A linear bound on the expected number of visibility events

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ABSTRACT

In this paper, we show that the expected number of maximal non-occluded line segments tangent to 4 balls, amongst n uniformly distributed unit balls in $\mathbb{R}^3$, is at most linear. This extends the result of Durand [5] who showed that the expected number of lines (possibly occluded) tangent to 4 unit balls, under the same model, is $O(n^{8/3})$. Using our techniques we show a linear bound on the expected size of the visibility complex, providing evidence that the storage requirement for this data structure is not prohibitive.

Our results generalize in two directions. First, we show that the linear bound on the expected number of visibility events occurring not too close to the boundary of the scene generalizes to polyhedra of bounded aspect ratio. This suggests that the linear bound holds for realistic polyhedral scenes. Second, our results provide new upper bounds on the expected size of other global visibility data structures, notably the aspect graph.

1. INTRODUCTION

Visibility computations are central in computer graphics applications. Computing the limits of the umbra and penumbra cast by an area light source, identifying the set of blockers between any two given polygons and determining the view from a given point are examples of visibility queries that are essential for the realistic rendering of 3D scenes.

In global illumination algorithms, where the flow of light in a scene is simulated according to the laws of optical physics, visibility computations are excessively costly. In fact, more than half of the overall computation time can routinely be spent on visibility queries in radiosity simulations [9].

One approach to speeding up rendering is to store global visibility information in a data structure which can then be efficiently queried. The visibility complex, a partition of the set of maximal free line segments, has been proposed as a unified data structure encoding the visibility information of a scene [17] and has been used for rendering purposes [5, 7]. Other related data structures include Pellegrini's ray-shooting structure [14], the aspect graph [16] and the visual hull [10]; see [6] for a recent survey.

One problem with these types of data structures which prevents their application in practice is their potentially enormous size; the size of the visibility complex of a set of n triangles in $\mathbb{R}^3$ is, in the worst case, $\Theta(n^4)$ [5], which is prohibitive even for scenes of relatively modest size. Durand [5] showed that the expected size of the visibility complex of n random unit balls is $O(n^{8/3})$, but even this improved bound is prohibitive. Durand, Drettakis and Puech [8] also provided empirical evidence indicating that these bounds are largely pessimistic in practical situations.

In this paper we investigate the expected size of the visibility complex, or equivalently, the expected number of visibility events, occurring in scenes in $\mathbb{R}^3$. A visibility event is a combinatorial change in the view of a moving observer; such an event occurs when the viewing direction becomes tangent to some objects. For sets of convex objects in general position in $\mathbb{R}^3$, the viewing direction can be tangent to at most four objects. Visibility events thus correspond to maximal non-occluded line segments tangent to at most four objects; combinatorially different visibility events correspond to the faces of the visibility complex.

Previous related results include those that bound the number of lines and the number of free (non-occluded) lines amongst different sets of objects. Agarwal, Aronov and Sharir [1] showed an upper bound of $O(n^{3+\epsilon})$ on the complexity of the space of line transversals of n balls, by studying the lower envelope of a set of functions. A study of the upper envelope of the same set of functions yields the same upper bound on the number of free lines tangent to four balls [4]. Agarwal et al. [1] also showed a lower bound on the complexity of the space of line transversals of n balls of $\Omega(n^3)$ for arbitrarily sized balls and $\Omega(n^2)$ for unit sized balls. The construction also shows the same bound for the number of lines tangent to four balls. De Berg, Everett and Guibas [2] showed a $\Omega(n^3)$ lower bound on the number of free lines (and thus free segments) tangent to four amongst n disjoint homothetic convex polyhedra. Recently,

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Devillers and Ramos presented a simple $\Omega(n^3)$ lower bound on the number of free segments tangent to 4 arbitrarily sized balls, which also holds for non-intersecting balls [4]. We also present a simple $\Omega(n^2)$ lower bound on the number of free segments tangent to 4 unit balls.

In this paper we prove that the expected number of maximal non-occluded line segments tangent to four balls, among n uniformly distributed unit balls in $\mathbb{R}^3$ is linear, extending the result of Durand [5]. Using our techniques we then show a linear bound on the expected size of the visibility complex of n uniformly distributed unit balls in $\mathbb{R}^3$.

Our results generalize in two directions. First, we show that the linear bound applies also to balls of varying but bounded radius and to polyhedral objects enclosed between two concentric balls of fixed radius. Second, since the size of the visibility complex dominates the number of combinatorially different visibility events, our results also provide new upper bounds on the expected size of other global visibility data structures, notably the aspect graph.

In the next section we carefully define the problem and state our main results. Section 3 proves the bound on the number of segments tangent to four balls and Section 4 extends this result to the visibility complex. We present in Section 5 a $\Omega(n^2)$ worst-case lower bound. In conclusion, we discuss the impact of our bounds on more general models.

