Triangulations in non-Euclidean spaces

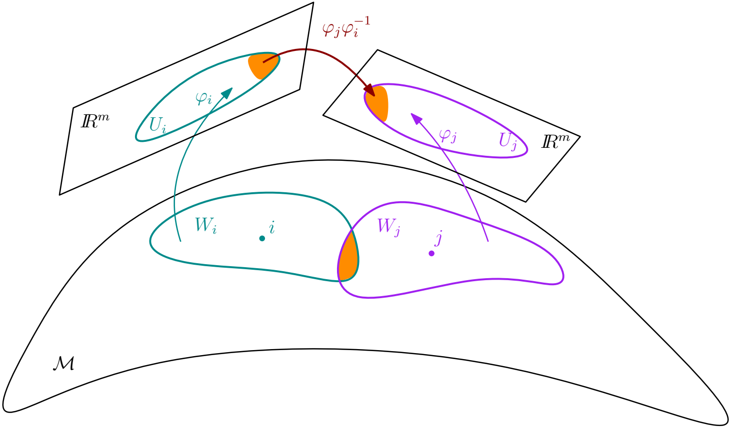
Monique Teillaud

Outline

Manifolds

Smooth manifold

Locally homeomorphic to a linear space.
Atlas of charts



Smooth manifold: transition maps C^∞

Bounded set $D \subset \mathbb{R}^m$. $P \subset \mathbb{R}^m$ finite set of points.

- P is ϵ-dense for D: $d(x, P) < \epsilon$ for all $x \in \mathcal{D}$
- P is $\mu\epsilon$-separated is $d(p, q) \geq \mu\epsilon$ for all $p, q \in P$

P is a (μ, ϵ)-net for D if it is $\mu\epsilon$-separated and ϵ-dense for D.
Usually $D = D_\epsilon(P) = \{x \in \text{conv}(P) | d(x, \partial\text{conv}(P)) \geq \epsilon\}$

Hypotheses

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Set of points in $\mathcal{M}$ indexed by $\mathcal{N} = \{i, \ldots, n\}$
For each $i \in \mathcal{N}$, subset of neighbors $\mathcal{N}_i \subset \mathcal{N}$.
$\phi_i : \mathcal{N}_i \to U_i \subset \mathbb{R}^m$ injective
Sampling condition $\epsilon_i > 0$
such that $P_i = \phi(\mathcal{N}_i)$ is a (μ, ϵ_i)-net for a small ball $B(p_i) \subset U_i, p_i = \phi(i)$.
(ϵ_i and ϵ_j close if p_i and p_j close)

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(ϵ_i and ϵ_j close if p_i and p_j close)

Low metric distortion
p_j close to $p_i \in U_i$
neighborhood $U_{ij} \subset U_i$ domain of the transition function φ_{ji}
$|d_i(x, y) - d_j(\varphi_{ji}(x), \varphi_{ji}(y))| \leq \xi d_i(x, y), \;\; \forall x, y \in U_{ij}$

Delaunay triangulation

For each $i \in \mathcal{N}$

- $DT(P_i)$ of the set of points P_i in $\mathbb{R}^m$
- $\text{star}(i, \mathcal{N}) \simeq \text{star}(p_i, DT(P_i))$

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Hypotheses

- Sampling condition
- Low distortion
- $d_i(\varphi_i(x), \varphi_i(y)) - d_{\mathcal{M}}(x, y)| \leq \eta d_i(\varphi_i(x), \varphi_i(y))$
  for $x, y \in \varphi_i^{-1}(B(p_i))$, where $B(p_i)$ and η small

then the union of the stars form the Delaunay complex of $\mathcal{N}$ in $\mathcal{M}$ with respect to $d_{\mathcal{M}}$

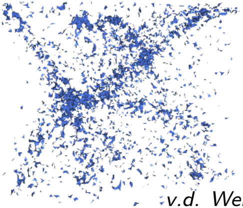
(*) more exactly some perturbed point sets P_i'

Flat manifolds

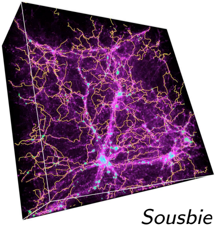
Flat manifolds — Motivation

Motivation

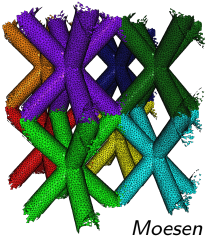
Needs for 3D periodic triangulations (3D flat torus)
in astronomy, material engineering, nano-structures . . .



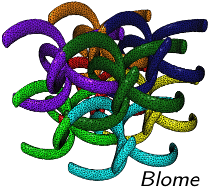
v.d. Weijgaert



Sousbie



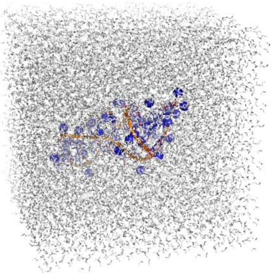
Moesen



Blome

Motivation

Crystallographic groups come up in structural molecular biology



"Biological structures and simulation are not living in a cubic box"

(Bernauer)

Flat manifolds — Closed Euclidean d-manifold

Definitions

Manifold $\mathbb{X}$

- closed: compact without boundary
- flat or Euclidean: every point has a neighborhood isometric to a neighborhood in $\mathbb{R}^d$

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where $\mathcal{G}$ group

- Bieberbach: discrete group of isometries of $\mathbb{R}^d$ with compact quotient
- torsion-free: no fixed point

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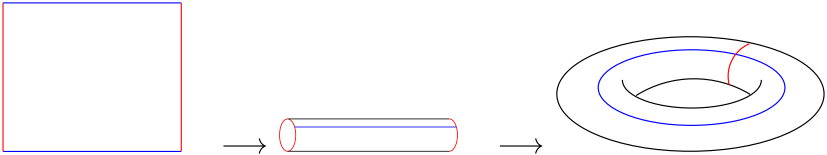
$\forall d$, only a finite number of dD Bieberbach groups up to isomorphism

Definitions

dimension 2:

- 17 wallpaper groups
- 2 closed Euclidean manifolds: torus and Klein bottle



$\mathbb{T}^2 = \mathbb{R}^2 / < t_x, t_y >$

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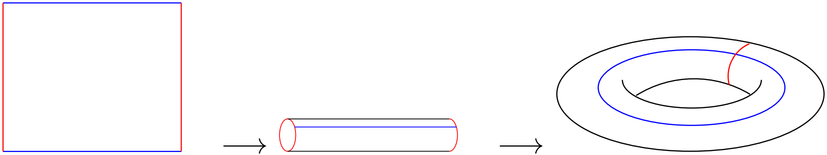

$\mathbb{T}^2 = \mathbb{R}^2 / < t_x, t_y >$

dimension 3:

- 230 crystallographic groups
- 10 closed Euclidean manifolds (4 non-orientable)

$\forall d$, only a finite number of dD Bieberbach groups up to isomorphism

Illustrations (2D)

Torus

$\mathbb{T}^2 = \mathbb{R}^2/\mathcal{G}$
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Klein bottle

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$\mathcal{G} =< r, t_y >$
r glide reflexion

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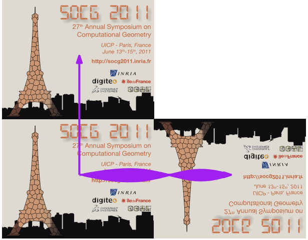

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Flat manifolds — Delaunay triangulation in $\mathbb{X}$: Definition

Definition

illustration

$\mathcal{G} =< t_x, t_y >$

$\mathbb{X} = \mathbb{T}^2 = \mathbb{R}^2/\mathcal{G}$

$\pi : \mathbb{R}^2 \rightarrow \mathbb{X}$

Definition

$\mathcal{P}$ finite point set

illustration

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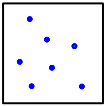


Definition

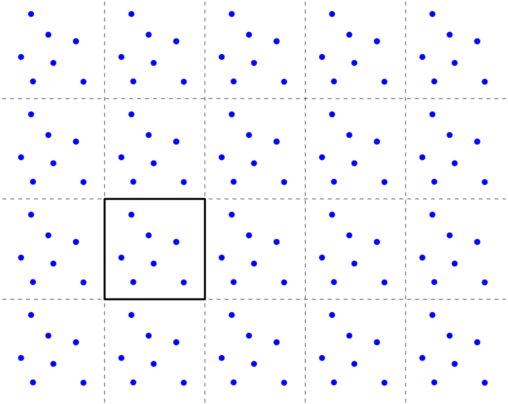
$\mathcal{GP}$ infinite point set

illustration

$\mathcal{G} = < t_x, t_y >$

$\mathbb{X} = \mathbb{T}^2 = \mathbb{R}^2 / \mathcal{G}$

$\pi : \mathbb{R}^2 \to \mathbb{X}$



Definition

$DT(\mathcal{GP})$

illustration

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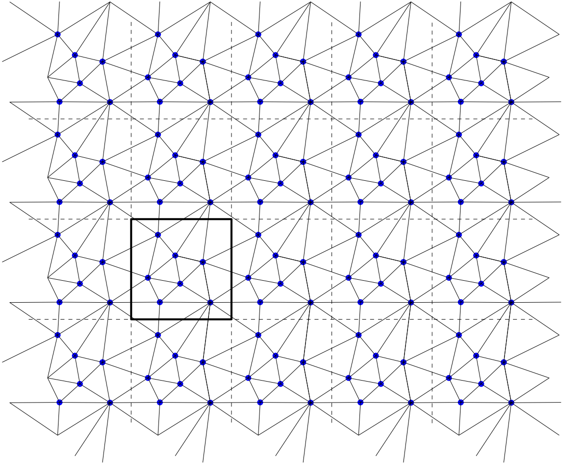

Definition

$$DT_{\mathbb{X}}(\pi(\mathcal{P})) = \pi(DT(\mathcal{G}\mathcal{P}))$$

if it is a simplicial complex

illustration

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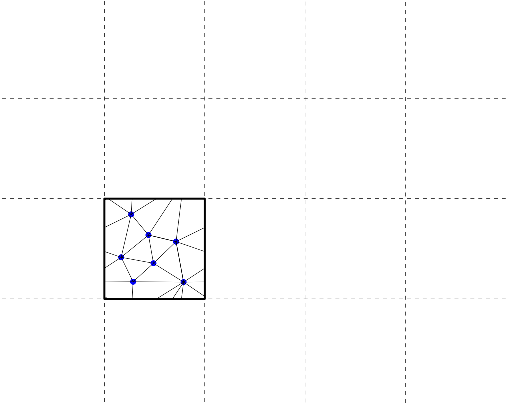
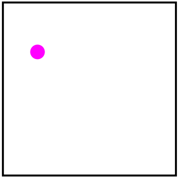


$\pi(DT(\mathcal{GP}))$ is not always a simplicial complex



One point in $\mathbb{T}^2$

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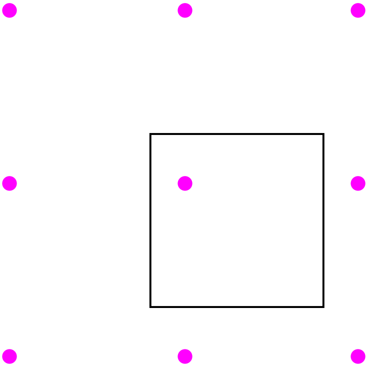

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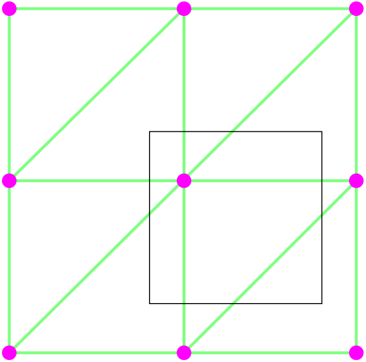

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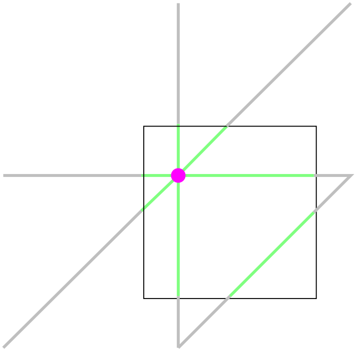
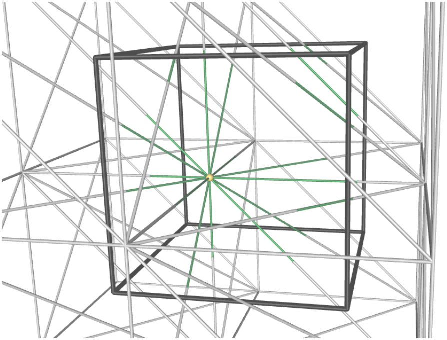


