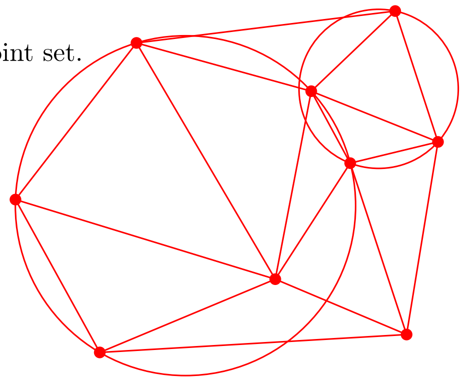
2 Delaunay triangulation: definitions, motivations, properties, classical algorithms.

2.1 Drawing

Draw the Delaunay triangulation of the attached point set.

2.1 Correction:



2.2 Nearest neighbor graphs

S a set of n points. $q_0 \in S$. Let q_1 denote the nearest neighbor of q_0 in $S \setminus \{q_0\}$. Let q_2 denote the second nearest neighbor of q_0 in S, i.e., the nearest neighbor in $S \setminus \{q_0, q_1\}$. Similarly q_i the i^{th} nearest neighbor.

The directed nearest neighbor graph of S is the graph whose vertices are the points in S and pq is an edge of the graph if q is the nearest neighbor of p.

Fact: The degree of the nearest neighbor graph is ≤ 6. (proof optional).

2.2.1 Nearest neighbor

Prove that q_0q_1 is an edge of the Delaunay triangulation of S.

2.2.2 Second nearest neighbor

Prove that q_0q_2 or q_1q_2 is an edge of the Delaunay triangulation of S.

2.2.3 k^{th} nearest neighbor

Prove that $\forall k \exists i < k$ such that q_kq_i is an edge of the Delaunay triangulation of S.

2.2.4 Nearest neighbor graph

Write an algorithm that takes the Delaunay triangulation of S and output the directed nearest neighbor graph of S.

You can write things like:

`for` v `enumerating all vertices of` $DT(S)$,

`for` w `enumerating the neighbor of` v `in` $DT(S)$,

or `output edge`(v, w),

or $v.color = red$ to add some information in a vertex (or edge or. . .)

What is the complexity of this algorithm?

2.2.5 Nearest neighbor graph

Write an algorithm that takes the Delaunay triangulation of S and output the directed second nearest neighbor graph of S.

What is the complexity of this algorithm?

2.2 Correction:

2.2.1 Nearest neighbor

The disk centered at q_0 passing through q_1 contains only q_0, thus the disk of diameter q_0q_1, which is included in the previous one is empty. By the empty circle property, q_0q_1 is a Delaunay edge.

2.2.2 Second nearest neighbor

The disk D_2 centered at q_0 passing through q_2 contains only q_0 and q_1, thus we consider the two disks Z_0 and Z_1 passing through q_2 tangent in q_2 to D_2 and respectively passing through q_0 and q_1. We have to cases:

— $Z_0 \subset Z_1 \subset D_2$ and Z_0 is empty, by the empty circle property, q_0q_2 is a Delaunay edge.

— $Z_1 \subset Z_0 \subset D_2$ and Z_1 is empty, by the empty circle property, q_1q_2 is a Delaunay edge.

2.2.3 k^{th} nearest neighbor

The disk of center q_0 through q_k verifies $D_k \cap S = \{q_0, q_1 \dots q_{k-1}\}$. Consider the pencil of circles through q_k tangent to D_k The bigest empty circle of that pencil inside D_k pass through a point inside D_k that is some q_i with $i < k$ and by the empty circle property, q_iq_k is a Delaunay edge.

2.2.4 Nearest neighbor graph

```
for u enumerating all vertices of DT(S) {
        d = ∞;
        for w enumerating the neighbor of u in DT(S) {
                if ||uw|| < d then {nn = w;   d = ||uw||;   }
        }
        output edge(u, nn),
}
```

The inside loop costs $d^\circ(u)$, thus the total cost of the algorithm is $\sum_{u\in S} d^\circ(u) < 6n$.

2.2.5 Nearest neighbor graph

```
for u enumerating all vertices of DT(S) {
        u.d = ∞;
        for w enumerating the neighbor of u in DT(S) {
                if ||uw|| < d then {u.nn = w;   d = ||uw||;   }
        }
}
for u enumerating all vertices of DT(S) {
        d = ∞;
        for w enumerating the neighbor of u in DT(S) {
                if (||uw|| < d and w ≠ u.nn) then {sn = w;   d = ||uw||;   }
        for w enumerating the neighbor of u.nn in DT(S) {
                if (||uw|| < d and w ≠ u) then {sn = w;   d = ||uw||;   }
        output edge(u, sn),
        }
}
```

The cost is

$$\begin{aligned}
\sum_{u\in S}(d^\circ_{DT}(u) + d^\circ_{DT}(u.nn)) &= \sum_{u\in S} d^\circ_{DT}(u) + \sum_{u\in S}\sum_{v\in\{u.nn\}} d^\circ_{DT}(v) \\
&= \sum_{u\in S} d^\circ_{DT}(u) + \sum_{v\in S}\sum_{u \text{ such that } v=u.nn} d^\circ_{DT}(v) \\
&= \sum_{u\in S} d^\circ_{DT}(u) + \sum_{v\in S} d^\circ_{NN}(v)\cdot d^\circ_{DT}(v) \\
&= \sum_{u\in S} d^\circ_{DT}(u) + \sum_{v\in S} 6d^\circ_{DT}(v) \le 42n
\end{aligned}$$