2. THE PROBLEM

We first describe our objects and their distribution. Let $n \in \mathbb{N}$ and μ a positive constant. A sample scene consists of n unit radius balls $B_1, \ldots, B_n$ whose centers p_i are uniformly distributed in a universal ball $\mathcal{U}$ of radius R centered at O. Since we distribute the centers p_i over $\mathcal{U}$, the balls B_i may intersect each other and are contained in the ball, denoted $\mathcal{U}^+$, whose radius is $R+1$ and whose center is that of $\mathcal{U}$.

We define the radius R of the universal ball $\mathcal{U}$ to be a function of n, with parameter μ, independent of n, satisfying

$$R^3 = n/\mu. \tag{1}$$

The parameter μ reflects the density of the balls in the sense that the expected number of centers lying in a solid of volume V taken randomly in the universe is $\frac{3}{4\pi}\mu V$.

The model is interesting only if n is asymptotically proportional to R^3. Indeed, if $\frac{n}{R^3}$ tends to infinity when n tends to infinity, then the universe gets entirely filled up with balls and visibility events only occur in $\mathcal{U}^+ \setminus \mathcal{U}$. Conversely, if $\frac{n}{R^3}$ tends to zero when n tends to infinity, then the balls get scattered so far apart that the probability that any four (or three) balls have a common tangent goes to zero.

We now define the *visibility complex* of a set of objects [17]. A free segment is an oriented line segment that does not intersect the interior of any object. A free segment is maximal if it is not properly contained in another one. Thus, the endpoints of a maximal free segment are either on an object or at infinity. We say that two maximal free segments are similar if their endpoints lie on the same objects (possibly at infinity). The visibility complex of a collection of objects is defined as the partition of the space of maximal free segments into connected components of similar segments. Its faces have dimension between 0 and 4; when the objects are in general position, a k-dimensional face corresponds to a connected set of similar maximal non-occluded line segments tangent to $4-k$ objects.

In order to bound the total number of faces of the visibility complex, we first bound the number of 0-faces, namely, when the objects are in general position, the maximal free segments tangent to 4 balls. To do this, we count the $T4$-*segments*, which are free segments tangent to 4 balls with endpoints on two of those balls. Since every 0-face contains a $T4$-segment, this provides a bound on the number of vertices of the visibility complex. Note that whereas the centers of the balls are contained in the universe $\mathcal{U}$, the balls themselves, and thus the $T4$-segments, are contained in $\mathcal{U}^+$.

Theorem 1 *The expected number of $T4$-segments amongst n uniformly distributed unit balls is $O(n)$.*

We then extend this result to the higher dimensional faces of the complex.

Theorem 2 *The expected size of the visibility complex of n uniformly distributed unit balls is $\Theta(n)$.*

3. THE EXPECTED NUMBER OF T4-SEGMENTS IS LINEAR

The general idea of the proof of Theorem 1 is to sum over all quadruples of balls B_i, B_j, B_k and B_l and all potential real tangent line segments to these four balls the probability that this tangent line exists and contains a $T4$-segment. For any ordered choice of four balls, we upper bound the probability that a line is tangent to these balls in the given order and is not occluded.

For any two points p and q, and a positive real number α, let $\mathcal{H}(p,q,\alpha)$ denote the union of all the balls of radius α centered on the line segment pq (see Figure 1). We first show that a line is tangent to B_i, B_j, B_k and B_l in that order only if p_j and p_k belong to $\mathcal{H}(p_i,p_l,2)$. Thus the volume of $\mathcal{H}(p_i,p_l,2) \cap \mathcal{U}$ gives an upper bound on the probability that a line tangent to the four balls exists.

We next show that a segment tangent to four balls B_i, B_j, B_k and B_l in that order, at points t_i, t_j, t_k and t_l, is not occluded if all centers of balls different from p_i, p_j, p_k and p_l are not in $\mathcal{H}(t_i,t_l,1)$. The volume of $\mathcal{U} \setminus \mathcal{H}(t_i,t_l,1)$ gives an upper bound on the probability that the tangent segment is not occluded Thus, to get an upper bound on that probability, we have to lower bound the volume of $\mathcal{H}(t_i,t_l,1) \cap \mathcal{U}$.

