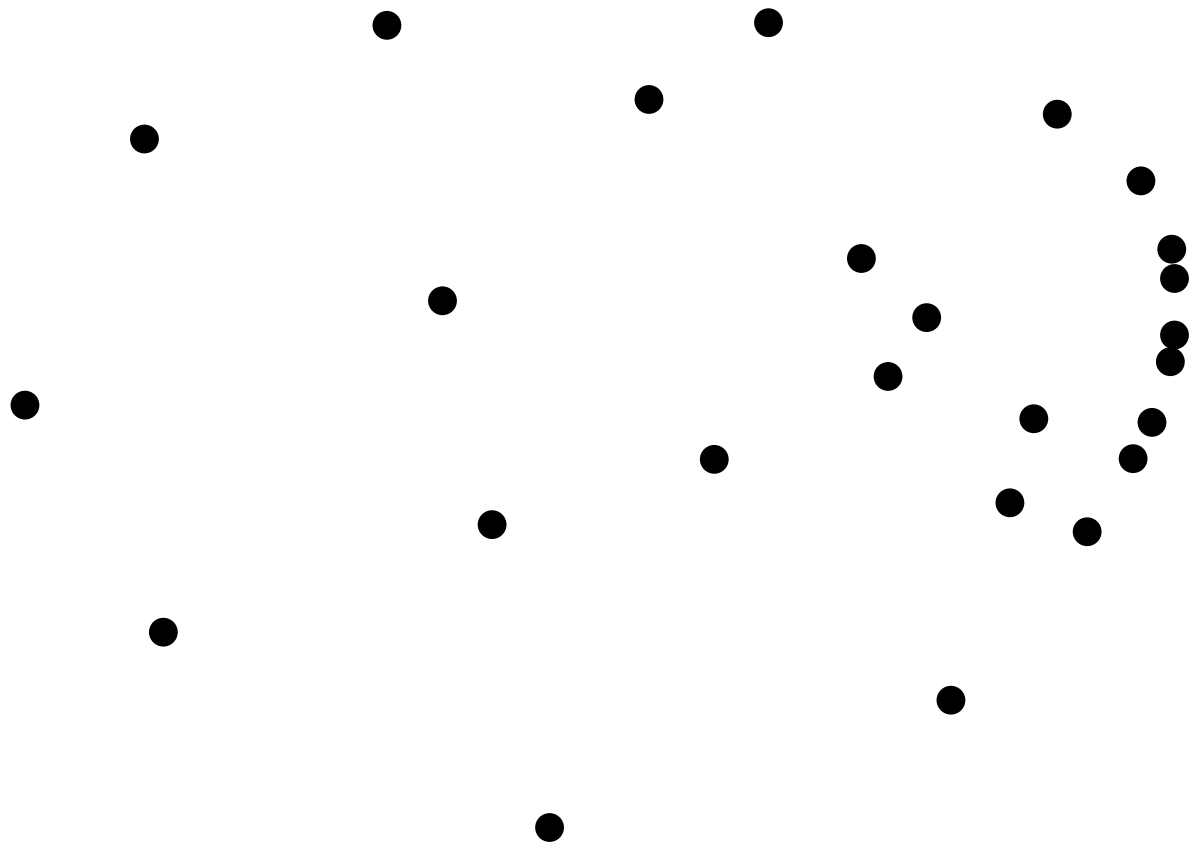
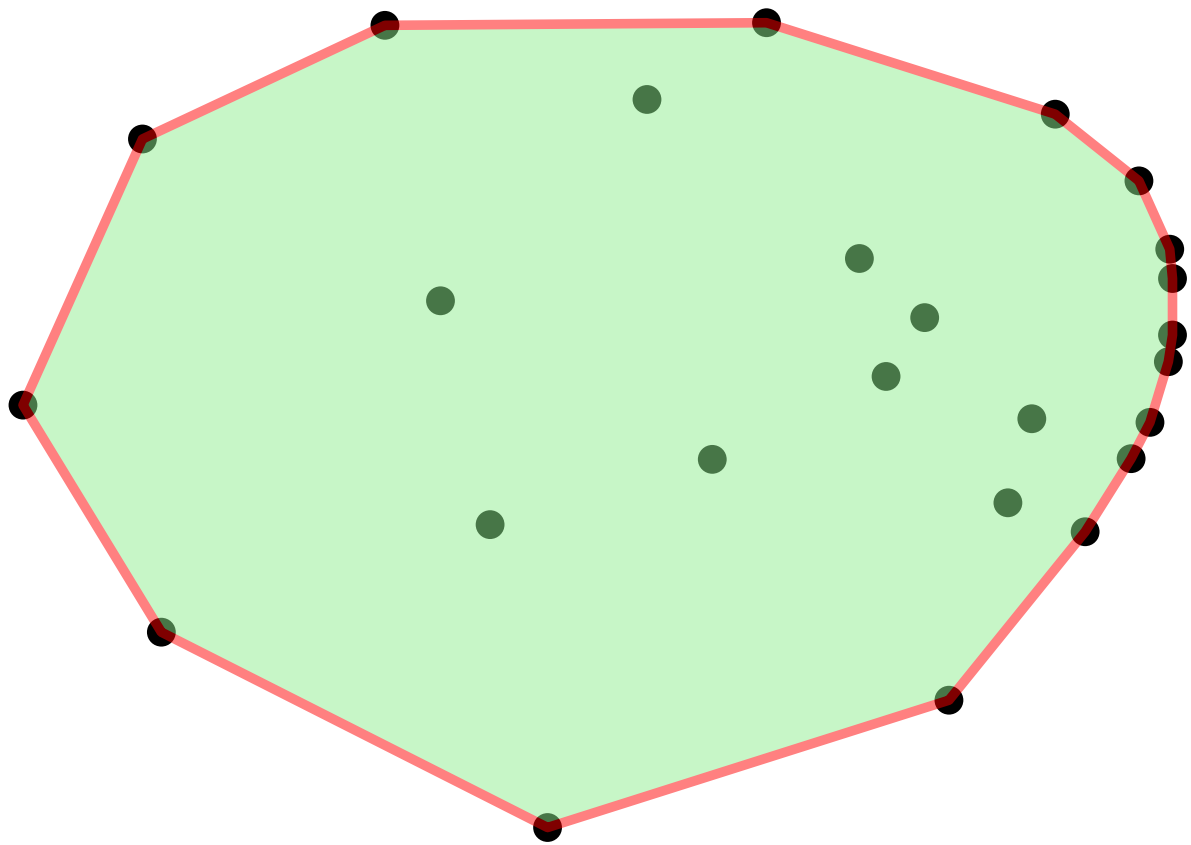
Convex hull

Convex hull



Convex hull



Convex hull

- Definition, extremal point
- Jarvis algorithm
- Orientation predicate
- Buggy degenerate example
- Real RAM model and general position hypothesis
- Graham algorithm
- Lower bound
- Other results
- Higher dimensions

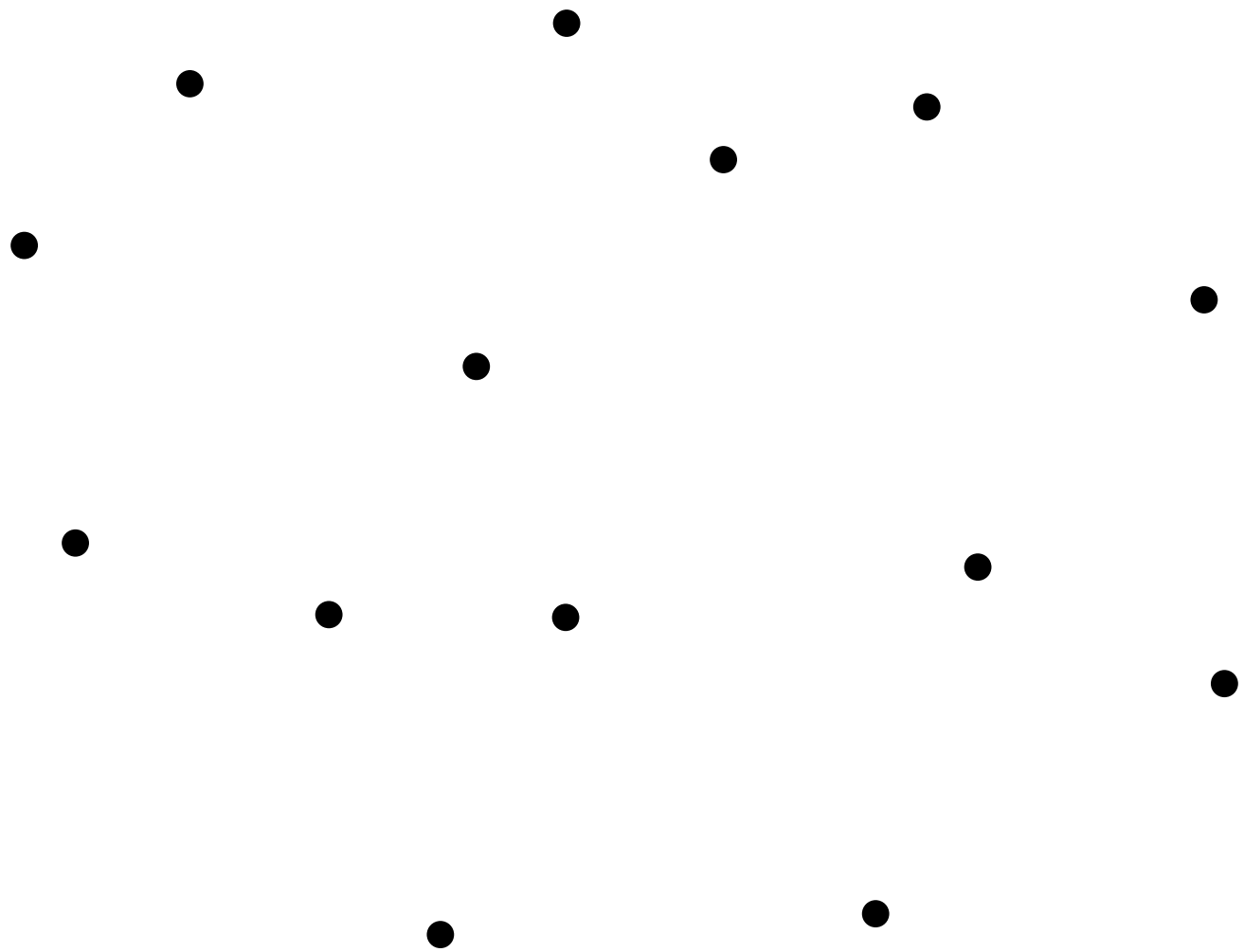
Convex hull

Definition, extremal point

Convex hull

Definition, extremal point

Set of points

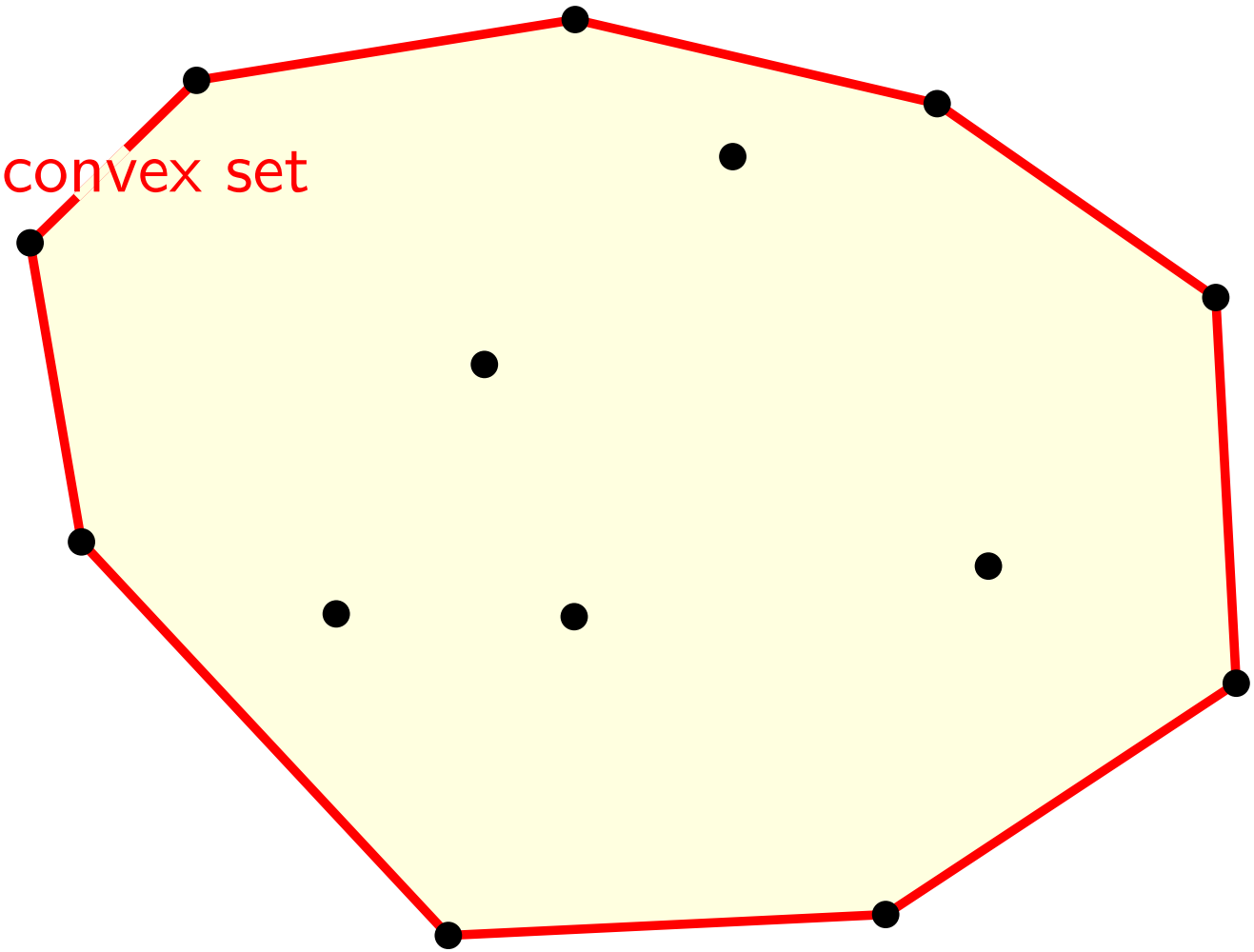


Convex hull

Definition, extremal point

Set of points

Smallest enclosing convex set

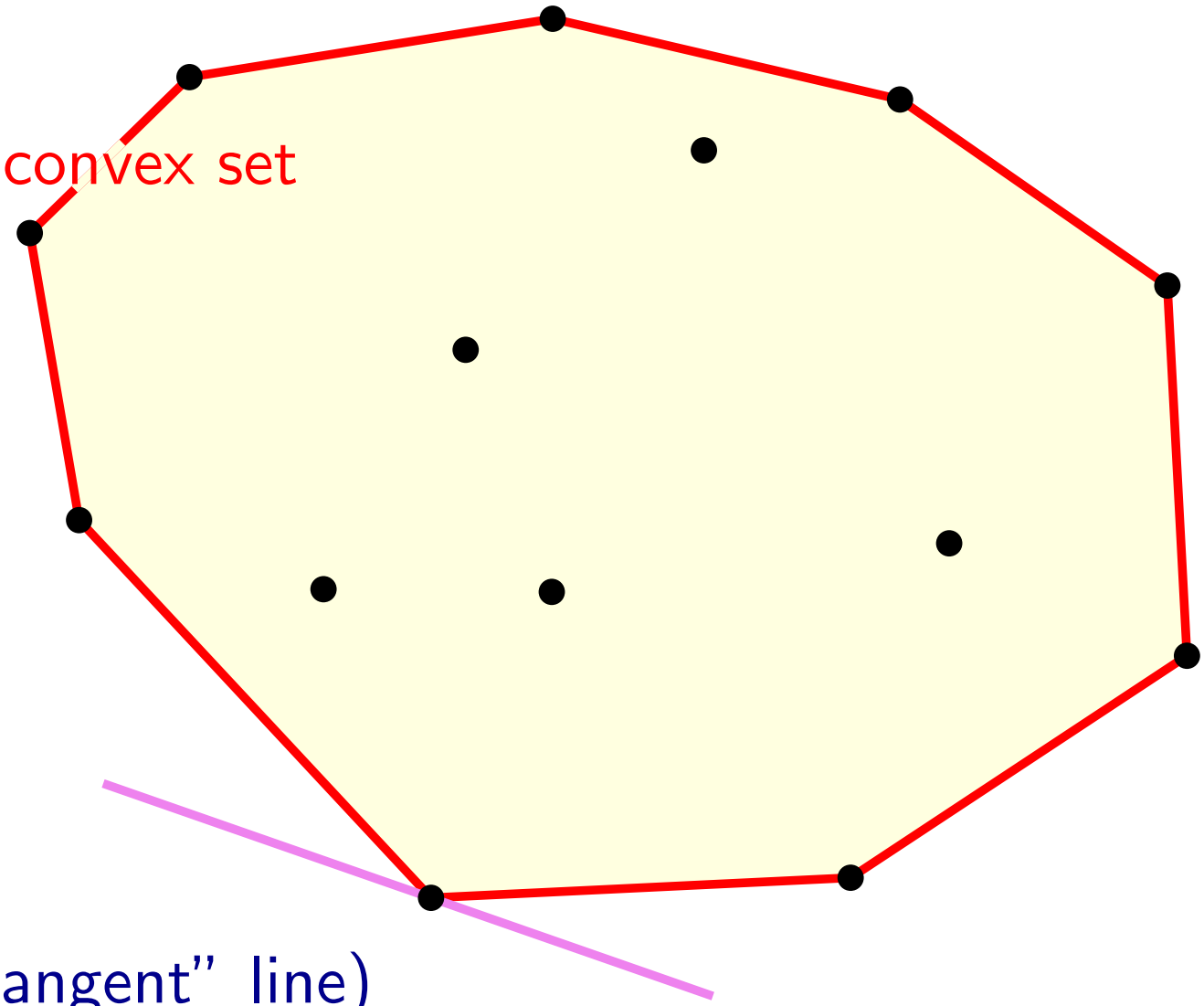


Convex hull

Definition, extremal point

Set of points

Smallest enclosing convex set

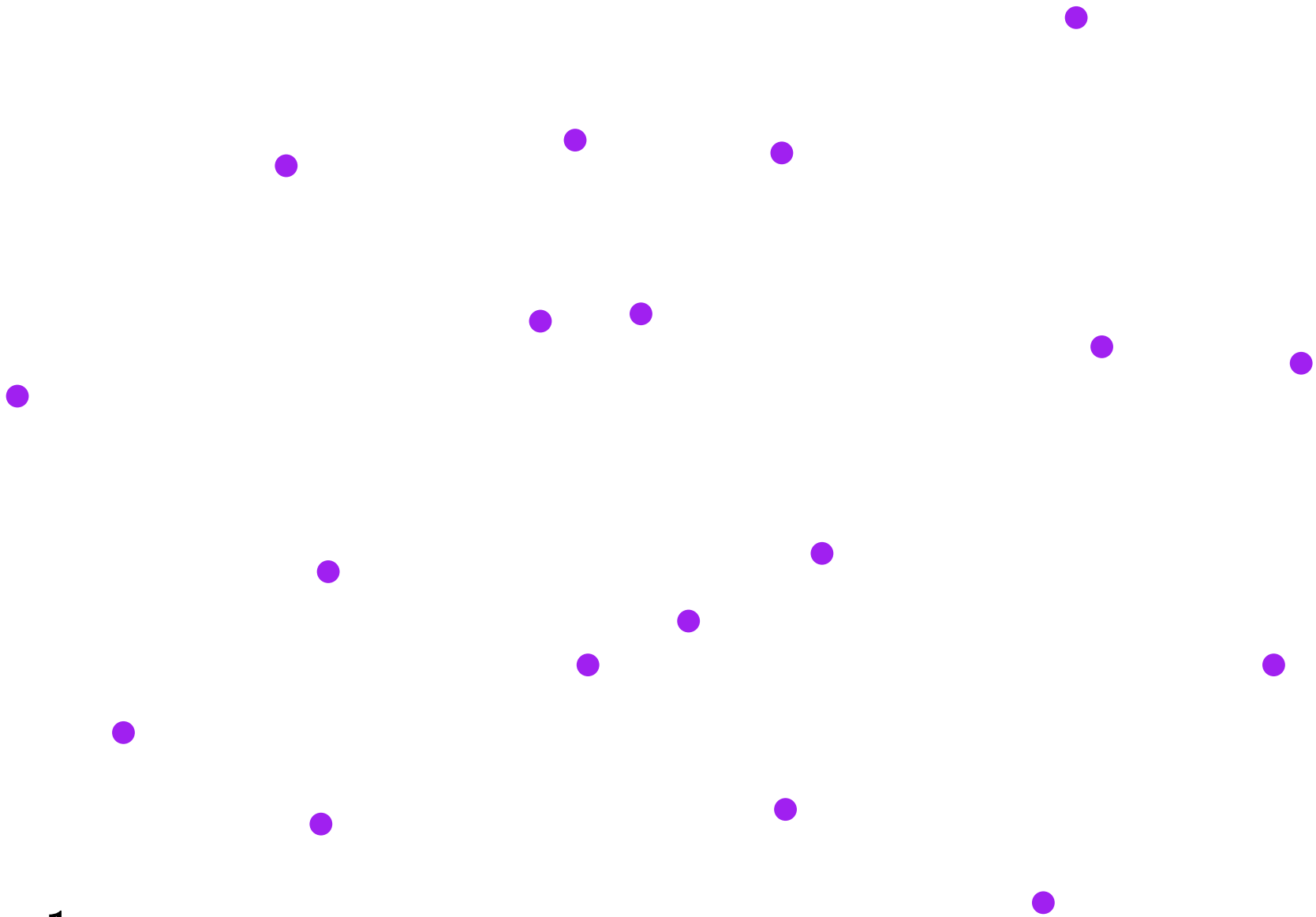


Extremal point

Supporting line ("tangent" line)

