Systolic inequalities, discrete or not

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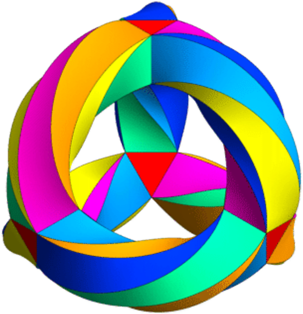
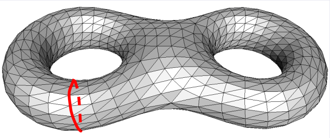
Based on joint work with Éric Colin de Verdière and Alfredo Hubard.



We deal with ***connected***, ***compact*** and ***orientable*** surfaces of ***genus*** g without boundary.

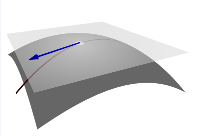
Discrete metric

Triangulation G.
Length of a curve $|\gamma|_G$:
Number of edges.



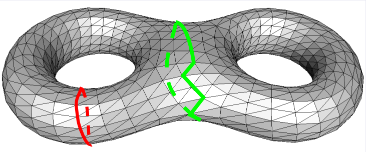
Riemannian metric

Scalar product m on the tangent space.
Riemannian length $|\gamma|_m$.

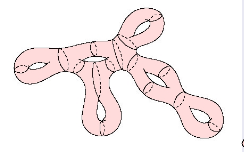


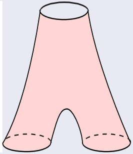
We study the length of topologically interesting curves for discrete and continuous metrics.

Non-contractible curves



Pants decompositions



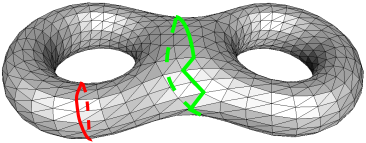


Motivations

Why should we care ?

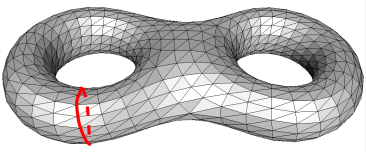
- ***Topological graph theory:*** If the shortest non-contractible cycle is ***long***, the surface is ***planar-like***.
  $\Rightarrow$ Uniqueness of embeddings, colourability, spanning trees.
- ***Riemannian geometry:***
  René Thom: ***"Mais c'est fondamental !"***.
  Links with isoperimetry, topological dimension theory, number theory.
- ***Algorithms for surface-embedded graphs:*** Cookie-cutter algorithm for surface-embedded graphs: Decompose the surface, solve the planar case, recover the solution.
- More practical sides: ***texture mapping***, ***parameterization***, ***meshing*** . . .

Part 1: Length of shortest curves

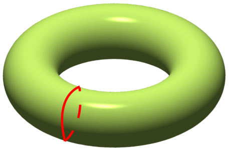


On shortest noncontractible curves

Discrete setting

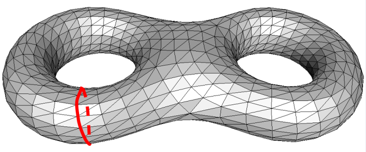


Continuous setting



What is the length of the red curve?

Discrete setting



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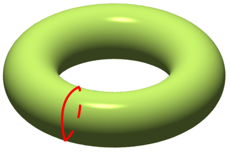

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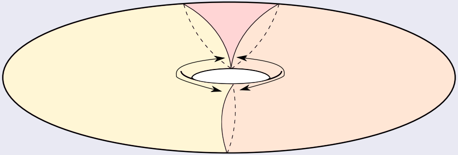
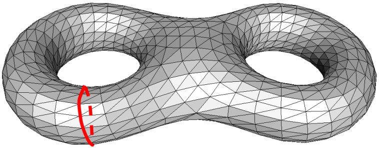
Intuition



It should have length $O(\sqrt{A})$ or $O(\sqrt{n})$, but what is the dependency on g ?

Discrete Setting: Topological graph theory

The ***edgewidth*** of a triangulated surface is the length of the shortest ***noncontractible*** cycle.



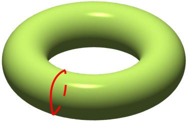
Theorem (Hutchinson '88)

The edgewidth of a triangulated surface with n triangles of genus g is $O(\sqrt{n/g}\log g)$.

- Hutchinson conjectured that the right bound is $\Theta(\sqrt{n/g})$.
- Disproved by Przytycka and Przytycki '90-97 who achieved $\Omega(\sqrt{n/g}\sqrt{\log g})$, and conjectured $\Theta(\sqrt{n/g}\log g)$.
- How about non-separating, or null-homologous non-contractible cycles ?