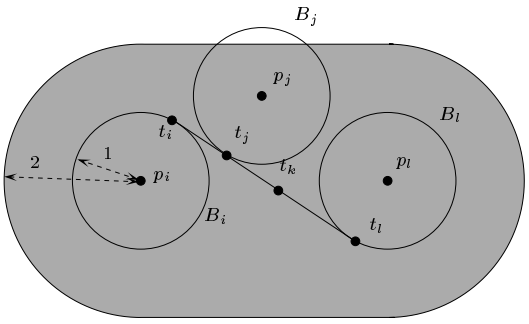
To bound the probability that a $T4$-segment exists, we integrate over the distance between p_i and p_l, and over the distance from p_i to the boundary of the universe $\mathcal{U}$. This integral is split into three parts corresponding to the cases where

(i) B_i and B_l are close to one another,
(ii) B_i and B_l are not close to one another and are not both close to the boundary of the universe,
(iii) B_i and B_l are not close to one another but are close to the boundary of the universe.

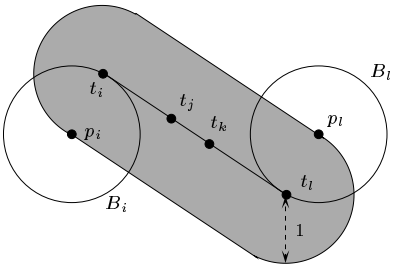
In each case we use different kinds of over-estimation for the volumes of $\mathcal{H}(p_i,p_l,2) \cap \mathcal{U}$ and under-estimation of $\mathcal{H}(t_i,t_l,1) \cap \mathcal{U}$.

3.1 Definitions

Let $\mathcal{N}$ be the set of ordered 4-tuples (i,j,k,l) chosen from $\{1,2,\ldots,n\}$ such that i,j,k,l are pairwise distinct. In our model, the probability that four centers are collinear is zero, so we may assume that any set of four balls admits at most 12 real common tangent lines [3, 11], which we can order



(a) $\mathcal{H}(p_i, p_l, 2)$



(b) $\mathcal{H}(t_i, t_l, 1)$

Figure 1. $\mathcal{H}(p_i, p_l, 2)$ and $\mathcal{H}(t_i, t_l, 1)$ are shown shaded.

arbitrarily. Given four balls B_i, B_j, B_k and B_l, we denote by $\mathcal{L}^{\omega}_{i,j,k,l}$, for ω in $\{1, \ldots, 12\}$, the event that the ω^{th} tangent line is real, that it is tangent to the four balls B_i, B_j, B_k and B_l in that order, and that p_i is farther from the boundary of $\mathcal{U}$ than p_l. Let t_i (resp. t_j, t_k, t_l), depending on i, j, k, l and ω, be the point of tangency of that line on B_i (resp. B_j, B_k, B_l). Let $\delta^{\omega}_{i,j,k,l}$ be the event that $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs and the line segment $t_i t_l$ is not occluded. Notice that if $\delta^{\omega}_{i,j,k,l}$ occurs, the balls B_i, B_j, B_k, B_l define a $T4$-segment, and that a $T4$-segment corresponds to a unique tuple i, j, k, l, ω.

Let $\mathbf{x}_{i,l}$ be the random variable representing the distance from p_i to p_l, and $\mathbf{y}_i$ (resp. $\mathbf{y}_l$) be the random variable denoting the distance from p_i (resp. p_l) to the boundary of the universe. In the sequel, a random point p denotes a point chosen in $\mathcal{U}$ with the uniform distribution.

3.2 Preliminaries

We first start by a straightforward lemma.

Lemma 3 *The expected number of $T4$-segments amongst n uniformly distributed unit balls is*

$$\sum_{(i,j,k,l)\in\mathcal{N}} \sum_{\omega=1}^{12} \Pr(\delta^{\omega}_{i,j,k,l}).$$

The following simple lemma provides necessary conditions on the relative locations of 4 ball centers generating a $T4$-segment.

Lemma 4 *Let S be a $T4$-segment tangent to the balls B_i, B_j, B_k, B_l, in that order, at points t_i, t_j, t_k, t_l, respectively. Then $p_j, p_k \in \mathcal{H}(p_i, p_l, 2)$ and the interior of $\mathcal{H}(t_i, t_l, 1)$ does not contain the center of any other ball.*

Proof. Segment $t_i t_l$ is contained in $\mathcal{H}(p_i, p_l, 1)$. Since t_j and t_k belong to that segment, t_j and t_k are also in $\mathcal{H}(p_i, p_l, 1)$. Thus p_j and p_k are either inside $\mathcal{H}(p_i, p_l, 1)$ or within distance 1 of its boundary. So p_j, p_k are both in $\mathcal{H}(p_i, p_l, 2)$. See Figure 1(a).

Suppose now that the interior of $\mathcal{H}(t_i, t_l, 1)$ contains the center p_γ of some ball, B_γ, $\gamma \neq i, j, k, l$. Then, the distance from p_γ to $t_i t_l$ is less than 1. So the interior of B_γ intersects $t_i t_l$, contradicting the fact that S is a $T4$-segment. See Figure 1(b). □

The next lemma turns those necessary conditions into an upper bound on the probability that a given $T4$-segment exists.