Convex hull

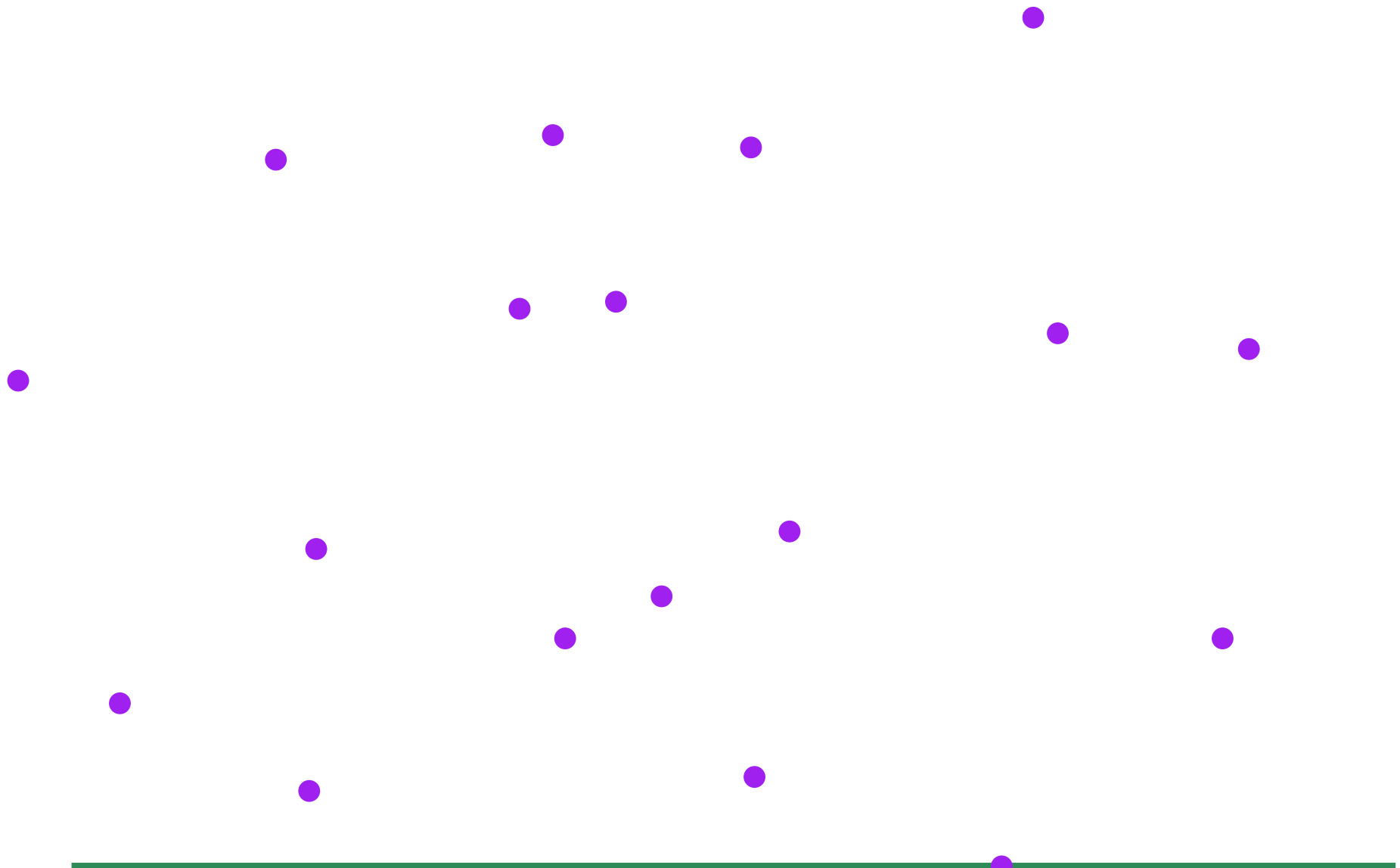
Jarvis algorithm



Convex hull

Jarvis algorithm

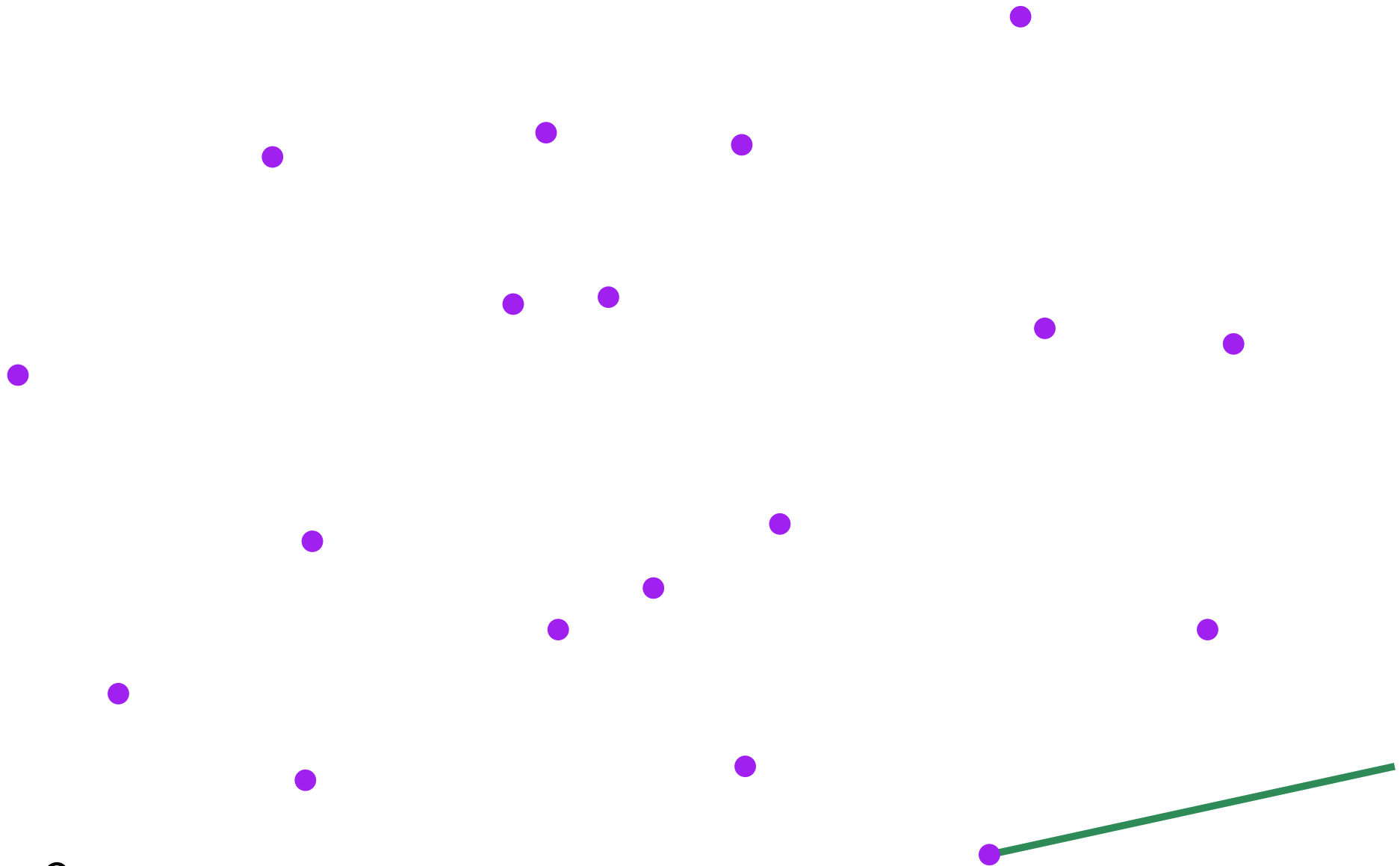
lowest point is extremal



Convex hull

rotate

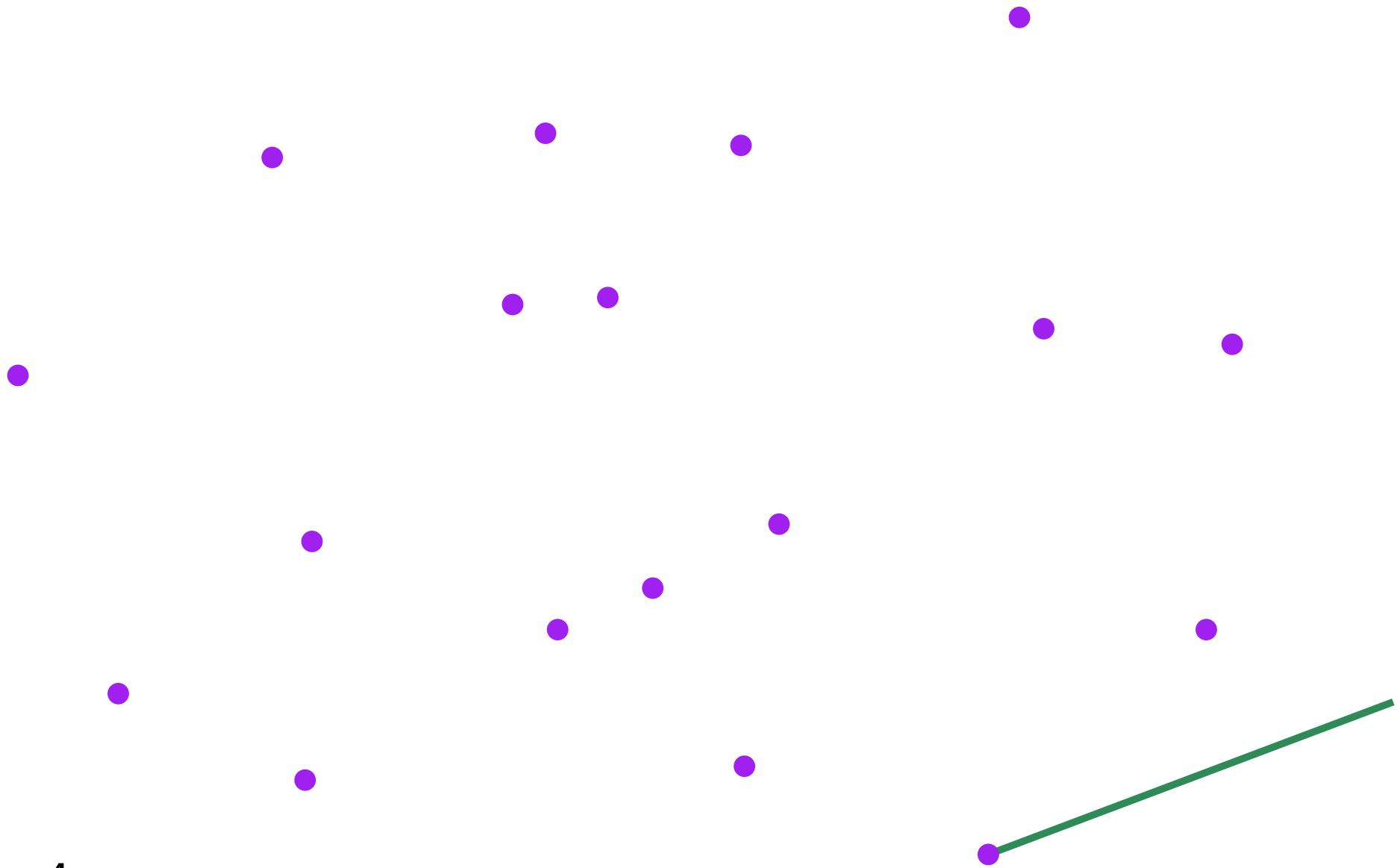
Jarvis algorithm



Convex hull

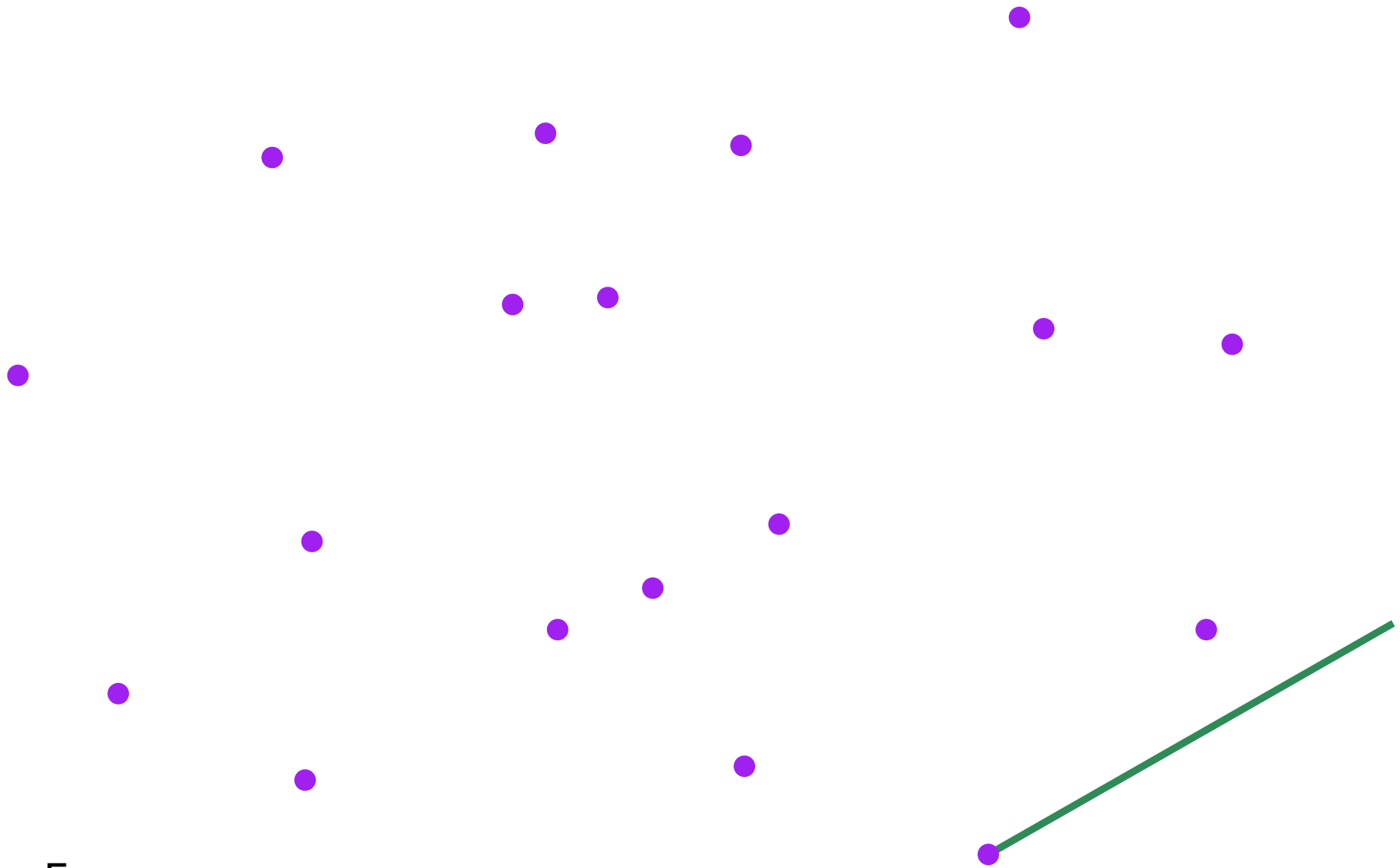
Jarvis algorithm

rotate



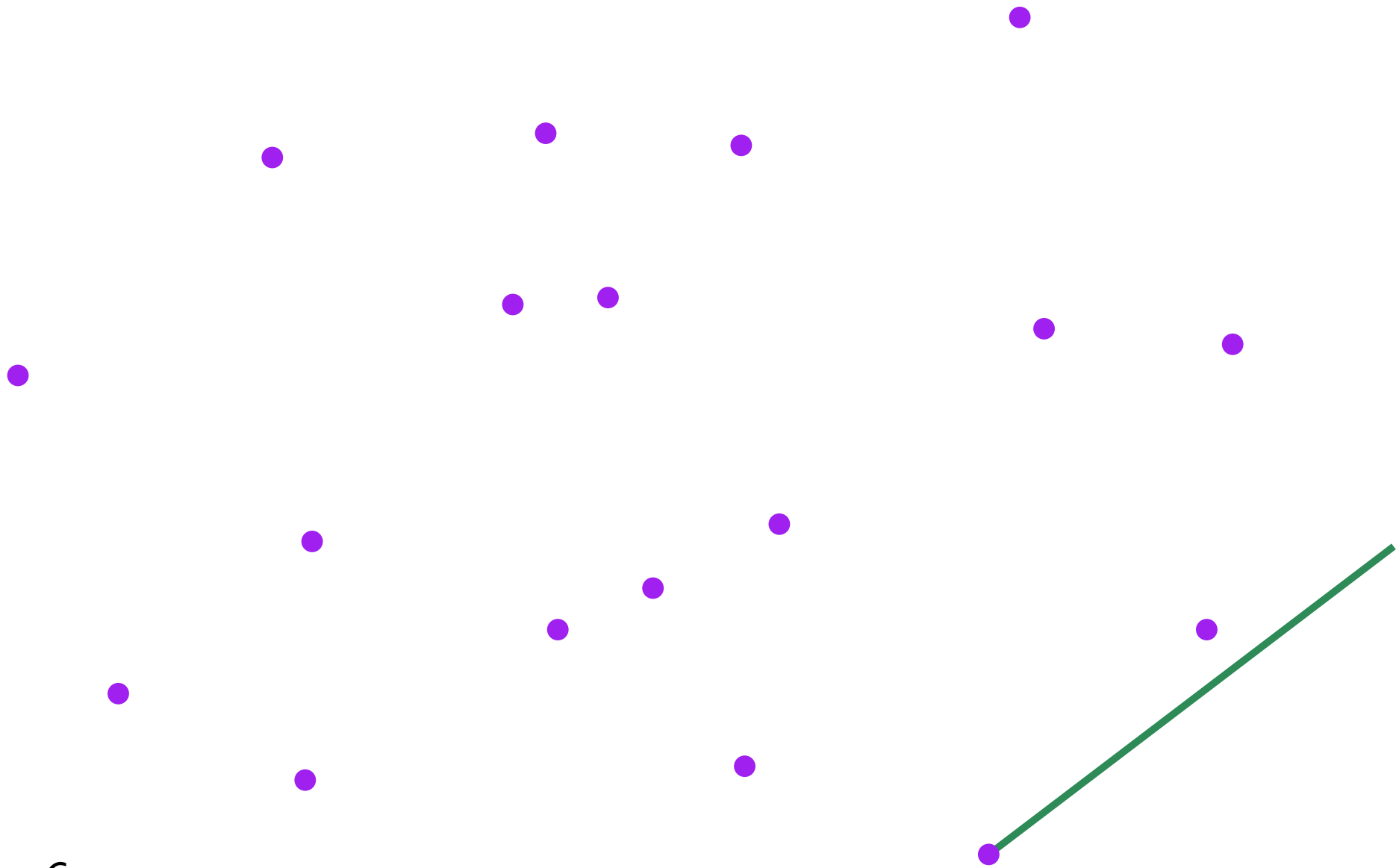
Convex hull

rotate



Convex hull

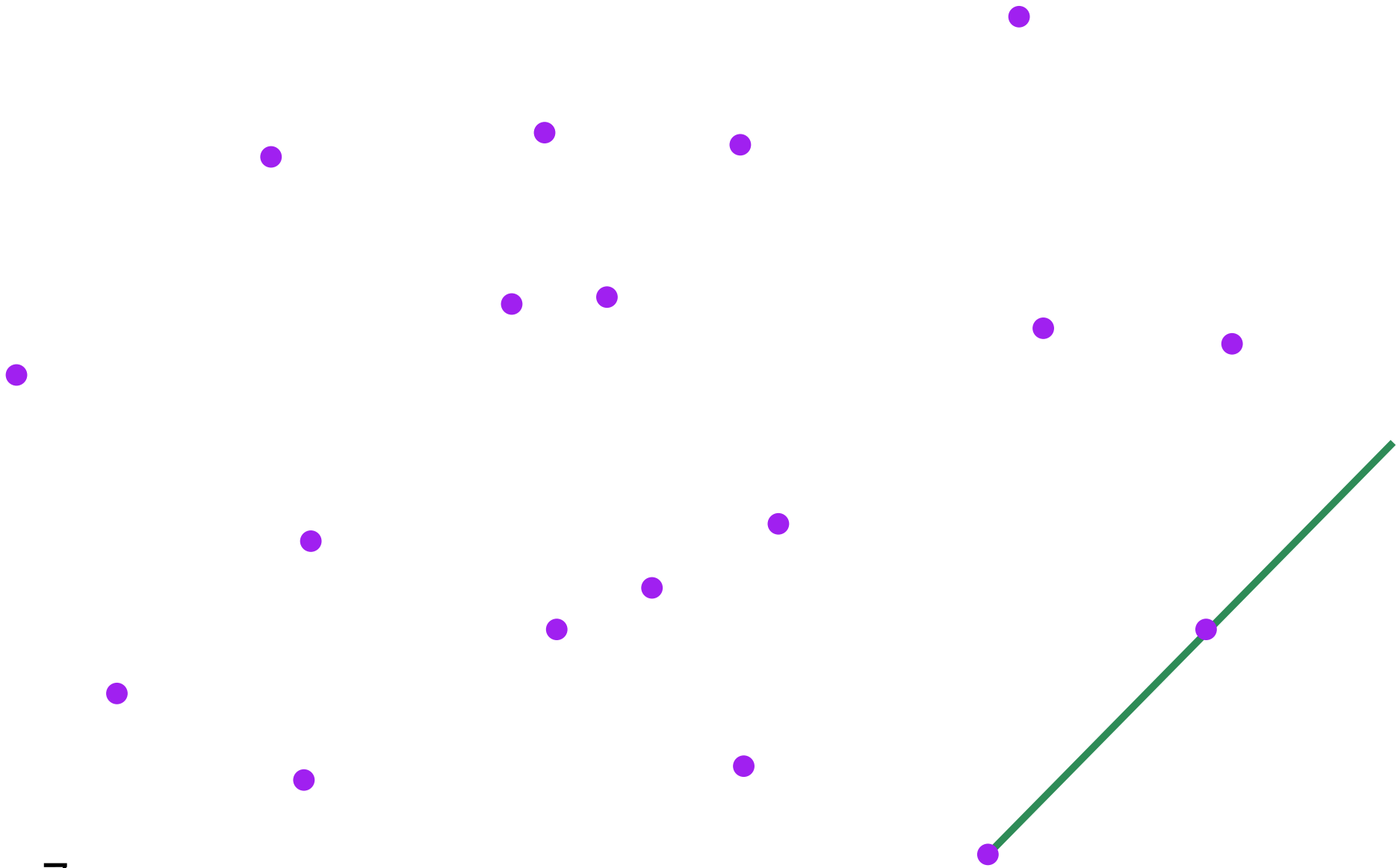
rotate



Convex hull

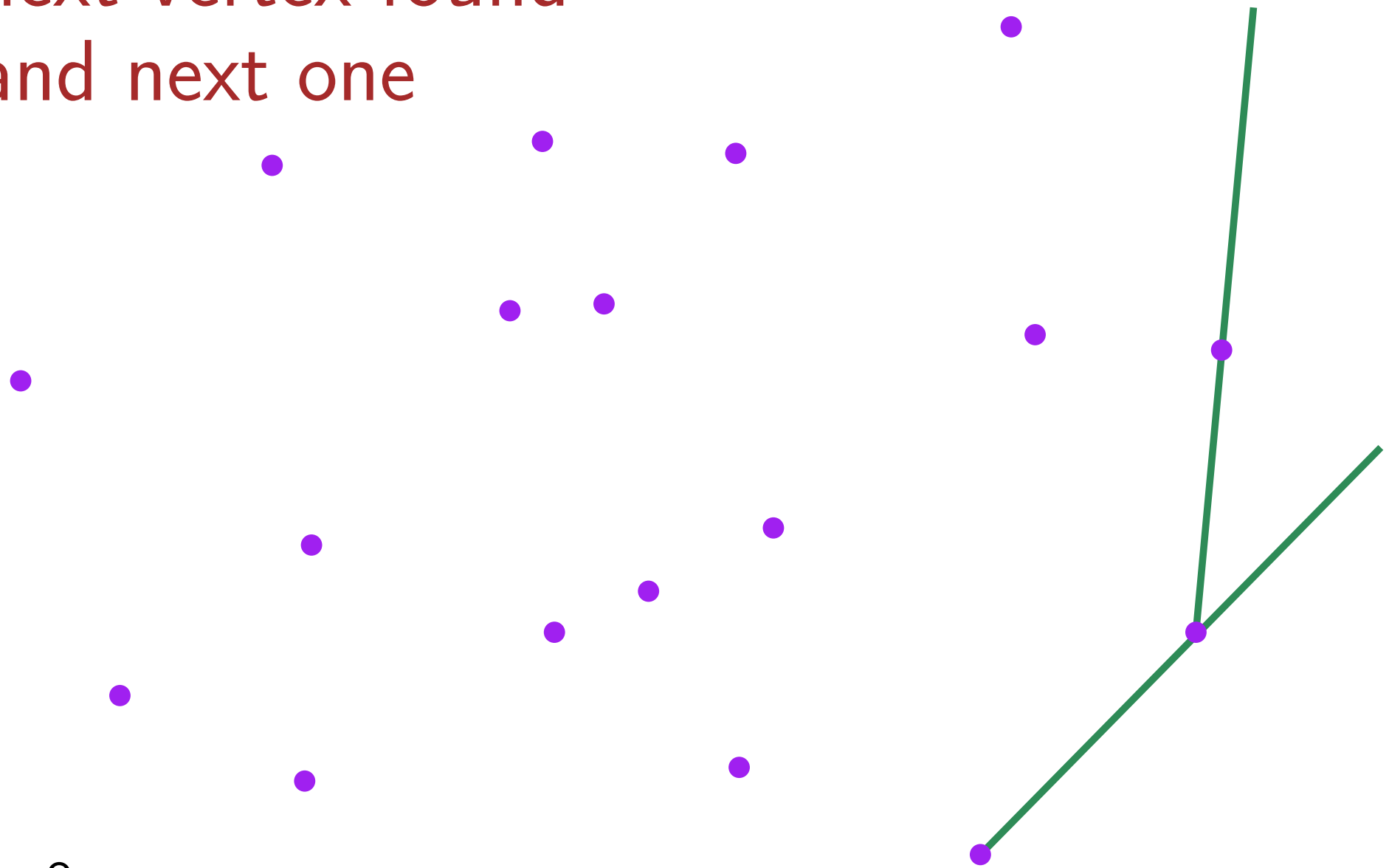
Jarvis algorithm

next vertex found



Convex hull

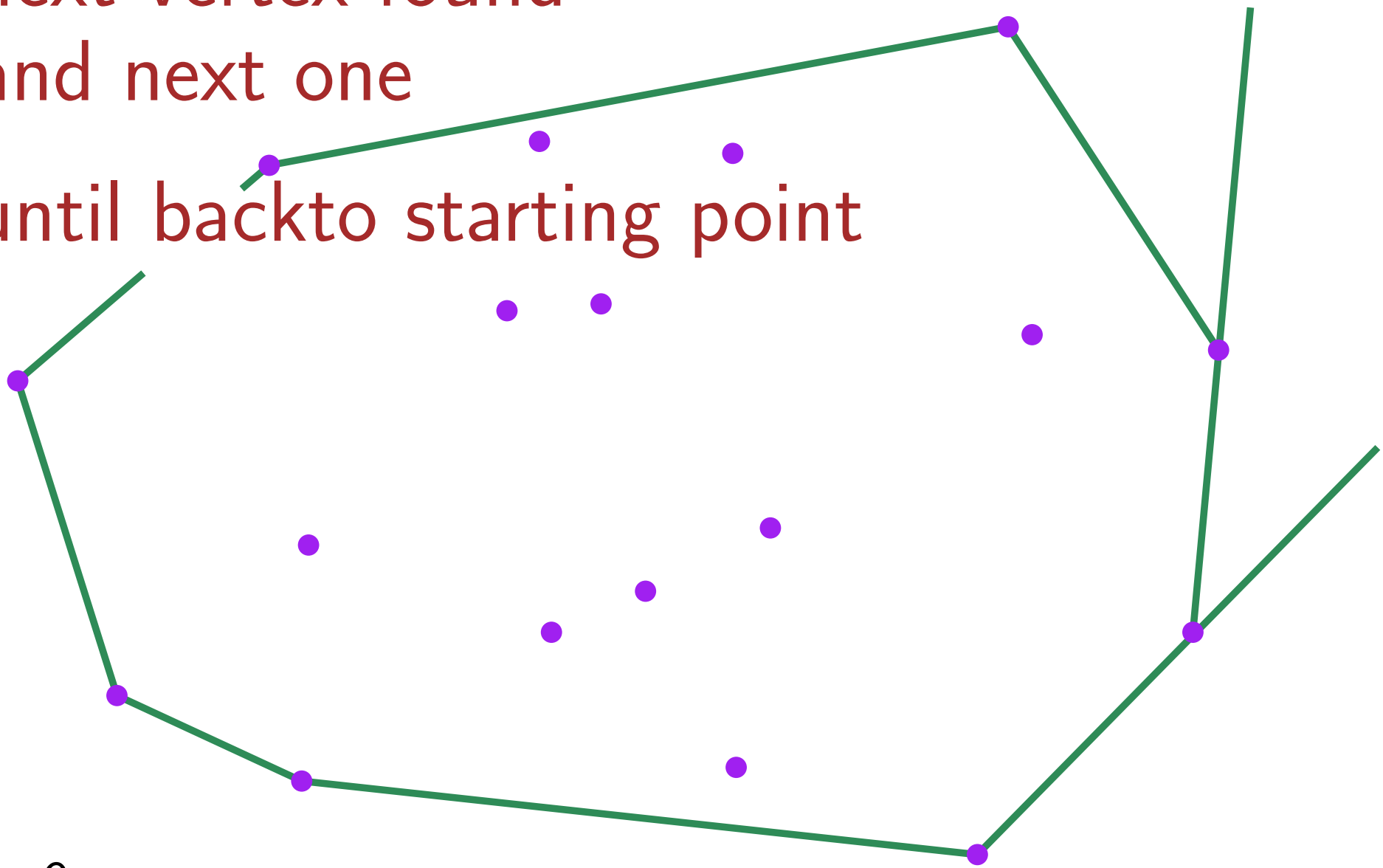
next vertex found
and next one



Convex hull

next vertex found
and next one

until backto starting point



Convex hull

Jarvis algorithm

Input : point set S
$u =$ lowest point in S; $min = \infty$
For each $w \in S \setminus \{u\}$
    if $angle(ux, uw) < min$
        then $min = angle(ux, uw)$; $v = w$;
$u.next = v$;
Do
    $S = S \setminus \{v\}$
    For each $w \in S$
        $min = \infty$
        if $angle(v.pred\ v, vw) < min$
            then $min = angle(v.pred\ v, vw)$; $v.next = w$;
    $v = v.next$;
While $v \neq u$

Convex hull

Jarvis algorithm

Complexity ?

Input : point set S
$u =$ lowest point in S; $min = \infty$
For each $w \in S \setminus \{u\}$
    if $angle(ux, uw) < min$
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Convex hull

Jarvis algorithm

Complexity ?

$O(n)$

Input : point set S
$u =$ lowest point in S; $min = \infty$
For each $w \in S \setminus \{u\}$
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Convex hull

Jarvis algorithm

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For each $w \in S \setminus \{u\}$

if $angle(ux, uw) < min$

then $min = angle(ux, uw)$; $v = w$;

$u.next = v$;

Do

$S = S \setminus \{v\}$

For each $w \in S$

$O(n)$

$min = \infty$

if $angle(v.pred\ v, vw) < min$

then $min = angle(v.pred\ v, vw)$; $v.next = w$;

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While $v \neq u$

Convex hull

Jarvis algorithm

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Do

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For each $w \in S$

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if $angle(v.pred\ v, vw) < min$

then $min = angle(v.pred\ v, vw)$; $v.next = w$;

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While $v \neq u$

Convex hull

Jarvis algorithm

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For each $w \in S \setminus \{u\}$

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Do

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$min = \infty$

if $angle(v.pred\ v, vw) < min$

then $min = angle(v.pred\ v, vw)$; $v.next = w$;

$v = v.next$;

While $v \neq u$

$O(n)$

Convex hull

Jarvis algorithm

Complexity ?

$O(n^2)$

Input : point set S
$u =$ lowest point in S; $min = \infty$
For each $w \in S \setminus \{u\}$
    if $angle(ux, uw) < min$
        then $min = angle(ux, uw)$; $v = w$;
$u.next = v$;
Do
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    For each $w \in S$
        $min = \infty$
        if $angle(v.pred\ v, vw) < min$
            then $min = angle(v.pred\ v, vw)$; $v.next = w$;
    $v = v.next$;
While $v \neq u$

Convex hull

Jarvis algorithm

Complexity ?

Input : point set S
$u =$ lowest point in S; $min = \infty$
For each $w \in S \setminus \{u\}$
    if $angle(ux, uw) < min$
        then $min = angle(ux, uw)$; $v = w$;
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Do
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    For each $w \in S$
        $min = \infty$
        if $angle(v.pred\ v, vw) < min$
            then $min = angle(v.pred\ v, vw)$; $v.next = w$;
    $v = v.next$;
While $v \neq u$

$O(n^2)$

$O(nh)$

Convex hull

Input : point set S

$u =$ lowest point in S; $min = \infty$

For each $w \in S \setminus \{u\}$

if $angle(ux, uw) < min$

then $min = angle(ux, uw)$; $v = w$;

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Do

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then $min = angle(v.pred\ v, vw)$; $v.next = w$;

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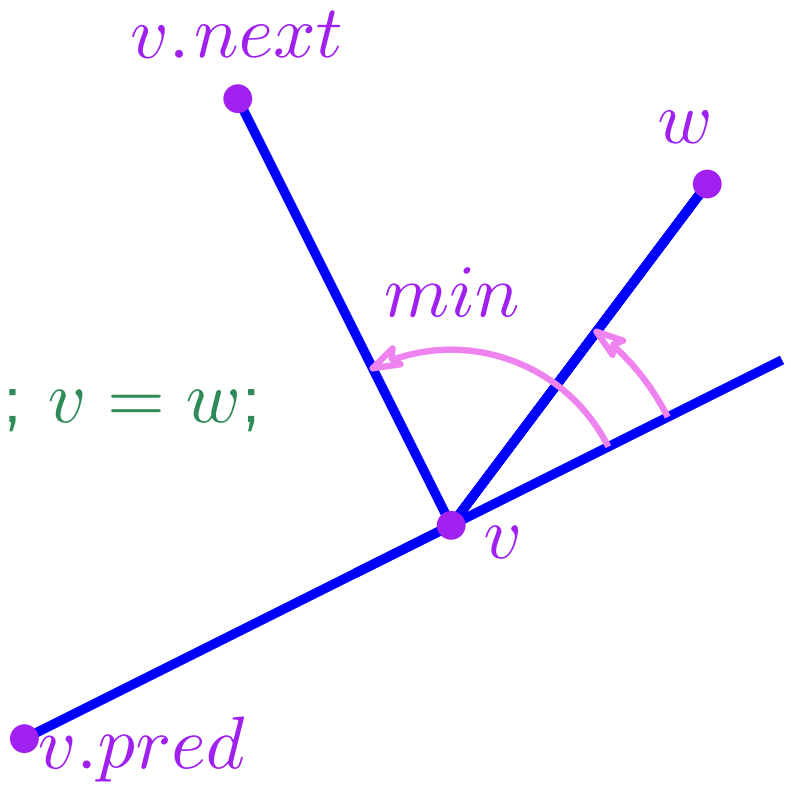
While $v \neq u$

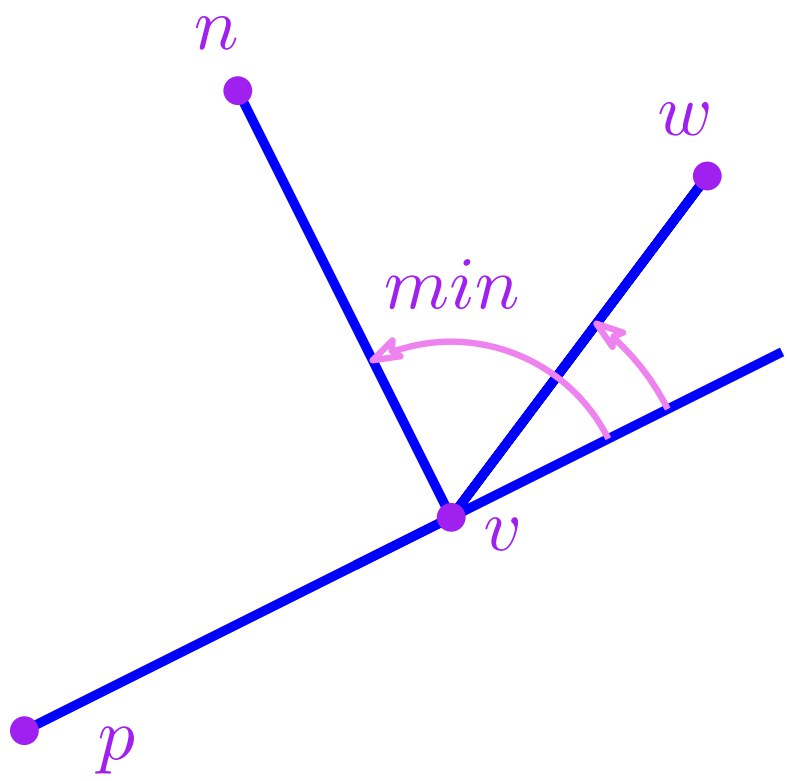
Orientation predicate



Convex hull

Orientation predicate

if $angle(pv, vw) < min$



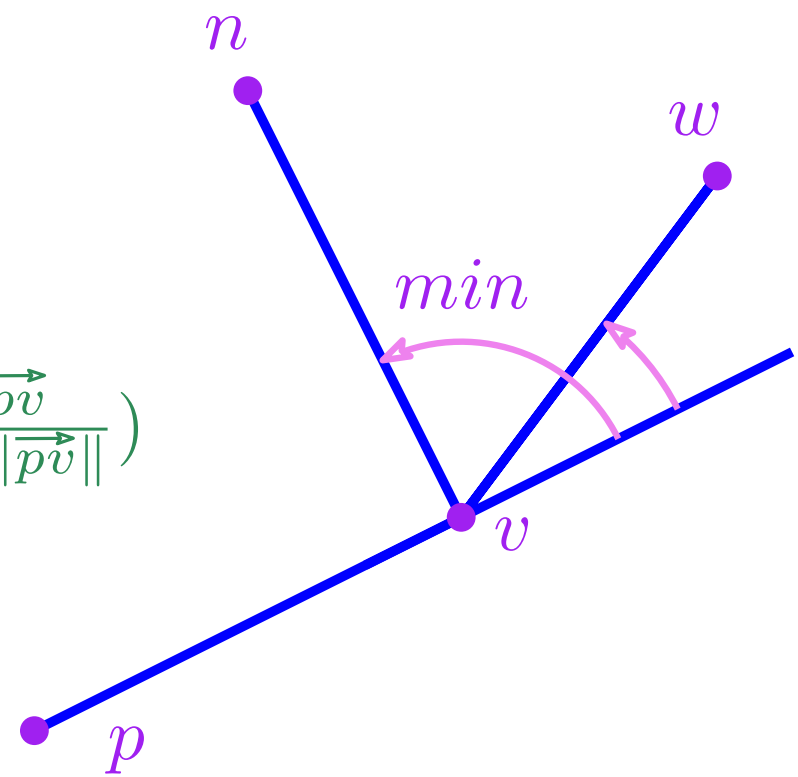
Convex hull

Orientation predicate

if $angle(pv, vw) < min$

$$angle(pv, vw) = \arccos\left(\frac{\overrightarrow{vw} \cdot \overrightarrow{pv}}{\|\overrightarrow{vw}\| \cdot \|\overrightarrow{pv}\|}\right)$$

?



Convex hull

Orientation predicate

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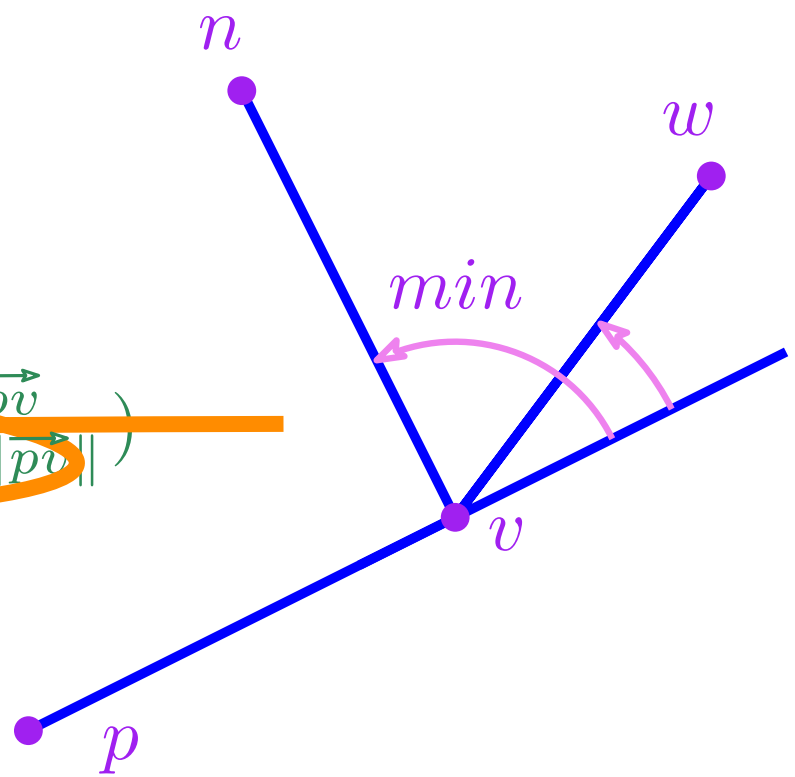
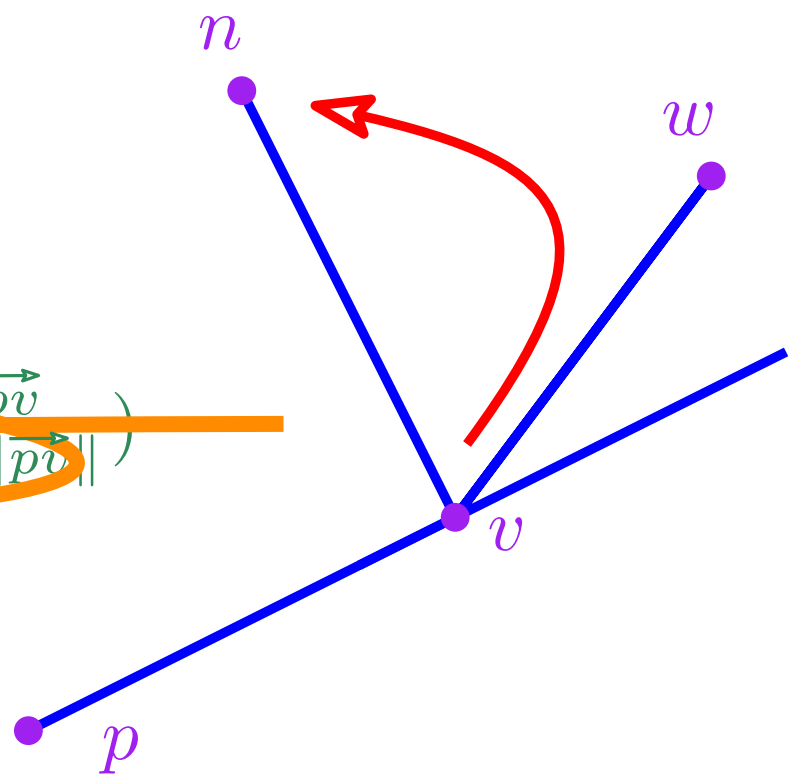


Convex hull

Orientation predicate

if $angle(pv, vw) < min$

$$angle(pv, vw) = \arccos\left(\frac{\overrightarrow{vw} \cdot \overrightarrow{pv}}{\|\overrightarrow{vw}\| \cdot \|\overrightarrow{pv}\|}\right)$$



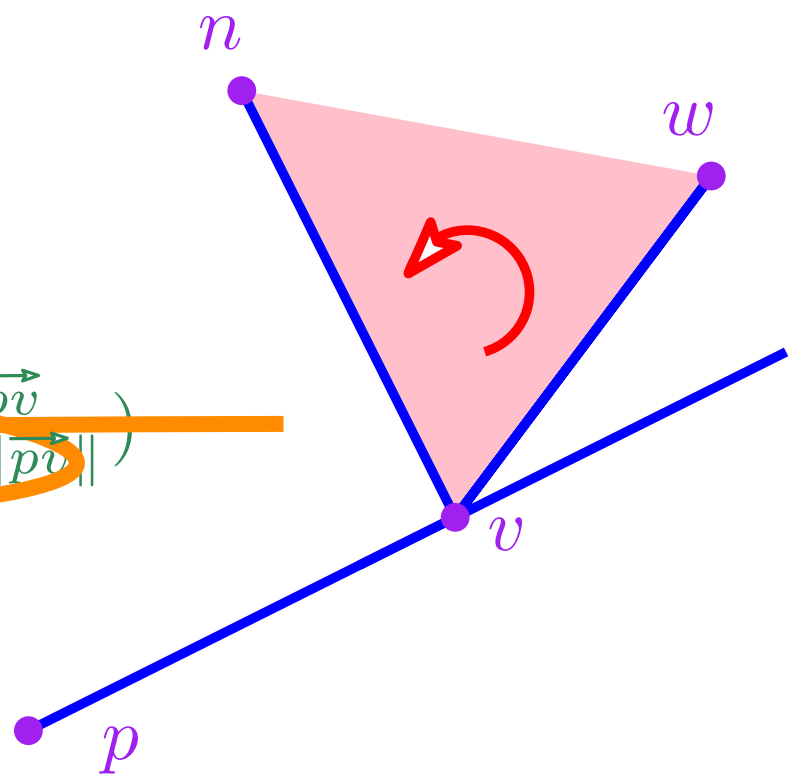
if vwn turn left

Convex hull

Orientation predicate

if $angle(pv, vw) < min$

$$angle(pv, vw) = \arccos\left(\frac{\overrightarrow{vw} \cdot \overrightarrow{pv}}{\|\overrightarrow{vw}\| \cdot \|\overrightarrow{pv}\|}\right)$$



if vwn turn left

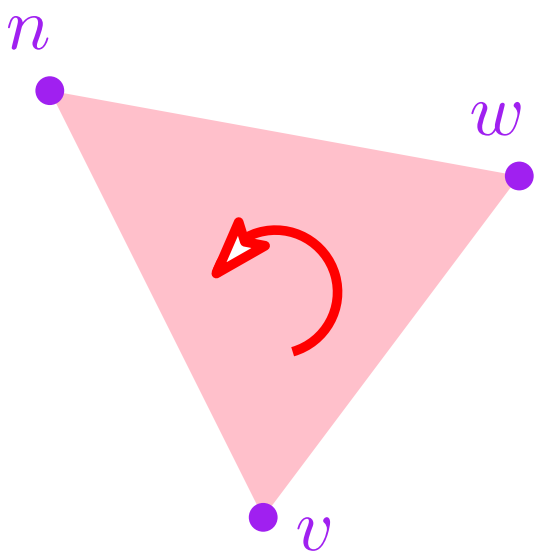
if triangle vwn counterclockwise (ccw)

if triangle vwn positively oriented

Convex hull

$vwn + ?$

Orientation predicate

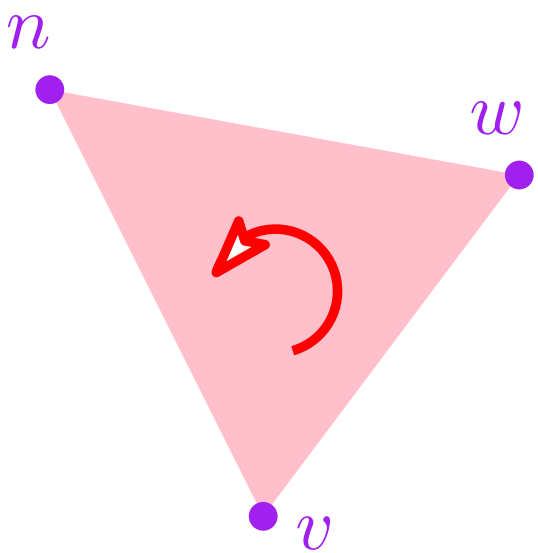


Convex hull

Orientation predicate

$vwn + ?$

$$\begin{vmatrix} x_w - x_v & x_n - x_v \\ y_w - y_v & y_n - y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} > 0$$

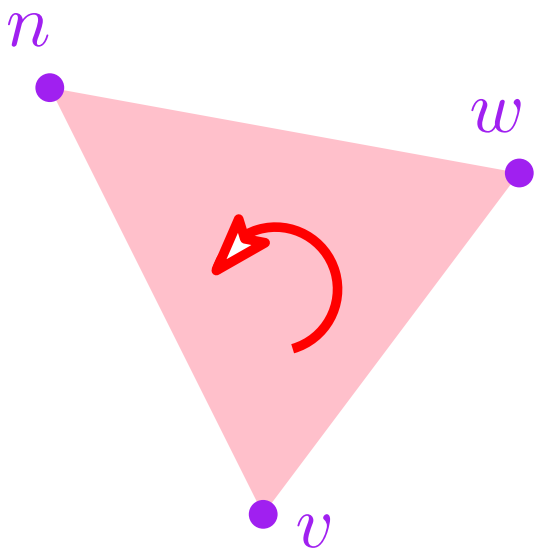


Convex hull

Orientation predicate

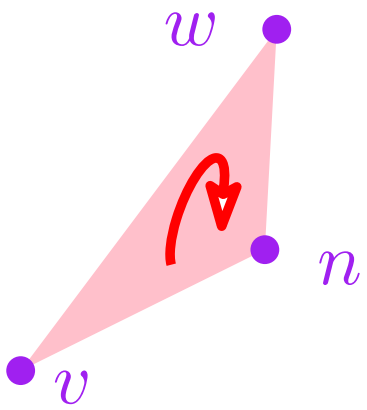
vwn + ?

$$\begin{vmatrix} x_w - x_v & x_n - x_v \\ y_w - y_v & y_n - y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} > 0$$



vwn - ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} < 0$$



Convex hull

Orientation predicate

vwn + ?