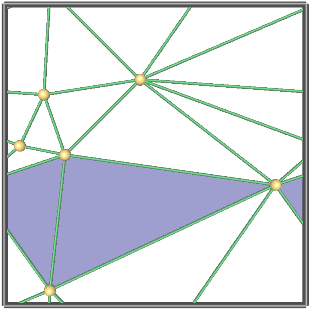
One point in $\mathbb{T}^2$: self-edges

$\pi(DT(\mathcal{GP}))$ is not always a simplicial complex



One point in $\mathbb{T}^3$: self-edges

$\pi(DT(\mathcal{GP}))$ is not always a simplicial complex



Cycle of length 2

Flat manifolds — Sufficient conditions

Sufficient conditions

If the 1-skeleton of $\pi(DT(\mathcal{GP}))$
does not contain cycles of length ≤ 2
then $\pi(DT(\mathcal{GP}))$ is a triangulation of $\mathbb{X}$

Sufficient conditions

If the 1-skeleton of $\pi(DT(\mathcal{G}\mathcal{P}))$ does not contain cycles of length ≤ 2

then $\pi(DT(\mathcal{G}\mathcal{P}))$ is a triangulation of $\mathbb{X}$

- $\delta(\mathcal{G}) = \min_{p\in\mathbb{R}^d, g\in\mathcal{G}, g\neq 1_\mathcal{G}} \text{dist}(p, gp)$
- $\Delta(\mathcal{P})$ diameter of the largest empty ball

If $\Delta(\mathcal{G}\mathcal{P}) < \frac{\delta(\mathcal{G})}{2}$

then $\pi(DT(\mathcal{G}\mathcal{P}'))$ is a triangulation of $\mathbb{X}$

for any finite $\mathcal{P}' \supseteq \mathcal{P}$

Flat manifolds — What if the conditions are not satisfied?

Covering spaces

$\mathbb{X}$ a topological space.

$\rho : \tilde{\mathbb{X}} \to \mathbb{X}$ is a covering map,
and $\tilde{\mathbb{X}}$ is a covering space of $\mathbb{X}$ if:

$\forall x \in \mathbb{X}$

- $\exists V_x$ open neighborhood of x
- $\exists$ a decomposition of $\rho^{-1}(V_x)$ as a family $\{U_{\alpha_x}\}$, $U_{\alpha_x} \subset \tilde{\mathbb{X}}$ pairwise disjoint

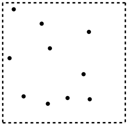
s.t. $\rho|_{U_{\alpha_x}}$ is a homeomorphism for each α_x.

If $h = max_{x \in \mathbb{X}} |U_{\alpha_x}|$ is finite, then $\tilde{\mathbb{X}} =$ h-sheeted covering space.

Covering spaces

$\mathcal{G} =< t_x, t_y >$

$\mathbb{X} = \mathbb{T}^2 = \mathbb{R}^2/\mathcal{G}$



1-sheeted covering

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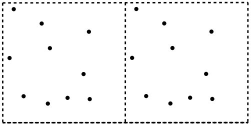

$\mathbb{R}^2/ < 2 \cdot t_x, t_y > =$ 2-sheeted covering

Covering spaces

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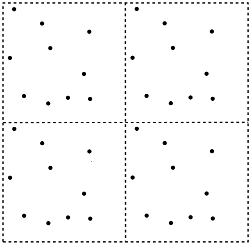

$\mathbb{R}^2/ < 2 \cdot t_x, 2 \cdot t_y > =$ 4-sheeted covering

Covering spaces

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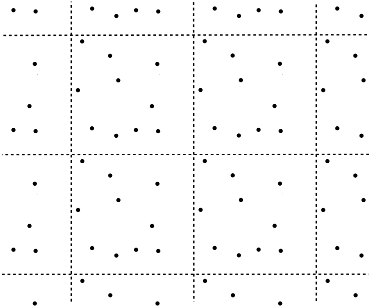

∞-sheeted covering $= \mathbb{R}^2 =$ universal covering

If the conditions are not satisfied

One can construct a finitely-sheeted covering space of $\mathbb{X}$:

$\mathbb{X}_C = \mathbb{R}^d / \mathcal{G}_C$ s.t. $\pi_C(DT(\mathcal{G}_C c_C(\mathcal{P})))$ is a triangulation for any $\mathcal{P}$

$c_C(\mathcal{P}) = \#$sheets copies of $\mathcal{P}$

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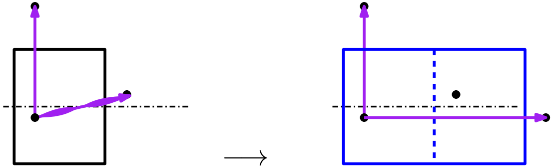
Proof uses

- each closed Euclidean *orbifold* has a d-torus as finitely-sheeted covering space
- it can be constructed



$\mathbb{T}^2 = 2$-sheeted covering space of $\mathbb{K}$

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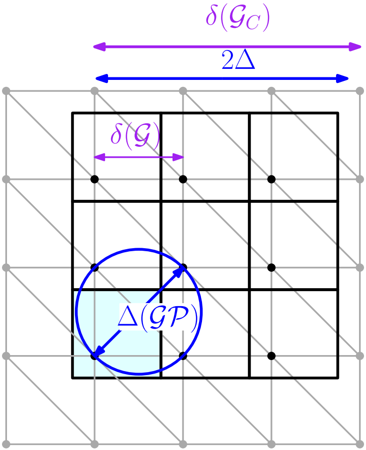
Then (illustration)

$\mathcal{G} =< t_x, t_y >$

$\mathbb{X} = \mathbb{T}^2 = \mathbb{R}^2/\mathcal{G}$

$\pi : \mathbb{R}^2 \to \mathbb{X}$



$\mathcal{G}_C =< 3t_x, 3t_y >$

$\mathbb{X}_C = \mathbb{R}^2/\mathcal{G}_C$

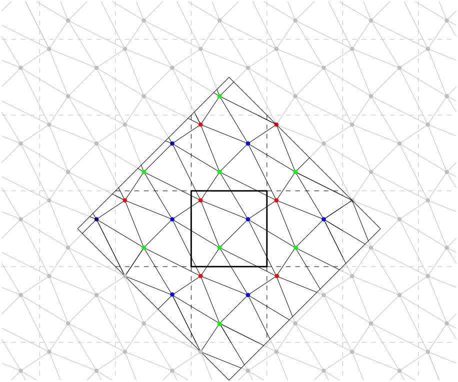
$\Delta(\mathcal{G}_C c_C(\mathcal{P})) < \frac{\delta(\mathcal{G}_C)}{2}$

If the conditions are not satisfied

$\mathcal{G} =< t_x, t_y >$, $\mathbb{X} = \mathbb{T}^2 = \mathbb{R}^2/\mathcal{G}$

9-sheeted covering space

improved to 8-sheeted covering space



Flat manifolds — Incremental algorithm

Incremental algorithm

$\mathbb{X}$ given

- compute $\mathbb{X}_C$

Incremental algorithm

$\mathbb{X}$ given

- compute $\mathbb{X}_C$

$\mathcal{P}$ given

- insert copies of points in $DT_{\mathbb{X}_C}$
- once diameter of largest empty ball $< \delta(\mathcal{G})/2$
  - remove copies
  - insert remaining points in $DT_{\mathbb{X}}$

Incremental algorithm

$\mathbb{X}$ given

- compute $\mathbb{X}_c$

$\mathcal{P}$ given

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- once diameter of largest empty ball $< \delta(\mathcal{G})/2$
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Computes
- $DT_{\mathbb{X}}(\mathcal{P})$ if possible
- $DT_{\mathbb{X}_c}(\mathcal{P})$ otherwise

Incremental algorithm

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Computes
- $DT_{\mathbb{X}}(\mathcal{P})$ if possible
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works for orbifolds

Complexity

optimal randomized worst-case

- $O\left(n^{\lceil \frac{d}{2} \rceil} + \log n\right)$ time
- $O\left(n^{\lceil \frac{d}{2} \rceil}\right)$ space

complexity

using the Delaunay hierarchy

Implementation 3D flat torus (cubic domain)

CGAL 3.5 and further

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- Spatial sorting from CGAL
- Optional insertion of dummy points to force 1-sheeted covering

Implementation 3D flat torus (cubic domain)

CGAL 3.5 and further

- Spatial sorting from CGAL
- Optional insertion of dummy points to force 1-sheeted covering

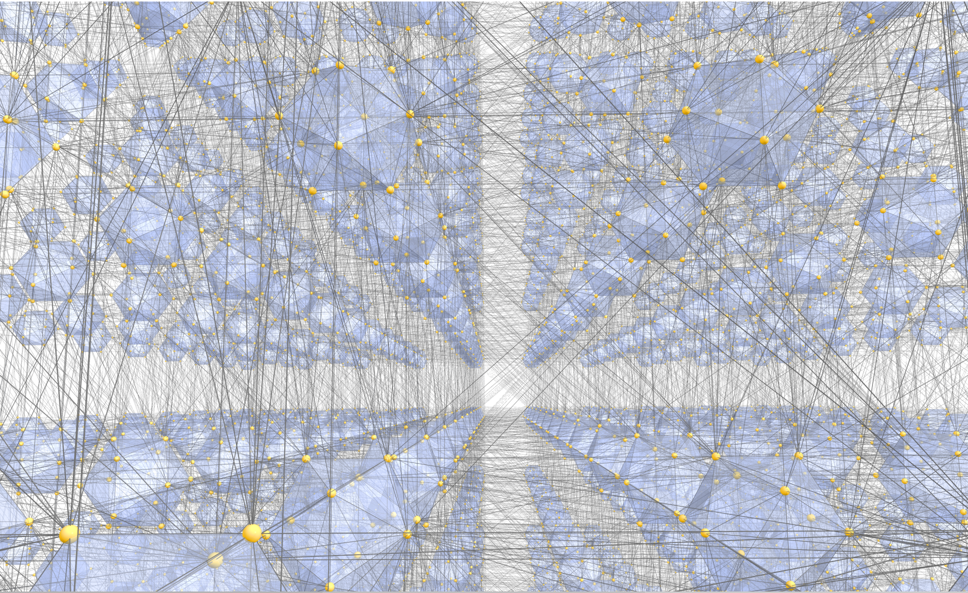
- Data from research in cosmology
- Random points

1 million points in 23 seconds

2.33 GHz Intel Core 2 Duo processor

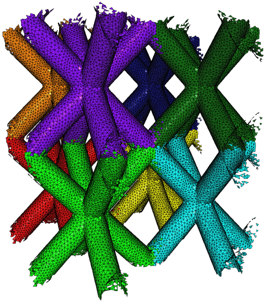
factor $\simeq 1.5$ compared to CGAL Delaunay triangulation in $\mathbb{R}^3$
on large data sets

Implementation 3D flat torus (cubic domain)



Implementation 3D flat torus (cubic domain)

- prototypes for periodic meshes

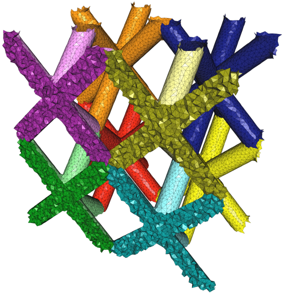


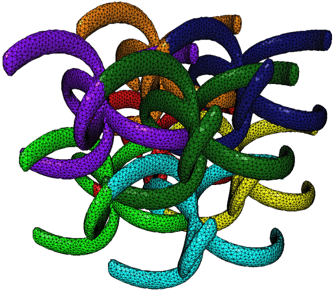
surface *(Fisikopoulos)*

using CGAL meshes

Implementation 3D flat torus (cubic domain)

- prototypes for periodic meshes





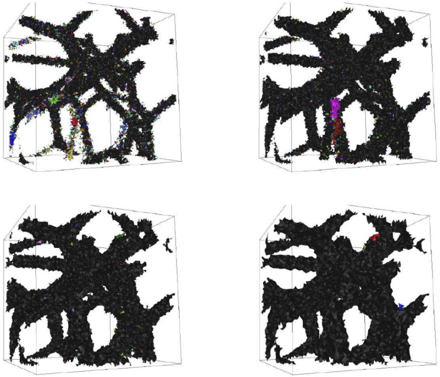
volume *(Bogdanov, Pellé)*

using CGAL meshes

Implementation 3D flat torus (cubic domain)

- CGAL periodic α-shapes (3.6 and further)
  using CGAL α-shapes

applications, eg. topology of the cosmic web

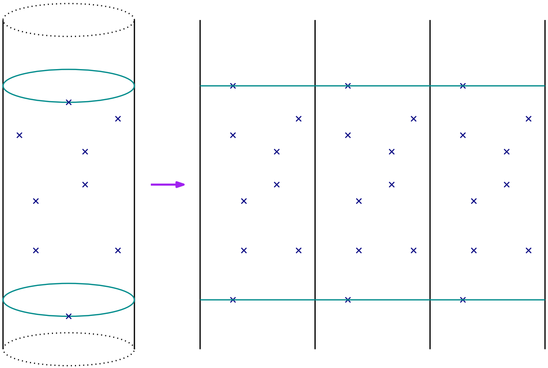


Flat manifolds — Open questions

Implementation

for more crystallographic groups. . .