Continuous Setting: Systolic Geometry

The ***systole*** of a Riemannian surface is the length of the shortest ***noncontractible*** cycle.



Theorem (Gromov '83, Katz and Sabourau '04)

The systole of a Riemannian surface of genus g and area A is $O(\sqrt{A/g}\log g)$.

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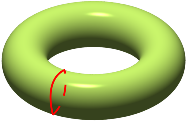

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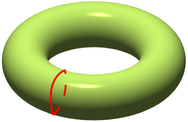

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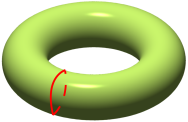

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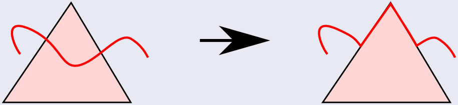
- Larry Guth: "Arithmetic hyperbolic surfaces are remarkably hard to picture."

A two way street: From discrete to continuous

How to switch from a discrete to a continuous metric ?

Proof.

- Glue equilateral triangles of area 1 on the triangles .
- Smooth the metric.
- 
- In the worst case the lengths double.

□

Theorem (Colin de Verdière, Hubard, de Mesmay '14)

Let (S, G) be a triangulated surface of genus g, with n triangles. There exists a Riemannian metric m on S with area n such that for every closed curve γ in (S, m) there exists a homotopic closed curve γ' on (S, G) with

$$|\gamma'|_G \leq (1+\delta)\sqrt[4]{3}\ |\gamma|_m \quad \text{for some arbitrarily small } \delta.$$

Corollary

Let (S, G) be a triangulated surface with genus g and n triangles.

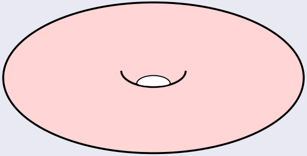
1. *Some non-contractible cycle has length* $O(\sqrt{n/g}\log g)$.
2. *Some non-separating cycle has length* $O(\sqrt{n/g}\log g)$.
3. *Some null-homologous non-contractible cycle has length* $O(\sqrt{n/g}\log g)$.

- (1) shows that Gromov $\Rightarrow$ Hutchinson and improves the best known constant.
- (2) and (3) are new.

A two way street: From continuous to discrete

How do we switch from a continuous to a discrete metric ?

Proof.



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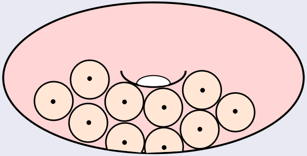
A two way street: From continuous to discrete

How do we switch from a continuous to a discrete metric ?

Proof.

Take a maximal set of balls of radius ε and perturb them a little.



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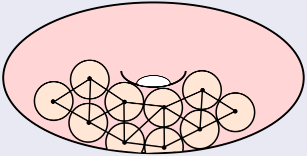

By [Dyer, Zhang and Möller '08], the Delaunay graph of the centers is a triangulation for ε small enough.

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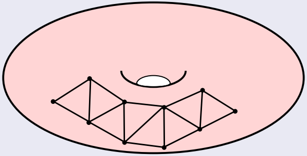

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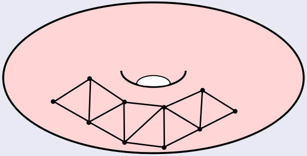

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$$|\gamma|_m \leq 4\varepsilon|\gamma|_G.$$

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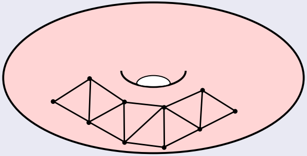

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$$|\gamma|_m \leq 4\varepsilon|\gamma|_G.$$

Each ball has radius $\pi\varepsilon^2 + o(\varepsilon^2)$, thus $\varepsilon = O(\sqrt{A/n})$.

□

Theorem (Colin de Verdière, Hubard, de Mesmay '14)

Let (S, m) be a Riemannian surface of genus g and area A. There exists a triangulated graph G embedded on S with n triangles, such that every closed curve γ in (S, G) satisfies

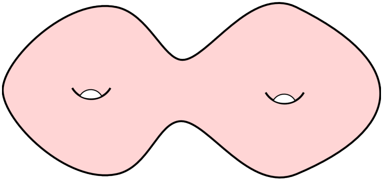
$|\gamma|_m \leq (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{A/n}\;|\gamma|_G$ *for some arbitrarily small* δ.