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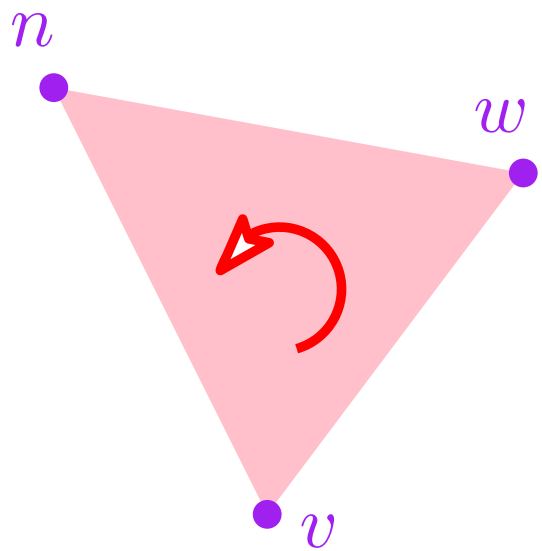
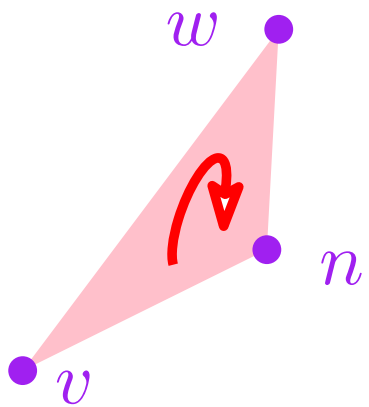


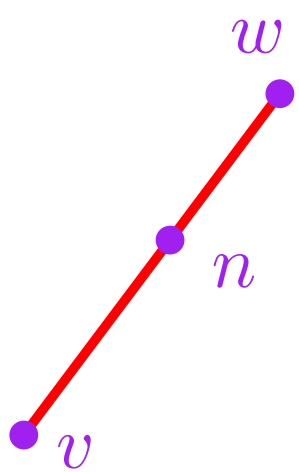
vwn - ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} < 0$$



vwn 0 ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} = 0$$

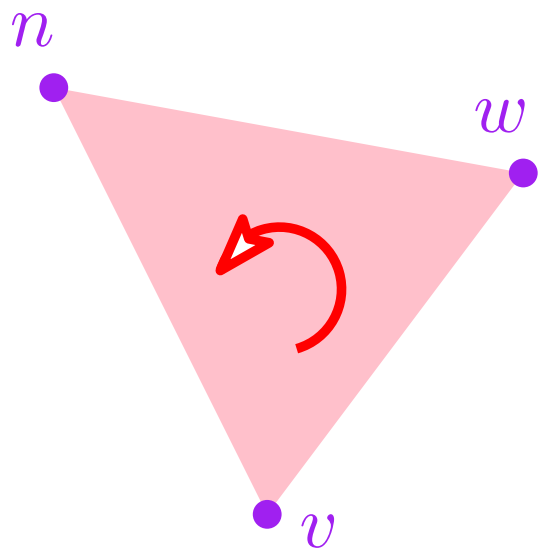


Convex hull

Orientation predicate

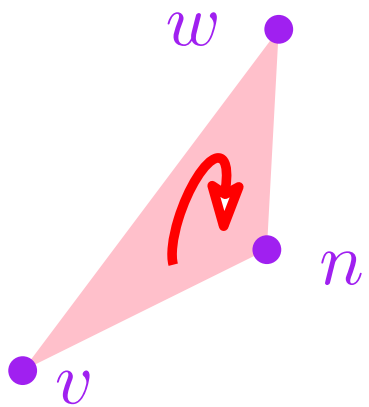
vwn + ?

$$\begin{vmatrix} x_w - x_v & x_n - x_v \\ y_w - y_v & y_n - y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} > 0$$



vwn - ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} < 0$$



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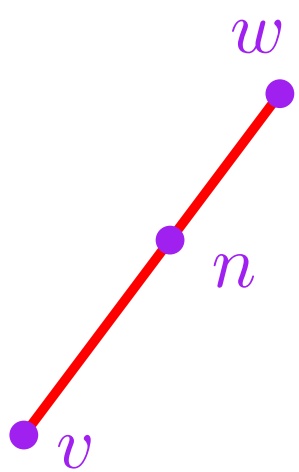


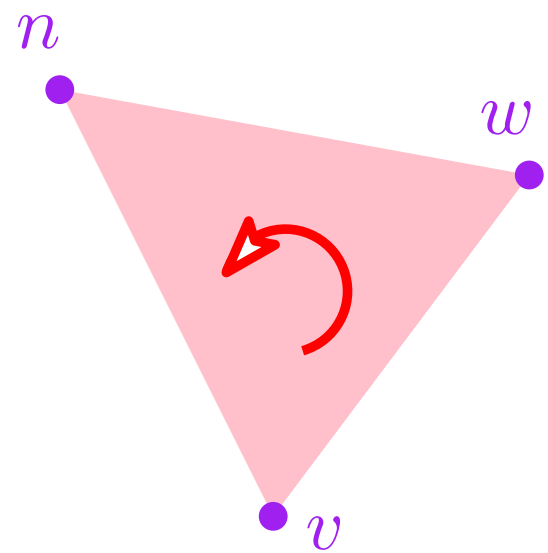
degenerate case

Convex hull

Orientation predicate

vwn + ?

$$\begin{vmatrix} x_w - x_v & x_n - x_v \\ y_w - y_v & y_n - y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} > 0$$



vwn - ?

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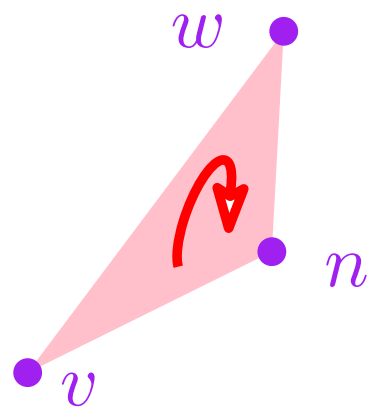
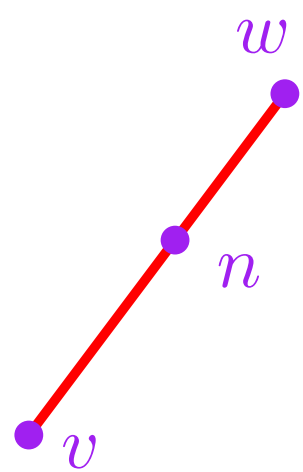


vwn 0 ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} = 0$$



rounding errors

Convex hull

Rounding errors possible

Orientation predicate

$p = (\frac{1}{2} + x.u, \frac{1}{2} + y.u)$

$0 \leq x, y \leq 256,\ u = 2^{-53}$

$q = (12, 12)$

$r = (24, 24)$

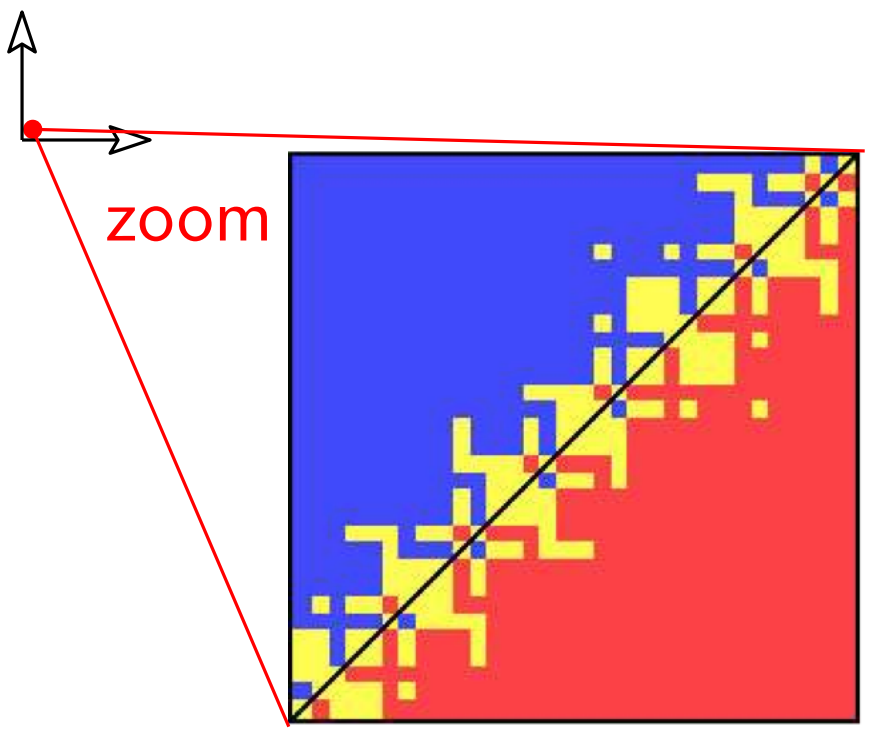
Teaser robustness lecture

$orientation(p, q, r)$

evaluated with double

Convex hull

Rounding errors possible



Orientation predicate

$p = (\frac{1}{2} + x.u, \frac{1}{2} + y.u)$

$0 \leq x, y \leq 256,\ u = 2^{-53}$

$q = (12, 12)$

$r = (24, 24)$

Teaser robustness lecture

$orientation(p, q, r)$

evaluated with double

≤ 0

0

≥ 0

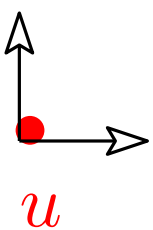
Convex hull

Teaser robustness lecture

Buggy degenerate example

(single precision)

$$w_1 = (12, 12)$$
$$w_2 = (24, 24)$$
$$w_3 = (30, 30.000001)$$
$$w_4 = (23, 36)$$
$$w_5 = (0.5000029, 0.5000027)$$



Convex hull

Teaser robustness lecture

Buggy degenerate example
(single precision)

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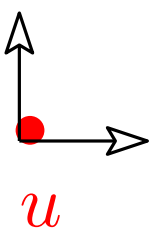

Input : point set S

$u = v =$ lowest point in S;

Jarvis

Convex hull

Teaser robustness lecture

Buggy degenerate example
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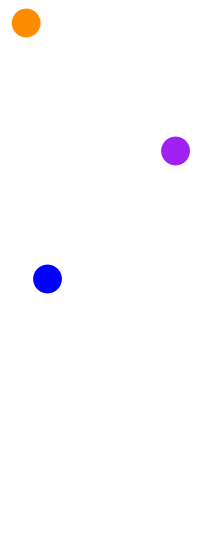

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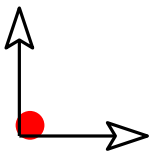

u

Do

$n =$ first in S;

For each $w \in S$

if vwn positive

then $n = w$;

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$S = S \setminus \{v\}$

While $v \neq u$

Convex hull

Teaser robustness lecture

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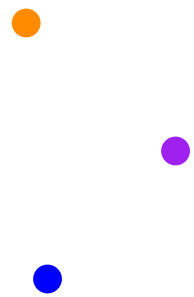

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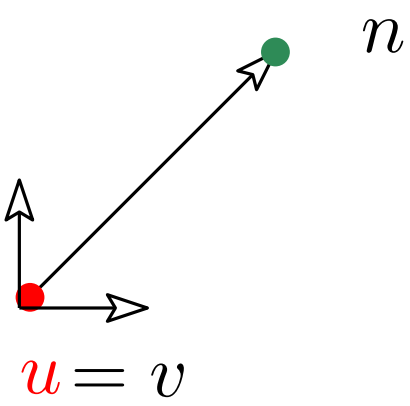

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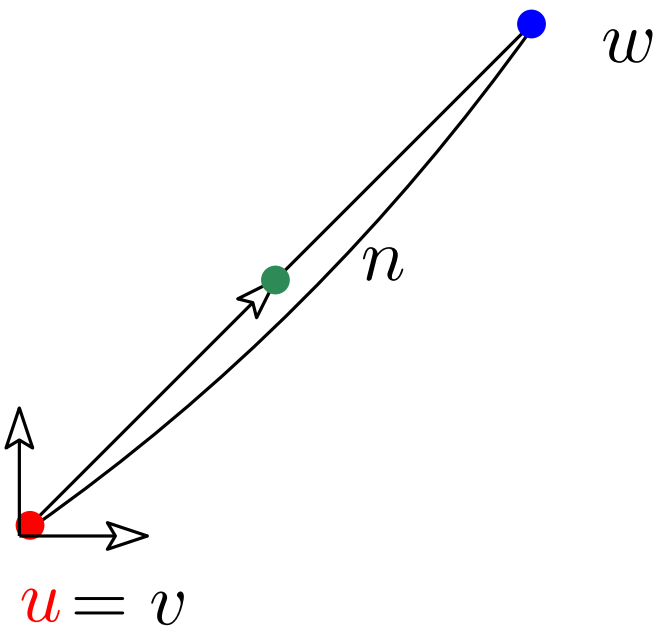

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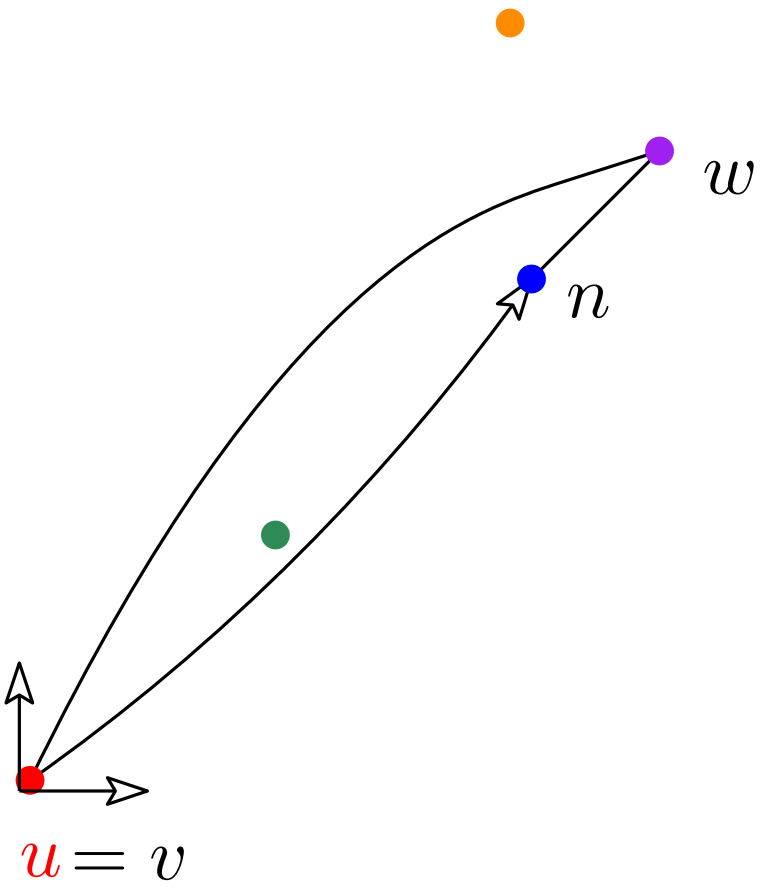

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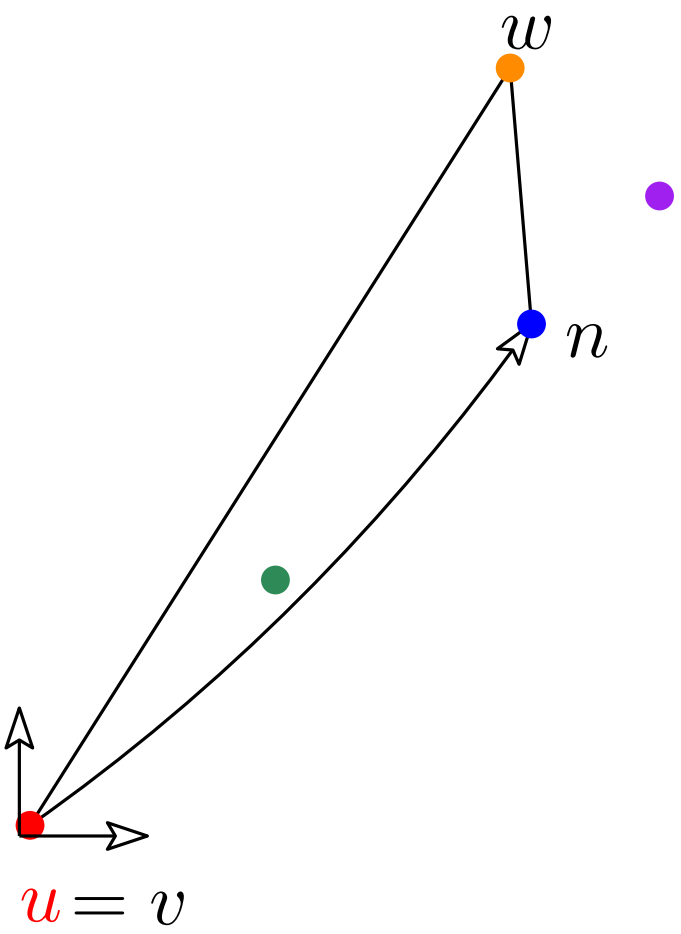

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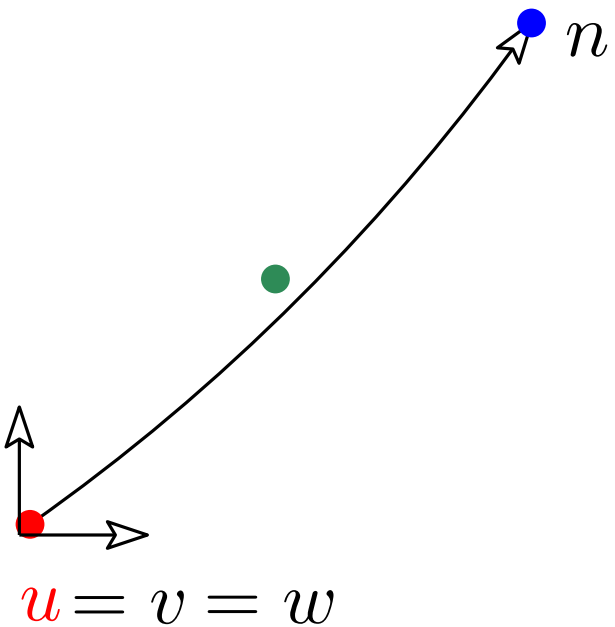

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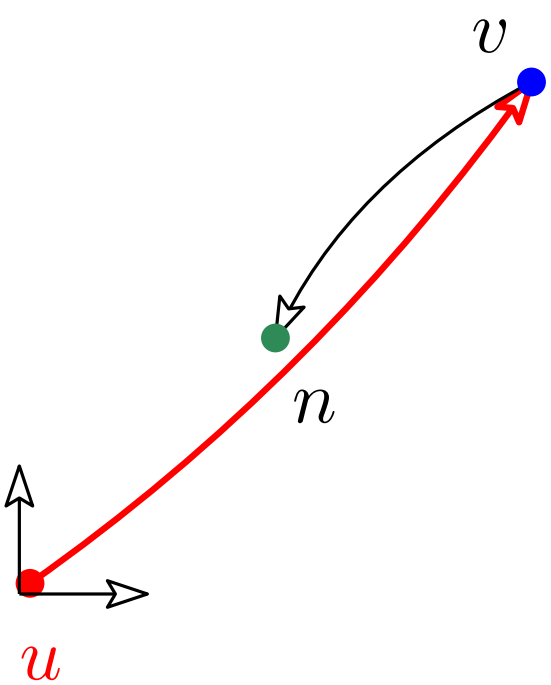

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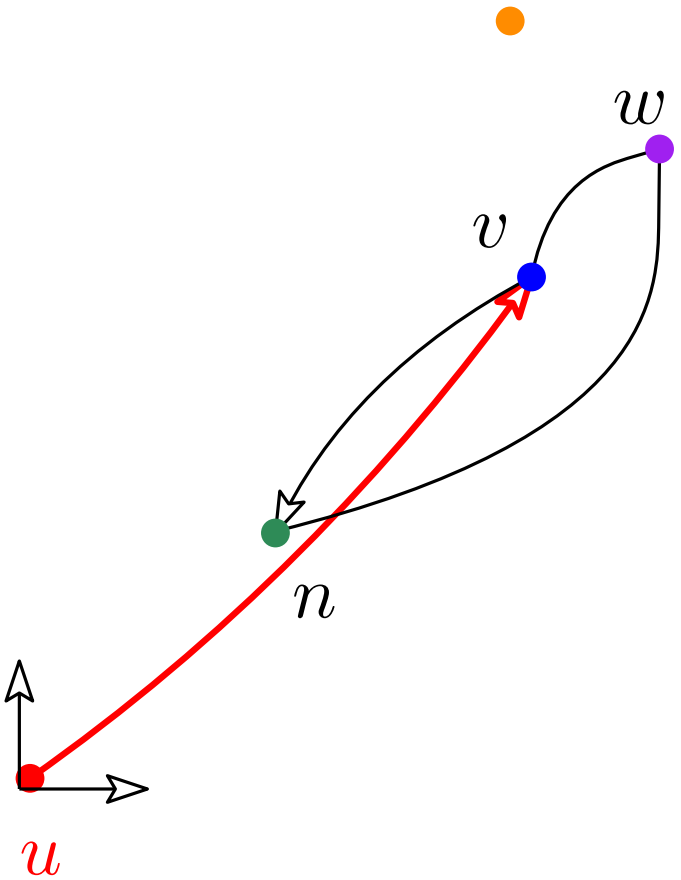

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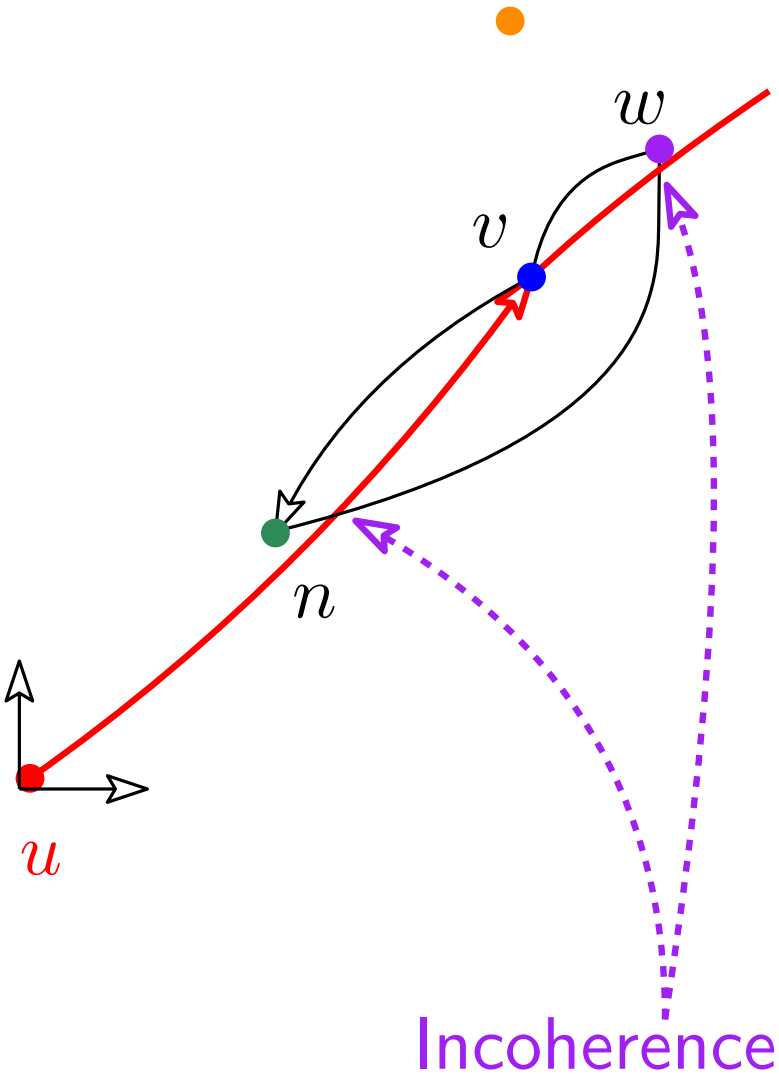

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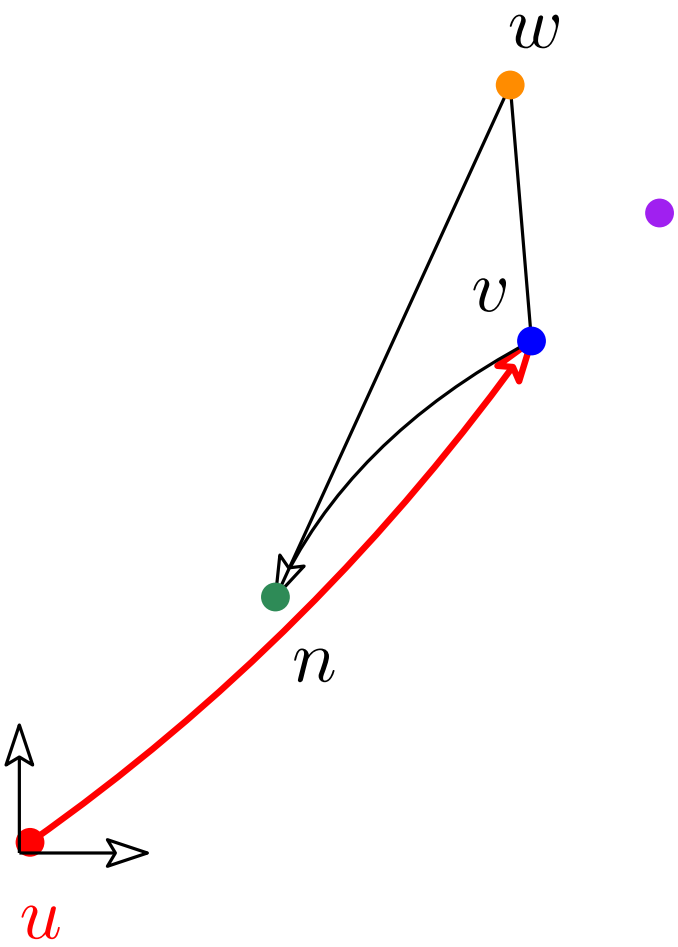

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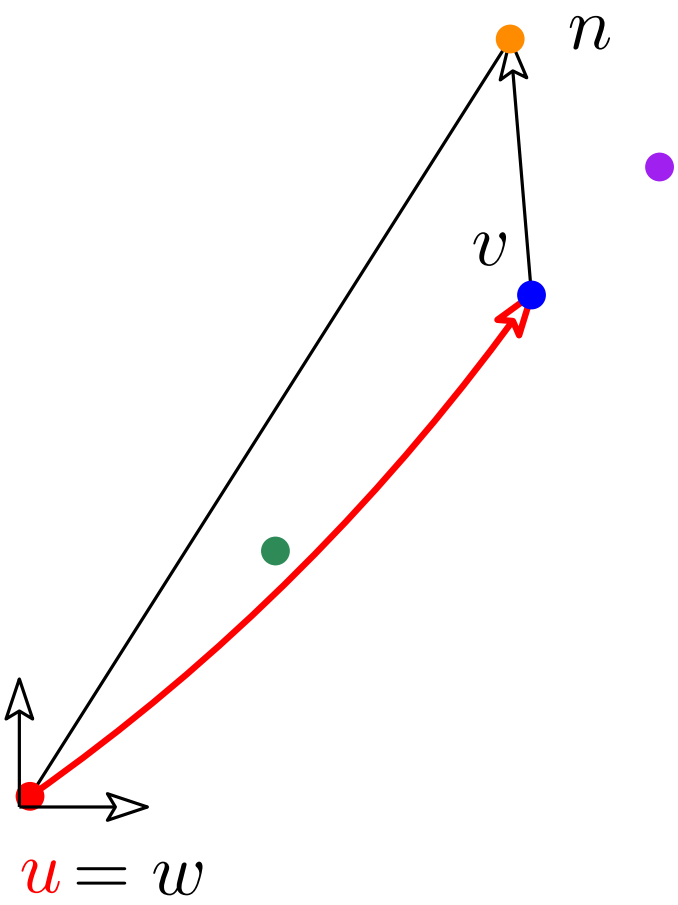

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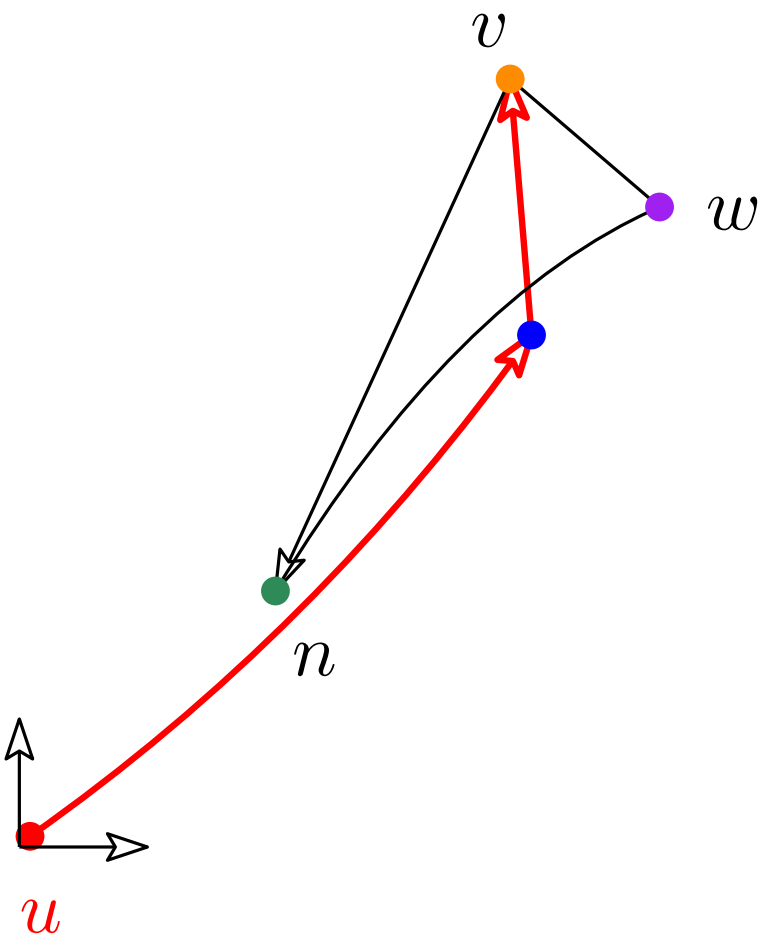

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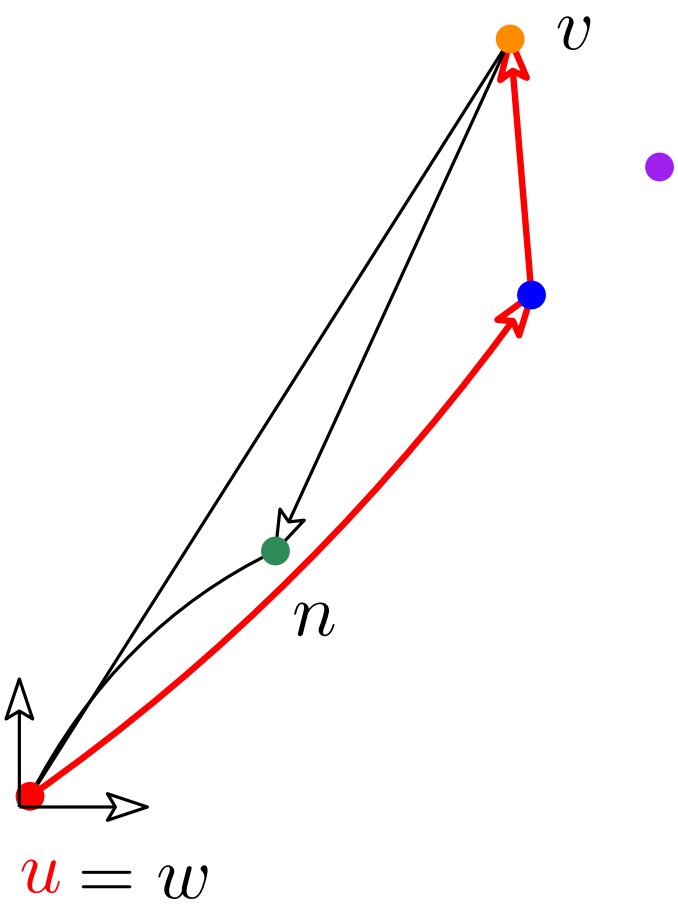

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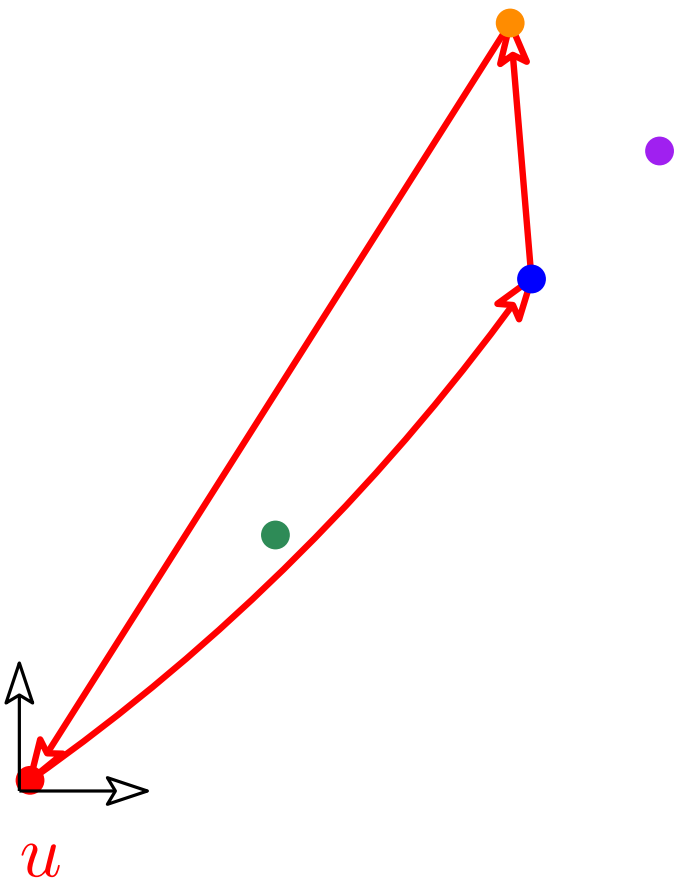

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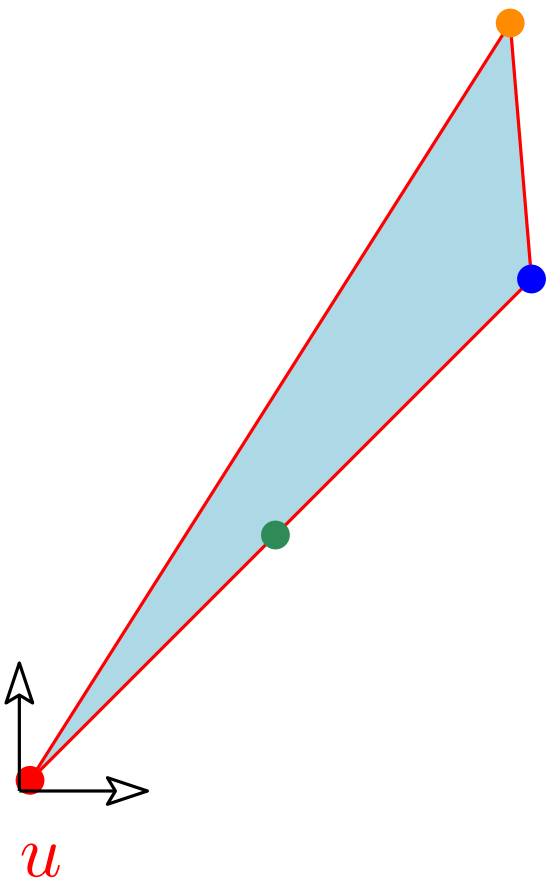

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Result is really wrong

Convex hull

Real RAM model and general position hypothesis

Real Random Access Memory model

Assume exact computation on real numbers

constant time for single operations: $+, -, \sqrt{}, \sin \ldots$

Convex hull

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General position hypotheses

Predicate: Input $\longmapsto \{-1, 0, 1\}$

Convex hull

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2D convex hull: no three points colinear