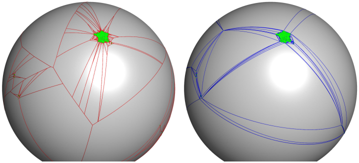
Non-compact quotients



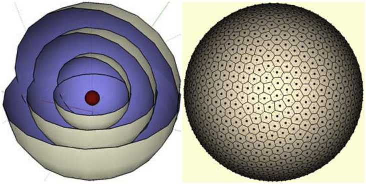
- always 1-cycles. . .
- data-structure?
  compactification of cylinder?
  in CGAL, $\mathbb{R}^2 \longrightarrow \mathbb{S}^2 \simeq \mathbb{R}^2 \cup \{\infty\}$

Sphere

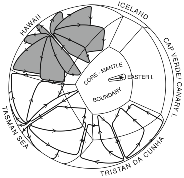
Motivation



Geographic Information System



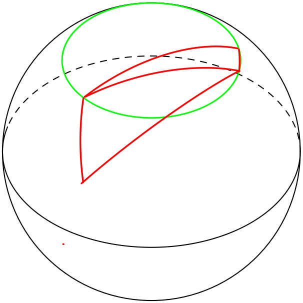
Information Visualization *(pictures Larrea et al.)*



Geophysics *(picture Fohlmeister)*

Delaunay triangulation on a sphere

Sphere $\mathcal{S} \in \mathbb{R}^3$.



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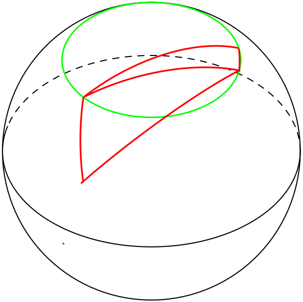


$=$ convex hull in 3D

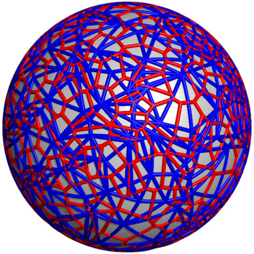
Use the fact that all points are lying on the convex hull

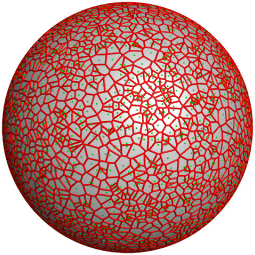
Software

- QHULL *(Barber et al.)*
  Quick Hull algorithm, 3D convex hull
- HULL *(Clarkson)*
  Randomized incremental construction, 3D convex hull
  exact arithmetic on integers
- SUG *(Sugihara)*
  divide-and-conquer, 3D convex hull
  exact arithmetic on integers
- CGAL 3D Delaunay triangulation
- STRIPACK *(Renka)*
  Incremental with flips, Delaunay on sphere
  `double` number type
  (Fortran)

Using a 2D data structure

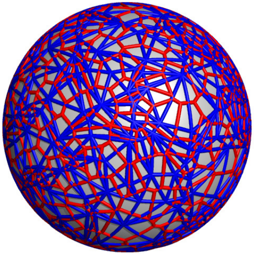
3D convex hull where all points are lying on the convex hull

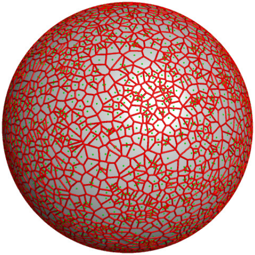




Using a 2D data structure

3D convex hull where all points are lying on the convex hull





In practice: input points close to the sphere $\longrightarrow$

- Delaunay triangulation of the points projected onto the sphere

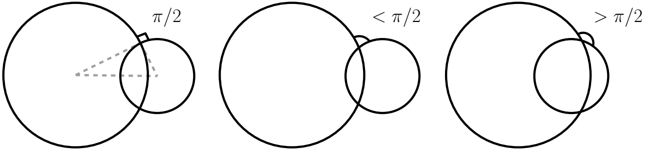
OR

- Regular triangulation on the sphere

Incremental algorithm

Regular triangulation in $\mathbb{R}^2$

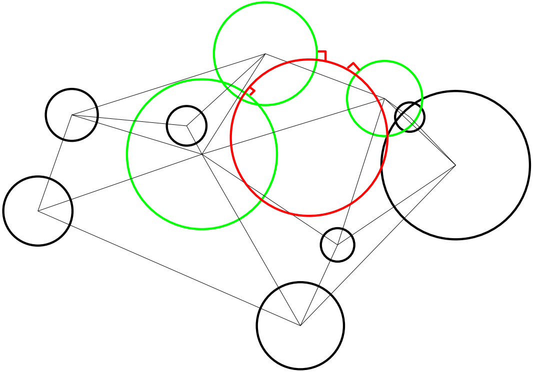
weighted-point: $(p, r^2) \mapsto$ circle c: center p, radius r
power product $pow(c, c') = \|pp'\|^2 - r^2 - r'^2$



$pow(c, c') = 0$ $pow(c, c') > 0$ $pow(c, c') < 0$

Regular triangulation in $\mathbb{R}^2$

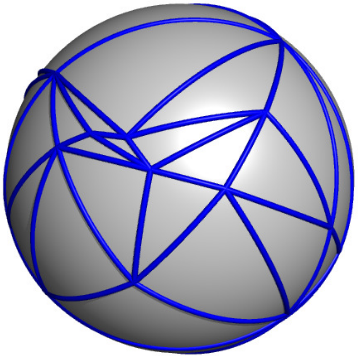
weighted-point: $(p, r^2) \mapsto$ circle c: center p, radius r
power product $pow(c, c') = \|pp'\|^2 - r^2 - r'^2$



null radii → Delaunay

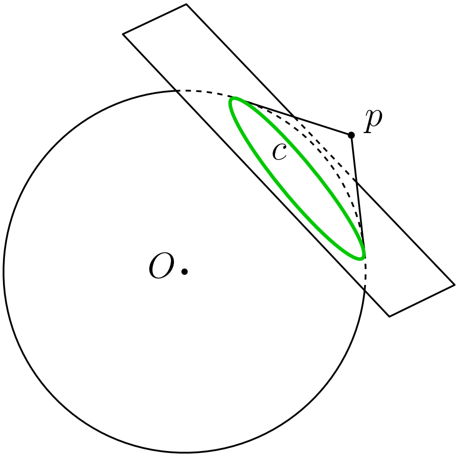
Regular triangulation on $\mathcal{S}$

- Triangle $\rightarrow$ Triangle on the sphere
- Edge $\rightarrow$ Arc of great circle

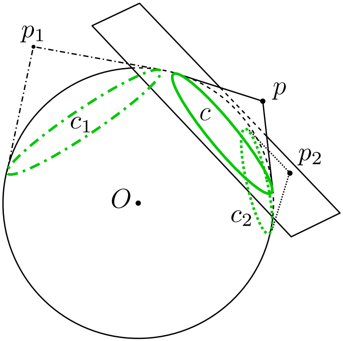


Regular triangulation on $\mathcal{S}$

$p \in \mathbb{R}^3 \mapsto$ circle c on $\mathcal{S}$.



Regular triangulation on $\mathcal{S}$



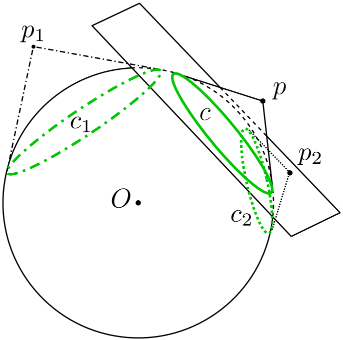
Power-test

$$\text{sign}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_{p_1} & x_{p_2} & x_{p_3} & x_{p'} \\ y_{p_1} & y_{p_2} & y_{p_3} & y_{p'} \\ z_{p_1} & z_{p_2} & z_{p_3} & z_{p'} \end{vmatrix}$$

power test $\leftrightarrow$ orientation test

regular triangulation on $\mathcal{S}$ $\leftrightarrow$ convex hull in $\mathbb{R}^3$

Regular triangulation on $\mathcal{S}$



Power-test

$$\text{sign}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_{p_1} & x_{p_2} & x_{p_3} & x_{p'} \\ y_{p_1} & y_{p_2} & y_{p_3} & y_{p'} \\ z_{p_1} & z_{p_2} & z_{p_3} & z_{p'} \end{vmatrix}$$

power test $\leftrightarrow$ orientation test

regular triangulation on $\mathcal{S}$ $\leftrightarrow$ convex hull in $\mathbb{R}^3$

null radii $\rightarrow$ Delaunay

Points on sphere

- null weights
- regular triangulation $\leftrightarrow$ Delaunay
- *in_circle* predicate = orientation

coordinates = algebraic numbers of degree 2

Orientation predicate:

$$\text{sign}(a_1\sqrt{\alpha_1} + a_2\sqrt{\alpha_2} + a_3\sqrt{\alpha_3} + a_4\sqrt{\alpha_4}),$$

a_i, α_i rational

sign of algebraic number of degree ≤ 16
can be reduced to evaluating sign of polynomial expression

see CGAL spherical kernel

Points close to the sphere

Regular triangulation $\longrightarrow$ hidden points?

Points close to the sphere

Regular triangulation $\longrightarrow$ hidden points?

sphere $\mathcal{S}$ of radius R
Points close to $\mathcal{S}$: $dist(p, \mathcal{S}) \leq \delta, \ \ \forall p \in P$.

$\forall p, q \in P, \ \ dist(p, q) > 2\sqrt{R\delta} \Longrightarrow$ no point is hidden.

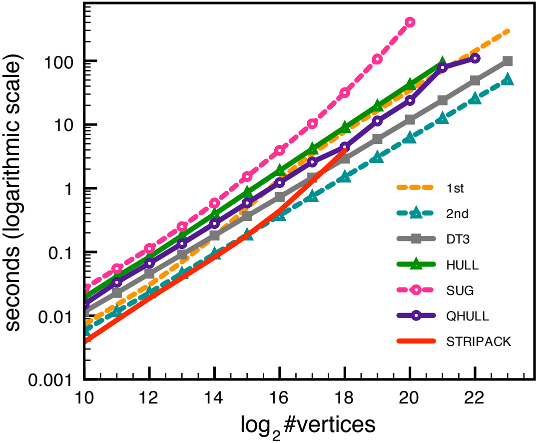
Implementation

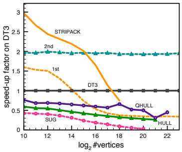
- Dt_on_sphere<geom_traits, Triangulation_data_structure_2>

- Two geometric traits
  - points on sphere ⟶ algebraic numbers
  - points close to sphere ⟶ rational numbers

Experiments

Random points

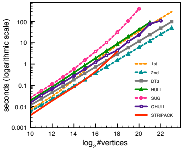


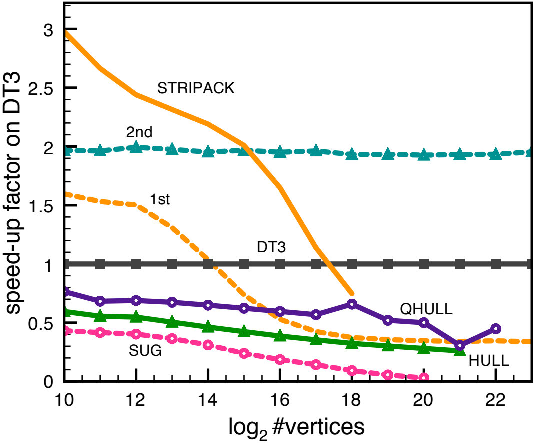


(aborted when running time > 10 min or failure)

Experiments

Random points

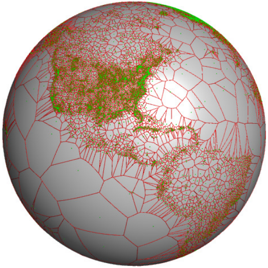




(aborted when running time $>$ 10 min or failure)

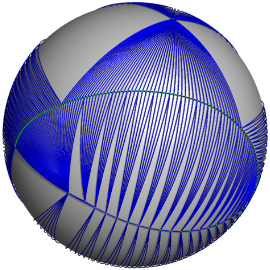
Experiments

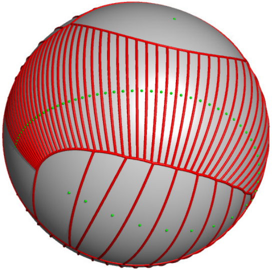
20,950 weather stations all around the world.