Lemma 5 *The probability that $\delta^{\omega}_{i,j,k,l}$ occurs is bounded from above by*

$$\Pr(p \in \mathcal{H}(p_i, p_l, 2))^2 \cdot \Pr(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathcal{L}^{\omega}_{i,j,k,l})^{n-4}$$

where p is a random point in $\mathcal{U}$.

Proof. If $\delta^{\omega}_{i,j,k,l}$ occurs, then $\mathcal{L}^{\omega}_{i,j,k,l}$ necessarily occurs, thus

$$\begin{aligned}\Pr(\delta^{\omega}_{i,j,k,l}) &= \Pr(\delta^{\omega}_{i,j,k,l} \cap \mathcal{L}^{\omega}_{i,j,k,l}) \\ &= \Pr(\mathcal{L}^{\omega}_{i,j,k,l}) \cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathcal{L}^{\omega}_{i,j,k,l}).\end{aligned}$$

By Lemma 4, p_j and p_k must belong to $\mathcal{H}(p_i, p_l, 2)$, and for all $\gamma \neq i, j, k, l$, p_γ must be outside $\mathcal{H}(t_i, t_l, 1)$. Since all the points are independent, we get the result. □

We now integrate the probability $\Pr(\delta^{\omega}_{i,j,k,l})$ over the distance x between p_i and p_l, and over the distance y from p_i to the boundary of the universe $\mathcal{U}$, and split it into three parts corresponding to the cases where (i) the balls B_i and B_l are close to one another, (ii) the balls B_i and B_l are far apart and p_i is at distance larger than 1 from the boundary of $\mathcal{U}$, and (iii) the balls B_i and B_l are far apart, p_i is at distance less than 1 from the boundary of $\mathcal{U}$. Note that in the last case, if $\delta^{\omega}_{i,j,k,l}$ occurs, then both ball centers p_i and p_l are within distance 1 from the boundary of $\mathcal{U}$. Two balls are considered far apart if they are farther apart than some sufficiently large constant; for technical reasons, which are embedded in the details of the proof of Lemma 16, we actually define far apart to mean distance at least 6.

Lemma 6 $\Pr(\delta^{\omega}_{i,j,k,l}) \leqslant I_{x<6} + I_{x>6,y>1} + I_{x>6,y<1}$, *where*

$$I_{x<6} = \int_{x=0}^{6} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x) \cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x + dx),$$

$$\begin{aligned}I_{x>6,y>1} = \int_{x=6}^{2R} &\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i > 1) \\ &\cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x + dx \mid \mathbf{y}_i > 1),\end{aligned}$$

$$\begin{aligned}I_{x>6,y<1} = \int_{x=6}^{2R}\int_{y=0}^{1} &\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y) \\ &\cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y) \\ &\cdot \Pr(y \leqslant \mathbf{y}_i < y + dy).\end{aligned}$$

PROOF. By the Total Probability Theorem (see e.g. [13]),

$$\Pr(\delta^{\omega}_{i,j,k,l}) = \int_{x=0}^{2R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x) \cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x + dx).$$

The integral can be split at $x = 6$, giving $I_{x<6}$. Then applying the Total Probability Theorem on what remains, we get

$$\int_{x=6}^{2R}\int_{y=0}^{R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y) \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y) \cdot \Pr(y \leqslant \mathbf{y}_i < y + dy) \tag{2}$$

which can be split at $y = 1$. The part corresponding to y between 1 and R is equal to

$$\int_{x=6}^{2R}\int_{y=1}^{R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_i > 1) \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y,\ \mathbf{y}_i > 1) \cdot \Pr(y \leqslant \mathbf{y}_i < y + dy)$$

$$\leqslant \int_{x=6}^{2R}\int_{y=0}^{R} \Pr(y \leqslant \mathbf{y}_i < y + dy) \cdot \Pr(\delta^{\omega}_{i,j,k,l} \cap (x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y,\ \mathbf{y}_i > 1).$$

Applying the Total Probability Theorem again, we get

$$\int_{x=6}^{2R} \Pr(\delta^{\omega}_{i,j,k,l} \cap (x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i > 1) = I_{x>6,y>1}.$$

Consider now the part of (2) for y between 0 and 1. Since, if $\mathbf{y}_l > \mathbf{y}_i$ then $\delta^{\omega}_{i,j,k,l}$ does not occur (by definition), and we have

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y) \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y)$$
$$= \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y) \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y).$$

Thus, the part of (2) for y between 0 and 1 is equal to $I_{x>6,y<1}$. □

We consider in turn the three integrals $I_{x<6}$, $I_{x>6,y>1}$ and $I_{x>6,y<1}$ in the following subsections, and prove that each of them is $O\left(\frac{1}{n^3}\right)$. This will complete the proof of Theorem 1 since, by Lemmas 3 and 6, the expected number of $T4$-segments is less than

$$12\binom{n}{4}(I_{x<6} + I_{x>6,y>1} + I_{x>6,y<1}).$$

3.3 B_i and B_l are close to one another

We prove here that $I_{x<6}$ is $O(\frac{1}{n^3})$ when B_i and B_l are close to one another. In this case, the probability that there are two other balls, B_j and B_k, defining a line tangent to B_i, B_j, B_k, B_l in that order, is small enough that we do not need to consider occlusions in order to get the bound we want.