- This shows that Hutchinson $\Rightarrow$ Gromov.
- Proof of the conjecture of Przytycka and Przytycki:

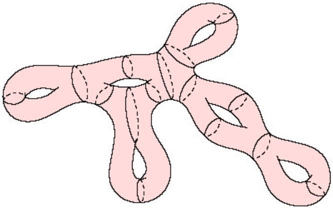
Corollary

There exist arbitrarily large g and n such that the following holds: There exists a triangulated combinatorial surface of genus g, with n triangles, of edgewidth at least $\frac{1-\delta}{6}\sqrt{n/g}\log g$ *for arbitrarily small* δ.

Part 2:
Pants decompositions



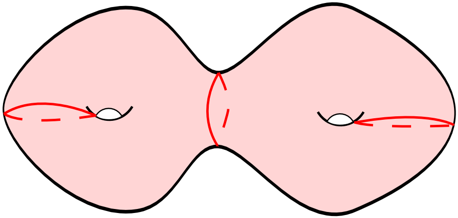
- A ***pants decomposition*** of a triangulated or Riemannian surface S is a family of cycles Γ such that cutting S along Γ gives pairs of pants, e.g., spheres with three holes.



- A pants decomposition has $3g - 3$ curves.
- Complexity of computing a shortest pants decomposition on a triangulated surface: in **NP**, not known to be **NP**-hard.

An algorithm to compute pants decompositions:

1. Pick a shortest non-contractible cycle.
2. Cut along it.
3. Glue a disk on the new boundaries.
4. Repeat $3g - 3$ times.



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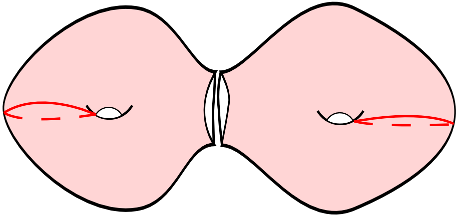

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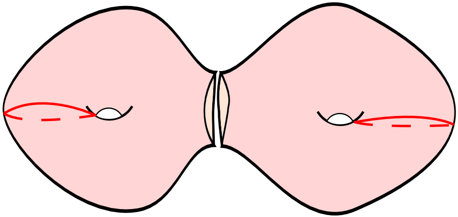

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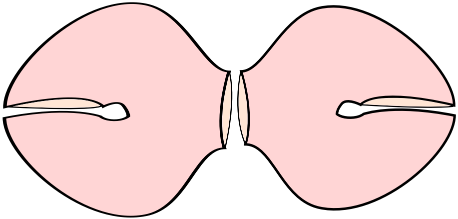

We obtain a pants decomposition of length

$$(3g-3)O(\sqrt{n/g}\log g) = O(\sqrt{ng}\log g).$$

An algorithm to compute pants decompositions:

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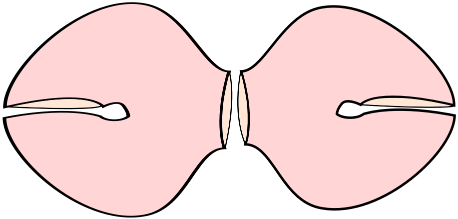
2. Cut along it.
3. Glue a disk on the new boundaries. ***This increases the area!***
4. Repeat $3g - 3$ times.



We obtain a pants decomposition of length

$$\cancel{(3g-3)O(\sqrt{n/g}\log g) = O(\sqrt{ng}\log g).}$$ ***Wrong!***

Doing the calculations correctly gives a subexponential bound.

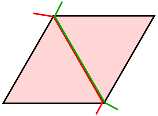
A correct algorithm

Denote by *PantsDec* the shortest pants decomposition of a triangulated surface.

- ***Best previous bound:*** $\ell(PantsDec) = O(gn)$.
  [Colin de Verdière and Lazarus '07]
- ***New result:*** $\ell(PantsDec) = O(g^{3/2}\sqrt{n})$.
  [Colin de Verdière, Hubard and de Mesmay '14]
- Moreover, the proof is algorithmic.

We "combinatorialize" a continuous construction of Buser.