Approach	Time (in secs)
1st	0.57
2nd	0.14
DT3	0.25
QHULL	0.35
STRIPACK	fails

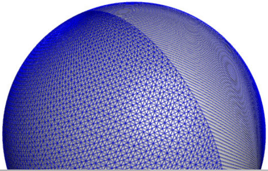
Hard cases

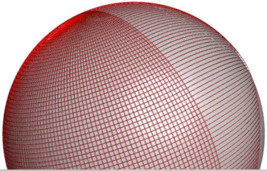




STRIPACK fails for e.g. $1,500$ points.

Degenerate cases:





Memory usage

Approach	Bytes per vertex
1st	113
2nd	113
DT3	174
QHULL	288

Hyperbolic space

Hyperbolic space — Motivation

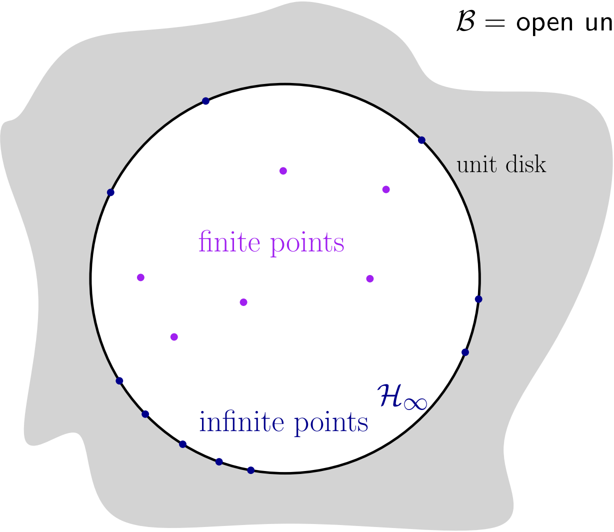
Delaunay triangulation of points lying in two parallel planes