Convex hull

Real RAM model and general position hypothesis

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Assume exact computation on real numbers

constant time for single operations: $+, -, \sqrt{}, \sin \ldots$

General position hypotheses

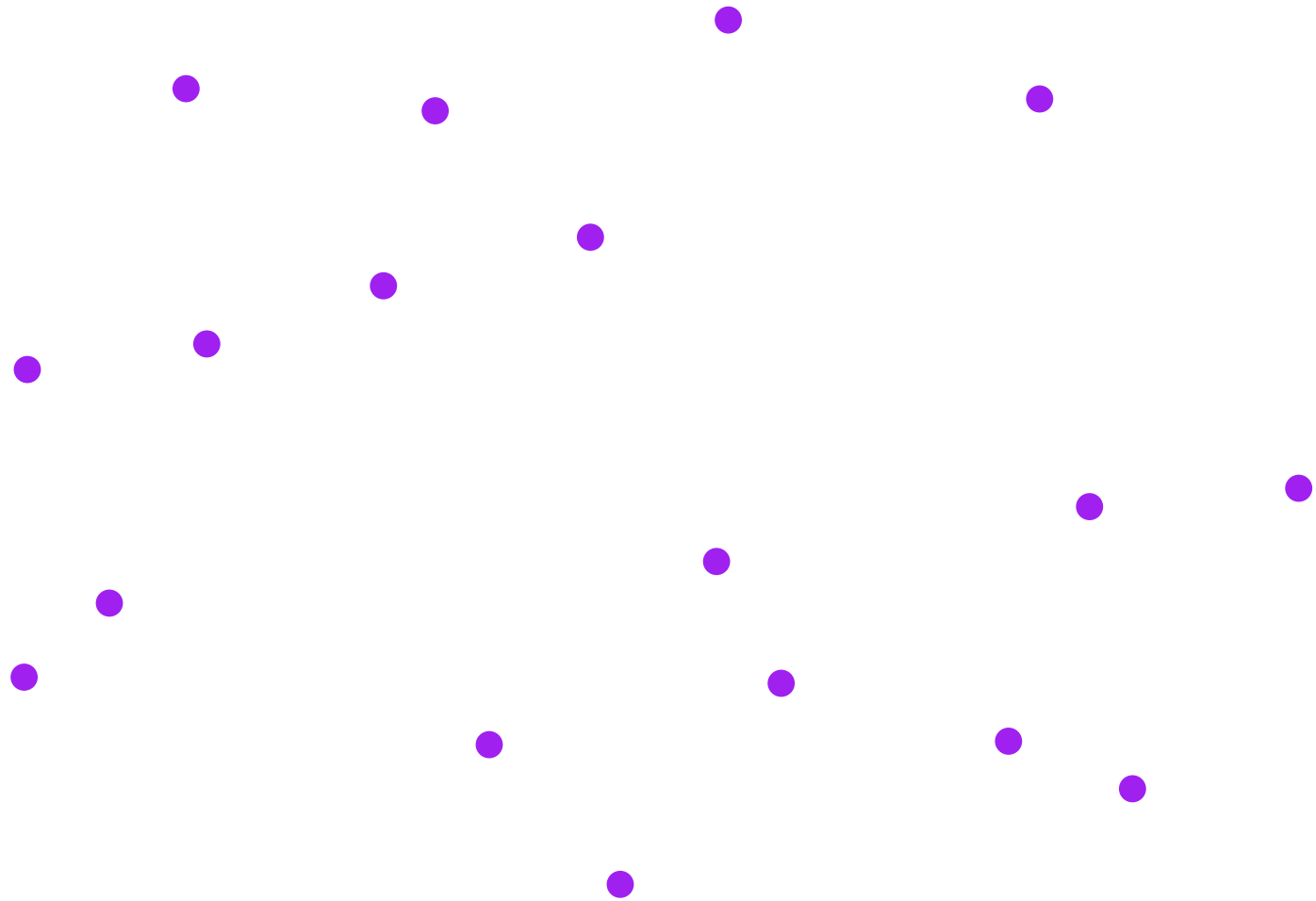
Predicate: Input $\longmapsto$ $\{-1, 0, 1\}$

2D convex hull: no three points colinear

possibly: no 2 points with same x

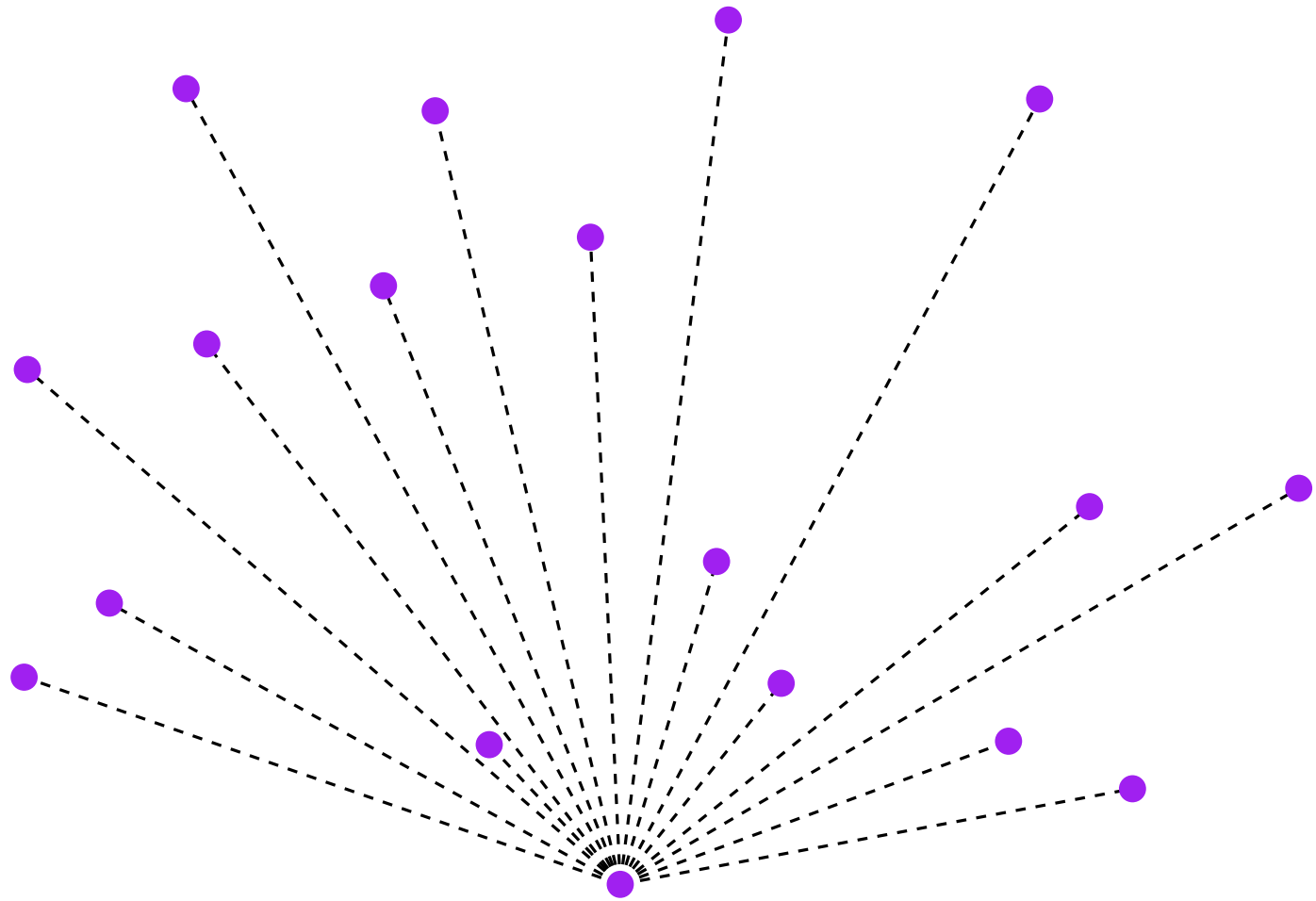
Convex hull

Graham algorithm



Convex hull

sort around a point (e.g. lowest point)



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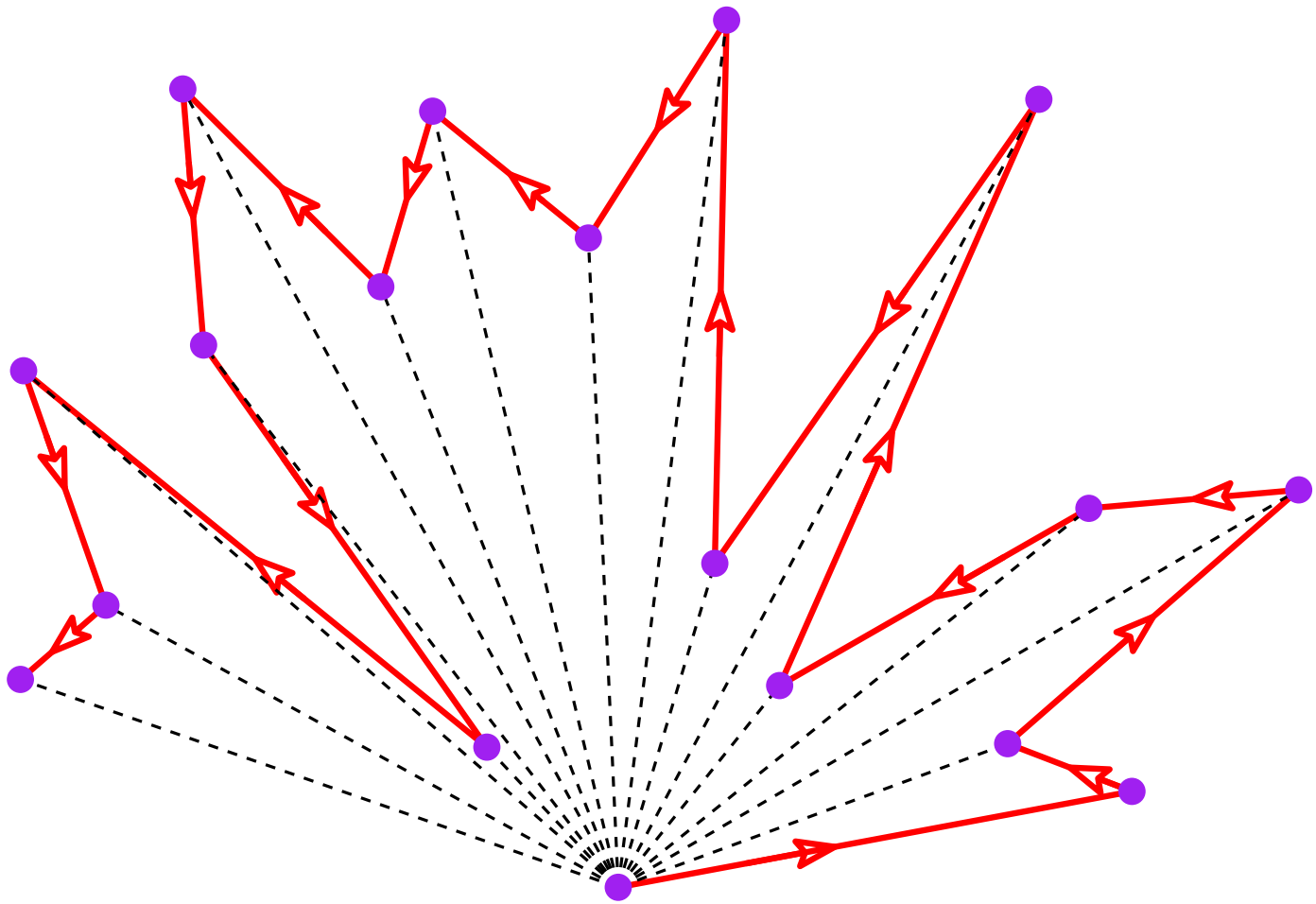

Convex hull

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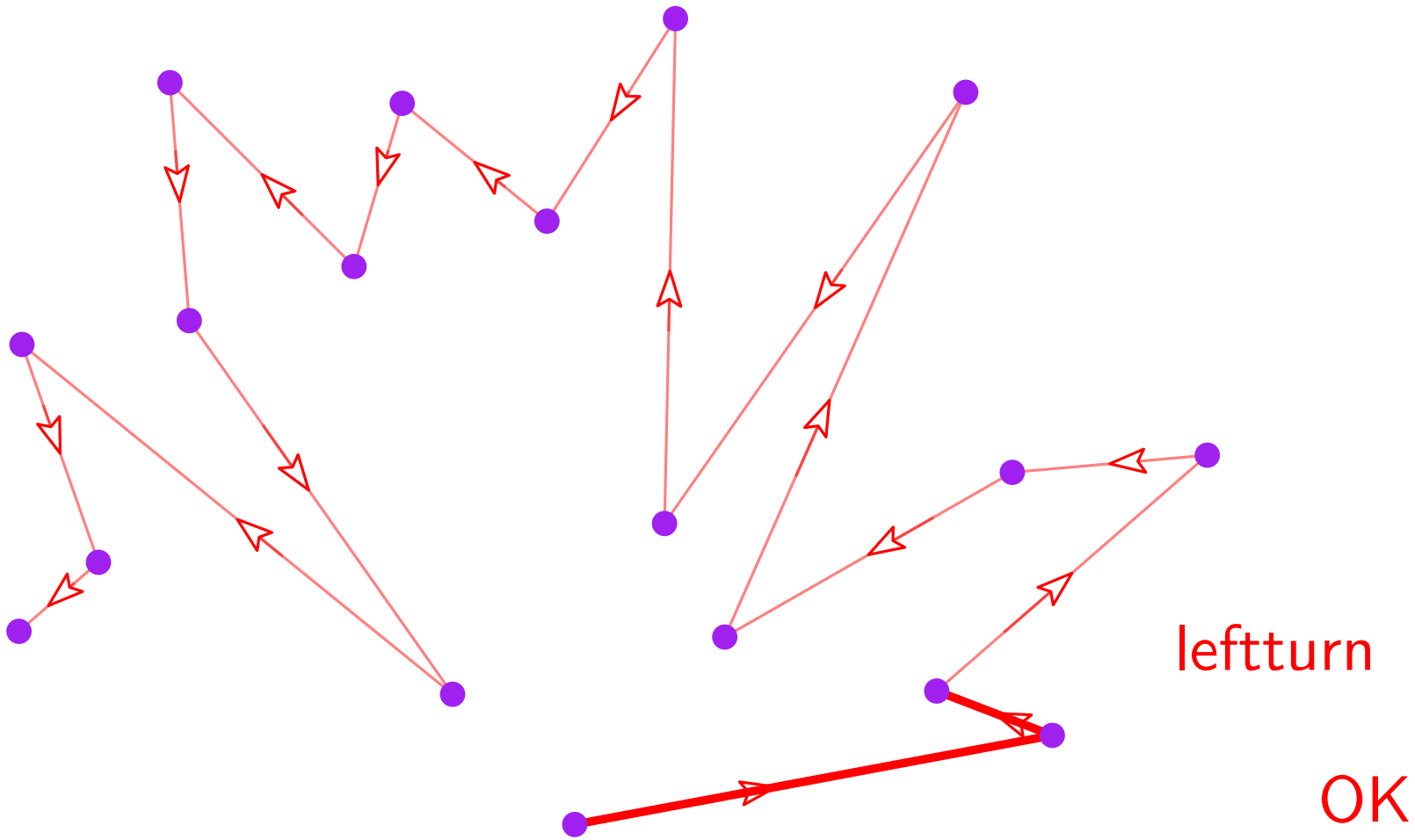

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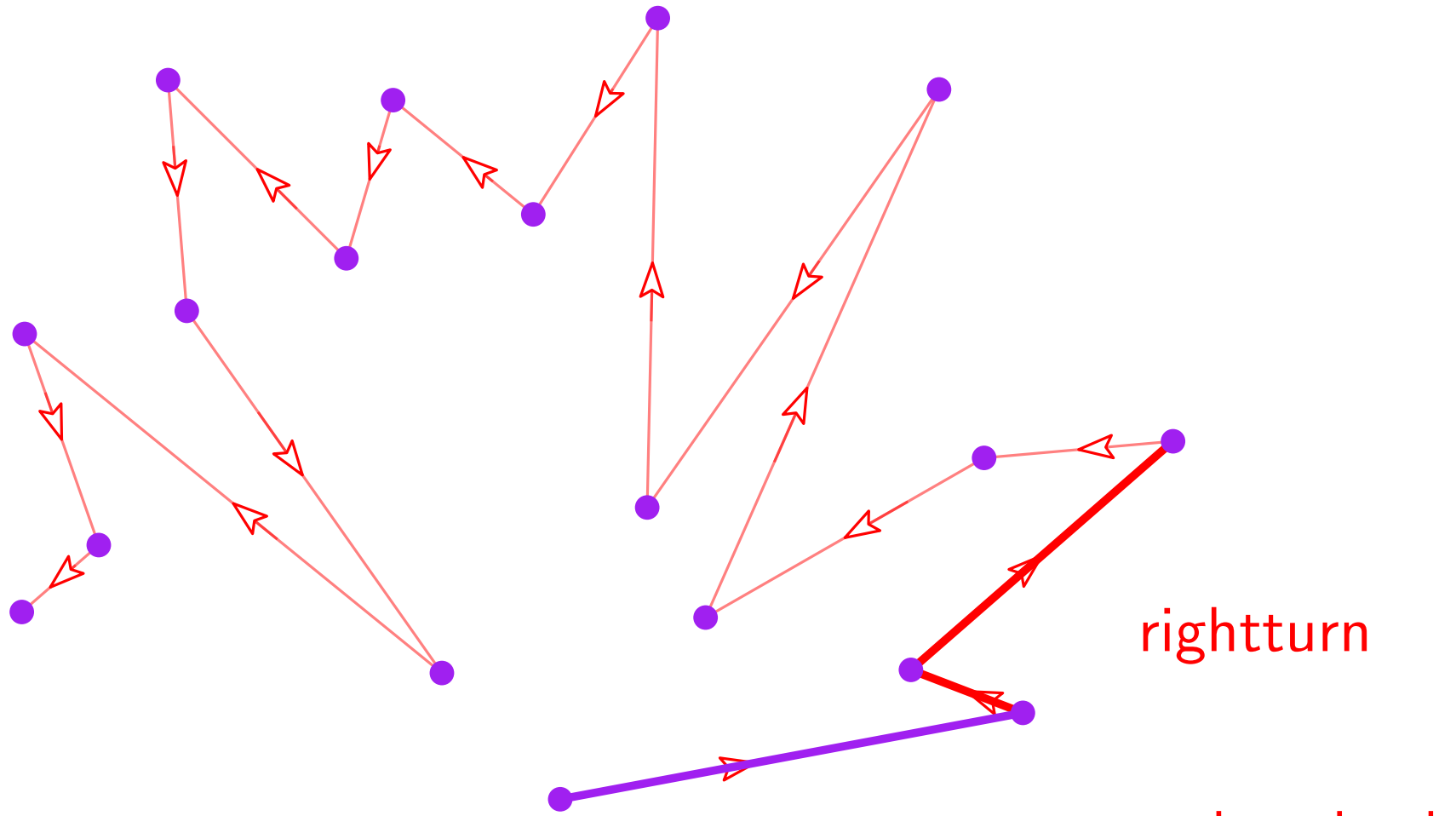

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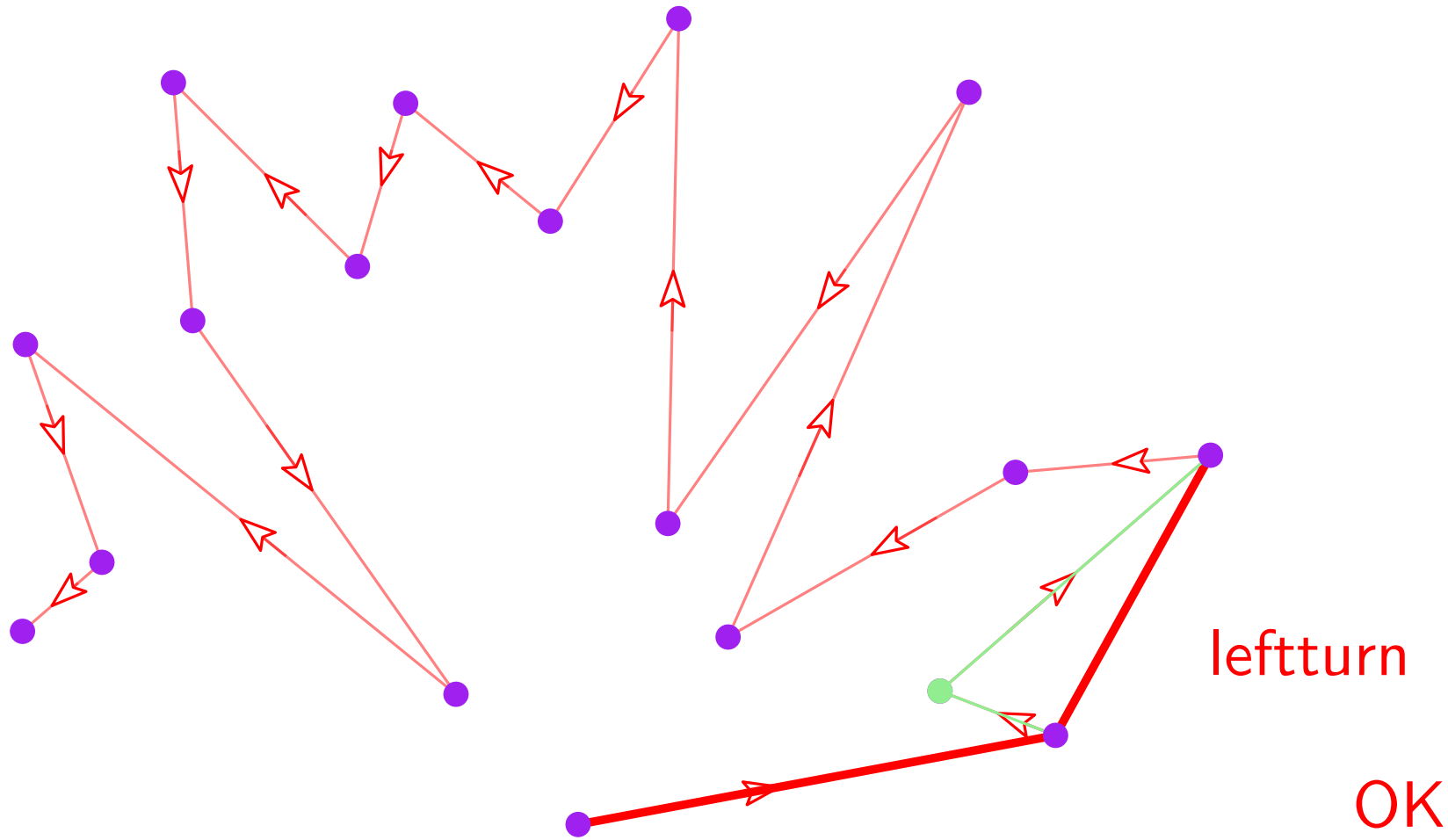

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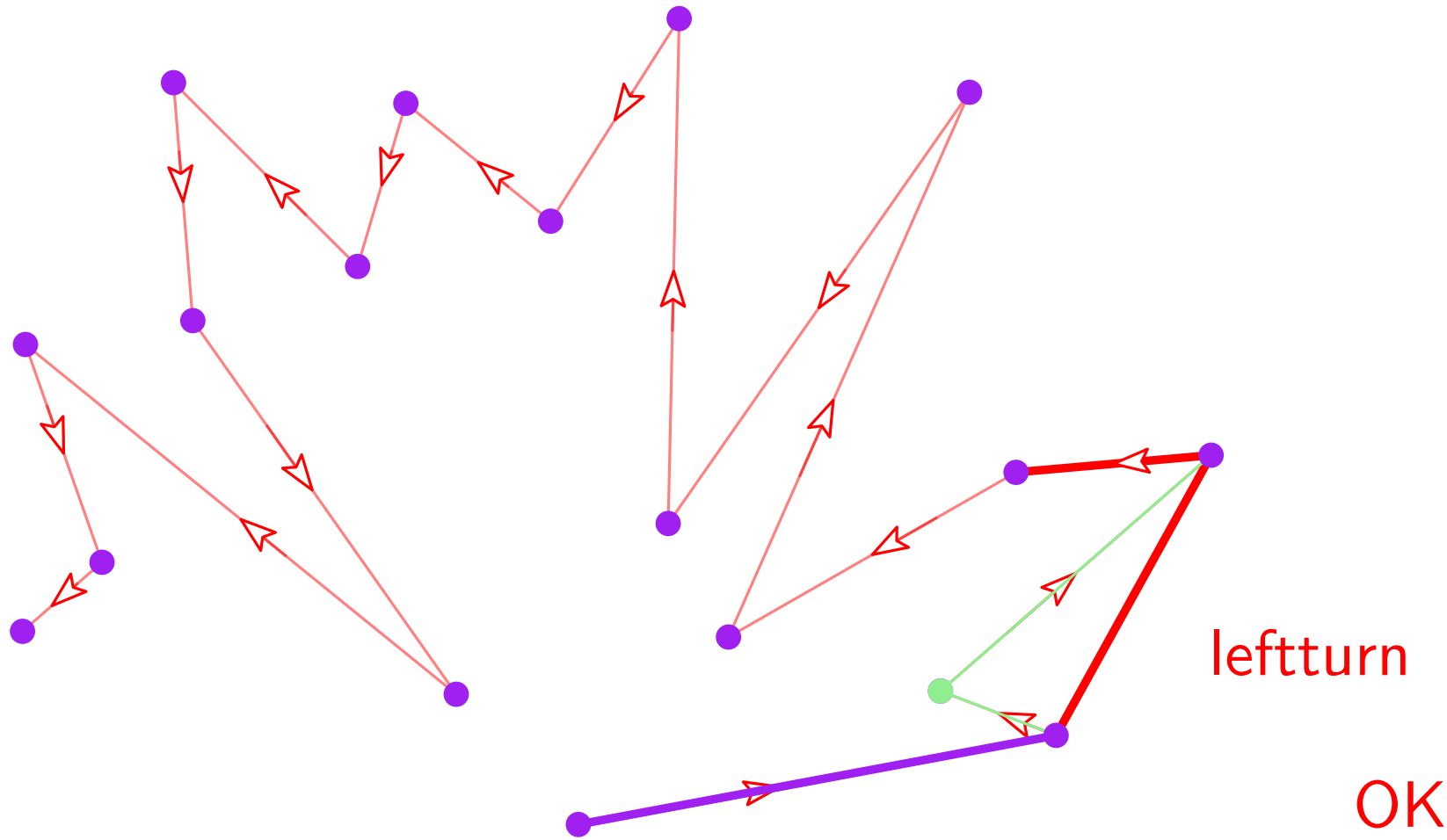

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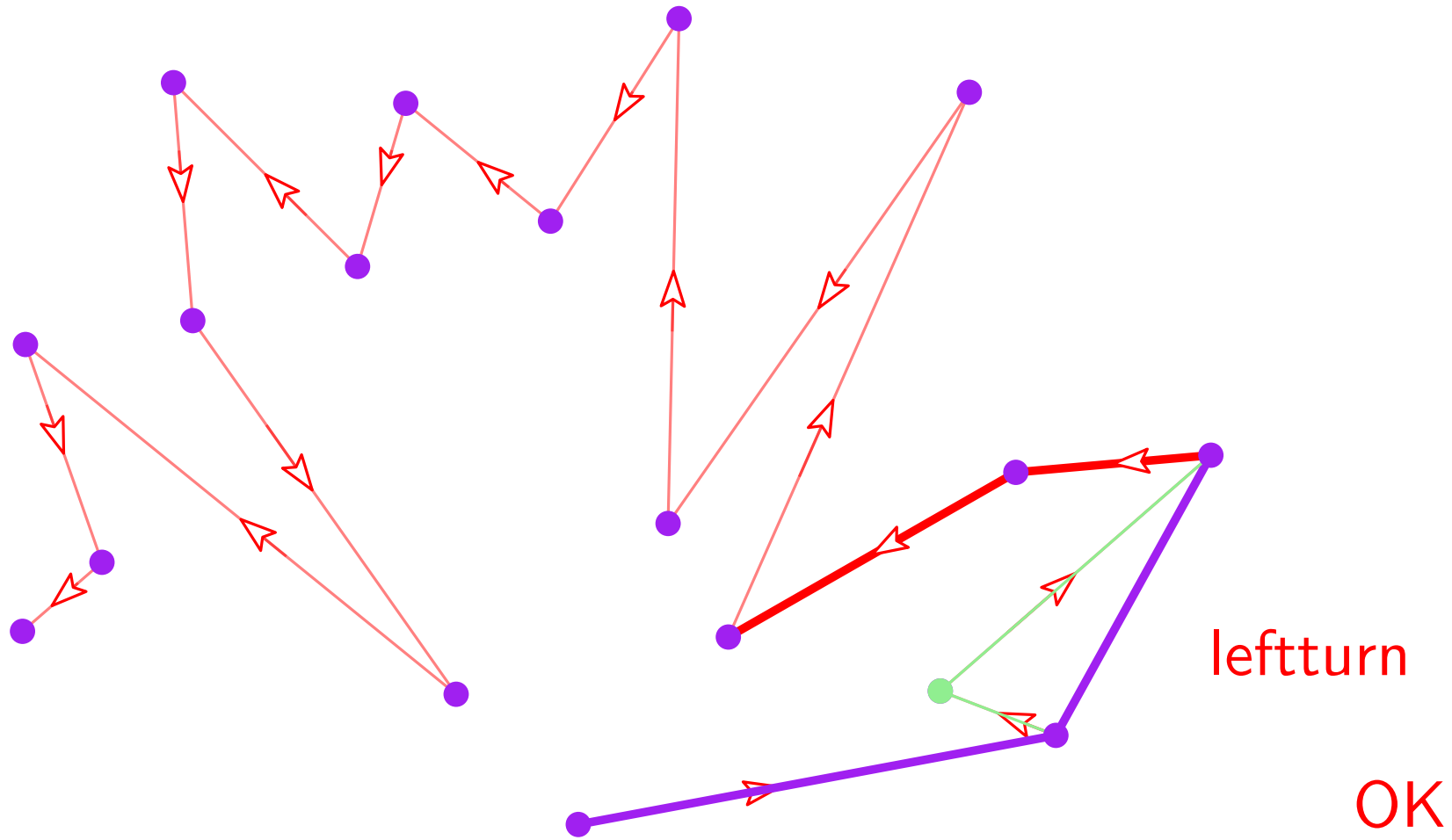

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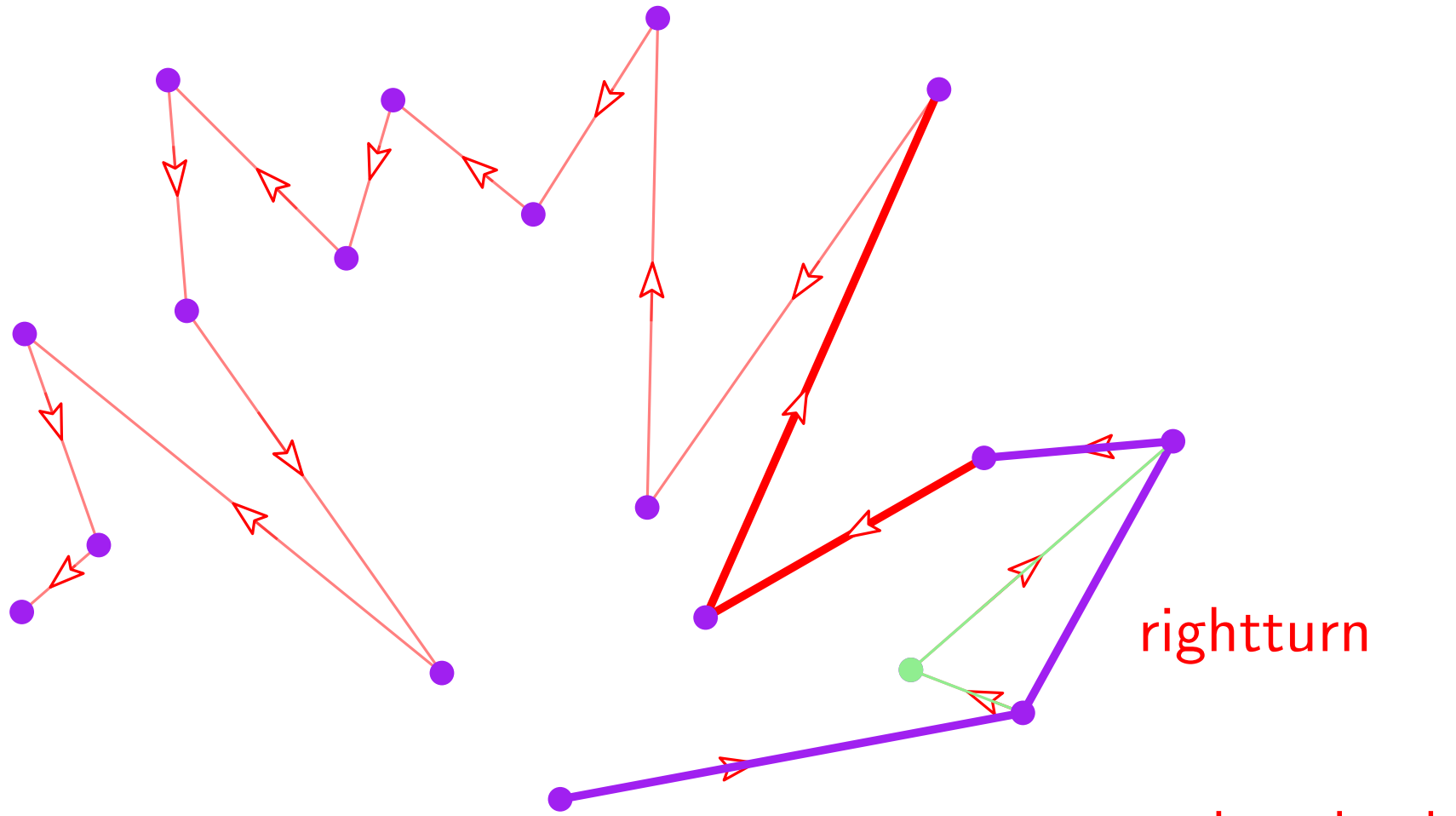

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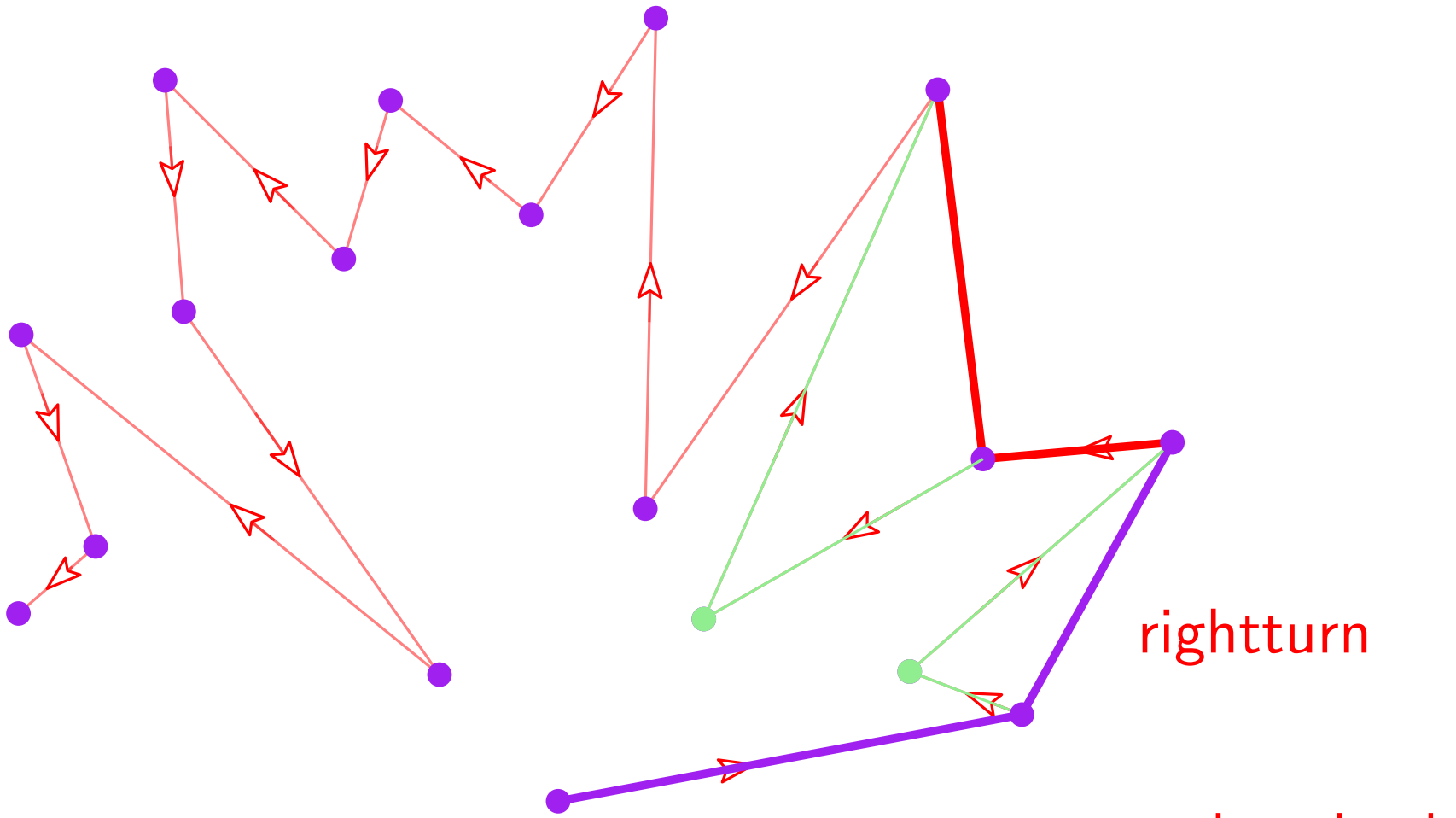