We first bound $\Pr(x < \mathbf{x}_{i,l} \leqslant x + dx)$ appearing in the integral $I_{x<6}$.

Lemma 7 $\Pr(x < \mathbf{x}_{i,l} \leqslant x + dx) \leqslant \dfrac{3x^2}{R^3}dx$.

PROOF. When p_i is chosen, p_l must belong to a spherical shell between two spheres of center p_i and radii x and $x+dx$. The probability, if p_i is known, is exactly the part of the spherical shell inside $\mathcal{U}$ that we upper bound by the total volume of the spherical shell which is $4\pi x^2 dx$ and since the volume of $\mathcal{U}$ is $\frac{4}{3}\pi R^3$ we get the claimed bound. The exact value of $\Pr(x < \mathbf{x}_{i,l} \leqslant x + dx)$ can be computed [12, 18] but that bound is enough for our purpose. □

The following lemma is used to bound the probability that there is a common tangent between B_i, a unit ball centered at a random point p, and B_l, in that order.

Lemma 8 $\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \mathbf{x}_{i,l} = x) \leqslant \dfrac{(3x+8)}{R^3}$.

PROOF.

$$\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \mathbf{x}_{i,l} = x) = \frac{\text{Volume of } \mathcal{H}(p_i, p_l, 2) \cap \mathcal{U}}{\text{Volume of } \mathcal{U}} \Big|_{\mathbf{x}_{i,l}=x} \leqslant \frac{\text{Volume of } \mathcal{H}(p_i, p_l, 2)}{\text{Volume of } \mathcal{U}} \Big|_{\mathbf{x}_{i,l}=x}.$$

Since the volumes of $\mathcal{H}(p_i, p_l, 2)$ and $\mathcal{U}$ are $\frac{4\pi}{3}(3x+8)$ and $\frac{4\pi}{3}R^3$, respectively,

$$\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \mathbf{x}_{i,l} = x) \leqslant \frac{3x+8}{R^3}$$

□

The following proposition now bounds $I_{x<6}$.

Proposition 9 *$I_{x<6}$ is $O(\frac{1}{n^3})$.*

PROOF. Recall that $I_{x<6} = \int_{x=0}^{6} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x) \cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x + dx)$ (see Lemma 6). Applying Lemma 5 conditioned by the knowledge of x, we have

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x) = \Pr(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathcal{L}^{\omega}_{i,j,k,l})^{n-4} \cdot \Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \mathbf{x}_{i,l} = x)^2.$$

We bound $\Pr(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathcal{L}^{\omega}_{i,j,k,l})$ by 1 and, by Lemma 8, $\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \mathbf{x}_{i,l} = x) \leqslant \frac{3x+8}{R^3}$. Since, by Lemma 7, $\Pr(x \leqslant \mathbf{x}_{i,l} < x + dx) \leqslant \frac{3x^2}{R^3}dx$, we get

$$I_{x<6} \leqslant \int_{x=0}^{6} \frac{(3x+8)^2}{R^6} \cdot \frac{3x^2}{R^3}dx = \frac{\mu^3}{n^3}\int_{x=0}^{6} 3x^2(3x+8)^2 dx = O\left(\frac{1}{n^3}\right).$$

□

3.4 $\mathbf{B}_i$ and $\mathbf{B}_l$ are not close to one another, and $\mathbf{B}_i$ is not close to the boundary

For the integral $I_{x>6,y>1}$, occlusions must be taken into account. To this aim, we bound from below the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ in the following lemma

Lemma 10 *For any i,j,k,l,ω such that $\mathbf{y}_i \geqslant 1$, the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is greater than $\frac{\pi}{12}\mathbf{x}_{i,l}$.*

PROOF. Let K be the ball having diameter p_it_i. Notice that K and p_l are both contained in $\mathcal{H}(t_i,t_l,1)$ and in $\mathcal{U}$. The convex hull of p_l and K is thus contained in $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$, and its volume is half the volume of the ball K, $\frac{\pi}{12}$, plus the volume of a cone of apex p_l, of base a disk whose boundary is a great circle of K, and of height greater than $\mathbf{x}_{i,l}-1$. The volume of that cone is at least $\frac{1}{3}\frac{\pi}{2^2}(\mathbf{x}_{i,l}-1) = \frac{\pi}{12}\mathbf{x}_{i,l} - \frac{\pi}{12}$. □

We now bound the probability that a tangent line segment t_it_l is not occluded by any of the other $n-4$ balls, given that the line segment t_it_l exists, p_i and p_l are distance $x \geqslant 1$ apart, and the ball B_i is entirely contained in $\mathcal{U}$.