Delaunay triangulation of points lying in two non-parallel planes

Hyperbolic space — Background

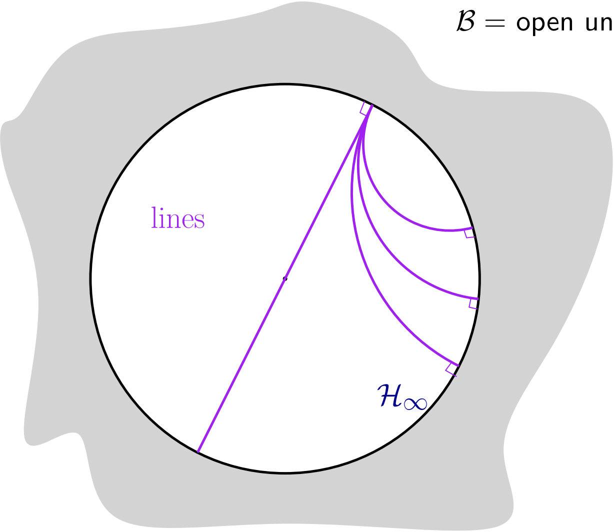
The Poincaré disk model

$\mathcal{B}$ = open unit disk in $\mathbb{R}^2$



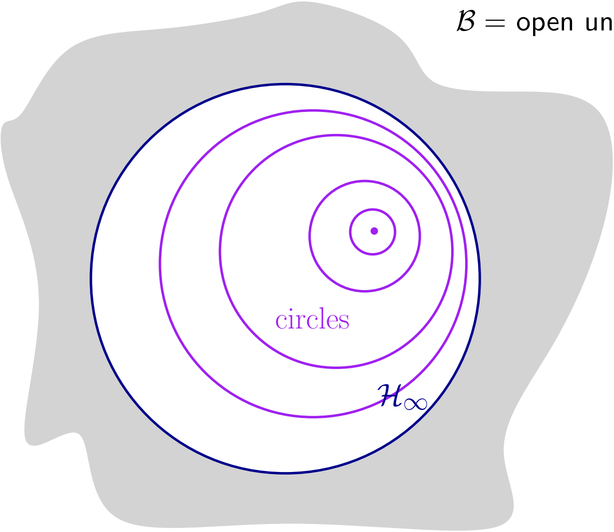
The Poincaré disk model

$\mathcal{B}$ = open unit disk in $\mathbb{R}^2$



The Poincaré disk model

$\mathcal{B}$ = open unit disk in $\mathbb{R}^2$



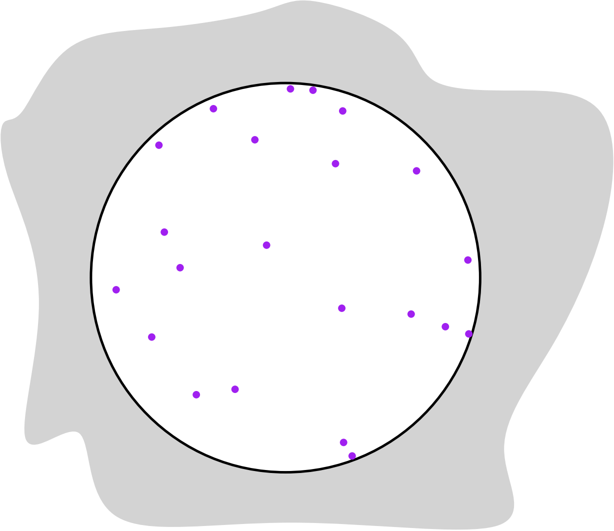
Delaunay triangulation

empty hyperbolic circles

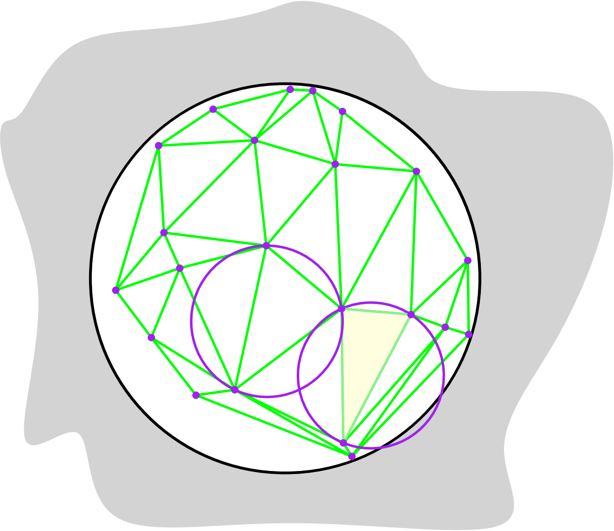
$=$

empty Euclidean circles
contained in $\mathcal{B}$

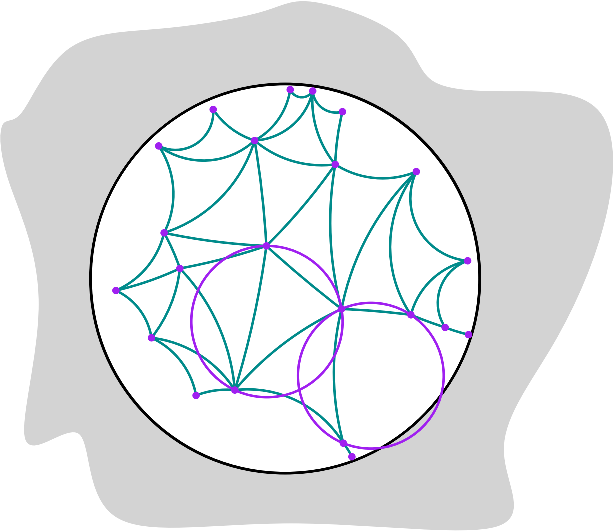
Delaunay triangulation



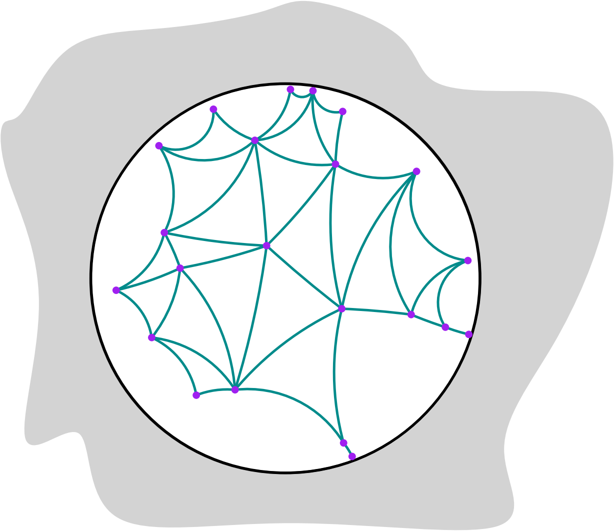
Delaunay triangulation



Delaunay triangulation

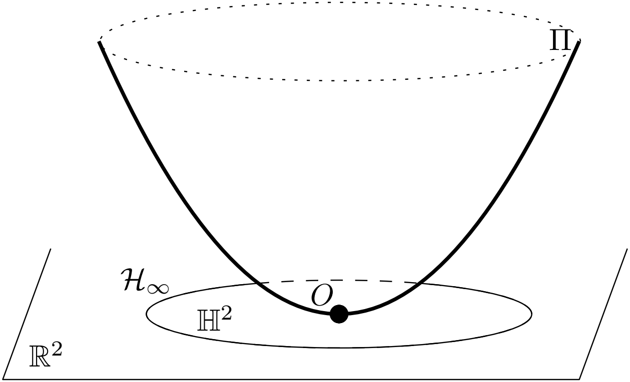


Delaunay triangulation

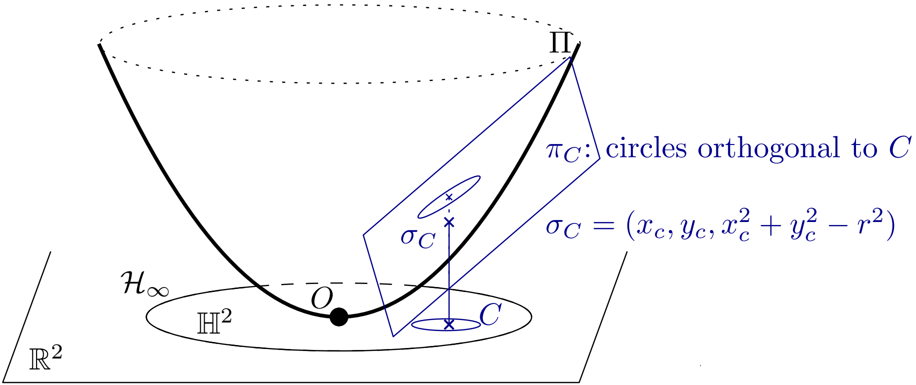


Hyperbolic space — The space of circles

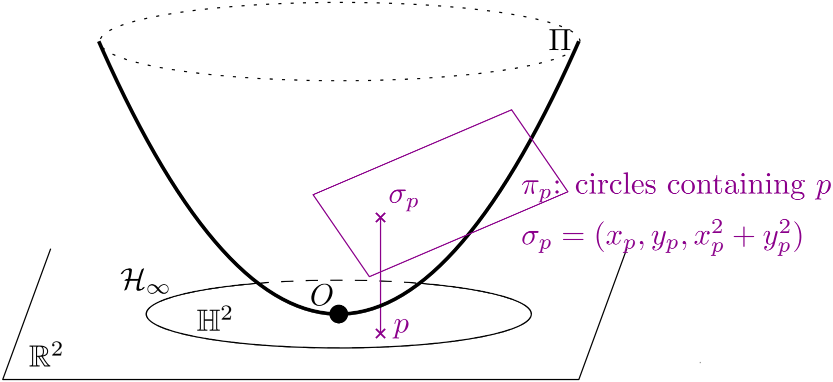
Definition



Definition

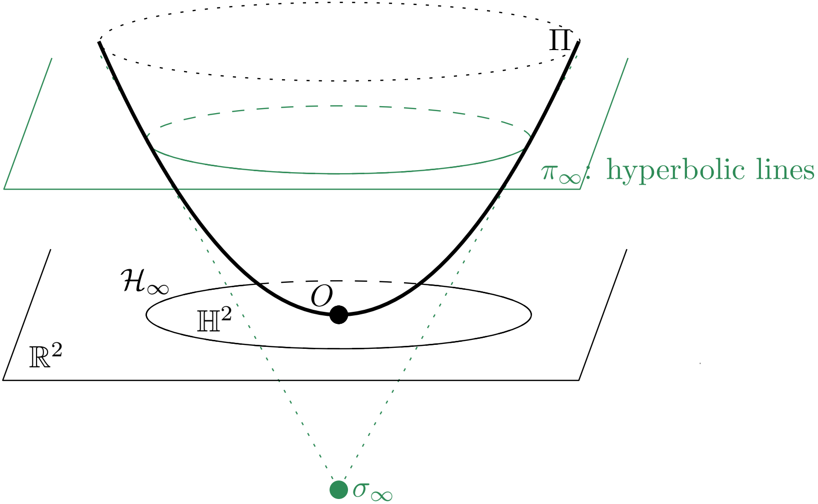

Π
π_C: circles orthogonal to C
$\sigma_C = (x_c, y_c, x_c^2 + y_c^2 - r^2)$
σ_C
$\mathcal{H}_\infty$
O
$\mathbb{H}^2$
C
$\mathbb{R}^2$

Definition



Π
σ_p
π_p: circles containing p
$\sigma_p = (x_p, y_p, x_p^2 + y_p^2)$
$\mathcal{H}_\infty$
O
$\mathbb{H}^2$
p
$\mathbb{R}^2$

Definition



Π
π_∞: hyperbolic lines
$\mathcal{H}_\infty$
O
$\mathbb{H}^2$
$\mathbb{R}^2$
σ_∞

Observation

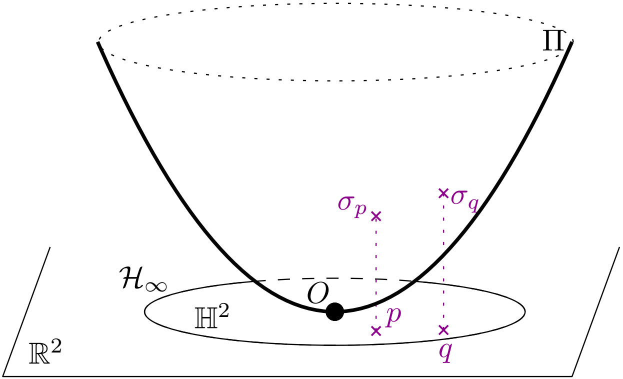
circle C has rational equation

iff

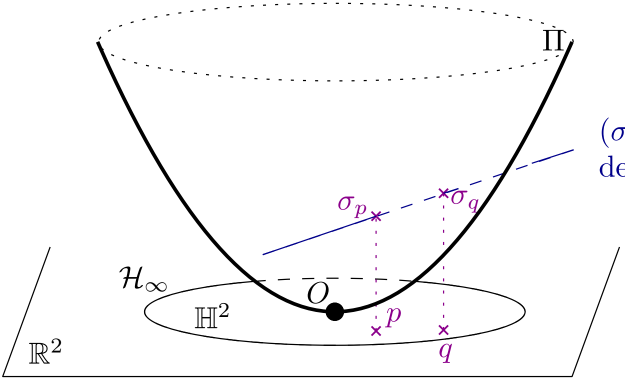
point $\sigma_C = (x_c, y_c, x_c^2 + y_c^2 - r^2)$ is rational

Hyperbolic space — Delaunay triangulation / Voronoi diagram

Bisector of two points

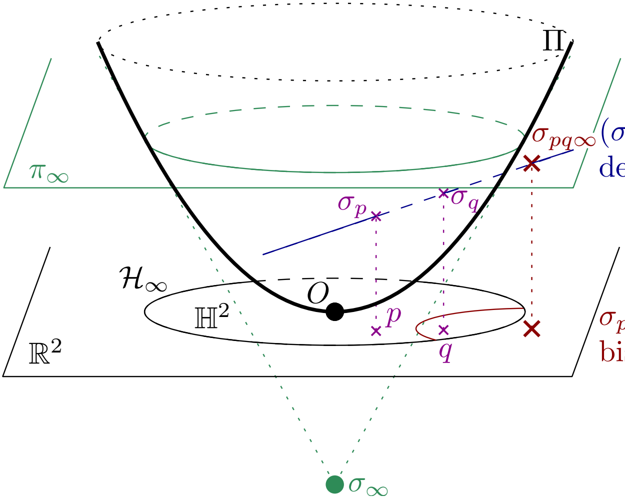


Bisector of two points



$(\sigma_p\sigma_q)$: pencil of circles defined by $\{p\}$ and $\{q\}$

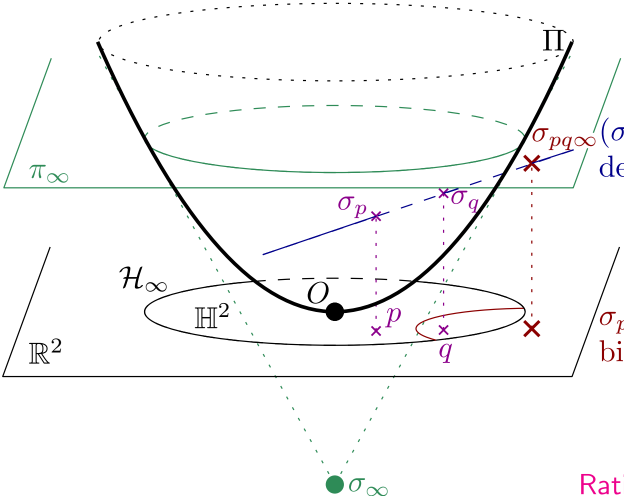
Bisector of two points



$(\sigma_p\sigma_q)$: pencil of circles defined by $\{p\}$ and $\{q\}$

$\sigma_{pq\infty} = (\sigma_p\sigma_q) \cap \pi_\infty$: bisector of p and q

Bisector of two points



$(\sigma_p\sigma_q)$: pencil of circles defined by $\{p\}$ and $\{q\}$

$\sigma_{pq\infty} = (\sigma_p\sigma_q) \cap \pi_\infty$: bisector of p and q

Rational computations only

More generally

- Delaunay triangulation
- Voronoi diagram

can be computed using only rational computations

(embedding of Voronoi vertices is algebraic)

More generally

- Delaunay triangulation
- Voronoi diagram

can be computed using only rational computations

(embedding of Voronoi vertices is algebraic)

in any dimension

Hyperbolic space — Algorithms

Extraction scheme

The graph of d-simplices (adjacency through facets) of

$$(DT_{\mathbb{R}}(\mathcal{P}) \setminus DT_{\mathbb{H}}(\mathcal{P})) \cup \{\text{infinite simplices of } DT_{\mathbb{R}}(\mathcal{P})\}$$

is connected.

algorithm digs into the triangulation from the outside

dangling k-faces detected (easy way in 2d)

Dynamic variant

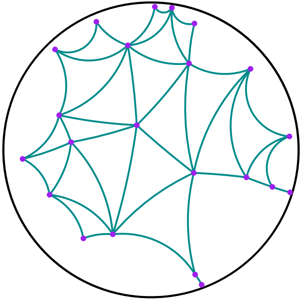
Insertion of p

simplices outside the link of p in $DT_{\mathbb{R}}(\mathcal{P}_{i-1} \cup \{p\})$
don't need to be tested.

$\longrightarrow$ optimal randomized worst-case
time $O\left(n^{\lceil \frac{d}{2} \rceil} + n \log n\right)$ and space $O\left(n^{\lceil \frac{d}{2} \rceil}\right)$

Implementation in 2D

Using CGAL 2D Delaunay triangulations



Exact

Degenerate cases handled

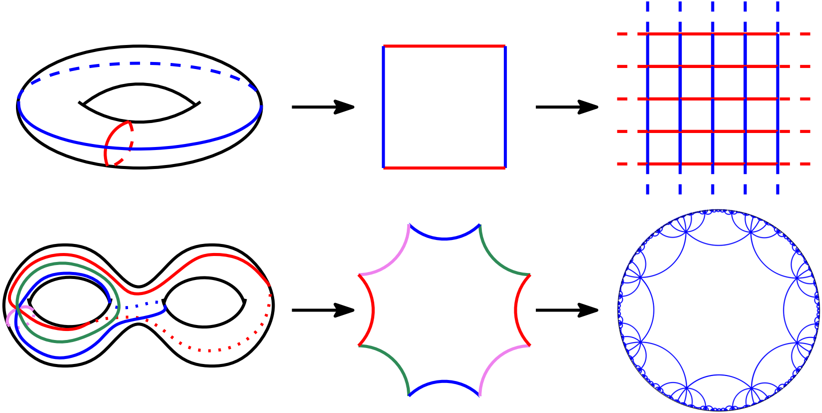
Extra cost $\simeq$ 2 to 20% to filter out non hyperbolic simplices

$\simeq$ 10 seconds for 10 million points (MacBookPro 2.6GHz)

Hyperbolic manifolds

Hyperbolic manifolds — Introduction

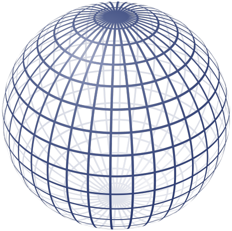
Flat vs. hyperbolic

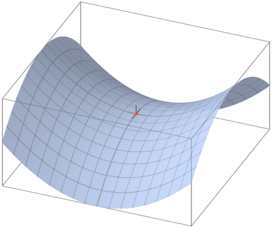


Flat vs. hyperbolic

Gaussian curvature

$$\kappa(x_0) = lim_{r \to 0} \; 12 \frac{\pi r^2 - A(r)}{\pi r^4}$$





Flat vs. hyperbolic

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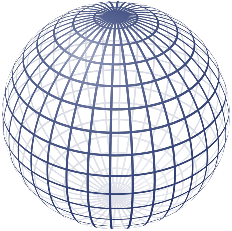
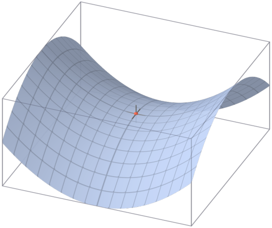




Gauss-Bonnet

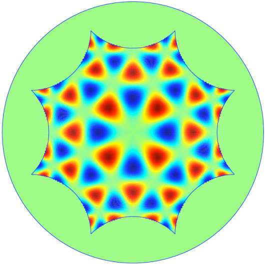
$$\int_S \kappa dA = 2\pi \cdot \chi(S)$$

Euler characteristics $\chi(S) = 2 - 2g$, where g = genus

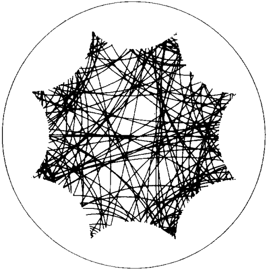
torus $g \geq 2 \longrightarrow$ locally hyperbolic

Motivation

[Sausset, Tarjus, Viot]

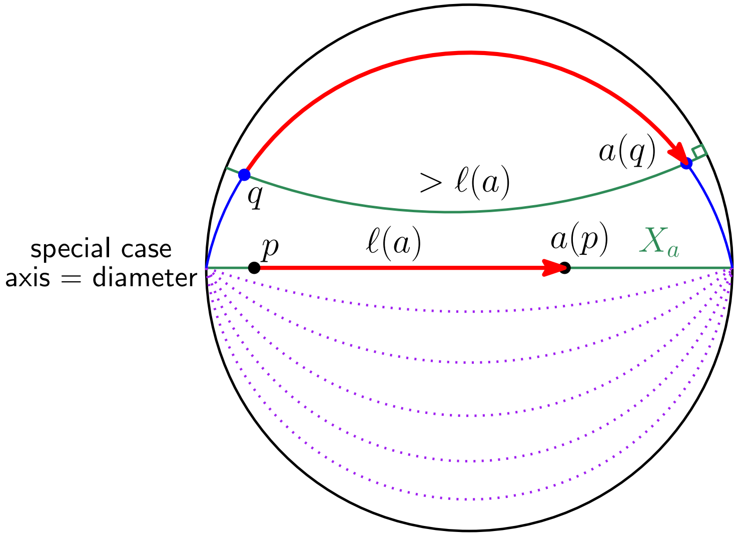


[Chossat, Faye, Faugeras]



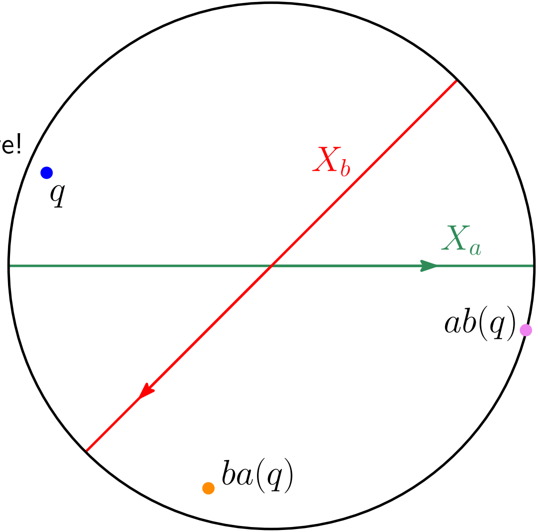
[Balazs, Voros]

Hyperbolic translations



Hyperbolic translations

non-commutative!