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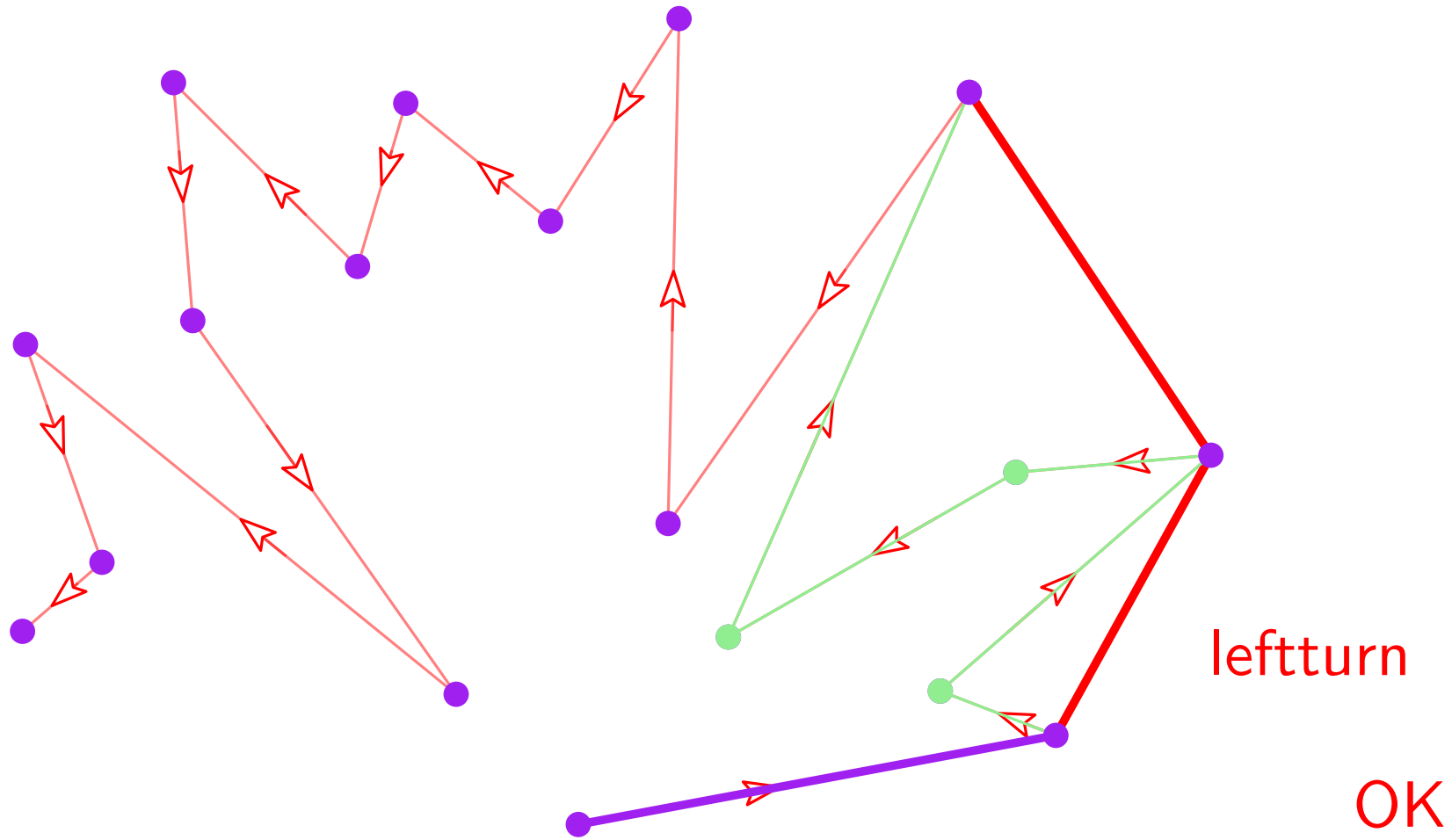

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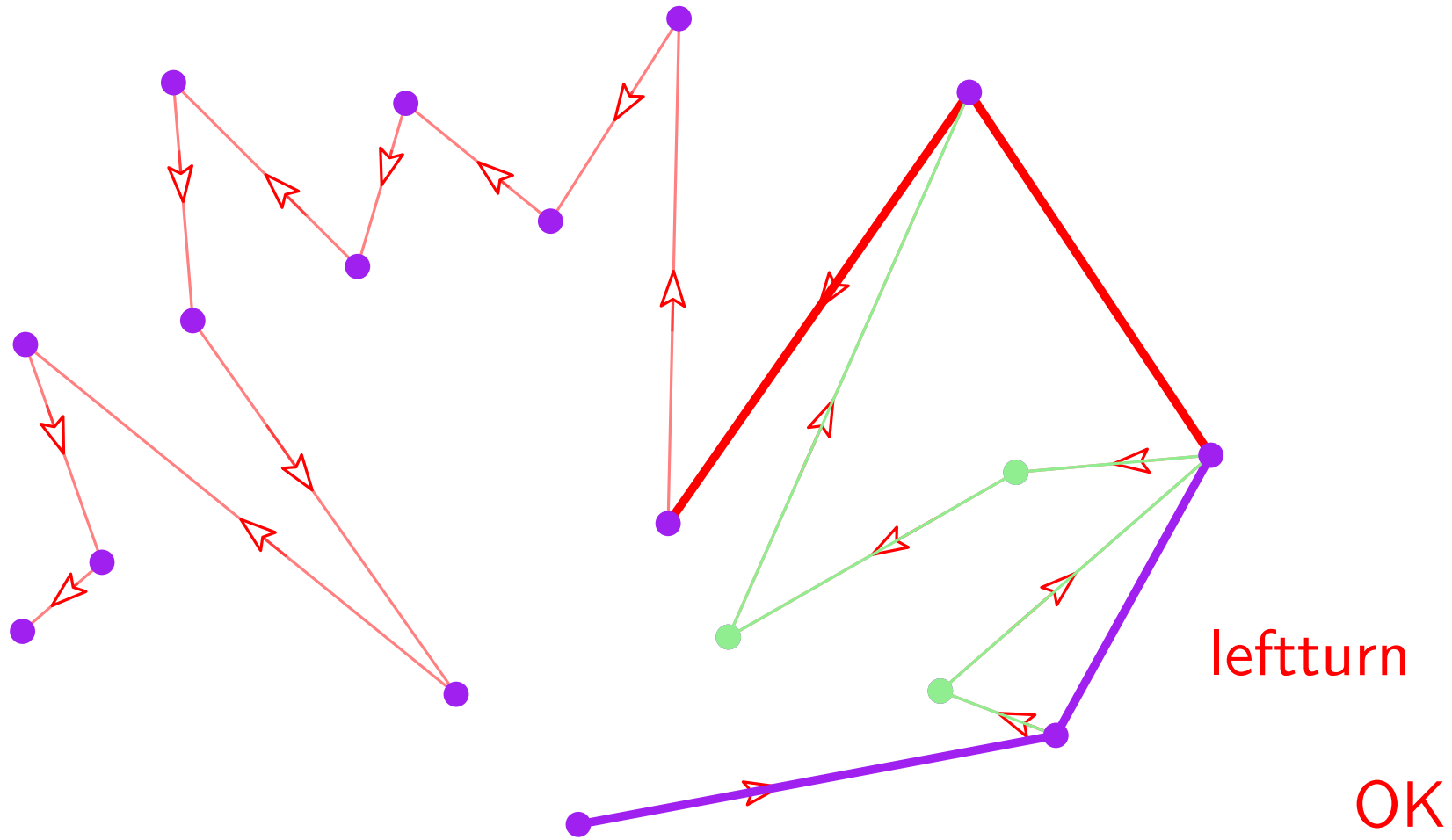
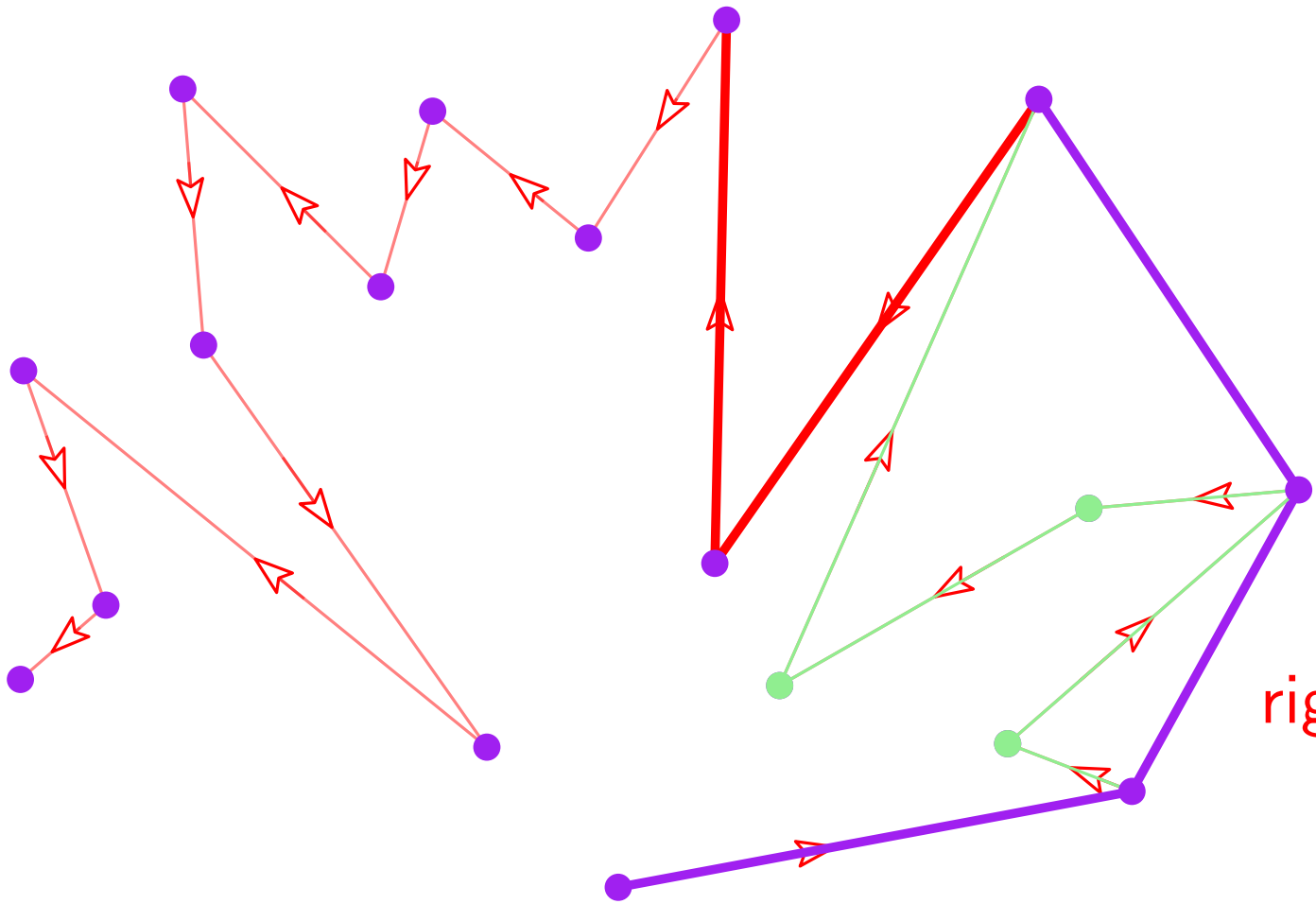


Convex hull

Graham algorithm

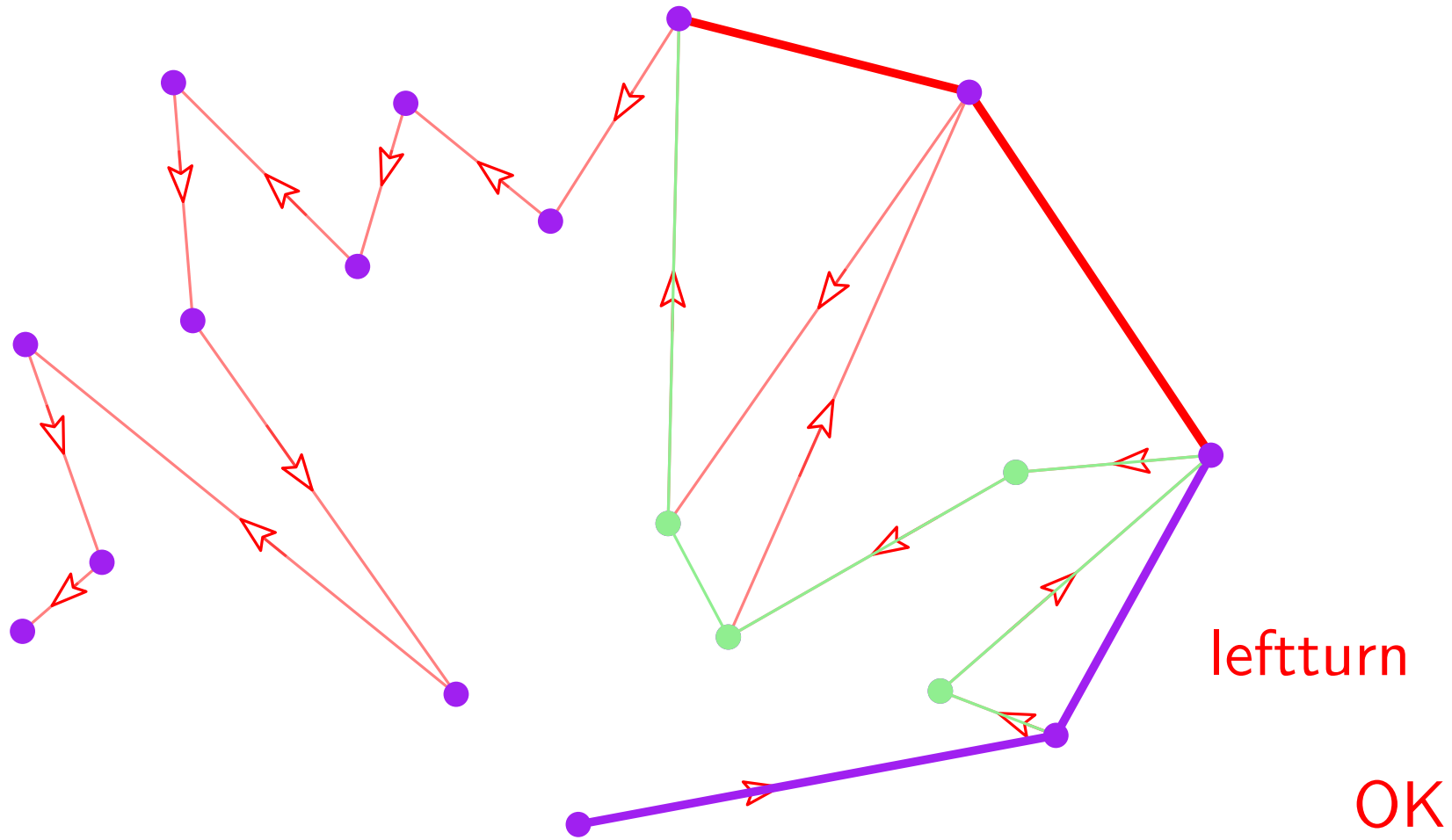


rightturn

remove and go back

Convex hull

Graham algorithm



Convex hull

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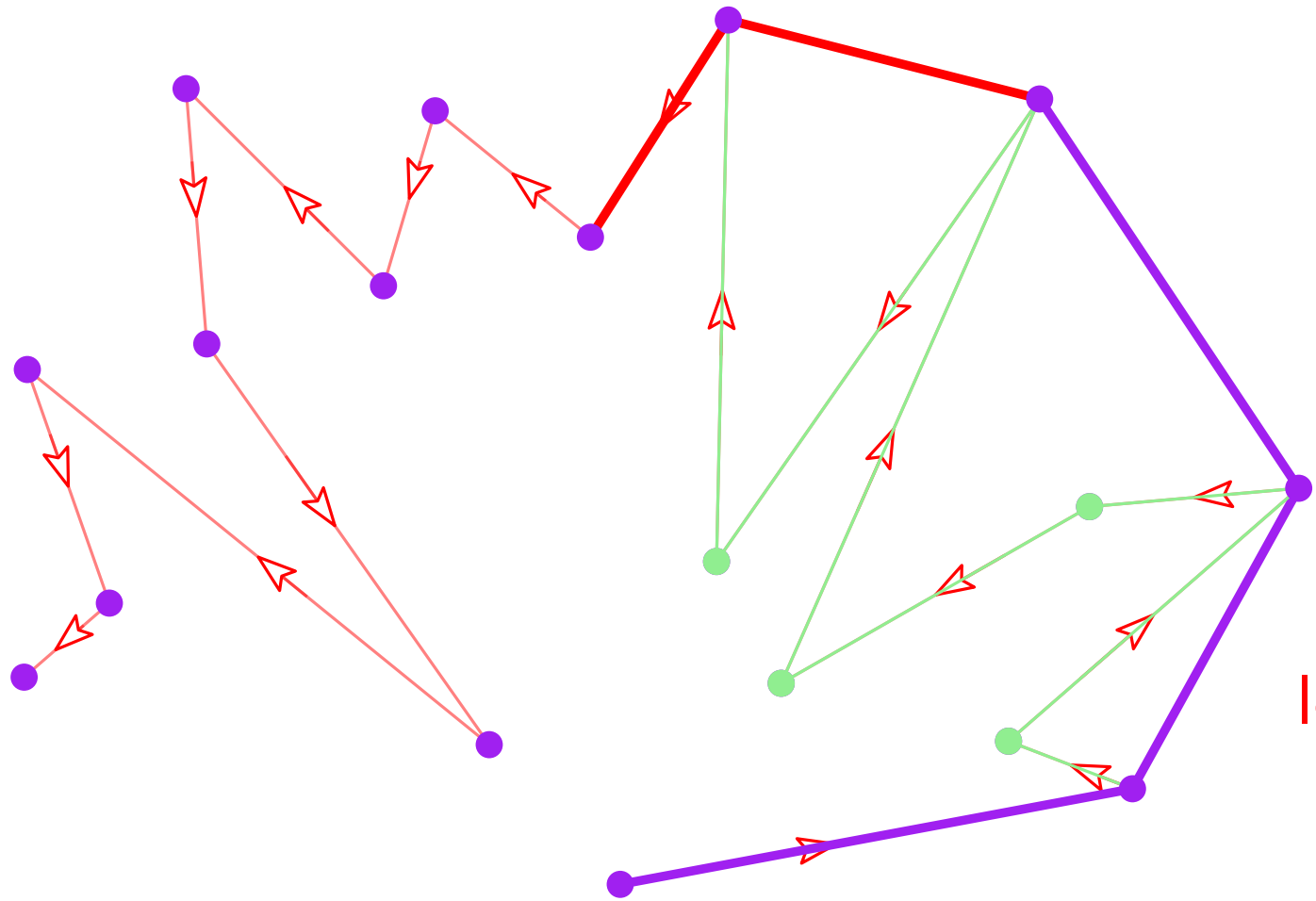
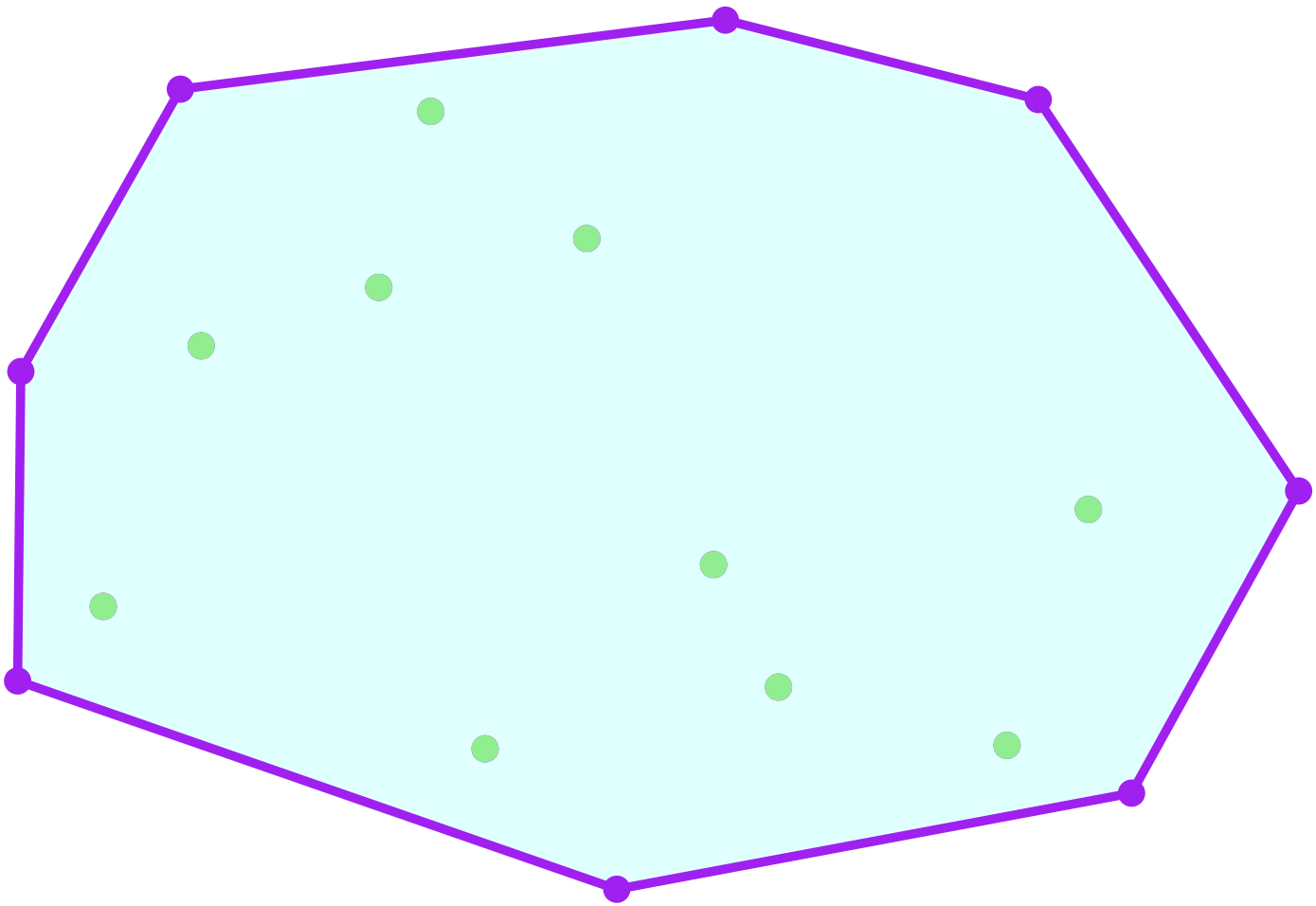


leftturn

OK

Convex hull

Graham algorithm



Convex hull

Complexity

Graham algorithm

Input: point set S
u lowest point of S;
sort S around u in a circular list including u;
$v = u$;

while $v.next \neq u$
    if $(v, v.next, v.next.next)$ ccw
        $v = v.next$;
    else
        $v.next = v.next.next$; $v.next.previous = v$;
        if $v \neq u$ $v = v.previous$;

Convex hull

Complexity

Graham algorithm

Input: point set S

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$O(n)$

sort S around u in a circular list including u;

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Convex hull

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$O(n \log n)$

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delete one point

at most n times

Convex hull

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while $v.next \neq u$
    if $(v, v.next, v.next.next)$ ccw
        $v = v.next$;
    else
        $v.next = v.next.next$; $v.next.previous = v$;
        if $v \neq u$ $v = v.previous$;

distance to u decreases at most n times

delete one point at most n times

Convex hull

Graham algorithm

Complexity

Input: point set S
u lowest point of S;
sort S around u in a circular list including u;
$v = u$;

$O(n)$

while $v.next \neq u$
    if $(v, v.next, v.next.next)$ ccw
        $v = v.next$;
    else
        $v.next = v.next.next$; $v.next.previous = v$;
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distance to u decreases at most n times

delete one point at most n times

Convex hull

Complexity

Input: point set S
u lowest point of S;
sort S around u in a circular list including u;
$v = u$;

$O(n \log n)$

while $v.next \neq u$
    if $(v, v.next, v.next.next)$ ccw
        $v = v.next$;
    else
        $v.next = v.next.next$; $v.next.previous = v$;
        if $v \neq u$ $v = v.previous$;

Convex hull

Problem lower bound is $\Omega(f(n))$

Iff there is NO algorithm

solving all size n problems

using less than $Cf(n)$ operations

$\forall n$

C constant independent of n

Sorting

Input: n real (positive) numbers

Sorting

Input: n real (positive) numbers

Output: sorting permutation

Sorting

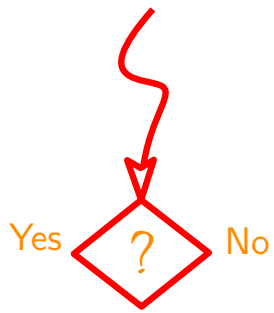
Input: n real (positive) numbers

Output: sorting permutation

Monitoring execution

Sorting

Input: n real (positive) numbers

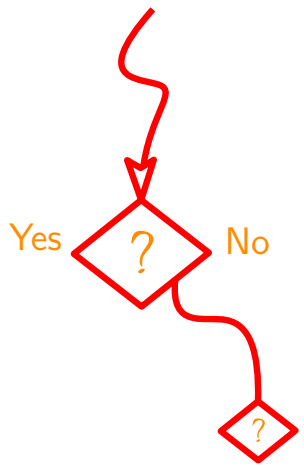


Output: sorting permutation

Monitoring execution

Sorting

Input: n real (positive) numbers

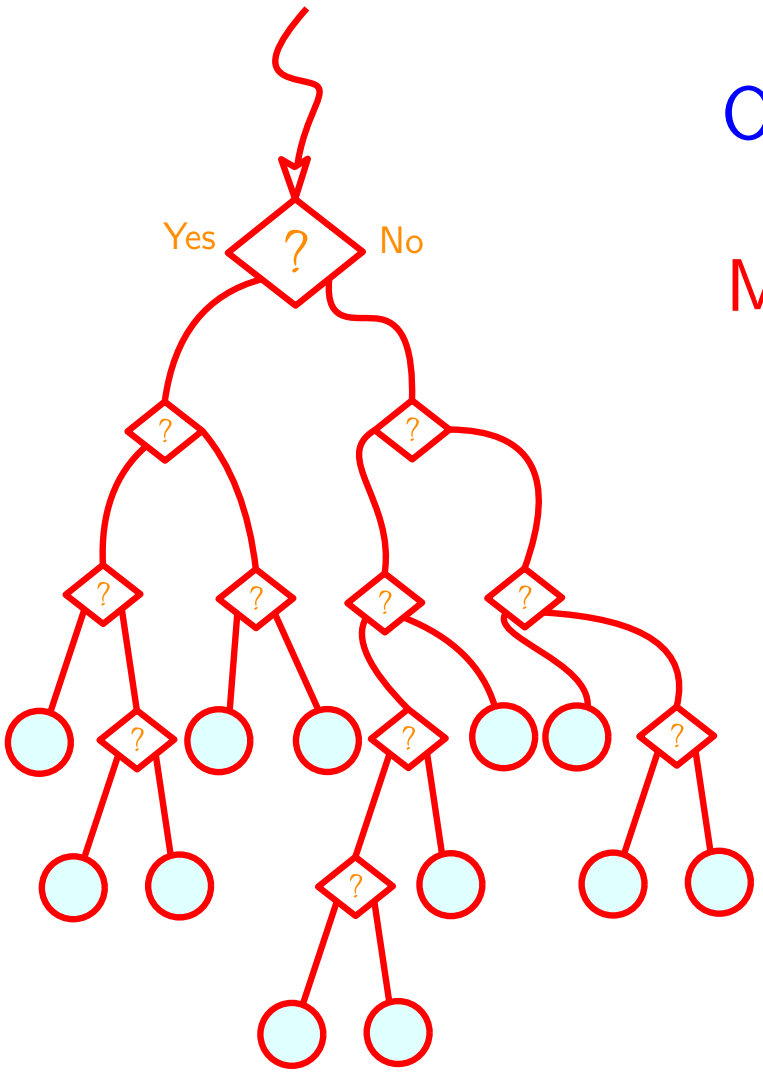


Output: sorting permutation

Monitoring execution

Sorting

Input: n real (positive) numbers

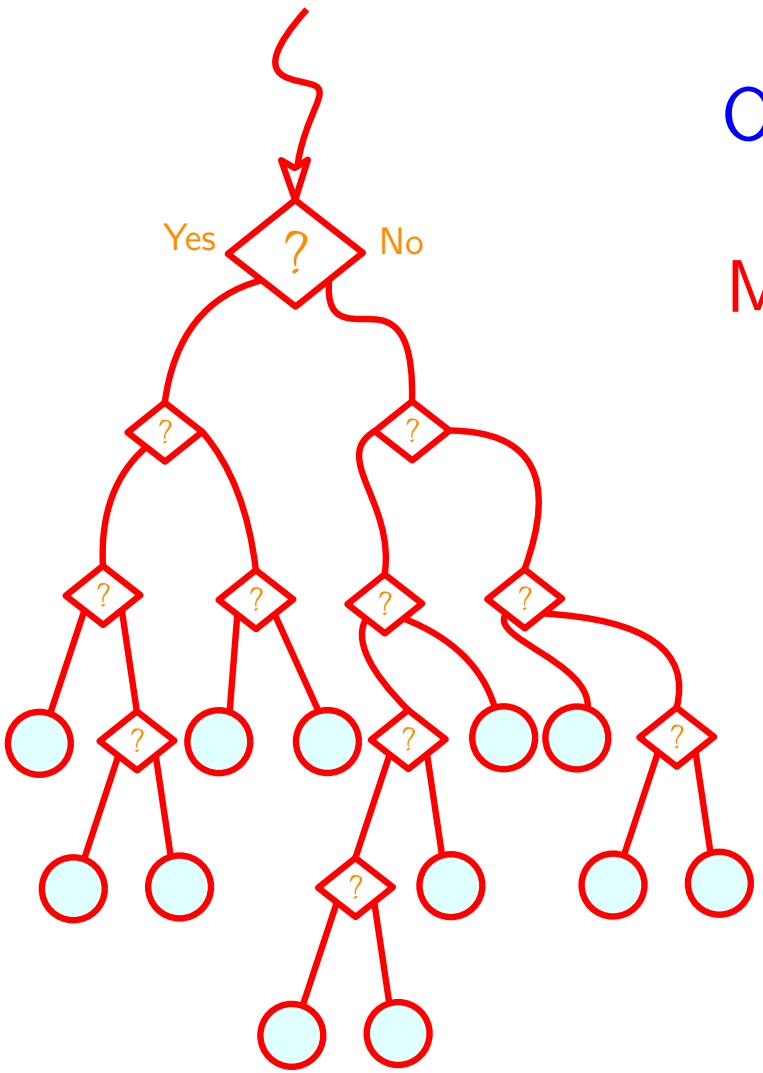


Output: sorting permutation

Monitoring execution

Sorting

Input: n real (positive) numbers



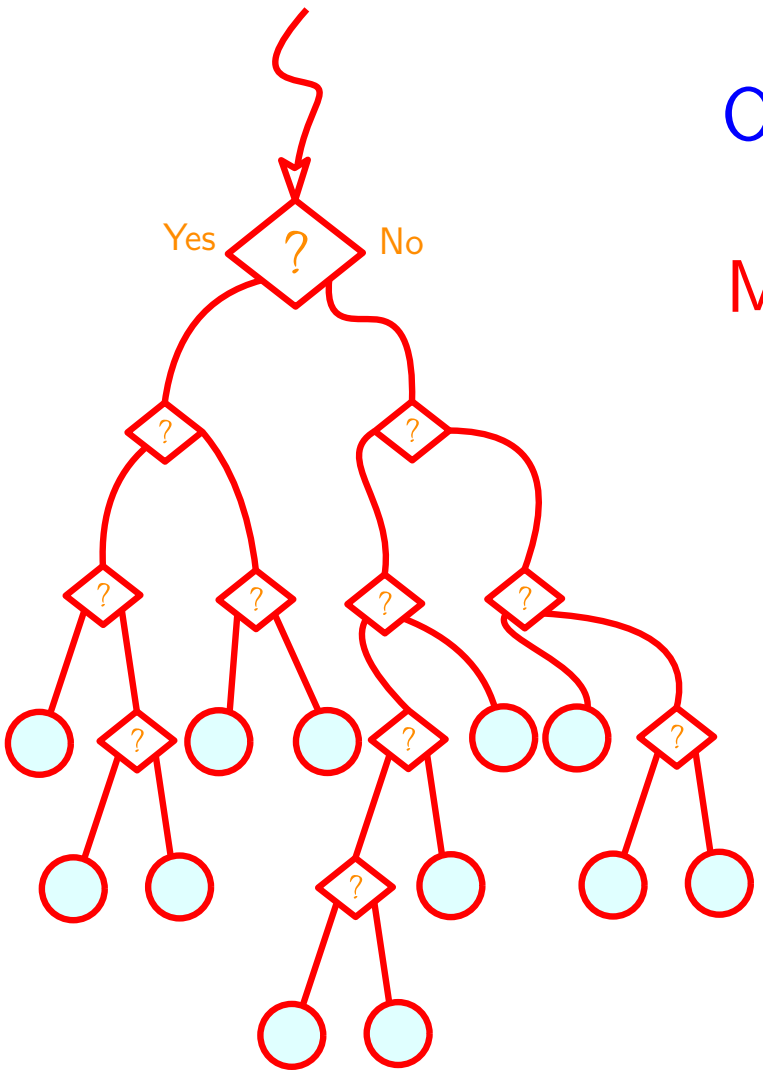
Output: sorting permutation

Monitoring execution

$\sharp$ leaves $\geq$ $\sharp$ permutations

Sorting

Input: n real (positive) numbers



Output: sorting permutation

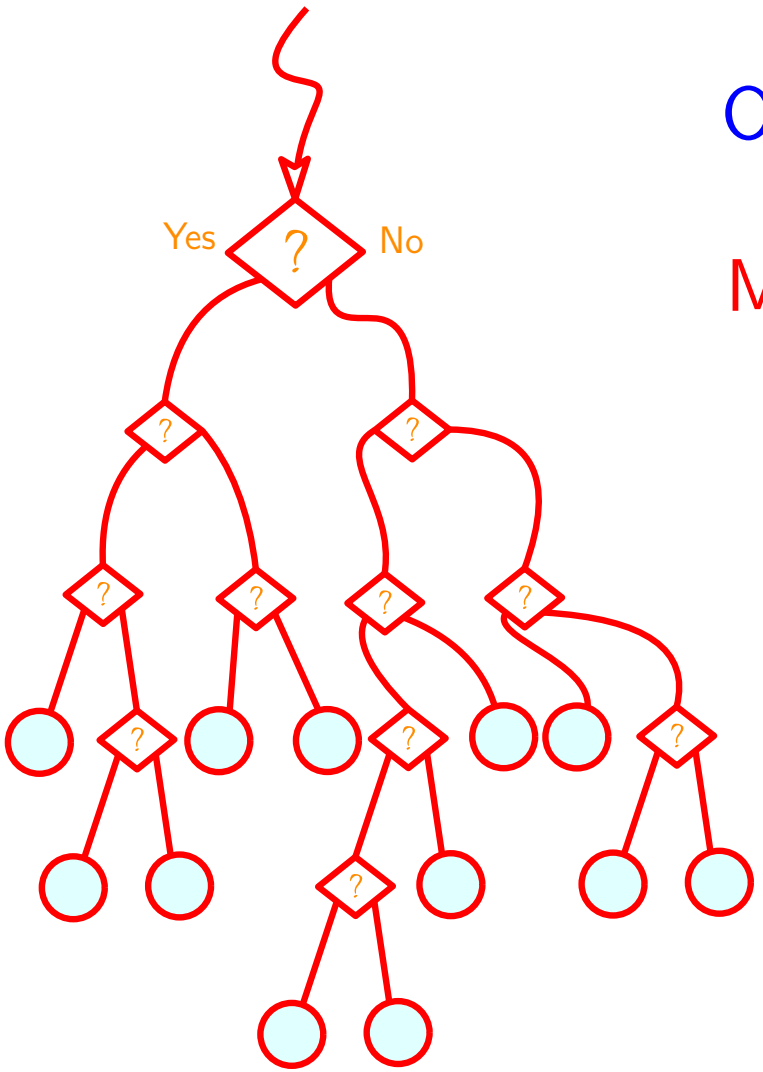
Monitoring execution

$\sharp$ leaves $\geq$ $\sharp$ permutations

There are $n!$ permutations

Sorting

Input: n real (positive) numbers



Output: sorting permutation

Monitoring execution

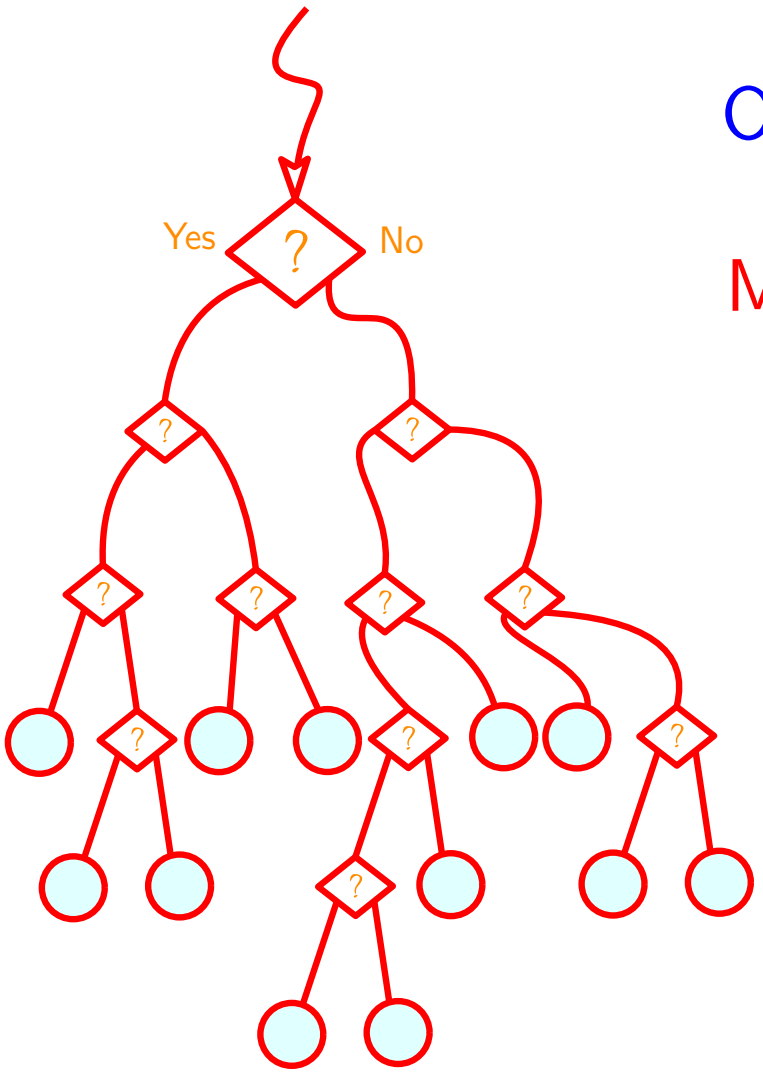
$\sharp$ leaves $\geq$ $\sharp$ permutations