Lemma 11 *For $x \geqslant 1$ and p a random point in $\mathcal{U}$,*

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l}\right)^{n-4} < 55\cdot\exp\left(-\frac{\mu\, x}{16}\right).$$

PROOF. First notice that

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l}\right) = 1 - \frac{\text{Volume of } \mathcal{H}(t_i,t_l,1)\cap\mathcal{U}}{\text{Volume of } \mathcal{U}}\,|_{\mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l}}.$$

By Lemma 10, the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is bounded from below by $\frac{\pi}{12}x$. Since the volume of $\mathcal{U}$ is $\frac{4}{3}\pi R^3$, we get

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x, \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l}\right) < 1 - \frac{x}{16R^3}.$$

For any $0 \leqslant t \leqslant 1$, we have $(1-t) \leqslant \exp(-t)$ thus

$$(1-t)^{n-4} \leqslant e^{-t(n-4)} = e^{-tn}e^{4t} \leqslant e^4 e^{-tn}.$$

On the other hand, $0 \leqslant \frac{x}{16R^3} \leqslant \frac{1}{8R^2} \leqslant 1$ for any x in $[1,2R]$, thus we have

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l}\right)^{n-4} < \exp(4)\cdot\exp\left(-\frac{n\,x}{16R^3}\right) < 55\cdot\exp\left(-\frac{\mu\, x}{16}\right).$$

□

The following proposition now bounds the integral $I_{x>6,y>1}$.

Proposition 12 *$I_{x>6,y>1}$ is $O(\frac{1}{n^3})$.*

PROOF. Recall that

$$I_{x>6,y>1} = \int_{x=6}^{2R} \Pr(\delta^\omega_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1) \cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x+dx \mid \mathbf{y}_i>1).$$

Similarly to the proof of Proposition 9, applying Lemma 5, we have

$$\Pr(\delta^\omega_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1) = \Pr(p \in \mathcal{H}(p_i,p_l,2) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1)^2 \cdot \Pr(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l})^{n-4}.$$

We bound

$$\Pr(p \in \mathcal{H}(p_i,p_l,2) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i>1)$$

by Lemma 8,

$$\Pr(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,,\ \mathbf{y}_i>1,\ \mathcal{L}^\omega_{i,j,k,l})^{n-4}$$

by Lemma 11, and the density of x by Lemma 7. Notice that the knowledge that $\mathbf{y}_i>1$ does not invalidate the proof of Lemmas 7 and 8. Then we get

$$\begin{aligned} I_{x>6,y>1} &\leqslant \int_{x=6}^{2R}\left(\frac{3x+8}{R^3}\right)^2\cdot 55\exp\left(-\frac{\mu\,x}{16}\right)\cdot\frac{3x^2}{R^3}dx \\ &\leqslant \frac{\mu^3}{n^3}\int_{x=6}^{+\infty} 3x^2(3x+8)^2\cdot 55\exp\left(-\frac{\mu\,x}{16}\right)dx. \end{aligned}$$

Changing $\frac{\mu\,x}{16}$ by z we get integrals of the kind

$$\int_0^\infty z^r\exp(-z)dz$$

which are bounded by constant and thus $I_{x>6,y>1}$ is $O\left(\frac{1}{n^3}\right)$. □

3.5 $\mathbf{B}_i$ and $\mathbf{B}_l$ are not close to one another, and $\mathbf{B}_i$ is close to the boundary

As in the previous case, to bound $I_{x>6,y<1}$, we need to bound from below the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$. Here, however, the tangent t_it_l can be outside $\mathcal{U}$, so the bound of Lemma 10 does not apply and a more intricate proof is needed. We need to distinguish two cases depending on the distance of segment t_it_l from O, the center of $\mathcal{U}$.

To this aim, we introduce two new types of events. For any $s \in \mathbb{R}$, let $\mathcal{F}^\omega_{i,j,k,l}(s)$ (resp. $\mathcal{N}^\omega_{i,j,k,l}(s)$) be the event that $\mathcal{L}^\omega_{i,j,k,l}$ occurs and the line segment t_it_l is at distance greater (resp. less) than $R+1-s$ from O. Note that the events $\mathcal{F}^\omega_{i,j,k,l}(s)$ and $\mathcal{N}^\omega_{i,j,k,l}(s)$ are disjoint, and their union is equal to $\mathcal{L}^\omega_{i,j,k,l}$. For reasons that will become clear in the proof of Lemma 17, we consider $s = y^{\frac{2}{3}}$.