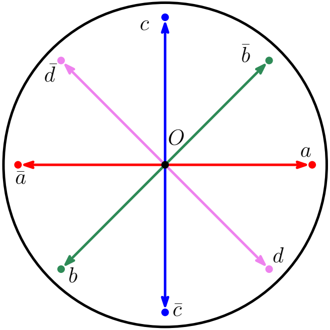


Hyperbolic manifolds — The Bolza surface

Properties

- Closed, compact, orientable surface of genus 2.
- Constant negative curvature $\longrightarrow$ locally hyperbolic metric.
- The most symmetric of all genus-2 surfaces.

Formal definition

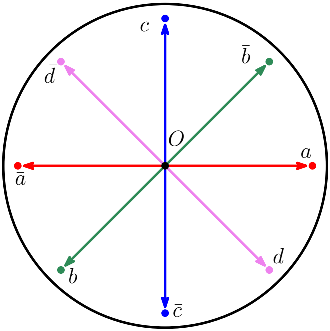


Fuchsian group $\mathcal{G}$ with finite presentation

$$\mathcal{G} = \langle a, b, c, d \mid abcd\overline{a}\overline{b}\overline{c}\overline{d} \rangle$$

$\mathcal{G}$ contains only translations (and $\mathbb{1}$)

Formal definition



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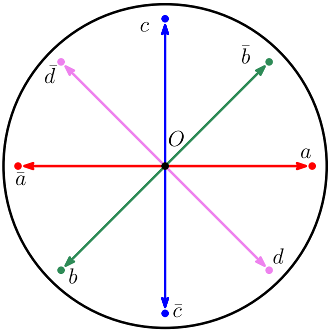
Bolza surface

$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \rightarrow \mathcal{M}$

Formal definition



Fuchsian group $\mathcal{G}$ with finite presentation

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Bolza surface

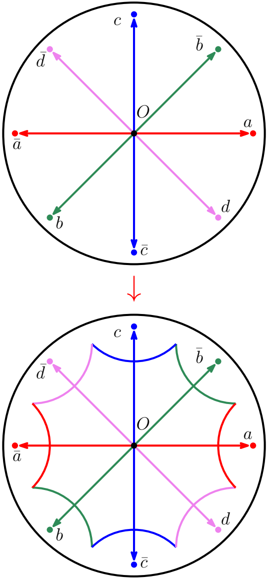
$$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$$

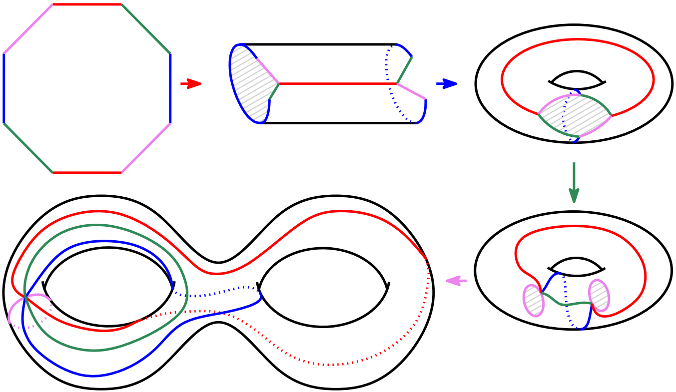
with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \rightarrow \mathcal{M}$

$$\mathcal{A} = [a, \overline{b}, c, \overline{d}, \overline{a}, b, \overline{c}, d] = [g_0, g_1, ..., g_7]$$

$$g_k = \begin{bmatrix} \alpha & \beta_k \\ \overline{\beta}_k & \overline{\alpha} \end{bmatrix}, \quad g_k(z) = \frac{\alpha z + \beta_k}{\overline{\beta}_k z + \overline{\alpha}}, \quad \alpha = 1 + \sqrt{2}, \quad \beta_k = e^{ik\pi/4}\sqrt{2\alpha}$$

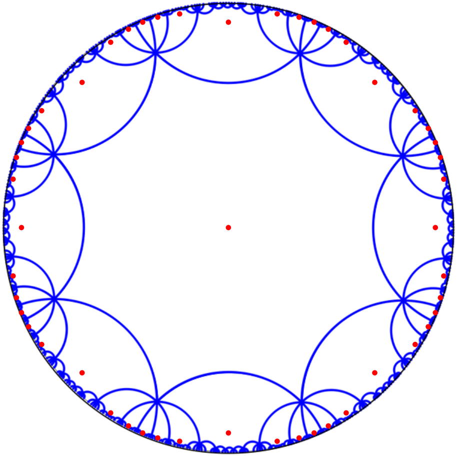
2-torus and octagon





(not embedded in $\mathbb{R}^3$)

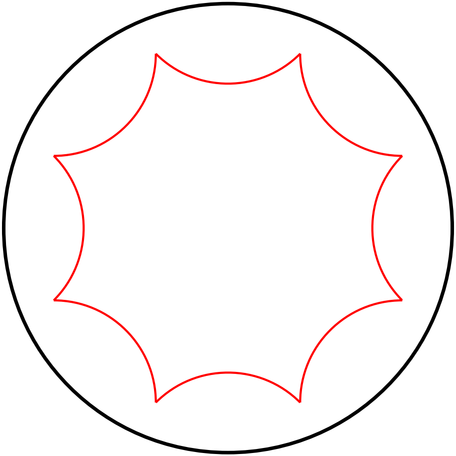
Hyperbolic octagon



Voronoi diagram of $\mathcal{G}O$

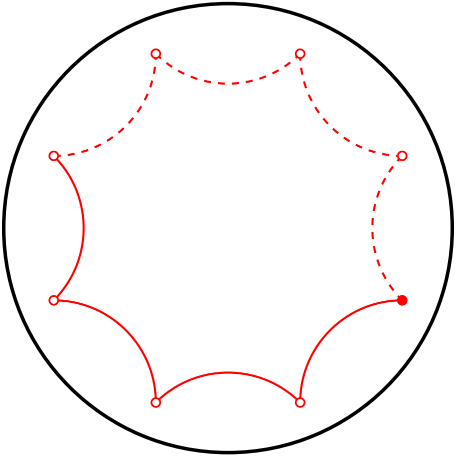
$\mathbb{H}^2$ = universal covering of the Bolza surface

Hyperbolic octagon



Fundamental domain $\mathcal{D}_O$ = Dirichlet region of O

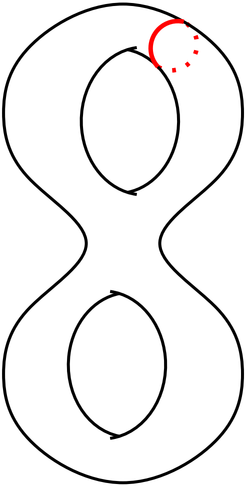
Hyperbolic octagon



“Original” domain $\mathcal{D}$: contains exactly one point of each orbit

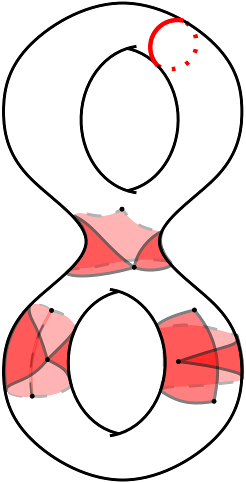
Hyperbolic manifolds — Delaunay triangulation

Sufficient condition



Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

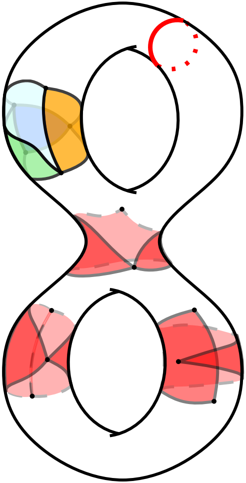
Sufficient condition



Systole $\mathrm{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

$$\pi_{\mathcal{M}}(\,\underline{\mathrm{DT}}_{\mathbb{H}}(\mathcal{G}S)\,)$$

Sufficient condition



Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

S set of points in $\mathbb{H}^2$

$\delta_S =$ diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\mathcal{G}S$

$\delta_S < \frac{1}{2}\,\text{sys}(\mathcal{M})$

$\Longrightarrow$ $\pi_{\mathcal{M}}(\,\underline{\text{DT}}_{\mathbb{H}}(\mathcal{G}S)\,) = \underline{\text{DT}}_{\mathcal{M}}(S)$
is a simplicial complex

$\Longrightarrow$ The usual incremental algorithm can be used

Sufficient condition

Two ways to satisfy $\delta_S < \frac{1}{2}\,\text{sys}\,(\mathcal{M})$

(similar to flat case)

1. Increase the systole
   - work in a covering space
   - $32 <$ number of sheets ≤ 128
   - genus increased!...

non-practical

Sufficient condition

Two ways to satisfy $\delta_S < \frac{1}{2}\,\mathsf{sys}\,(\mathcal{M})$

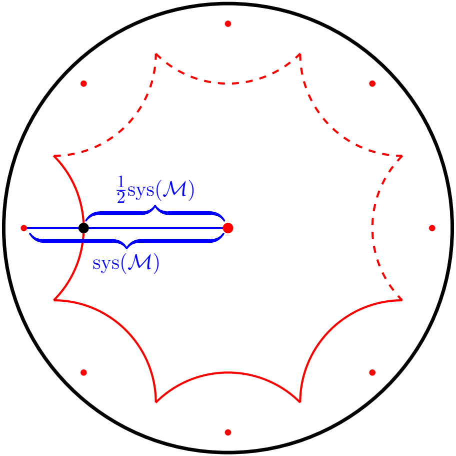
(similar to flat case)

1. Increase the systole
   - work in a covering space
   - $32 <$ number of sheets ≤ 128
   - genus increased!...

   non-practical

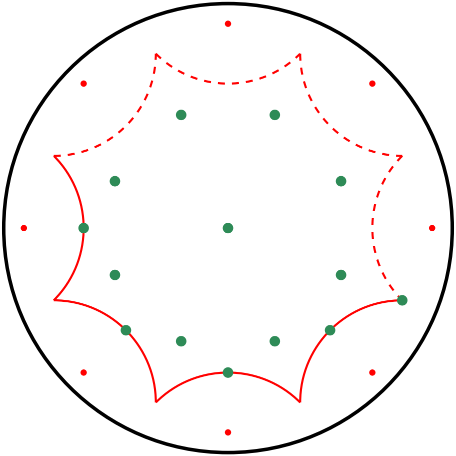
2. Reduce the size of the largest empty disk
   - use dummy points