- Several curves may run along the same edge:



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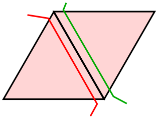
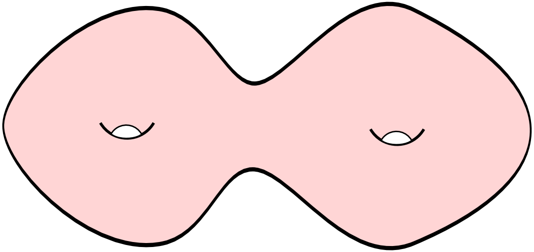


First idea



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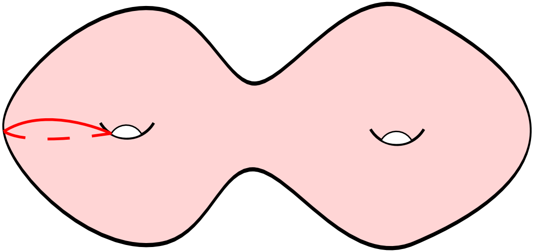

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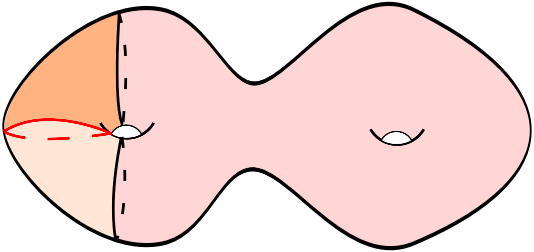

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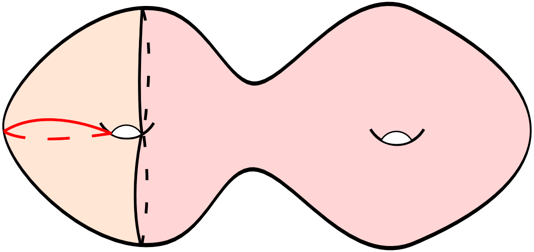
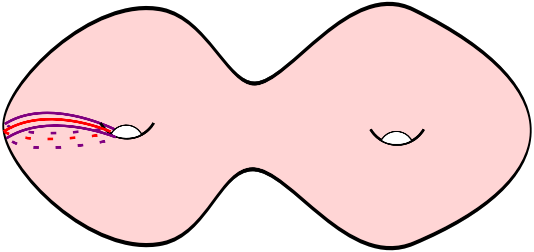


If the torus is fat, this is too long.

~~***First idea***~~

Second idea



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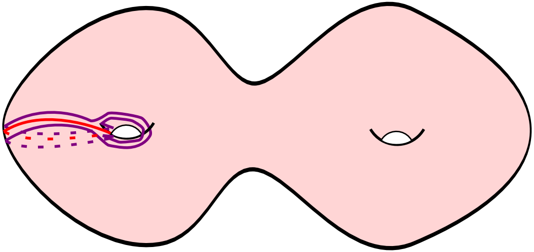

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~~*First idea*~~

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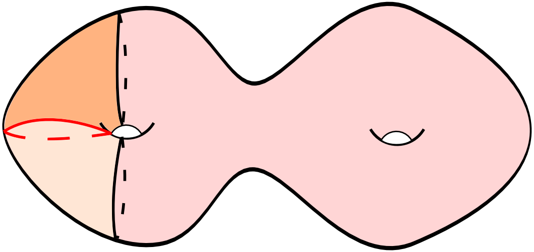
Both at the same time



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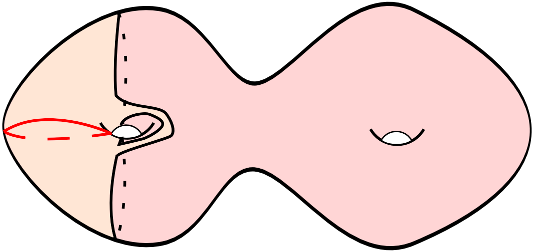

We take a trade-off between both approaches: As soon as the length of the curves with the first idea exceeds some bound, we switch to the second one.

Lower bounds

- Arithmetic surfaces (Buser, Sarnak), once discretized, yield systoles of size $\Omega(\sqrt{n/g}\log g)$.
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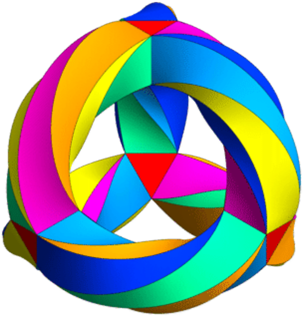
Random surfaces: Sample uniformly at random among the triangulated surfaces with n triangles.

Theorem (Guth, Parlier and Young '11)

If (S, G) is a random triangulated surface with n triangles, and thus $O(n)$ edges, the length of the shortest pants decomposition of (S, G) is $\Omega(n^{7/6-\delta})$ w.h.p. for arbitrarily small δ

We extend this lower bound to other decompositions than pants decompositions: cut-graphs with fixed combinatorial structure (skipped).