Lemma 13 *For any random point p in $\mathcal{U}$,*

$$\Pr(\delta^\omega_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y)$$

is equal to

$$\begin{aligned} &\Pr\left(\mathcal{F}^\omega_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right) \\ &\cdot \Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{F}^\omega_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} \\ &+ \Pr\left(\mathcal{N}^\omega_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right) \\ &\cdot \Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{N}^\omega_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}. \end{aligned}$$

PROOF. Since $\delta^{\omega}_{i,j,k,l}$ implies $\mathcal{L}^{\omega}_{i,j,k,l}$ which can be partitioned in $\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ and $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$,

$$\begin{aligned}&\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y)\\ &\quad = \Pr(\delta^{\omega}_{i,j,k,l} \cap \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y)\\ &\qquad + \Pr(\delta^{\omega}_{i,j,k,l} \cap \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y)\end{aligned}$$

which can be expanded into

$$\begin{aligned}&\Pr(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y)\\ &\cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}))\\ &+ \Pr(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y)\\ &\cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})).\end{aligned}$$

When $\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ occurs, the probability

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}))$$

reduces to the probability that the tangent is not occluded, that is p_γ does not belong to $\mathcal{H}(t_i, t_l, 1)$ for all the $n-4$ values of $\gamma \neq i, j, k, l$. The same argument holds for $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$. Since the p_γ are independent, we get the result. □

We now bound the different terms of the integral $I_{x>6,y<1}$.

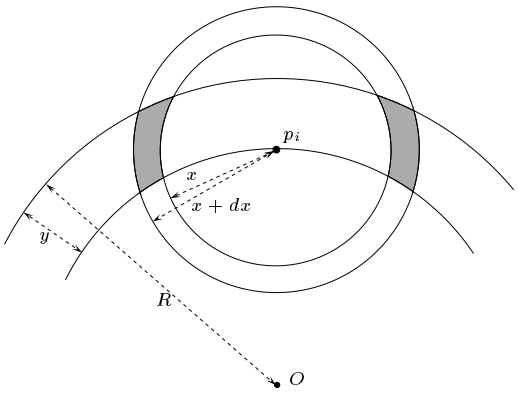


Figure 2. For the proof of Lemma 15.

Lemma 14 $\Pr(y \leqslant \mathbf{y}_i < y + dy) \leqslant \dfrac{3\,dy}{R}$.

PROOF. The event $(y \leqslant \mathbf{y}_i < y + dy)$ occurs if p_i lies in the spherical shell delimited by the two spheres centered at O of radii $R-y$ and $R-y-dy$ whose volume is smaller than $4\,\pi\,R^2\,dy$. Dividing by the volume of $\mathcal{U}$ proves the result. □

Lemma 15 *For $6 \leqslant x \leqslant 2R$ and $y \leqslant 1$, we have*

$$\Pr((x < \mathbf{x}_{i,l} \leqslant x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y) \leqslant \frac{9\,x\,y\,dx}{R^3}.$$

SKETCH OF PROOF:[1] The probability

$$\Pr((x < \mathbf{x}_{i,l} \leqslant x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y)$$

is equal to the volume of the region (shown in grey in Figure 2) which is the intersection of the region in between the two spheres centered at p_i and of radii x and $x+dx$, and the region in between the two spheres centered at O and of radii R and $R-y$, divided by the volume of $\mathcal{U}$. Roughly speaking, the volume bounded by the four spheres is $x\,y\,dx$ because, its "thickness" is dx, its "height" is y and its "radius" is x. Since the volume of $\mathcal{U}$ is $\frac{4}{3}\pi R^3$, we get that the probability $\Pr((x < \mathbf{x}_{i,l} \leqslant x+dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y)$ is bounded from above by $\frac{x\,y\,dx}{R^3}$, up to some constant. □

In order to bound the two terms of the form

$$\Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4},$$

we need to bound the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ from below.