Dummy points



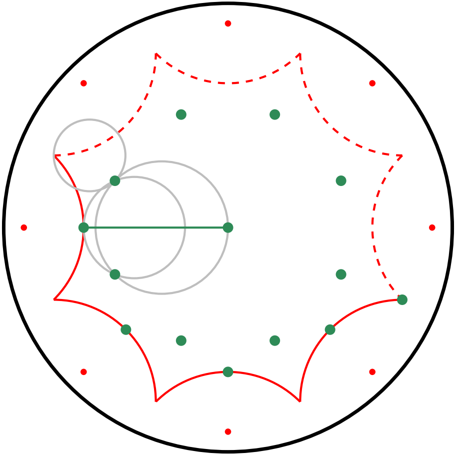
Systole on the octagon

Dummy points



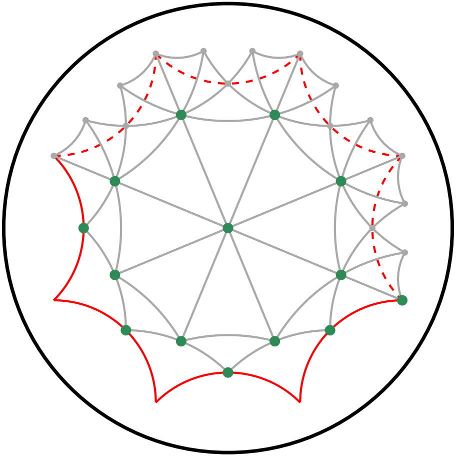
Set of dummy points

Dummy points



Sufficient condition

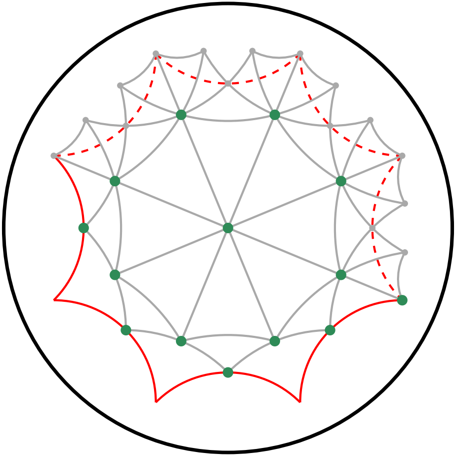
Dummy points



Delaunay triangulation of the dummy points

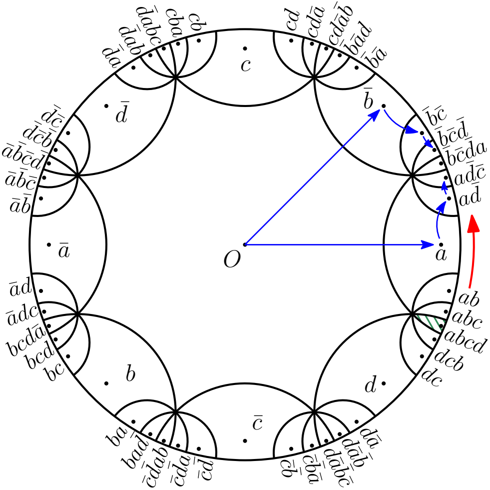
Algorithm

1. initialize with dummy points
2. insert points
3. remove dummy points



Hyperbolic manifolds — Implementation

Notation



$g(O),\ g \in \mathcal{G}$, denoted as g

$\mathcal{D}_g = g(\mathcal{D}_O),\ g \in \mathcal{G}$

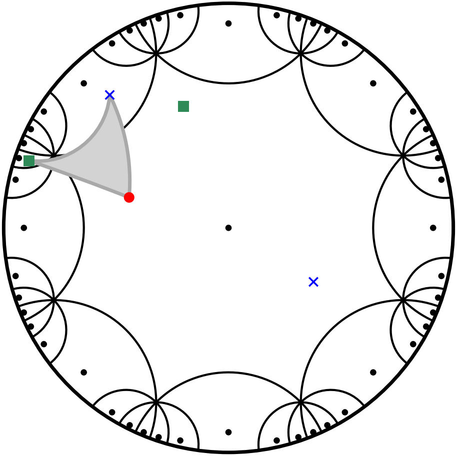
$\mathcal{N} = \{g \in \mathcal{G} \mid \mathcal{D}_g \cap \mathcal{D}_O \neq \emptyset\}$

$$\mathcal{D}_{\mathcal{N}} = \bigcup_{g \in \mathcal{N}} \mathcal{D}_g$$

Property of $\underline{\mathrm{DT}}_{\mathbb{H}}(\mathcal{G}S)$

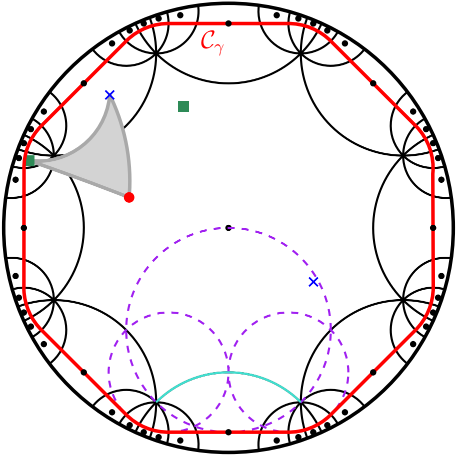
$S \subset \mathcal{D}$ input point set
s.t. criterion $\delta_S < \frac{1}{2}\,\mathrm{sys}(\mathcal{M})$ holds

σ face of $\underline{\mathrm{DT}}_{\mathbb{H}}(\mathcal{G}S)$ with at least one vertex in $\mathcal{D}$
$\longrightarrow$ σ is contained in $\mathcal{D}_{\mathcal{N}}$



Proof of property

Fact:
$\text{sys}\,(\mathcal{M}) \leq \ell(g_k),\ k = 0, \ldots, 8$



Proof of property

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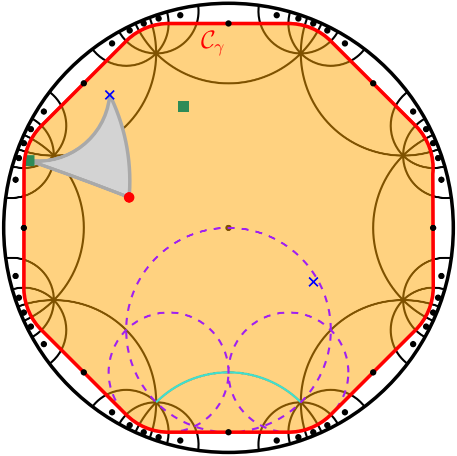
Get disks with radius $\frac{1}{2}\ell(g_k)$.

$$\delta_S < \frac{1}{2}\,\text{sys}\,(\mathcal{M}) \leq \frac{1}{2}\ell(g_k)$$



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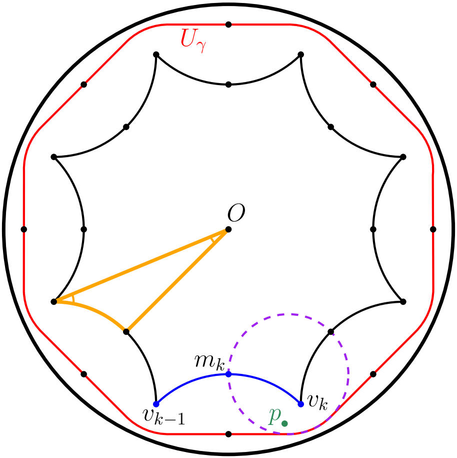
- case 1: center on vertex



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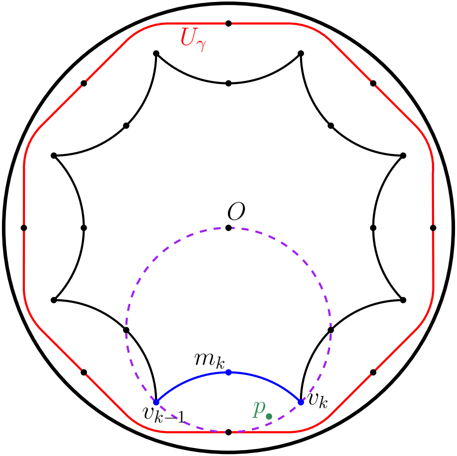
- case 1: center on vertex
- case 2: center on side midpoint



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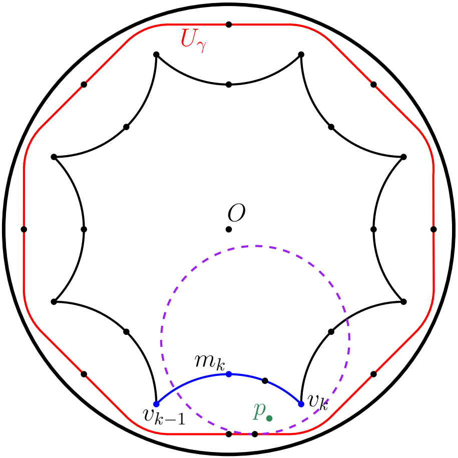
- case 1: center on vertex
- case 2: center on side midpoint
- case 3: center elsewhere on side



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Get disks with radius $\frac{1}{2}\ell(g_k)$.

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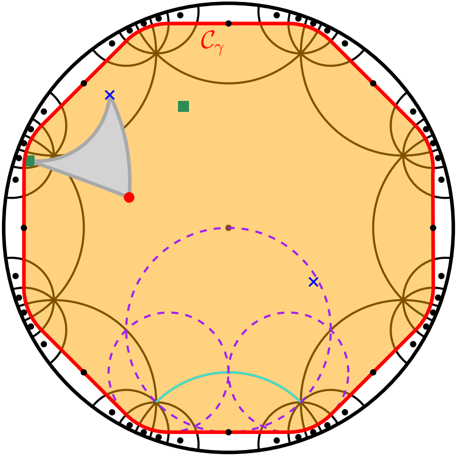
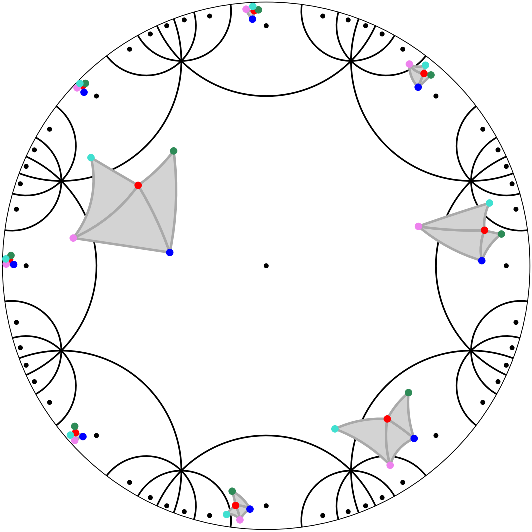
- case 1: center on vertex
- case 2: center on side midpoint
- case 3: center elsewhere on side

face $\subset$ disk $\subset U_\gamma \subset \mathcal{D}_\mathcal{N}$



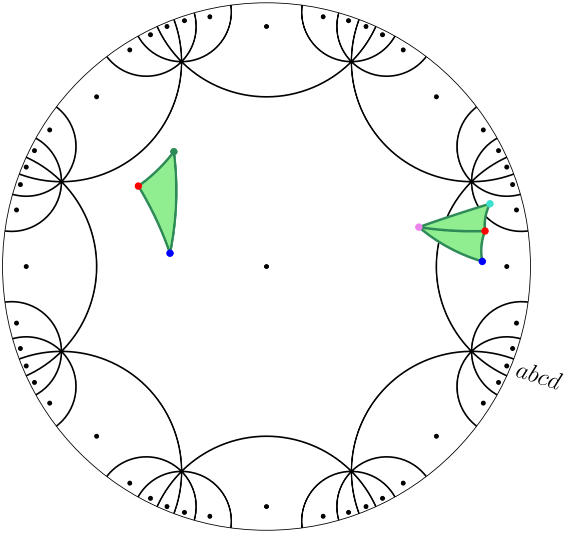
Canonical representative of a face

Each face of $\underline{\mathrm{DT}}_{\mathcal{M}}(S)$ has infinitely many pre-images in $\underline{\mathrm{DT}}_{\mathbb{H}}(\mathcal{G}S)$