Part 3:
A glimpse into arithmetic surfaces



"Arithmetic hyperbolic surfaces are remarkably hard to picture"

- The embedded graphs built in Part 1 are not very natural.
- Maybe arithmetic surfaces yield better lower bounds for Part 2.
- They provide lower bounds on the systoles of covers of small genus surfaces.

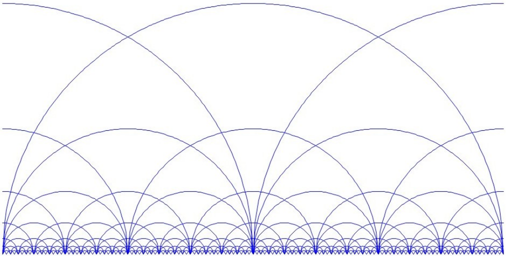
Theorem (Buser-Sarnak '94)

There exists a hyperbolic surface S and an infinite family of covers S_i of S such that

$$sys(S_i) = \Omega(\log g(S_i)).$$

Hyperbolic surfaces have area $4\pi(g-1)$.

- A hyperbolic surface is a quotient of the hyperbolic plane $\mathbb{H}^2$ by a subgroup Γ of its isometry group $Isom^+(\mathbb{H}^2)$.



- We view $\mathbb{H}^2$ in the upper half-plane model, then $Isom^+(\mathbb{H}^2) \equiv PSL_2(\mathbb{R})$, which acts by homographies:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

- We pick two integers a, b and look at the subgroup

$$\Gamma = \left\{ \begin{pmatrix} X_0 + X_1\sqrt{a} & X_2 + X_3\sqrt{a} \\ b(X_2 - X_3\sqrt{a}) & X_0 - X_1\sqrt{a} \end{pmatrix} \mid \begin{array}{c} X_i \in \mathbb{Z} \\ X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 = 1 \end{array} \right\}.$$

- as well as its ***congruence subgroups*** for a prime p.

$$\Gamma(p) = \left\{ \begin{pmatrix} X_0 + X_1\sqrt{a} & X_2 + X_3\sqrt{a} \\ b(X_2 - X_3\sqrt{a}) & X_0 - X_1\sqrt{a} \end{pmatrix} \mid \begin{array}{c} X_i \in \mathbb{Z} \\ X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 = 1 \\ X_0 \equiv 1[p]; X_1, X_2, X_3 \equiv 0[p] \end{array} \right\}.$$

- ***Claim:*** For well chosen a and b (eg $a = 2$ and $b = 3$), $\Gamma/\mathbb{H}^2$ and $\Gamma(p)/\mathbb{H}^2$ are compact surfaces S and $S(p)$.

- ***Claim:*** $g(S(p)) = O(p^3)$.
  - Indeed, $g(S(p)) = O([\Gamma, \Gamma(p)]g(S))$, where $[\Gamma, \Gamma(p)]$ is the index of $\Gamma(p)$ in Γ, i.e., the order of

$$\Gamma/\Gamma(p) = \left\{ \begin{pmatrix} X_0 + X_1\sqrt{a} & X_2 + X_3\sqrt{a} \\ b(X_2 - X_3\sqrt{a}) & X_0 - X_1\sqrt{a} \end{pmatrix} \mid \begin{array}{c} X_i \in \mathbb{Z}_p \\ X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 \equiv 1[p] \end{array} \right\}.$$

- ***Claim:*** $sys(S(p)) = \Omega(\log p)$.
  - Indeed, for a non-trivial element g of $\Gamma(p)$, $1 = X_0^2 - aX_1^2 - bX_2^2 + abX_3^2$, and $p|X_1, X_2, X_3$, thus

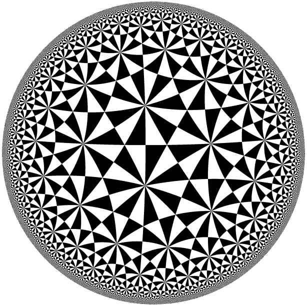
$$X_0^2 \equiv 1[p^2] \Rightarrow X_0 \equiv \pm 1[p^2].$$

  - Furthermore, $X_0 \neq \pm 1$, thus $|X_0| = \Omega(p^2)$, and $Trace(g) = 2X_0 = \Omega(p^2)$.
  - The ***translation length*** of g is controlled by its trace: $l(g) = 2argch(1/2\, Trace(g))$, and thus $sys(S(p)) = \Omega(\log p)$.