There are $n!$ permutations

Tree height is at least $\log_2 \sharp$ leaves

Sorting

Input: n real (positive) numbers



Output: sorting permutation

Monitoring execution

$\sharp$ leaves $\geq$ $\sharp$ permutations

There are $n!$ permutations

Tree height is at least $\log_2 \sharp$ leaves

$\sharp$ comparisons $\leq \log_2 n! \simeq n \log_2 n$

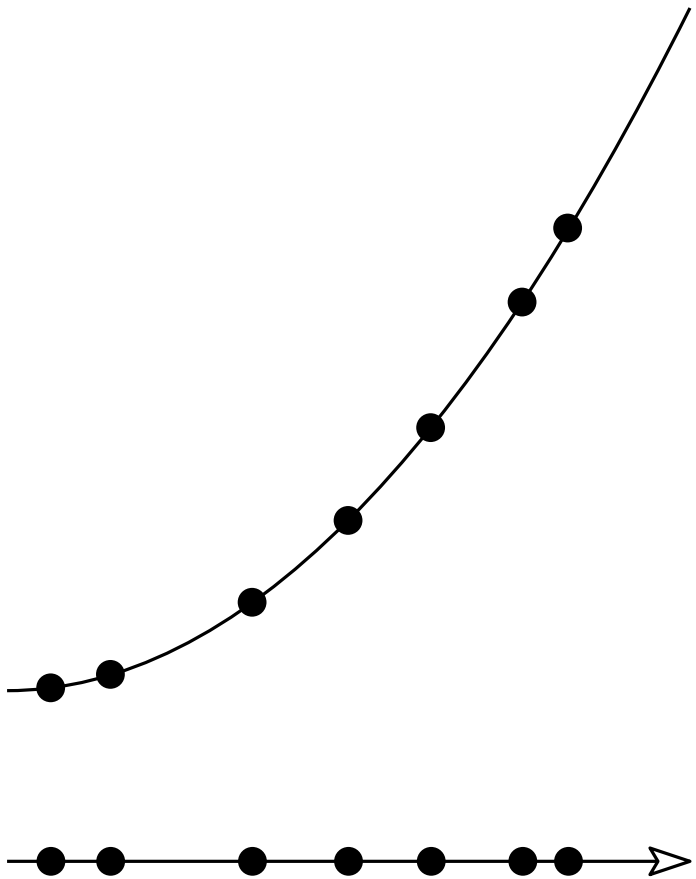
Convex hull

A stupid algorithm for sorting numbers

Convex hull

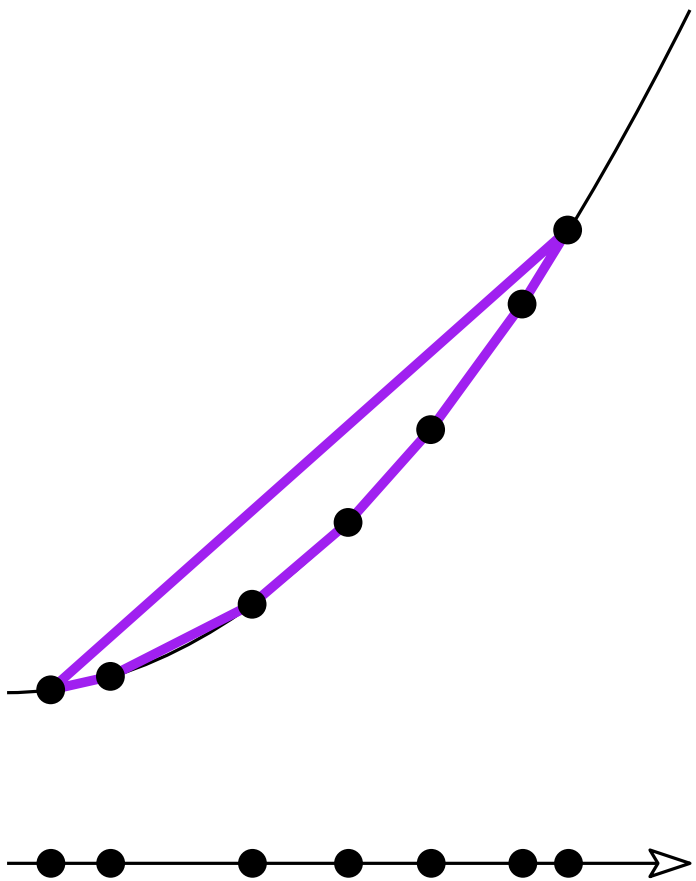
A stupid algorithm for sorting numbers

project on parabola



Convex hull

A stupid algorithm for sorting numbers

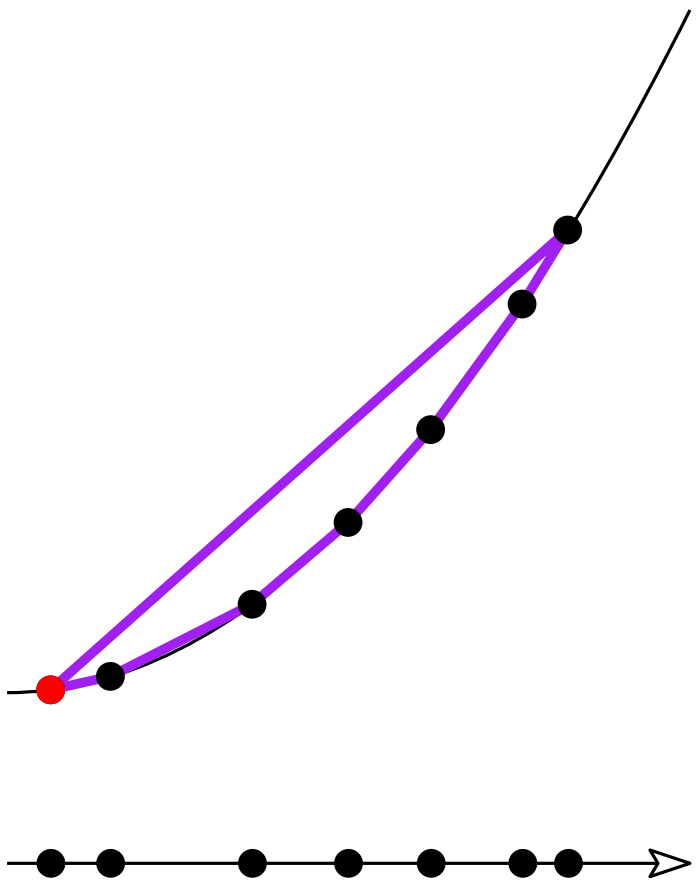


project on parabola

compute convex hull

Convex hull

A stupid algorithm for sorting numbers



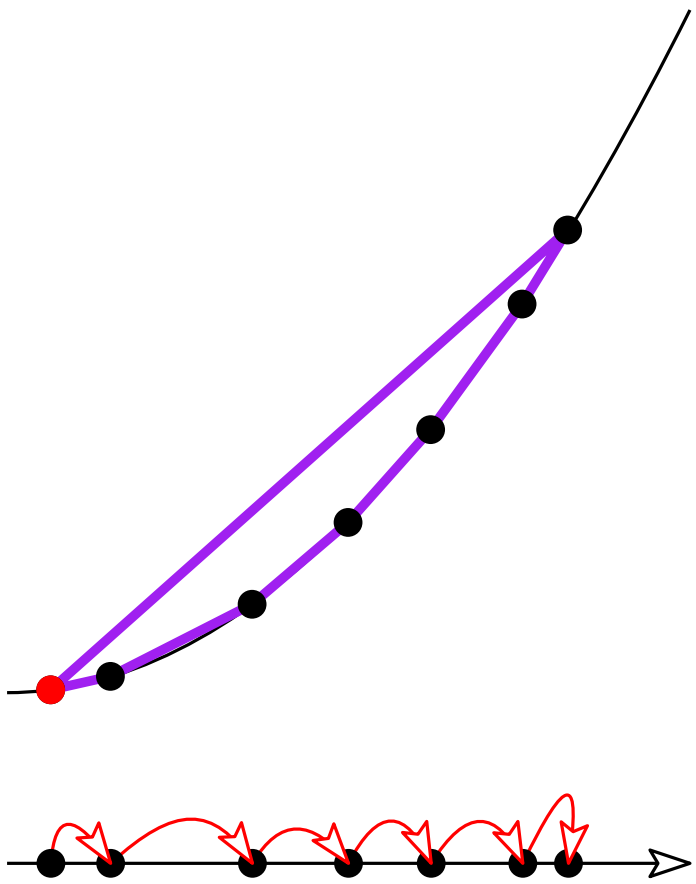
project on parabola

compute convex hull

find lowest point

Convex hull

A stupid algorithm for sorting numbers



project on parabola

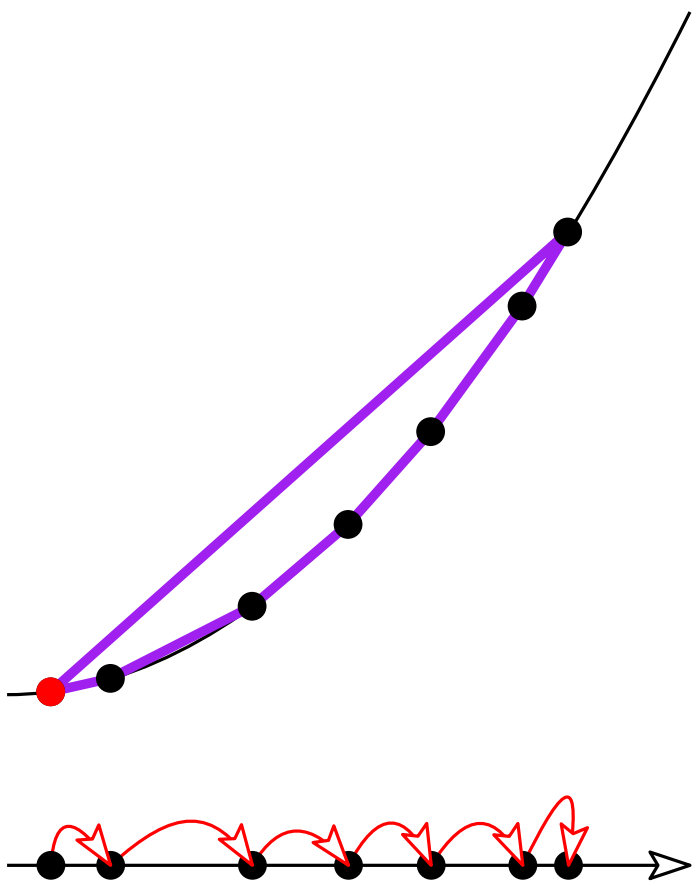
compute convex hull

find lowest point

enumerate x coordinates in ccw CH order

Convex hull

A stupid algorithm for sorting numbers



$O(n)$ project on parabola

$f(n)$ compute convex hull

$O(n)$ find lowest point

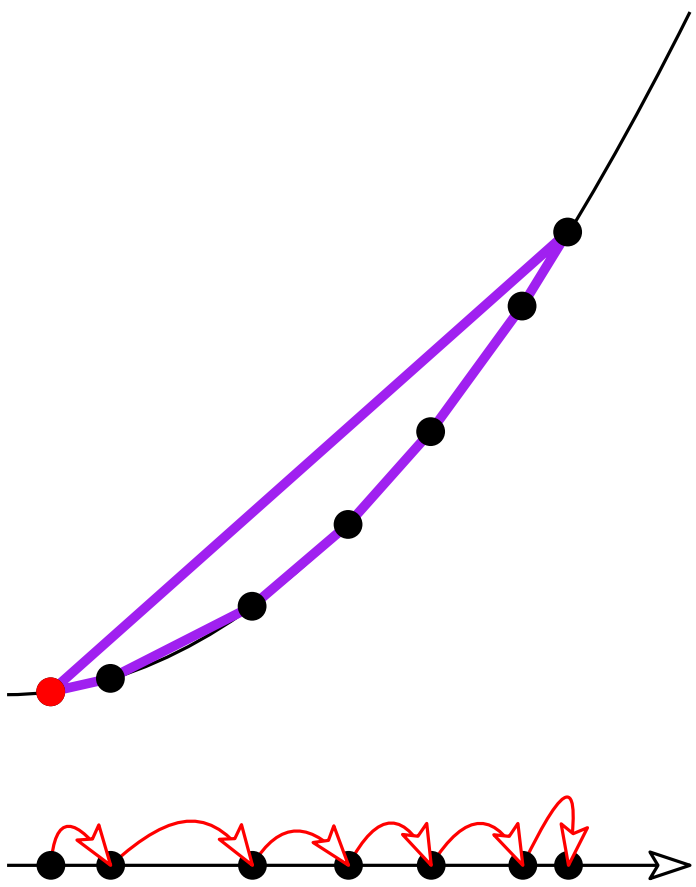
$O(n)$ enumerate x coordinates in ccw CH order

Lower bound on sorting

$$\Longrightarrow f(n) + O(n) \geq \Omega(n \log n)$$

Convex hull

A stupid algorithm for sorting numbers



$O(n)$ project on parabola

$f(n)$ compute convex hull

$O(n)$ find lowest point

$O(n)$ enumerate x coordinates in ccw CH order

Lower bound on sorting

$$\Longrightarrow f(n) + O(n) \geq \Omega(n \log n)$$

Convex hull

Expected size of the convex hull

Convex hull

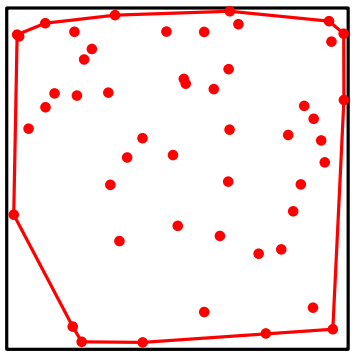
Other results

Expected size of the convex hull

expected

n random points in a square

$\Theta(\log n)$



Convex hull

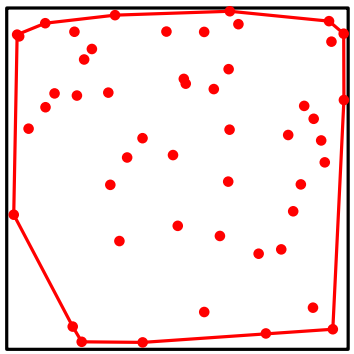
Other results

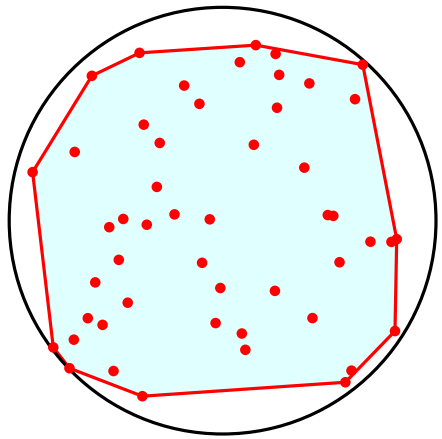
Expected size of the convex hull

expected

n random points in a square $\Theta(\log n)$

n random points in a disk $\Theta(n^{\frac{1}{3}})$





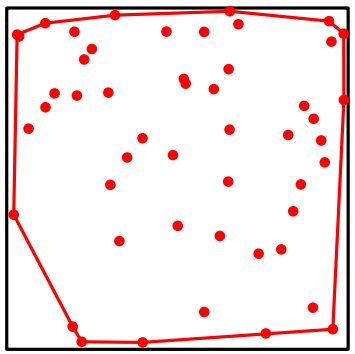
Convex hull

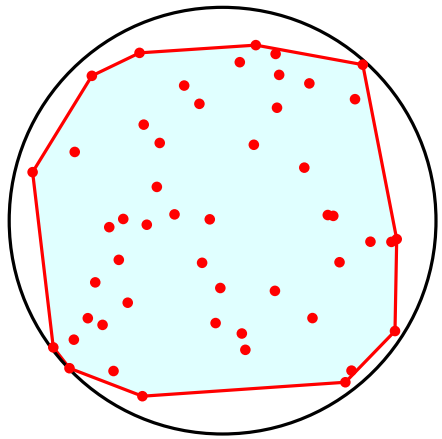
Other results

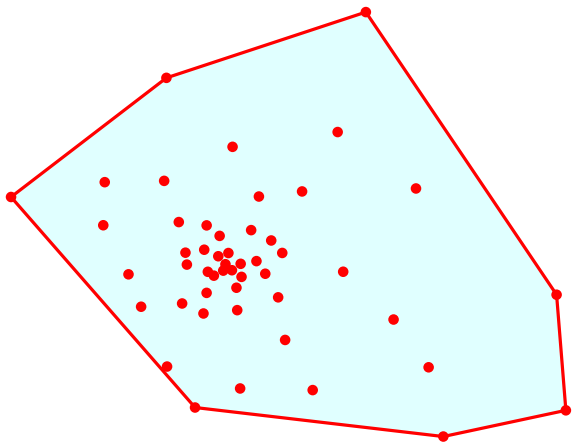
Expected size of the convex hull

expected

n random points in a square	$\Theta(\log n)$
n random points in a disk	$\Theta(n^{\frac{1}{3}})$
n random points, Gaussian distribution	$\Theta(\sqrt{\log n})$