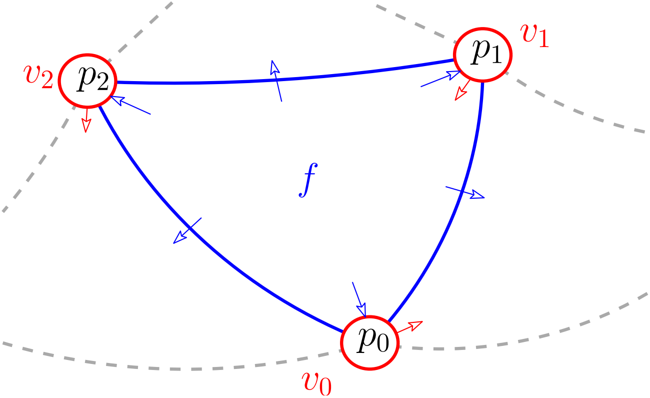
Canonical representative of a face

Triangle closest to *abcd* (in ccw order on $\mathcal{D}_\mathcal{N}$



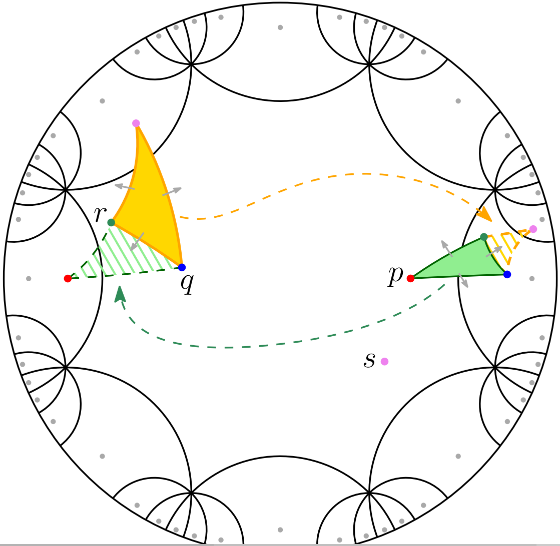
Representation

CGAL Triangulations



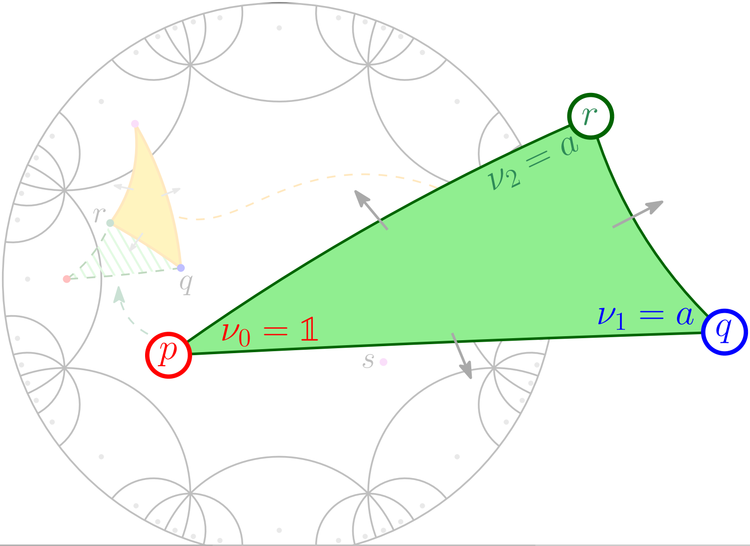
Representation

Face of $\underline{\mathrm{DT}}_{\mathcal{M}}(S)$

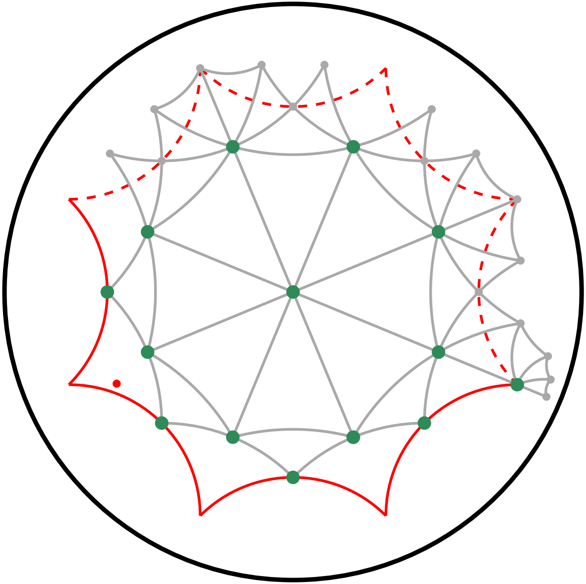


Representation

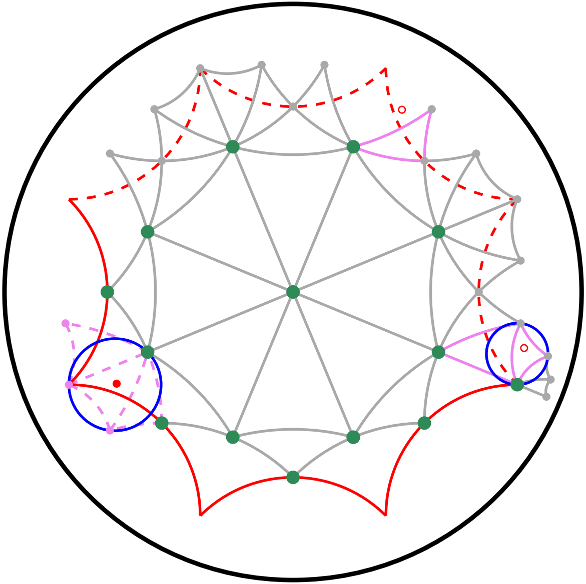
Face of $\underline{\mathrm{DT}}_{\mathcal{M}}(S)$



Point Location



Point Location



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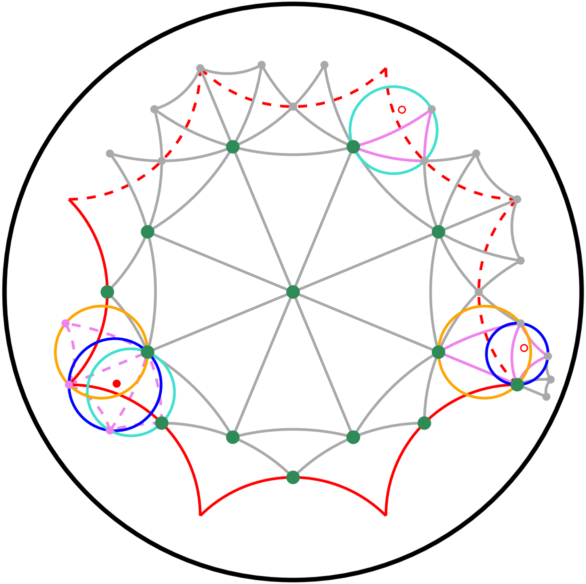

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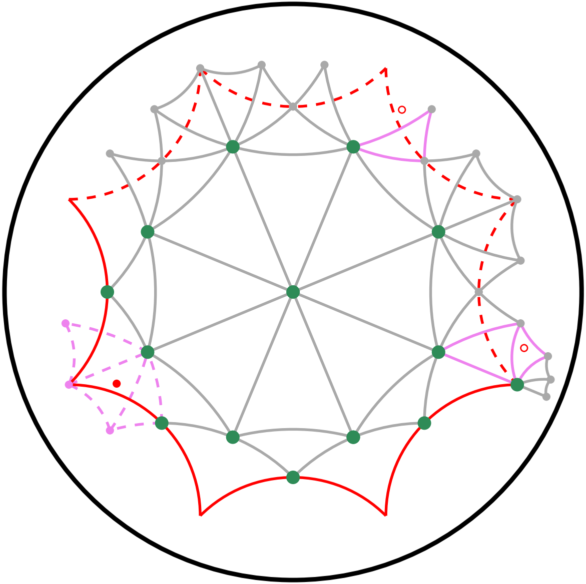
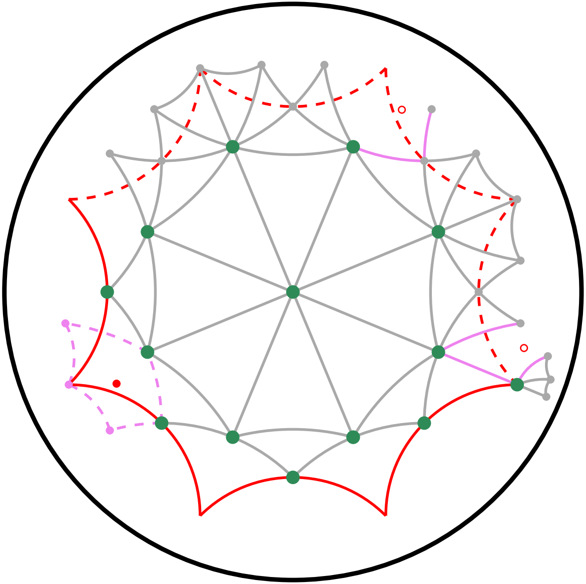


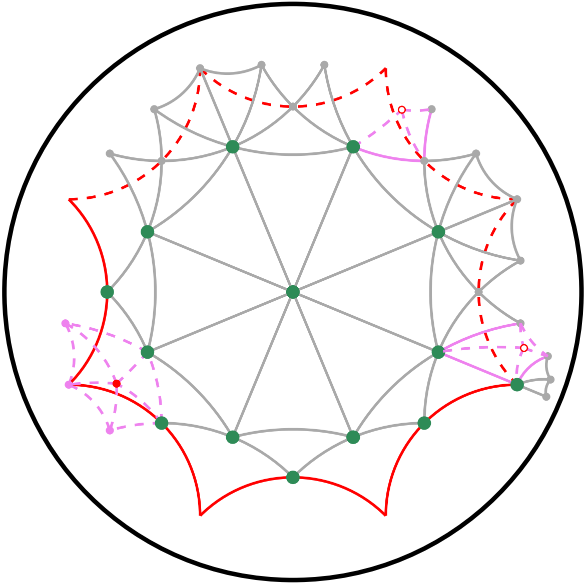
Point Insertion

“hole” $=$ topological disk



Point Insertion

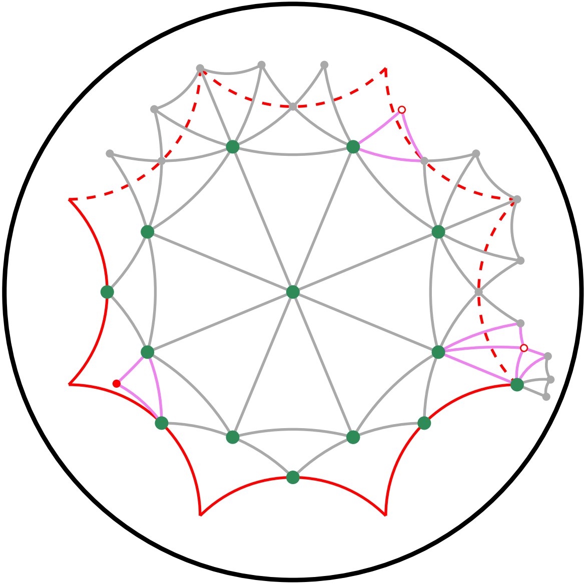
"hole" = topological disk



Point Insertion

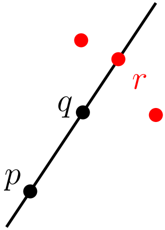
Computations
on translations

Dehn's algorithm
(slightly modified)

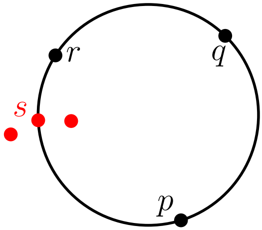


Predicates

$$\underline{\text{Orientation}}(p, q, r) = \text{sign}\begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$



$$\underline{\text{InCircle}}(p, q, r, s) = \text{sign}\begin{vmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{vmatrix}$$



Predicates

Suppose that the points in S are rational.

Input of the predicates can be images of these points under $\nu \in \mathcal{N}$.

$$g_k(z) = \frac{\alpha z + e^{ik\pi/4}\sqrt{2\alpha}}{e^{-ik\pi/4}\sqrt{2\alpha}z + \alpha}, \quad \alpha = 1 + \sqrt{2}, \quad k = 0, 1, ..., 7$$

- the Orientation predicate has algebraic degree at most 20
- the InCircle predicate has algebraic degree at most 72

Point coordinates represented with `CORE::Expr`
$\longrightarrow$ (filtered) exact evaluation of predicates

Hyperbolic manifolds — Open questions

Open questions

- Higher genus
- Other metrics
- ...

Non-trivial mathematics