Zooming out

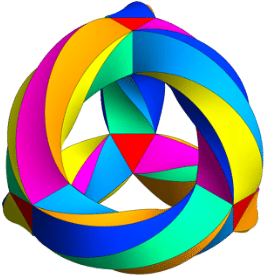
- Starting from a well-chosen (i.e., arithmetic) surface S, we can find covers $S(p)$ using congruences for which the systole grows logarithmically.
- To get discrete systolic lower bounds, it is enough to triangulate the first surface S and lift the triangulation.

Can we start with a surface that is already naturally triangulated, for example with triangles of angles $\pi/2, \pi/3$ and $\pi/7$?



Hurwitz surfaces

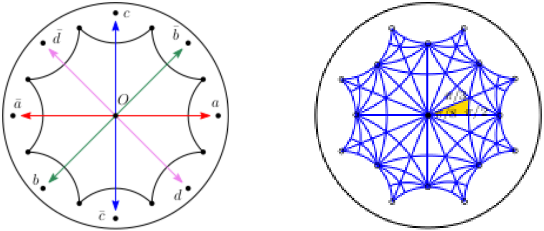
- Yes [Katz, Schaps, Vishne '07] but this requires taking the X_i in $\mathbb{Z}[\cos 2\pi/7]$ instead of just $\mathbb{Z}$.
  $\rightarrow$ The number theory is more involved (***orders*** in ***quaternion algebras***).
- ***Hurwitz surfaces*** are hyperbolic surfaces with the maximal number of symmetries (automorphisms). They are obtained from $(2, 3, 7)$ triangles.
  $\rightarrow$ Surfaces with maximal symmetry have big systoles.



Animation by Greg Egan

The Bolza surface

- The Bolza surface is the genus 2 surface obtained from an equilateral octagon.



- Its fundamental group is a subgroup of the $(2, 3, 8)$ triangle group.
- [Katz, Katz, Schein, Vishne '16] show that it is an arithmetic surface (using $\mathbb{Z}[\sqrt{2}]$) and use congruence subgroups to compute covers with the systole growing logarithmically.

- (M, T) : triangulated d-manifold, with $f_d(T)$ facets and $f_0(T)$ vertices.
- Supremum of $\frac{\mathrm{sys}^d}{f_d}$ or $\frac{\mathrm{sys}^d}{f_0}$?

Theorem (Gromov)

For every d, there is a constant C_d such that, for any Riemannian metric on any ***essential*** *compact d-manifold M without boundary, there exists a non-contractible closed curve of length at most $C_d vol(m)^{1/d}$.*

- We follow the same approach as for surfaces:
  - Endow the metric of a regular simplex on every simplex.
  - Smooth the metric.
  - Push curves inductively to the 1-dimensional skeleton.

Appendix: Discrete systolic inequalities in higher dimensions

- (M, T) : triangulated d-manifold, with $f_d(T)$ facets and $f_0(T)$ vertices.
- Supremum of $\frac{\mathrm{sys}^d}{f_d}$ or $\frac{\mathrm{sys}^d}{f_0}$?

Theorem (Gromov)

For every d, there is a constant C_d such that, for any ***piecewise*** *Riemannian metric on any* ***essential*** *compact d-manifold M without boundary, there exists a non-contractible closed curve of length at most $C_d \, vol(m)^{1/d}$.*

- We follow the same approach as for surfaces:
  - Endow the metric of a regular simplex on every simplex.
  - ~~Smooth the metric.~~ ***Non-smoothable triangulations*** [Kervaire '60]
  - Push curves inductively to the 1-dimensional skeleton.
- Corollary: $\frac{\mathrm{sys}^d}{f_d}$ is upper bounded by a constant for essential triangulated manifolds.

In the other direction, starting from a Riemannian manifold:

- Take an ε-separated net and its Delaunay complex.
- Hope that it will be a triangulation

Appendix: Discrete systolic inequalities in higher dimensions

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This allows us to translate discrete systolic inequalities w.r.t. the number of vertices to continuous systolic inequalities.

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- Take an ε-separated net and its Delaunay complex.
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This allows us to translate discrete systolic inequalities w.r.t. the number of vertices to continuous systolic inequalities.

But are there any?

Question: Are there manifolds M of dimension $d \geq 3$ for which there exists a constant c_M such that, for every triangulation (M, T), there is a non-contractible closed curve in the 1-skeleton of T of length at most $c_M f_0(T)^{1/d}$?