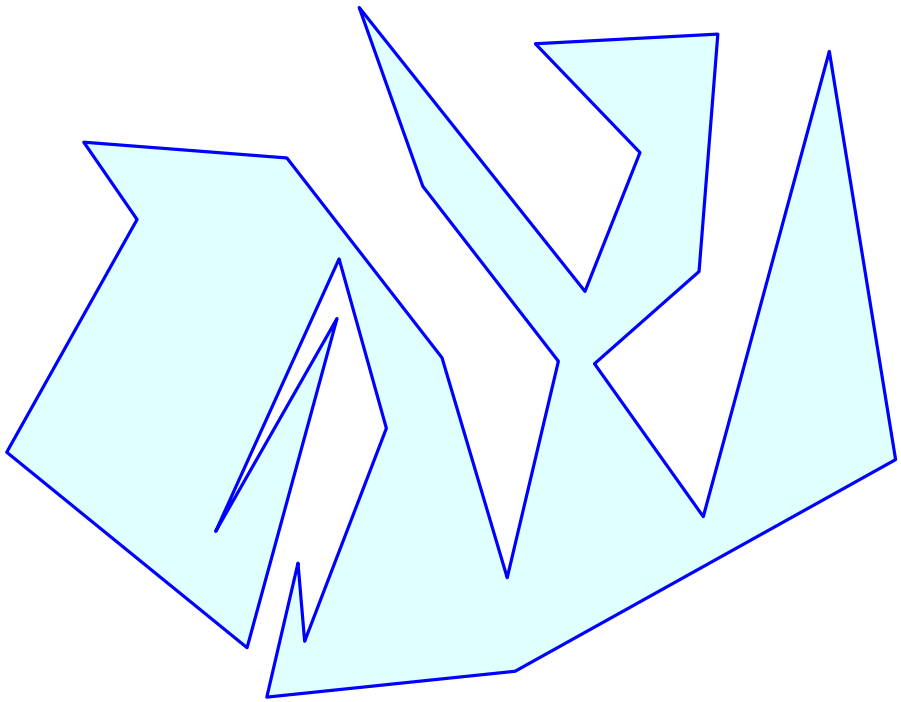






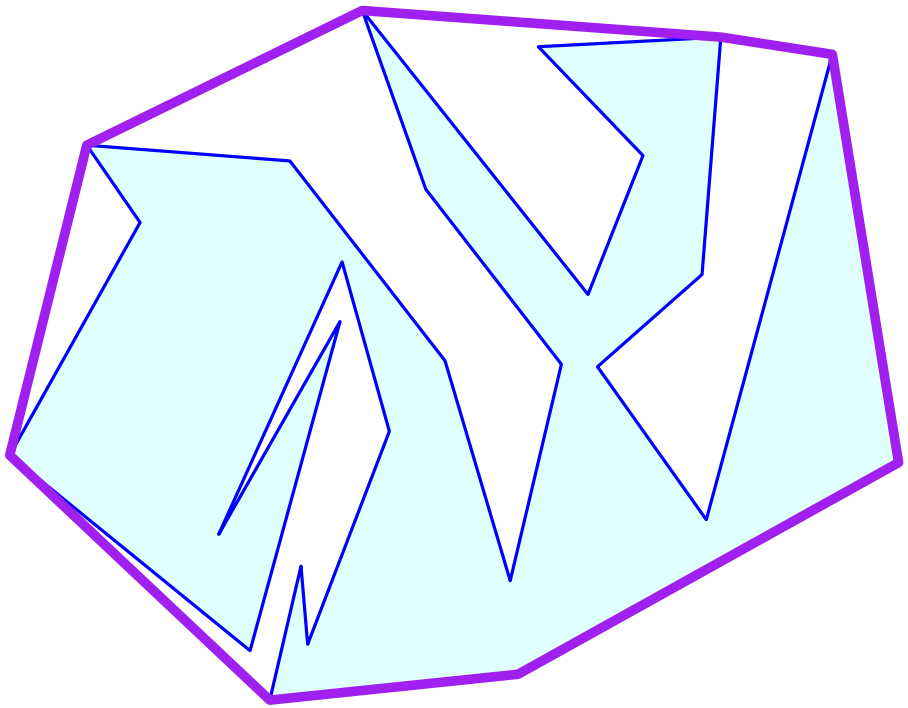
Convex hull

Simple polygon



Convex hull

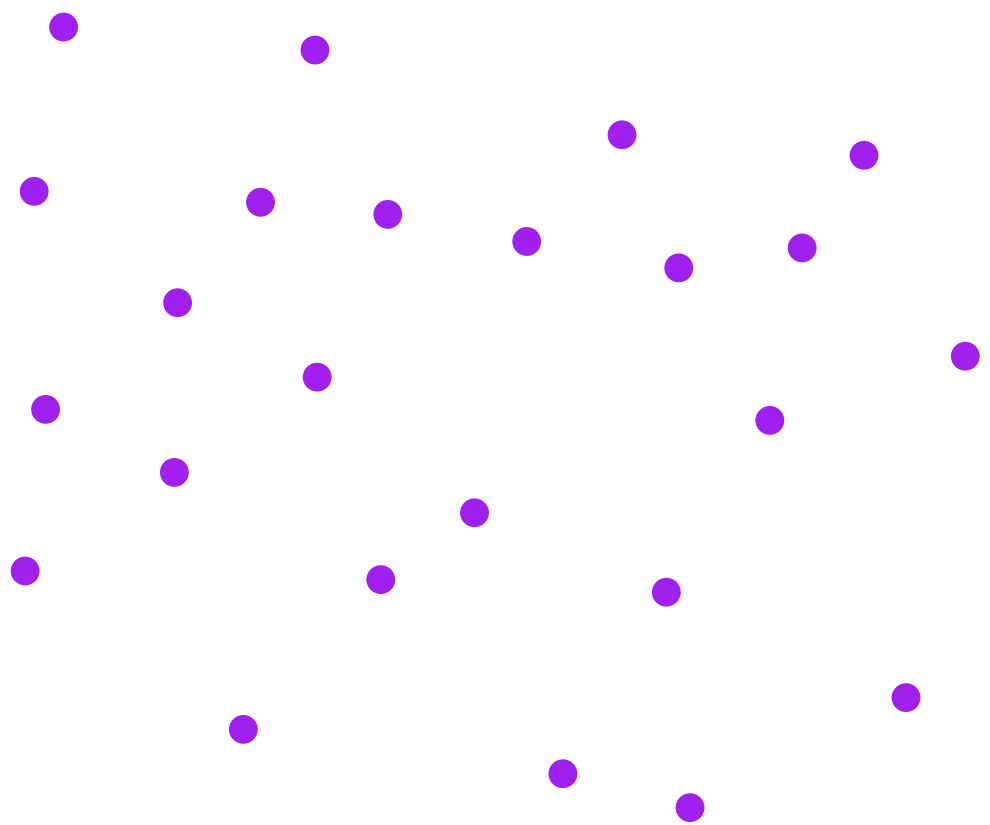
Simple polygon



$O(n)$

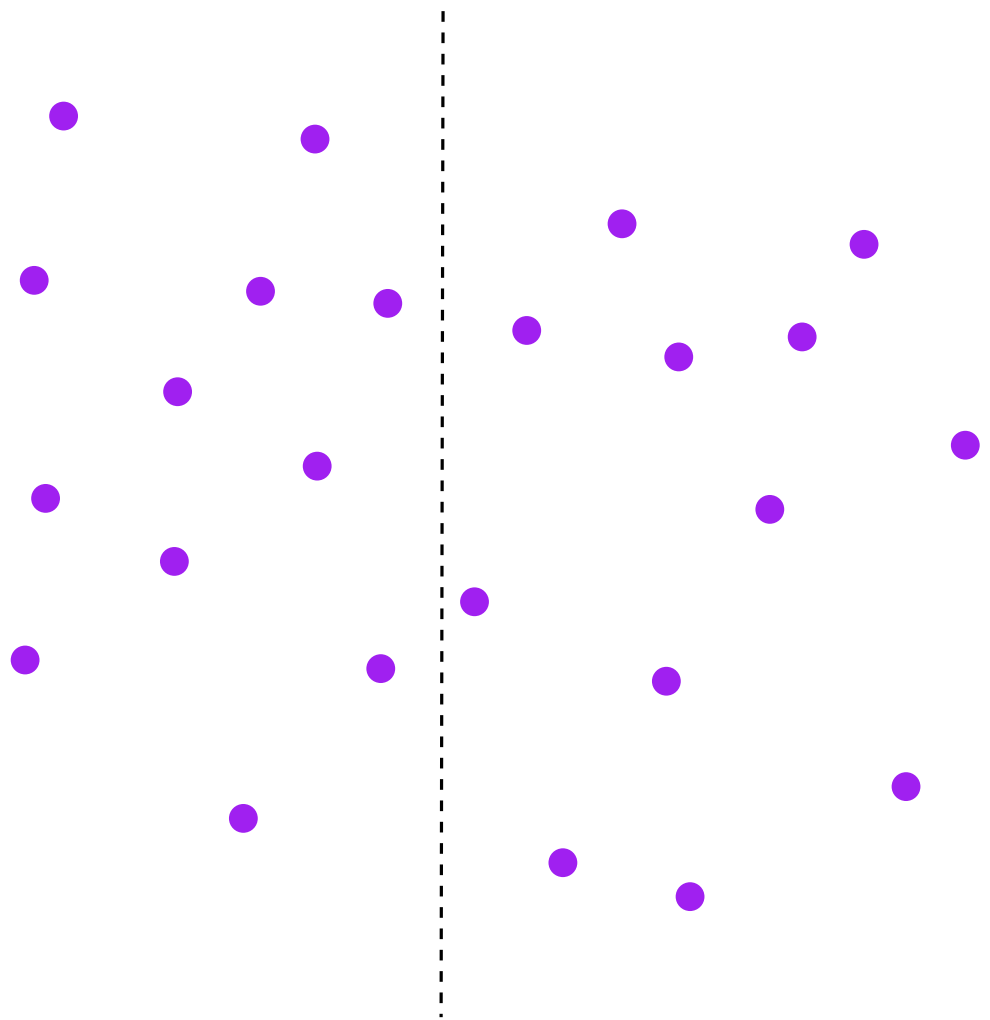
Convex hull

Divide and conquer



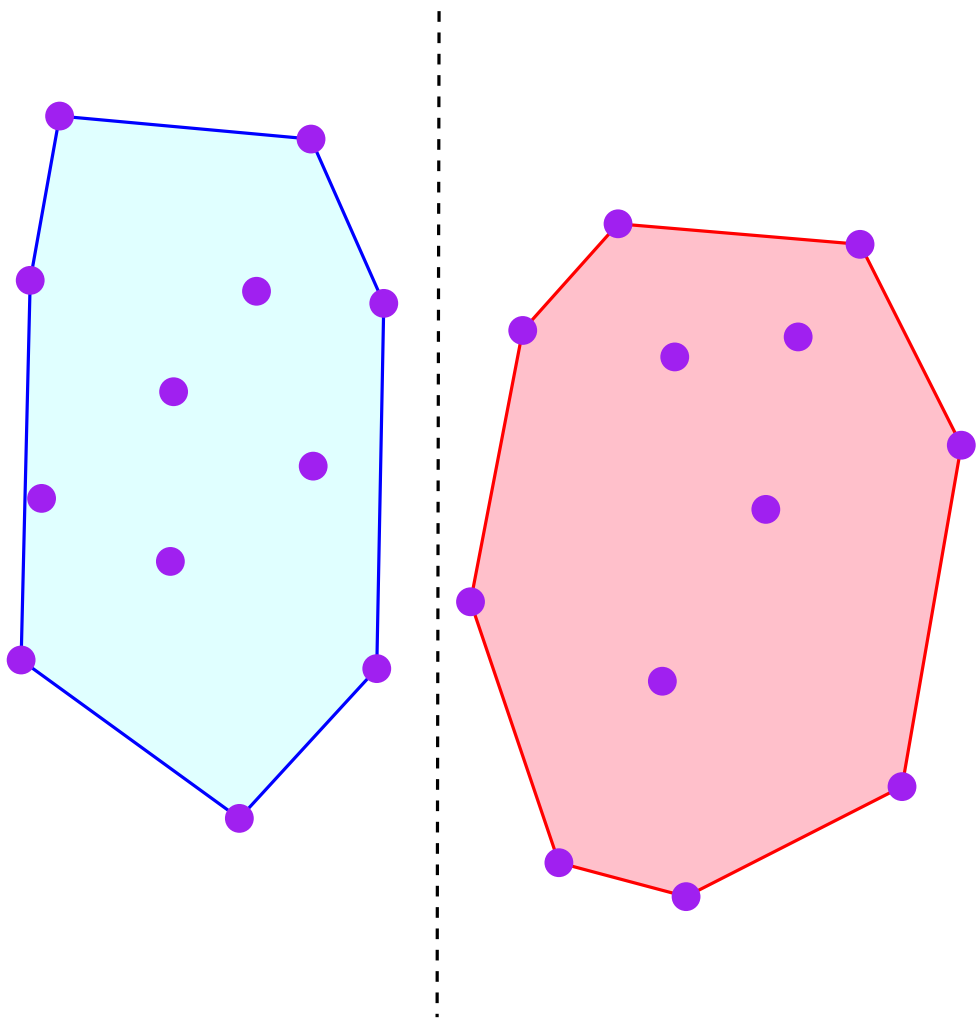
Convex hull

Divide and conquer



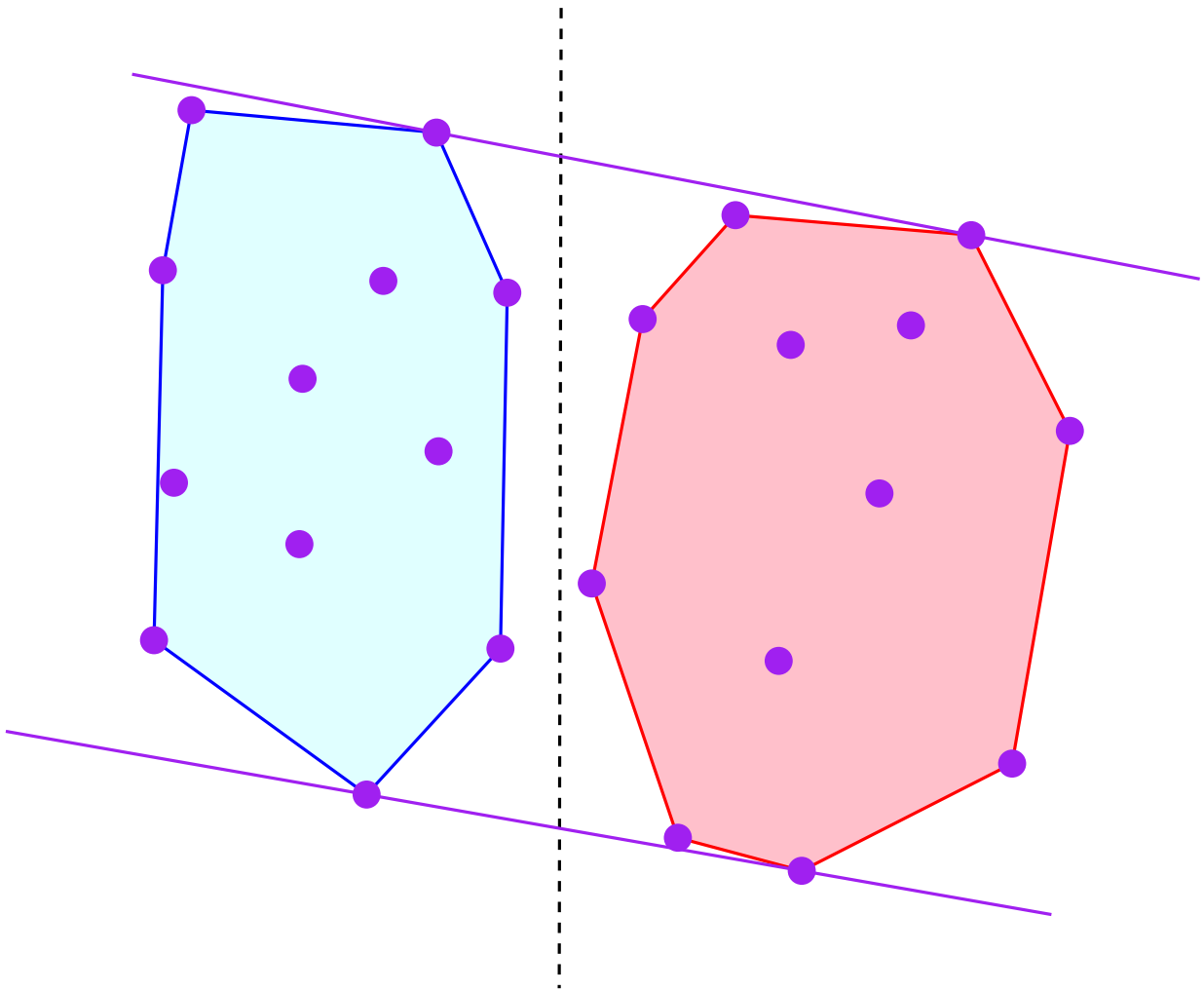
Convex hull

Divide and conquer



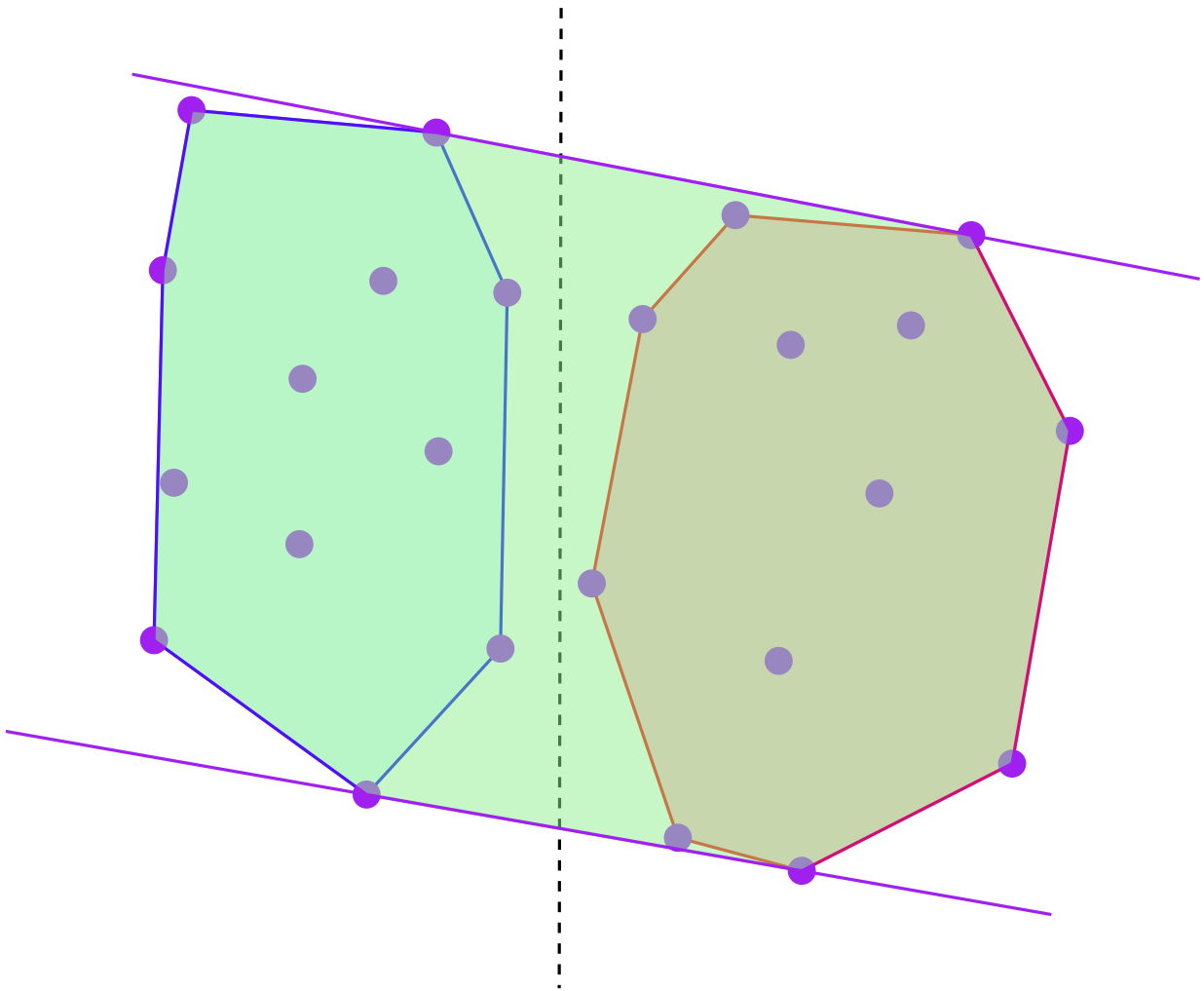
Convex hull

Divide and conquer



Convex hull

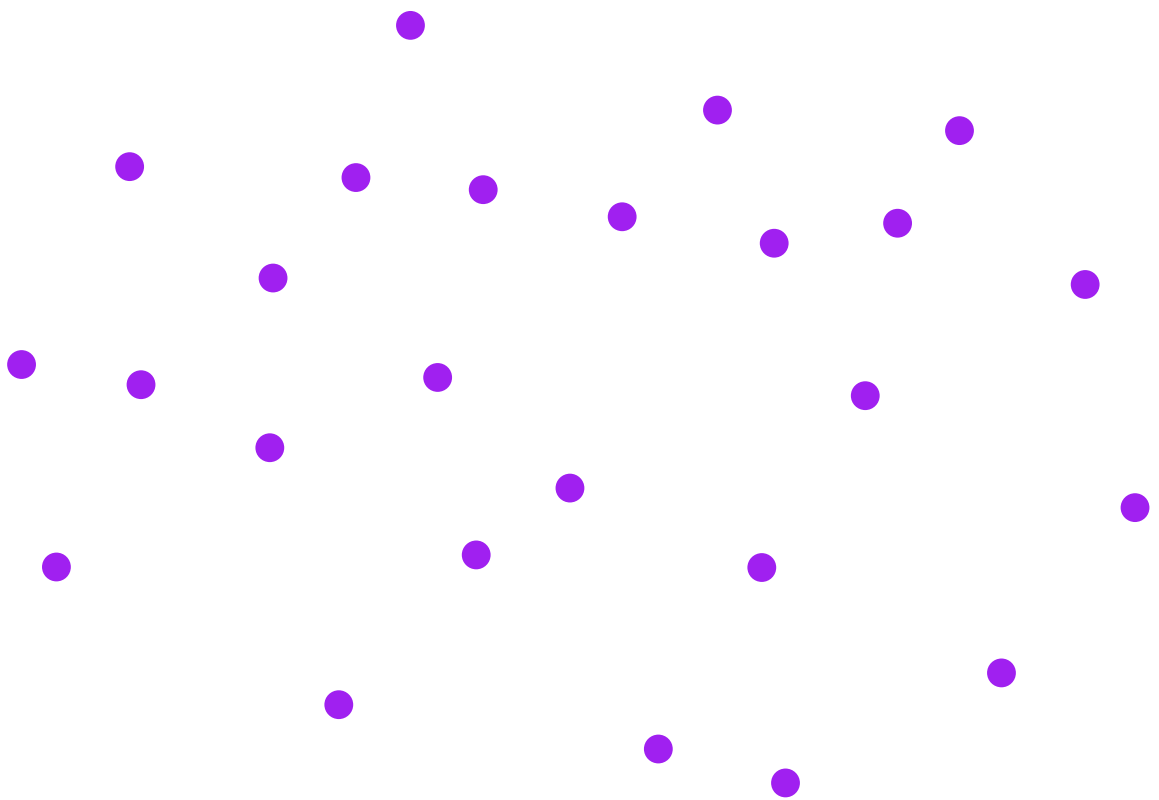
Divide and conquer



$O(n \log n)$

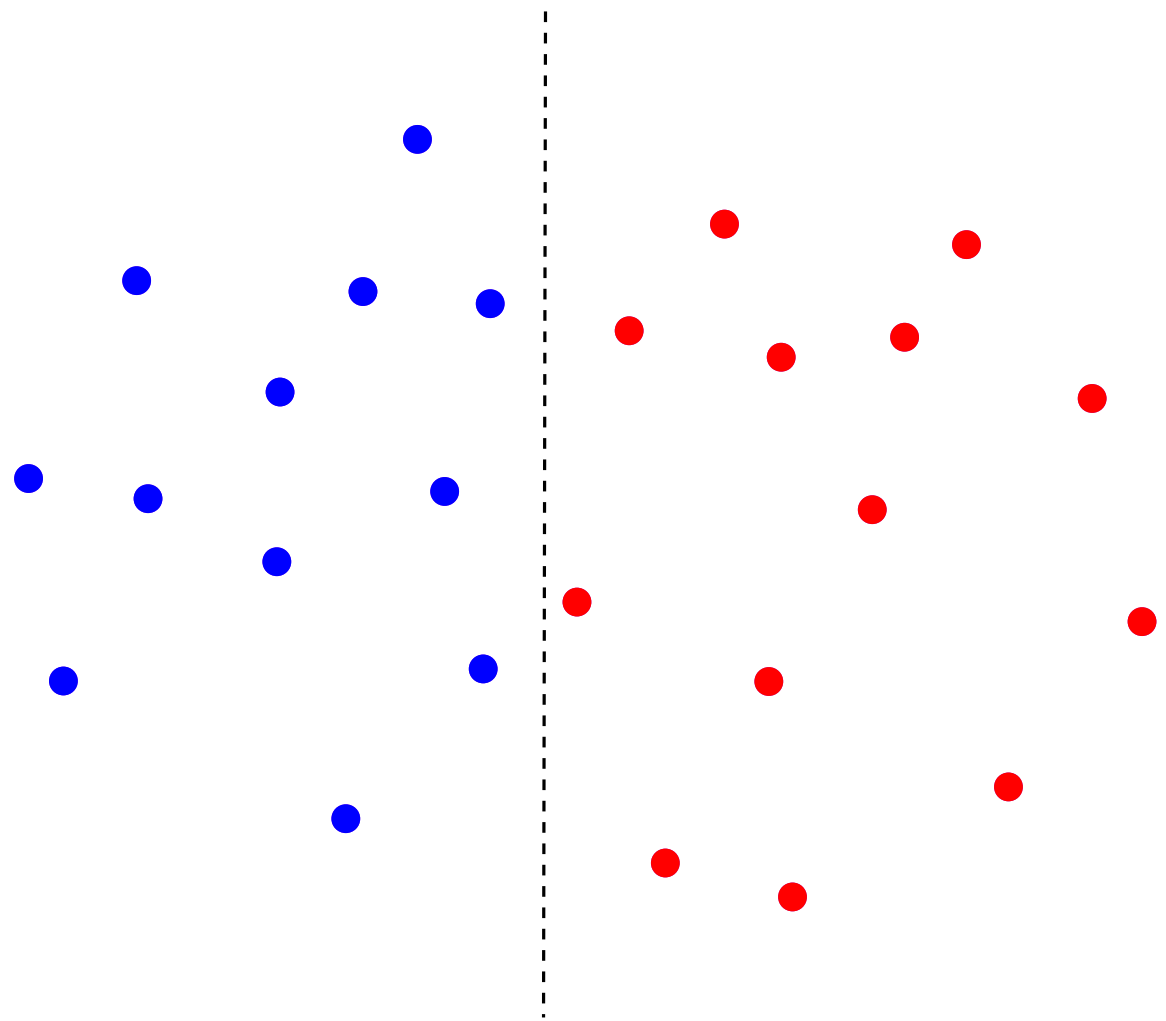
Convex hull

Mariage before conquest



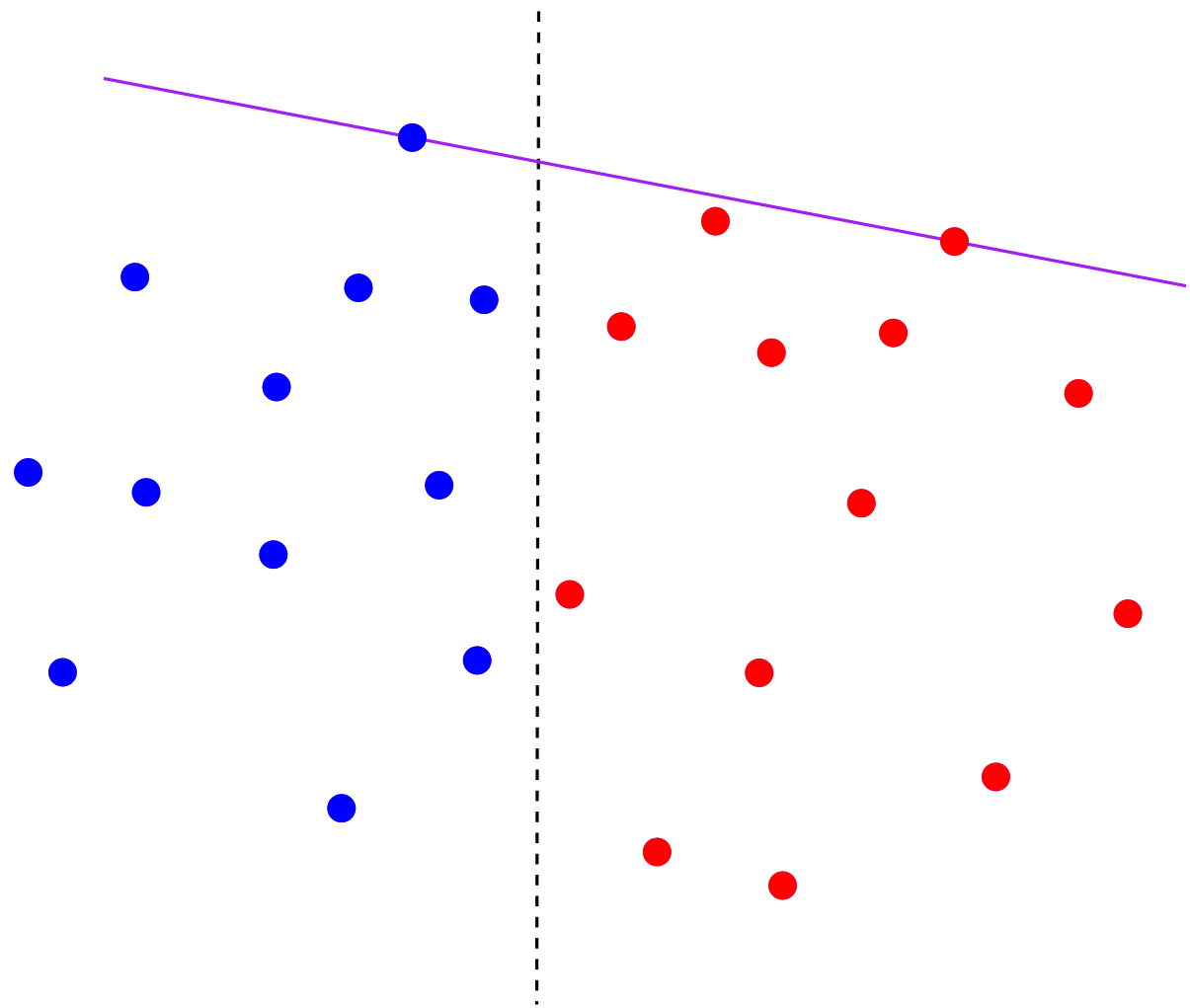
Convex hull

Mariage before conquest



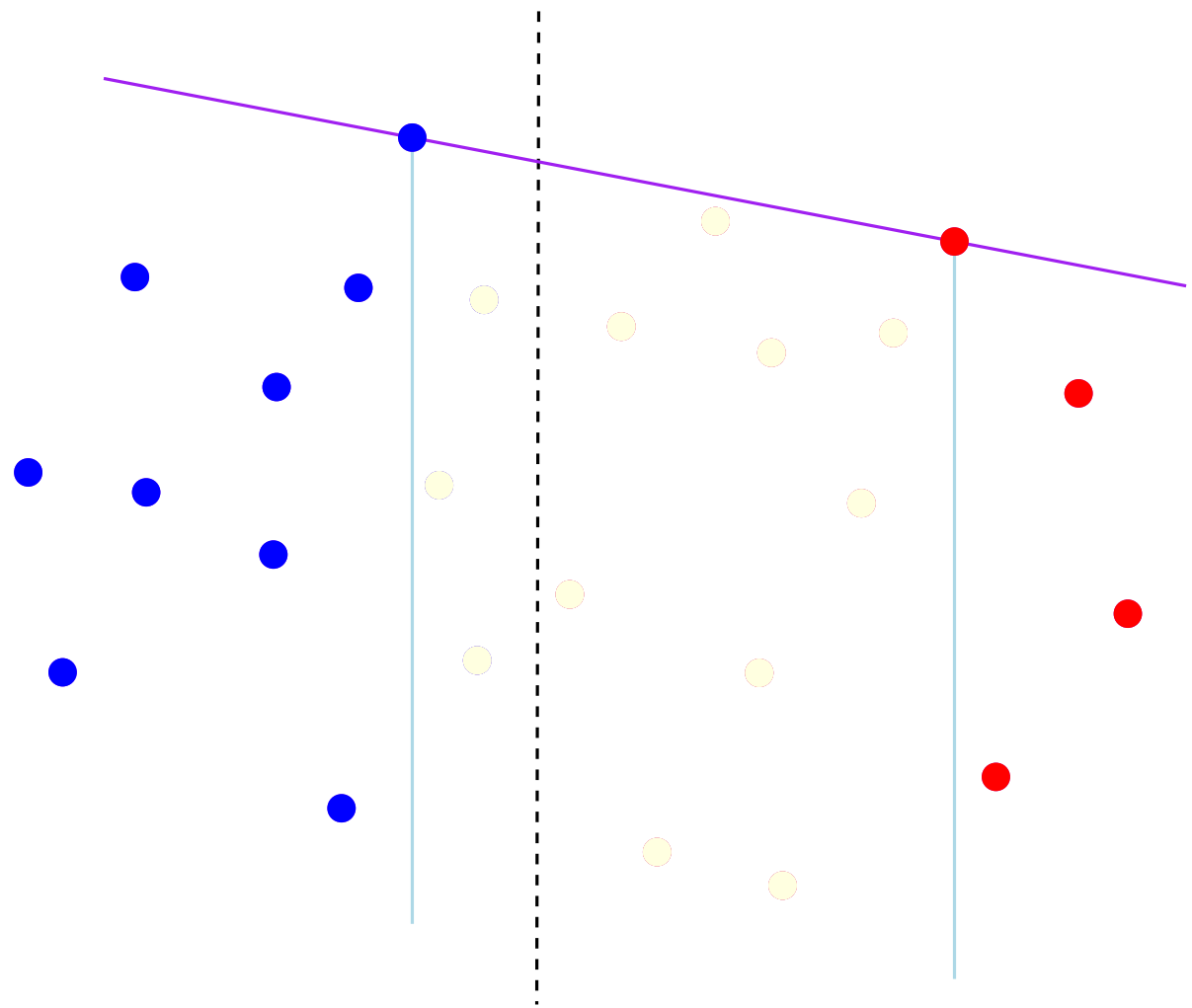
Convex hull

Mariage before conquest



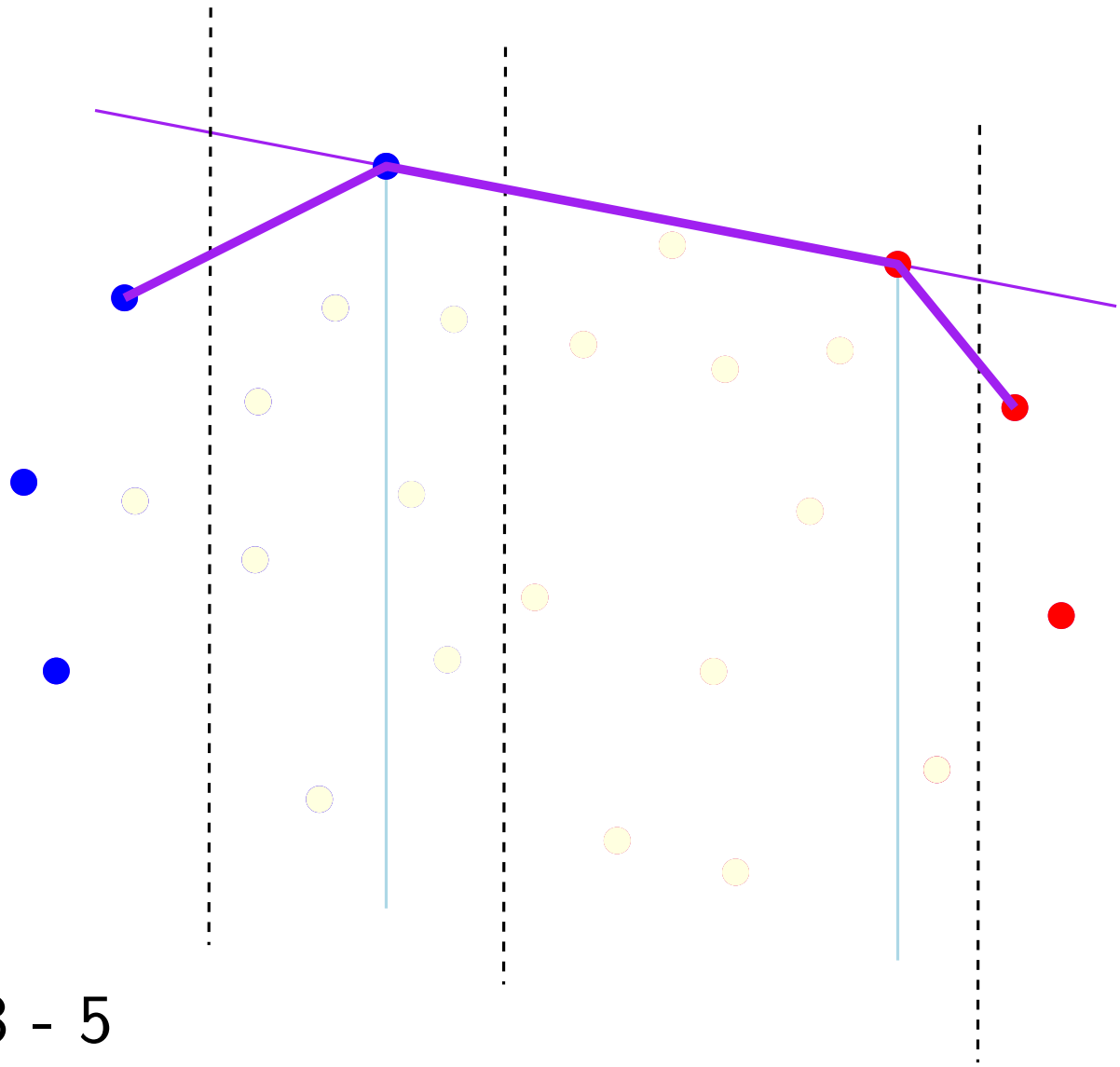
Convex hull

Mariage before conquest



Convex hull

Mariage before conquest

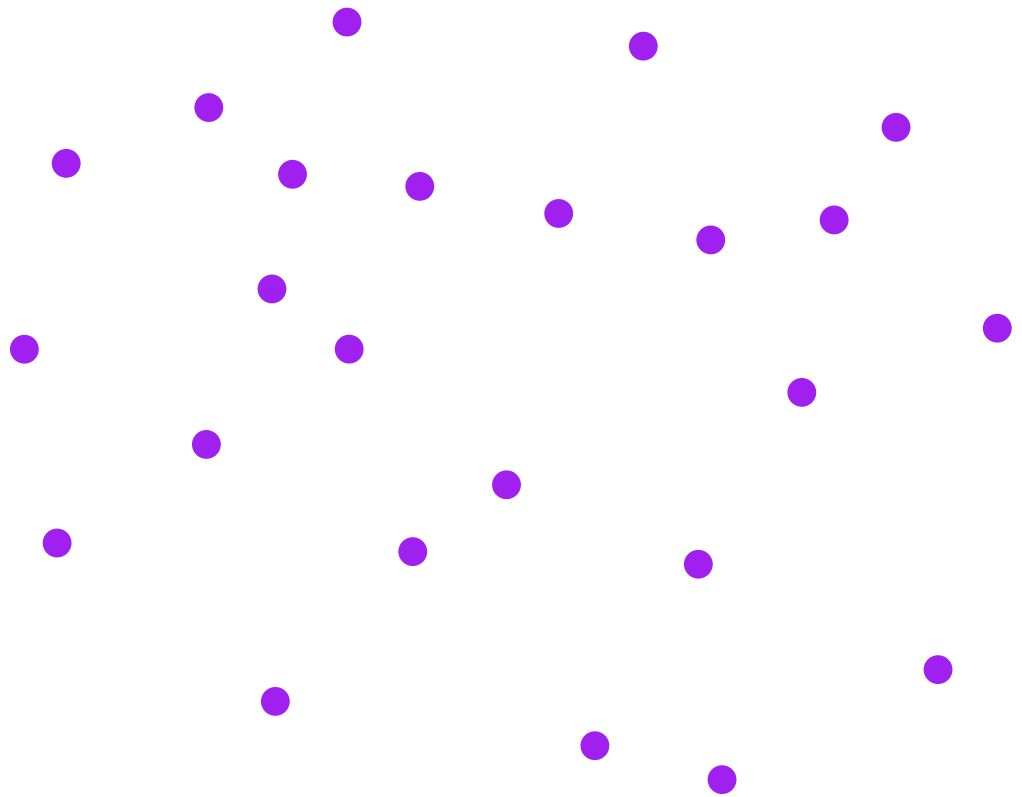


recursion

$O(n \log h)$

Convex hull

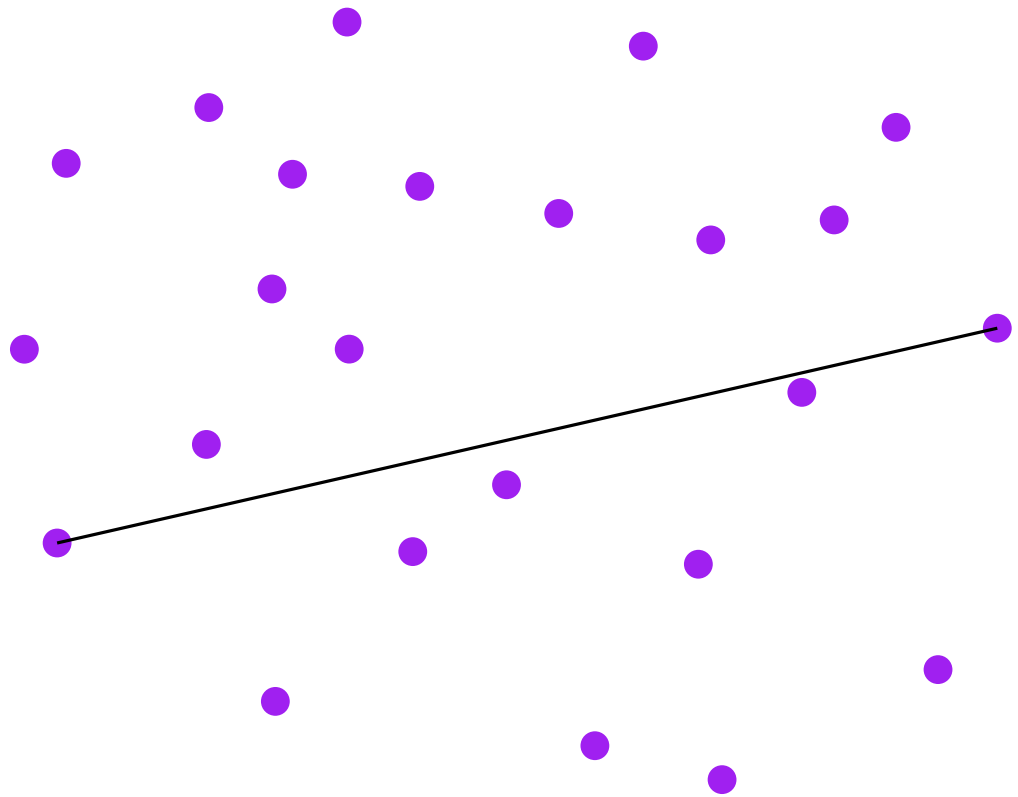
Quickhull



Convex hull

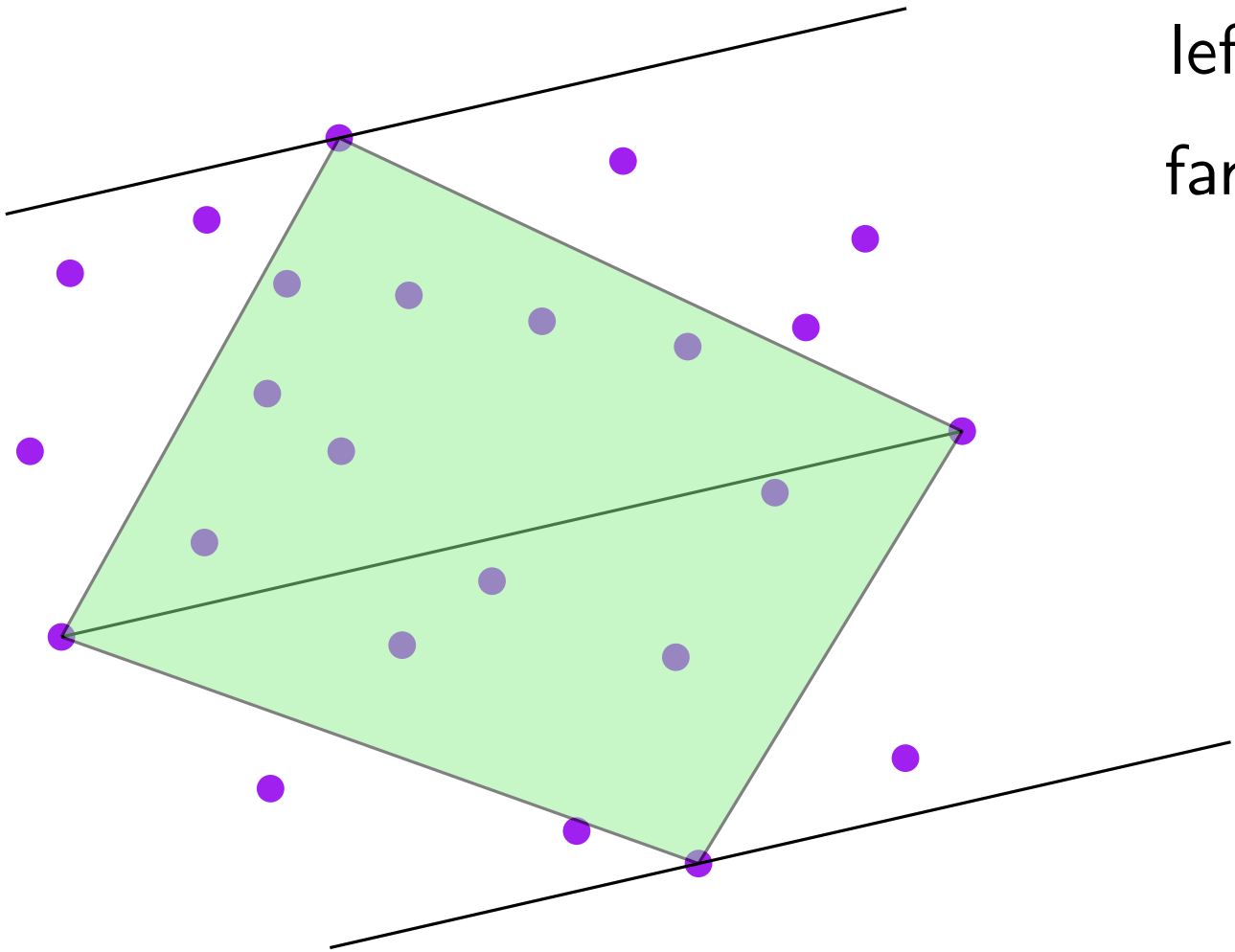
Quickhull

leftmost rightmost



Convex hull

Quickhull

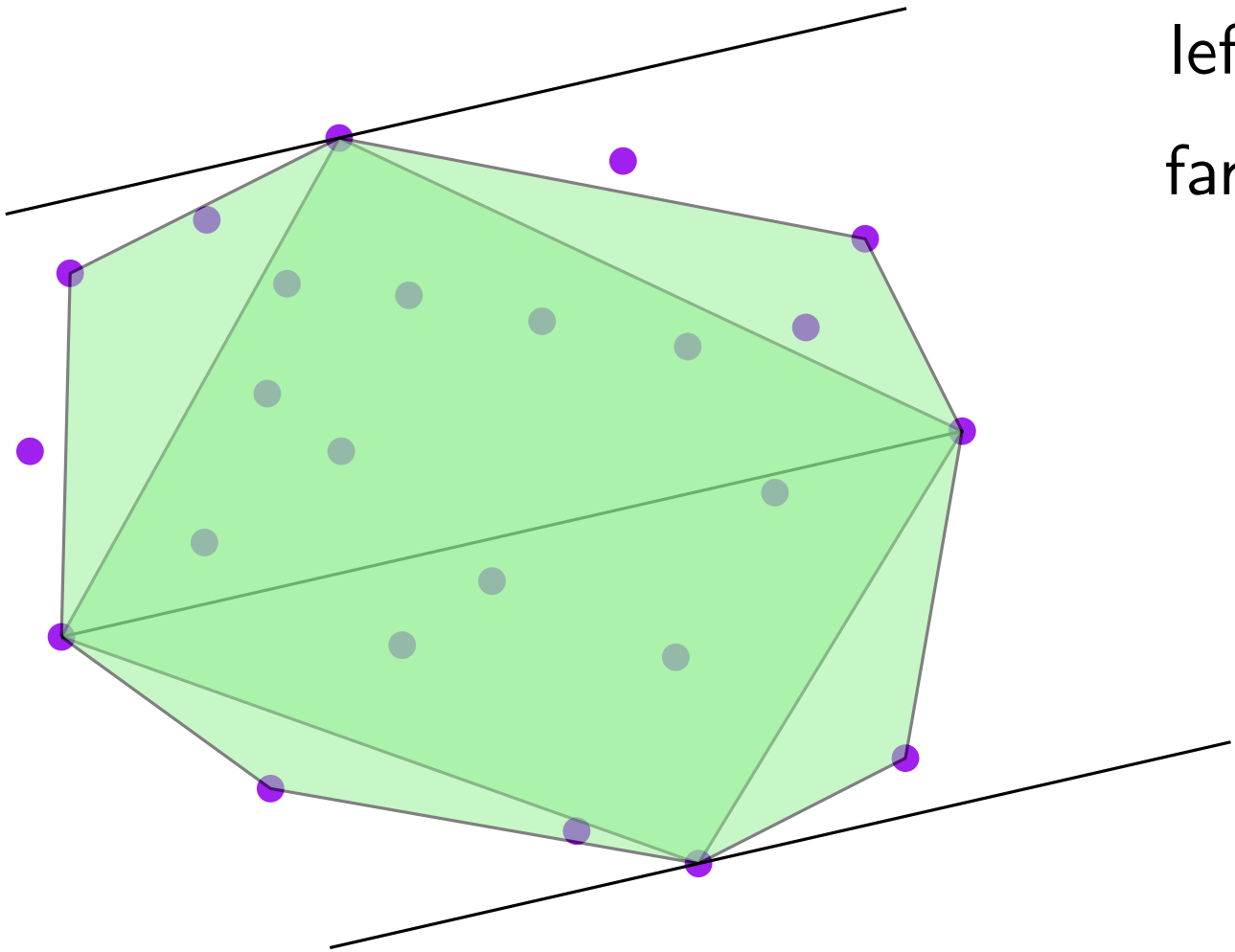


leftmost rightmost

farthest points

Convex hull

Quickhull

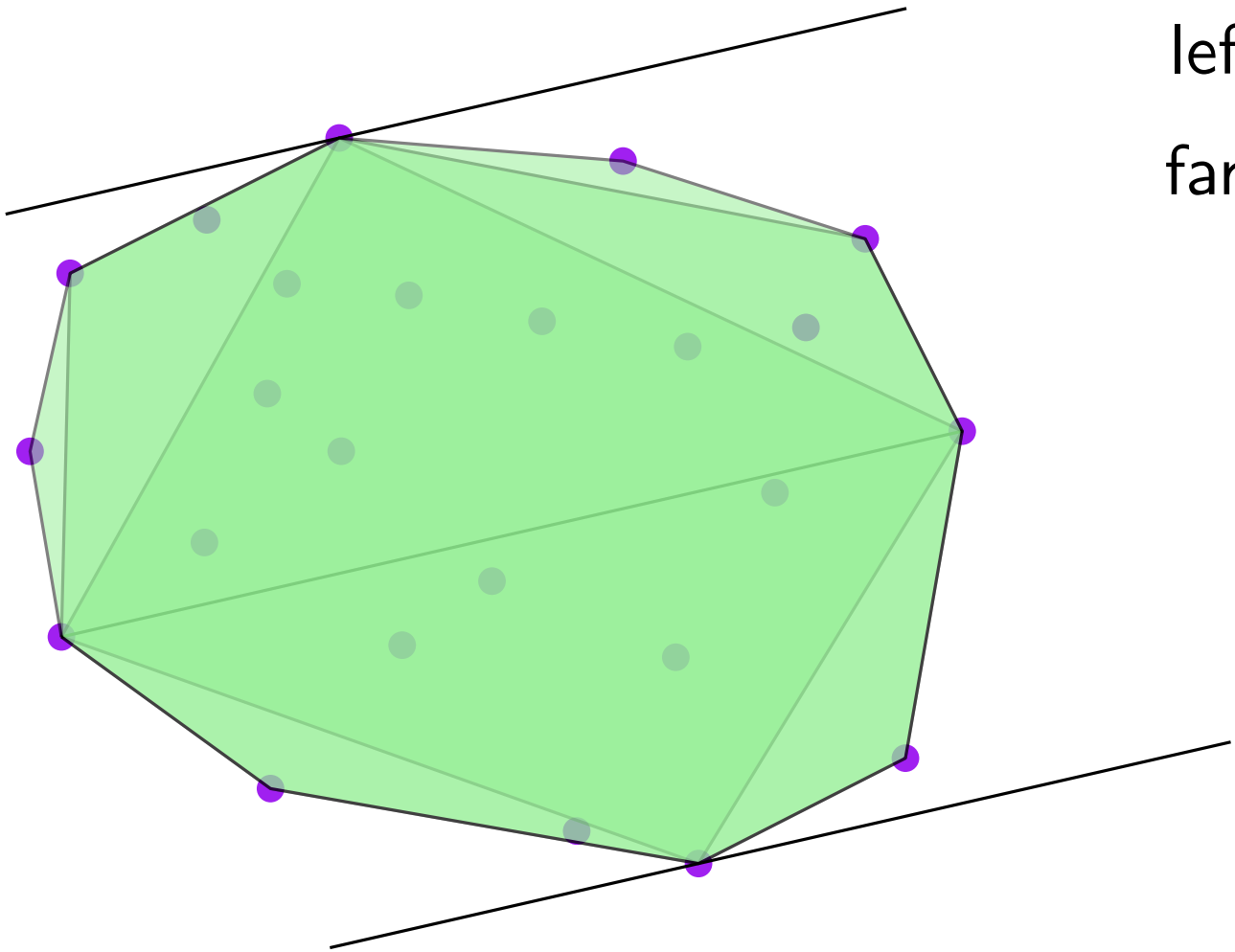


leftmost rightmost

farthest points

Convex hull

Quickhull

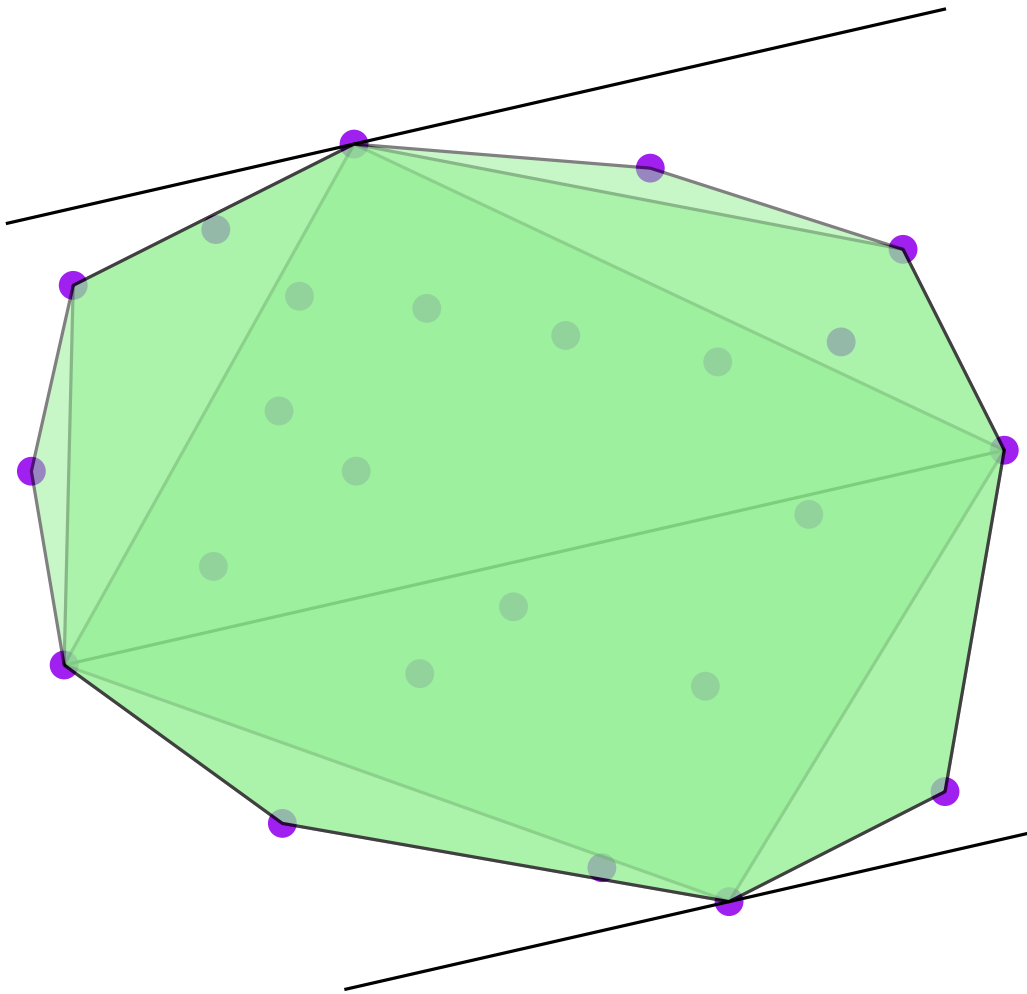


leftmost rightmost

farthest points

Convex hull

Quickhull



leftmost rightmost

farthest points

$O(n \log n)$ expected

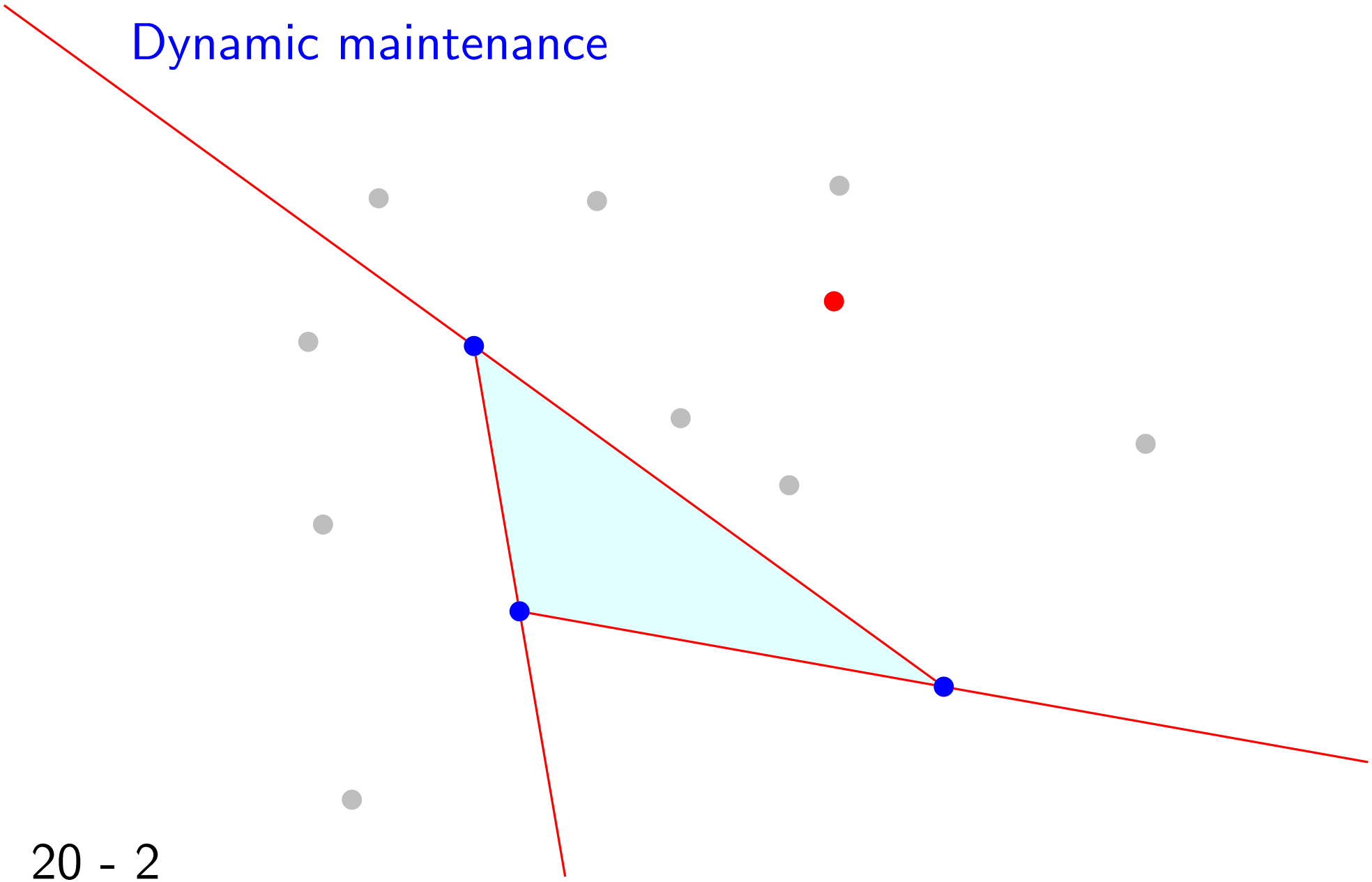
$O(n^2)$ worst-case

Convex hull

Dynamic maintenance

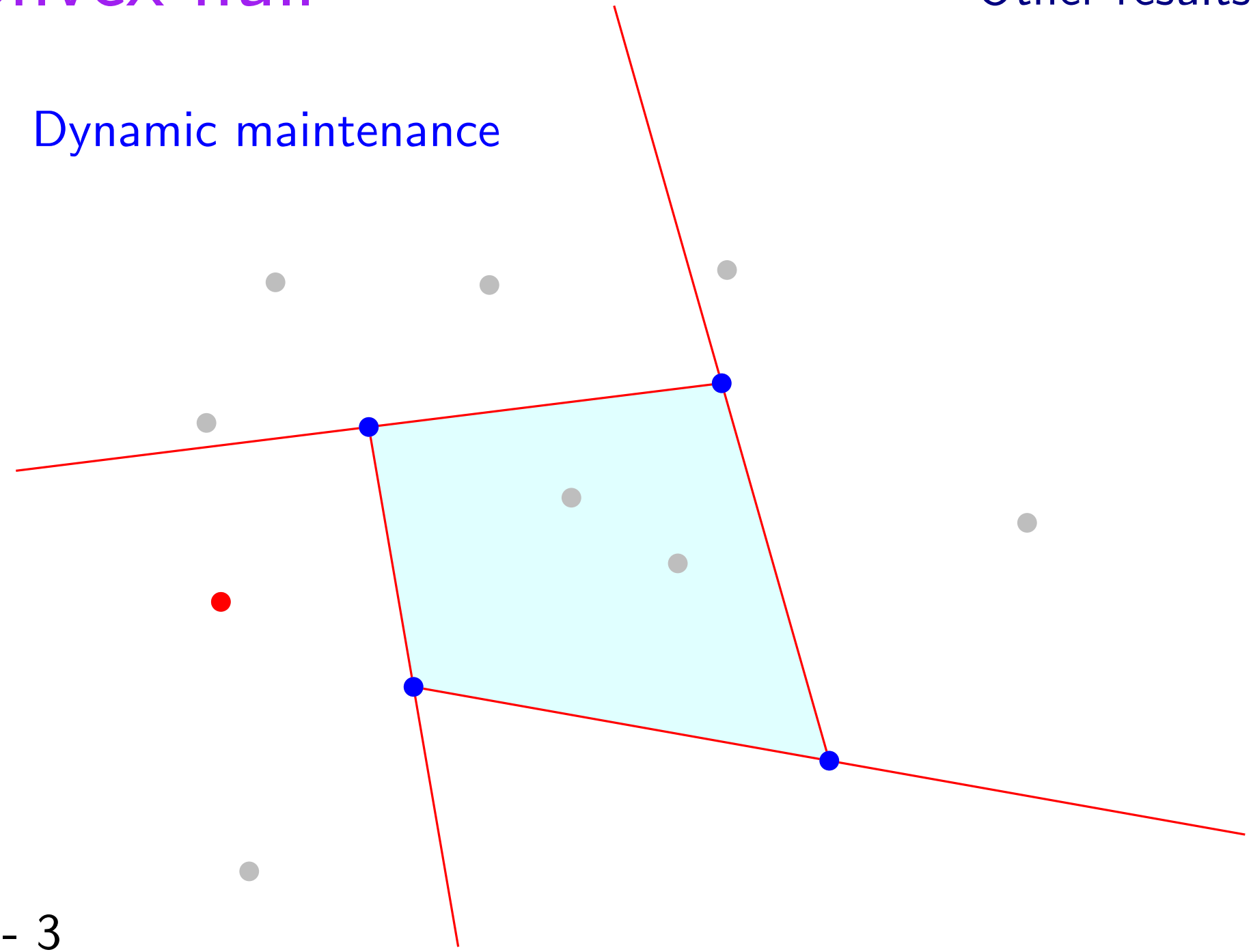
Convex hull

Dynamic maintenance



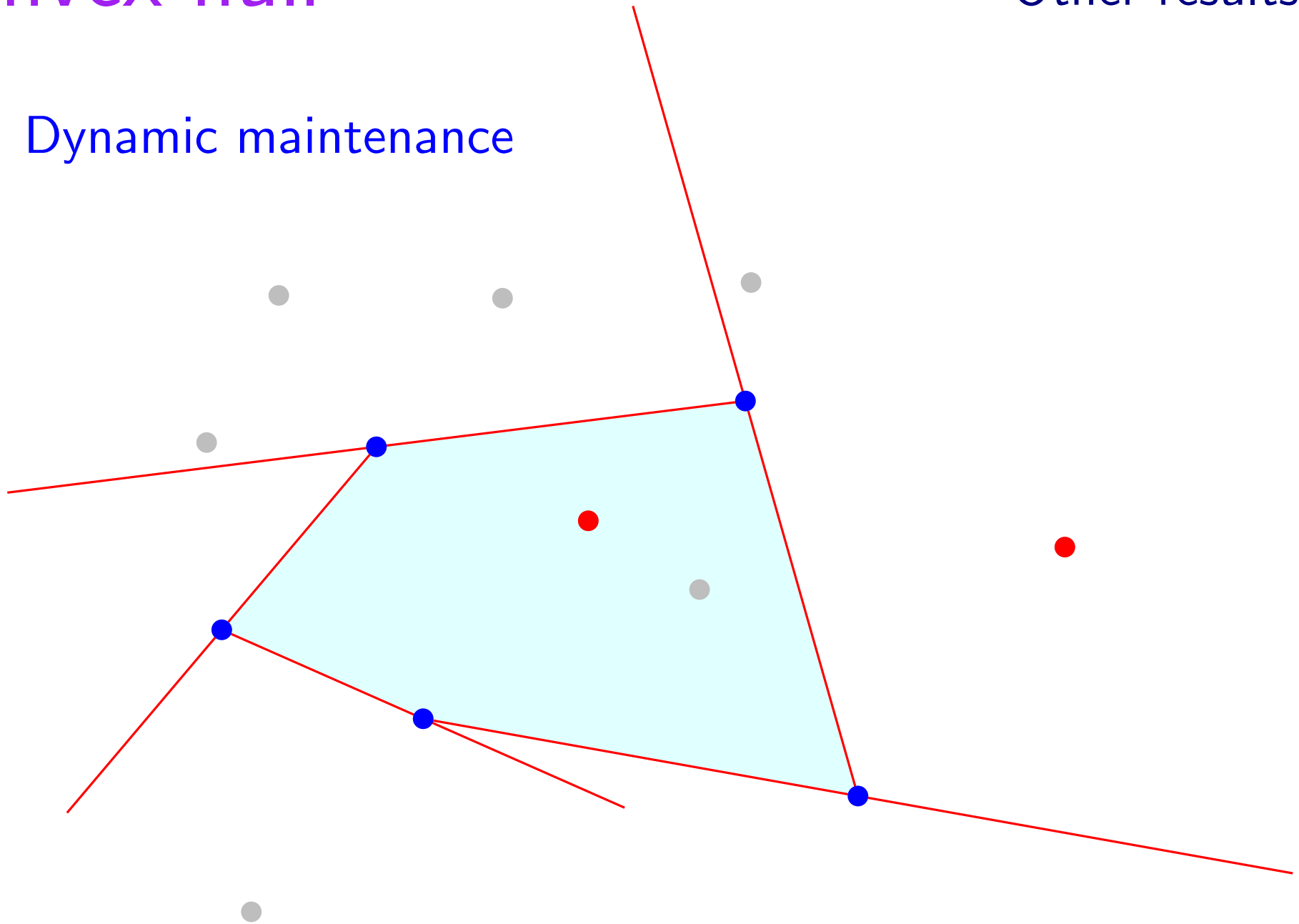
Convex hull

Dynamic maintenance



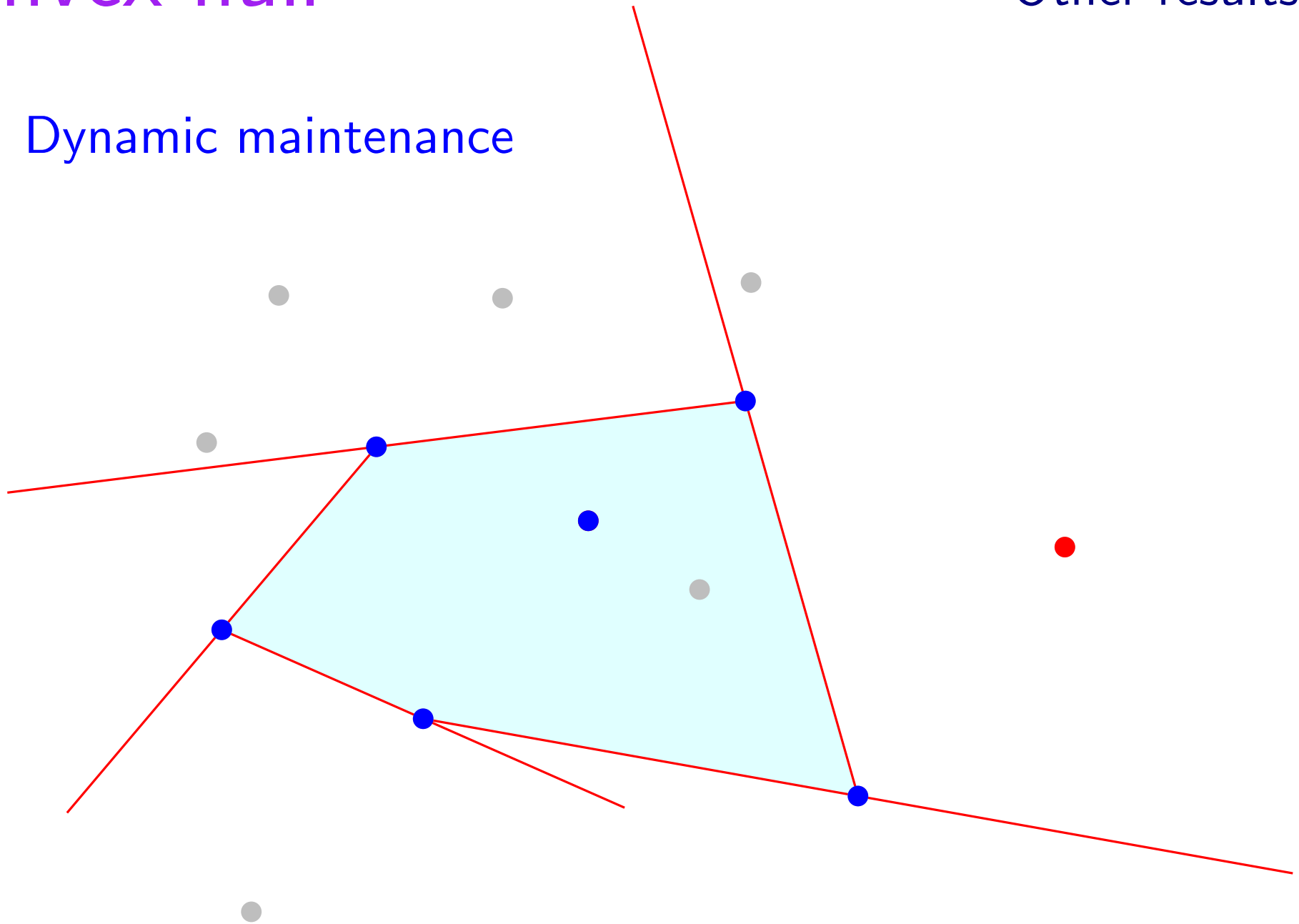
Convex hull

Dynamic maintenance



Convex hull

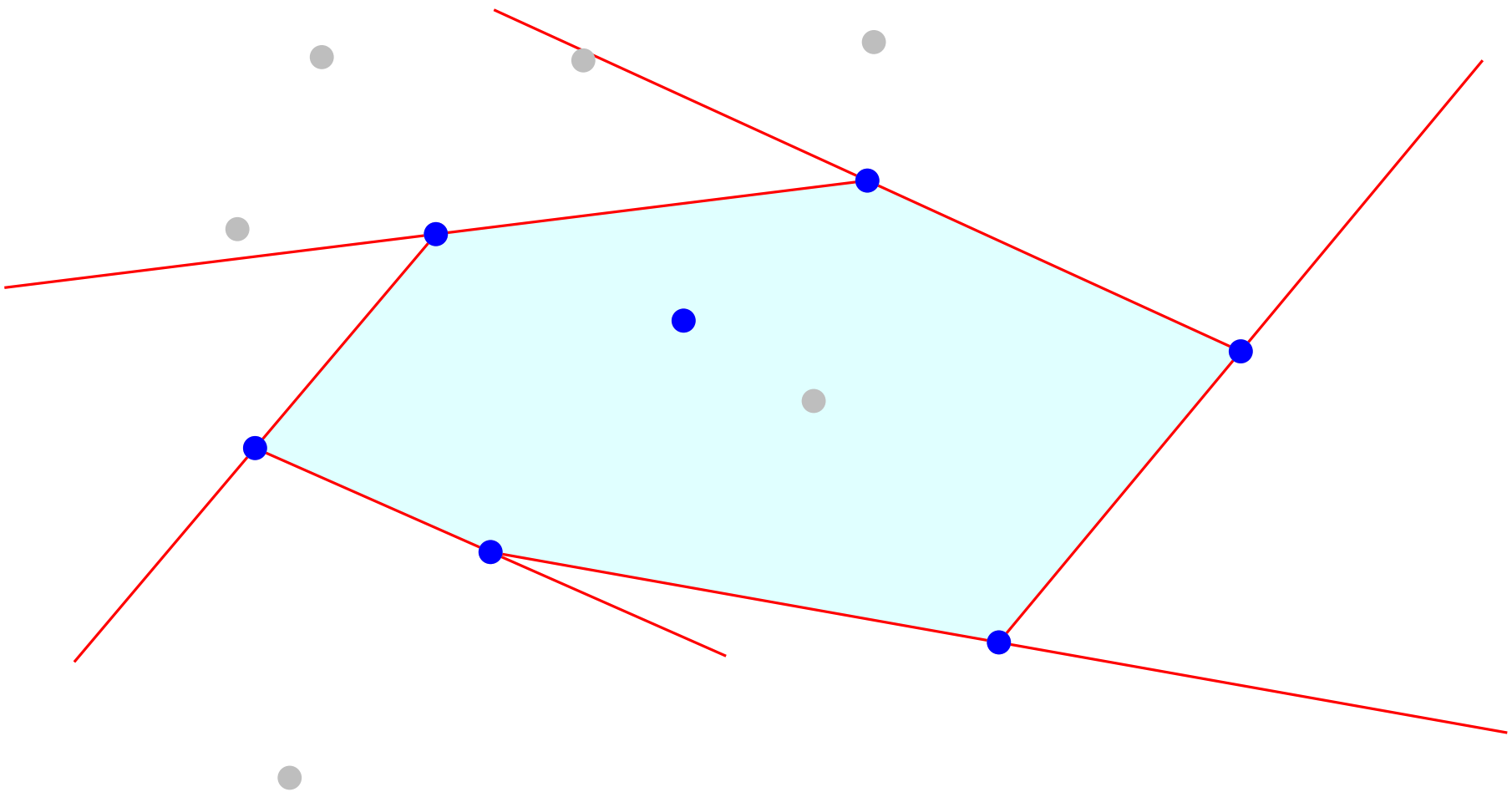
Dynamic maintenance



Convex hull

Dynamic maintenance

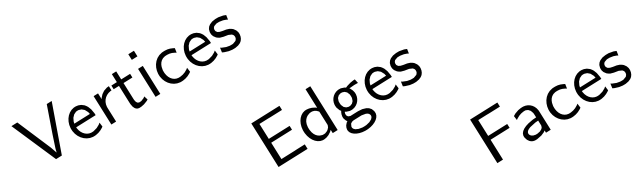
$O(\log n)$ per insertion



Convex hull

Euler relation

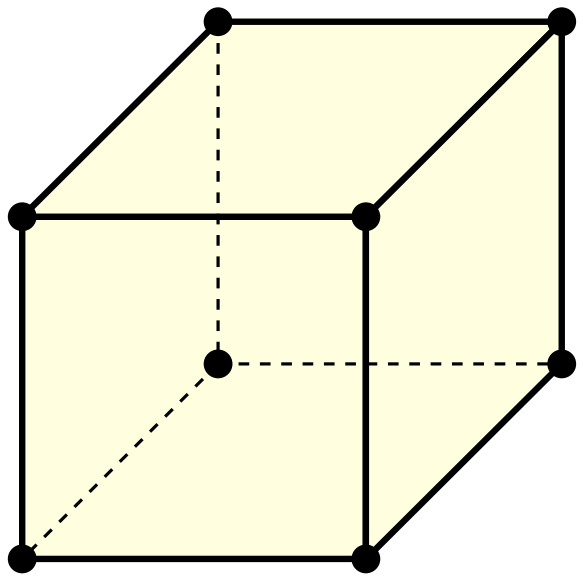
Polytope boundary



Convex hull

Euler relation

Polytope boundary



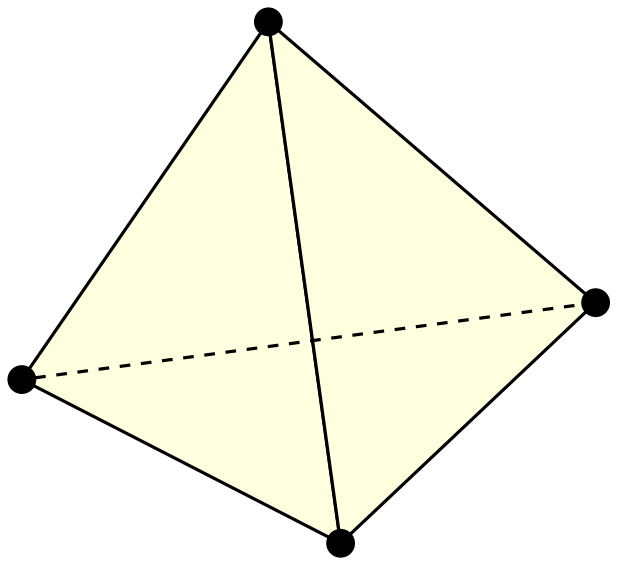
Vertices Edges Faces

$$8 - 12 + 6 = 2$$

