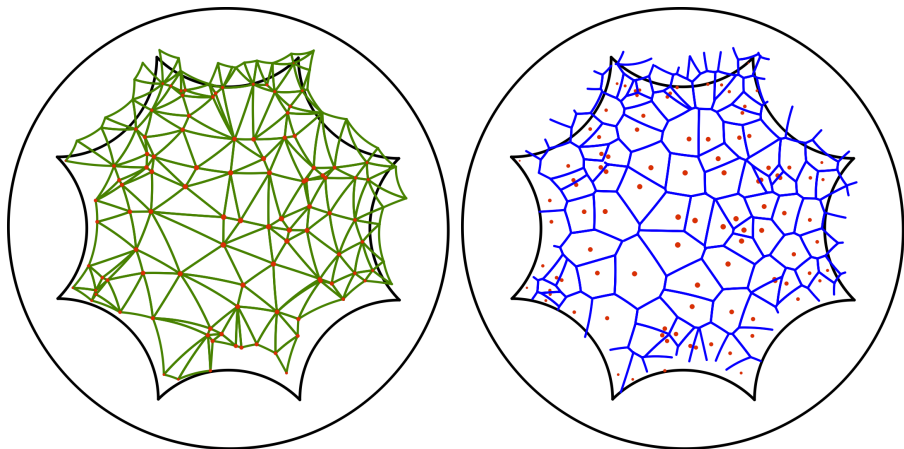
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Future work

- Implement 2D periodic hyperbolic mesh
- Algorithm for:
 - More general genus-2 surfaces
 - Surfaces of genus > 2

THANK YOU!



Source code and Maple sheets available online:

https://members.loria.fr/Monique.Teillaud/DT_Bolza_SoCG17/