Convex hull

Euler relation

Polytope boundary

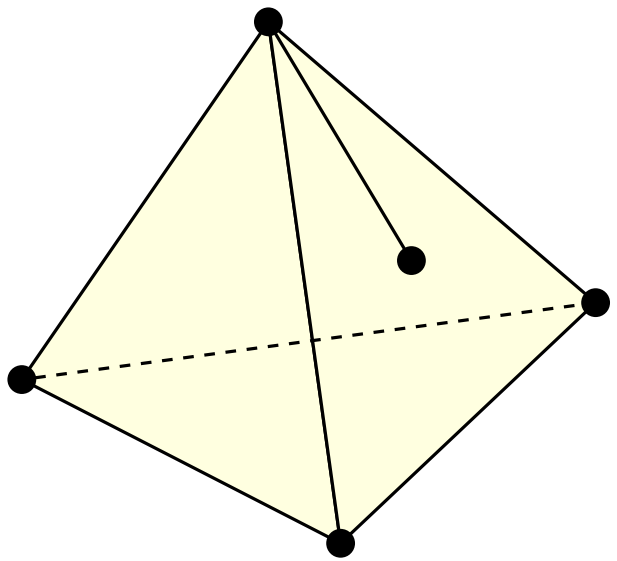


Vertices		Edges		Faces	
8	$-$	12	$+$	6	$= 2$
4	$-$	6	$+$	4	$= 2$

Convex hull

Euler relation

Polytope boundary

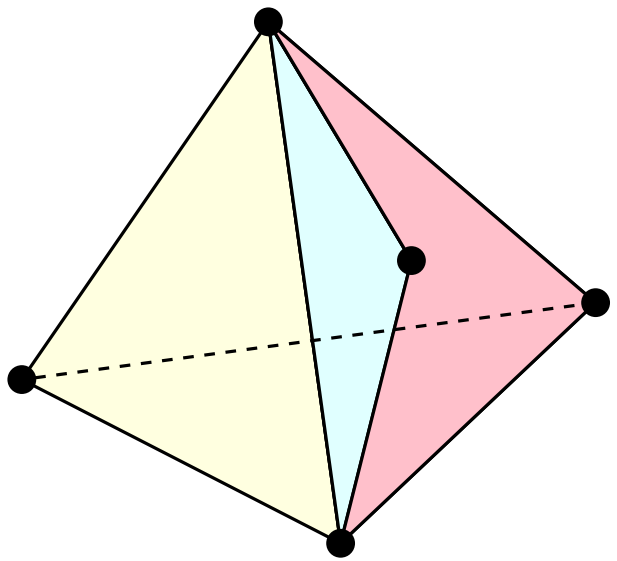


Vertices		Edges		Faces	
8	$-$	12	$+$	6	$= 2$
4	$-$	6	$+$	4	$= 2$
$+1-$		$+1$	$+$	0	$= +0$

Convex hull

Euler relation

Polytope boundary

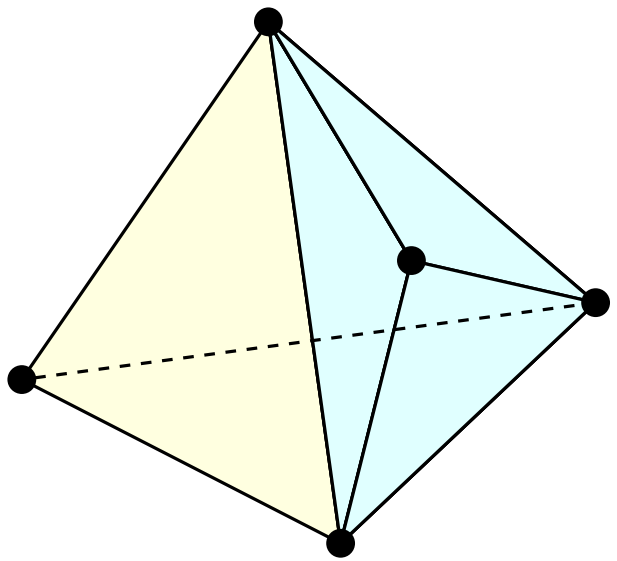


Vertices		Edges		Faces	
8	$-$	12	$+$	6	$= 2$
4	$-$	6	$+$	4	$= 2$
$+1$	$-$	$+1$	$+$	0	$= +0$
0	$-$	$+1$	$+$	$+1$	$= +0$

Convex hull

Euler relation

Polytope boundary



Vertices		Edges		Faces	
8	$-$	12	$+$	6	$= 2$
4	$-$	6	$+$	4	$= 2$
$+1$	$-$	$+1$	$+$	0	$= +0$
0	$-$	$+1$	$+$	$+1$	$= +0$

Convex hull

Euler relation

Polytope boundary

Vertices Edges Faces

$$n - e + f = 2$$

Convex hull

Euler relation

Polytope boundary

Vertices Edges Faces

$$n - e + f = 2$$

triangular faces $3f = 2e$

Convex hull

Euler relation

Polytope boundary

Vertices Edges Faces

$$n - e + f = 2$$

triangular faces $3f = 2e$

$$f = 2n - 4$$

$$e = 3n - 6$$

Convex hull

Linear size

$O(n \log n)$ divide and conquer algorithm

$O(nh)$ gift wraping algorithm

Convex hull

Dehn Sommerville relations $f_i = \sharp(\text{faces of dim } i)$

Euler: $f_0 - f_1 + f_2 - \dots f_{d-1} = (-1)^{d-1} + 1$

Convex hull

Dehn Sommerville relations $f_i = \sharp(\text{faces of dim } i)$

Euler: $f_0 - f_1 + f_2 - \dots f_{d-1} = (-1)^{d-1} + 1$

$$\sum_j = k^{d-1} - 1^j \begin{pmatrix} j+1 \\ k+1 \end{pmatrix} f_j = (-1)^{d-1} f_k$$

$-1 \leq k \leq d-2$ $\qquad$ $f_{-1} = f_d = 1$

$\left\lfloor \frac{d+1}{2} \right\rfloor$ independent equations

Convex hull

Dehn Sommerville relations $f_i = \sharp(\text{faces of dim } i)$

If $f_0, f_1, \ldots f_{\lfloor \frac{d-1}{2} \rfloor}$ are known

$f_{\lfloor \frac{d+1}{2} \rfloor}, \ldots f_{d-1}$ can be deduced

Convex hull

Dehn Sommerville relations $f_i = \sharp(\text{faces of dim } i)$

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$$f_{\lfloor \frac{d-1}{2} \rfloor} = O(n^{\lfloor \frac{d+1}{2} \rfloor})$$

$$\Longrightarrow \forall i \;\; f_i = O(n^{\lfloor \frac{d+1}{2} \rfloor})$$

Convex hull

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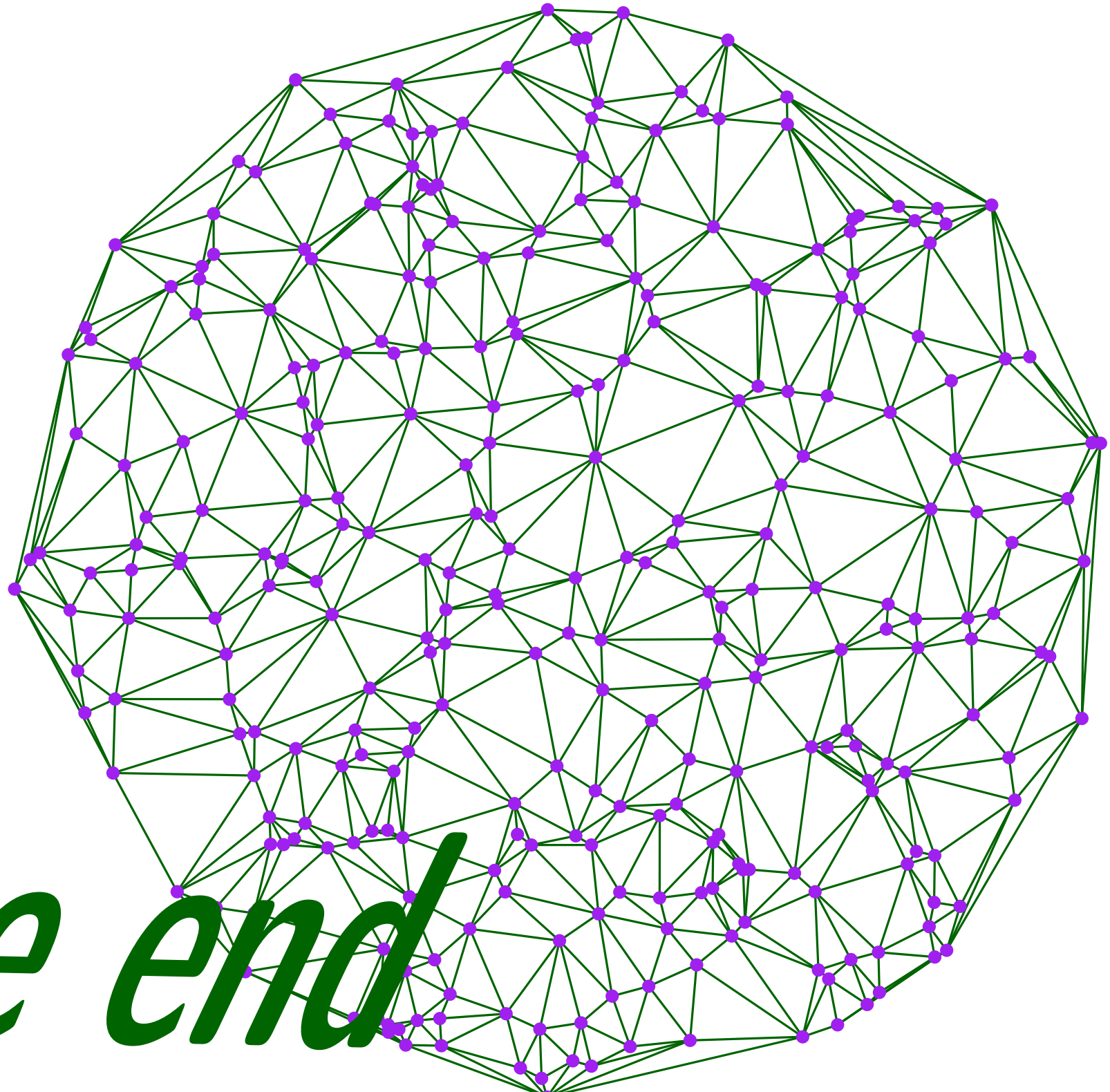
$\exists$ an optimal algorithm



The end