Lemma 16 *For any i, j, k, l, ω and $0 \leqslant s \leqslant 1$ such that $\mathbf{x}_{i,l} \geqslant 6$, $\mathbf{y}_l \leqslant \mathbf{y}_i \leqslant 1$ and segment $t_i t_l$ is at distance less than $R+1-s$ from the center of $\mathcal{U}$, the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is larger than $\frac{1}{9\sqrt{2}}\,(x-5)\,s\,\sqrt{s}$.*

SKETCH OF PROOF:[2] Let t be the closest point on segment $t_i t_l$ from O, and D be a unit radius disk centered at t in a plane containing O, the center of $\mathcal{U}$. We define a quadrilateral with vertices a, b, a', b' such that a and a' are the closest and the farthest points in $D \cap \mathcal{U}$ from O, and b and b' are the points of intersection of ∂D and the perpendicular bisector of segment aa' (see Figure 3). Let v be equal to $R+1$ minus the distance from O to segment $t_i t_l$. We prove that the convex hull of a, b, a', b' and p_l, which is included in $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$, has volume greater than $\frac{1}{9\sqrt{2}}\,(x-5)\,\min(2\sqrt{2}, v\sqrt{v})$. It follows that, for any $0 \leqslant s \leqslant 1$, if segment $t_i t_l$ is at distance less than $R+1-s$ from O, then $v \geqslant s$ and the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{9\sqrt{2}}\,(x-5)\,s\,\sqrt{s}$. □

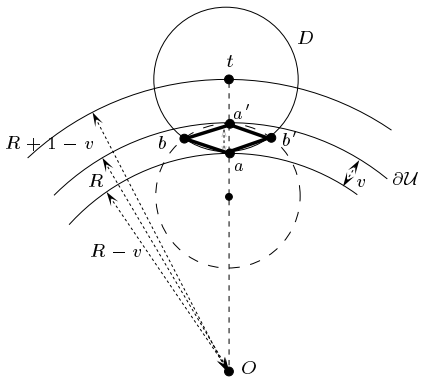


Figure 3. For the sketch of the proof of Lemma 16 ($v \in (0,1)$).

Lemma 17 *For any random point p in $\mathcal{U}$, $x \geqslant 6$ and $0 \leqslant$*

[1] A detailed proof is given in Appendix A.

[2] We prove this lemma in detail in Appendix B.

$y \leqslant 1$,

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} \leqslant 55\cdot\exp\left(-\frac{\mu}{12\sqrt{2}\,\pi}(x-5)\,y^2\right)$$

and

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} \leqslant 55\cdot\exp\left(-\frac{\mu}{12\sqrt{2}\,\pi}(x-5)\,y\right).$$

Furthermore, if $x \geqslant 6\sqrt{R}$ *then*

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} \leqslant 55\cdot\exp\left(-\frac{\mu}{12\sqrt{2}\,\pi}(x-5)\right).$$

PROOF. Since p_i is at distance $R-y$ from O, the segment t_it_l is at distance less than $R+1-y$ from O, and thus, by Lemma 16, the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is greater than $\frac{1}{9\sqrt{2}}(x-5)\,y\sqrt{y}$, which is bigger than $\frac{1}{9\sqrt{2}}(x-5)\,y^2$ since $0\leqslant y\leqslant 1$ (We bound $y\sqrt{y}$ from below by y^2 only so that we can actually compute the integral $\mathcal{I}_1$ in the proof of Proposition 20). We now follow the proof of Lemma 11, except that the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is now bounded from below by $\frac{1}{9\sqrt{2}}(x-5)\,y^2$ instead of $\frac{\pi}{12}x$. We get

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\cdot\exp\left(-\frac{\mu\,(x-5)\,y^2}{12\sqrt{2}\,\pi}\right).$$

Similarly, when $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ occurs, the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is, by Lemma 16, bounded from below by $\frac{1}{9\sqrt{2}}(x-5)\,y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}=\frac{1}{9\sqrt{2}}(x-5)\,y$ and we get

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\cdot\exp\left(-\frac{\mu\,(x-5)\,y}{12\sqrt{2}\,\pi}\right).$$

Now, if $x\geqslant 6\sqrt{R}$, the length of the tangent t_it_l is at least $6\sqrt{R}-2$. A simple computation shows that, since $R>3$ ($x\geqslant 6$), $6\sqrt{R}-2$ is bigger than $2\sqrt{2R+1}$, the length of the longest line segment that may entirely lie inside $\mathcal{U}^+\setminus\mathcal{U}$. Thus, $dist(O,t_it_l)\leqslant R=R+1-s$ with $s=1$, and by Lemma 16, the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is greater than $\frac{1}{9\sqrt{2}}(x-5)$. Then, as before, we get,

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\cdot\exp\left(-\frac{\mu\,(x-5)}{12\sqrt{2}\,\pi}\right).$$

□

Lemma 18

$$\Pr(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y) \leqslant \left(\frac{3x+8}{R^3}\right)^2.$$

PROOF. We apply Lemmas 4 and 8. □

Lemma 19 *If* $y<1$ *then*

$$\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right) \leqslant 81\frac{(x+4)^2y^2}{2R^6}.$$

SKETCH OF PROOF:[3] A "far" tangent t_it_l is at distance at least $R+1-y^{\frac{2}{3}}$ from the center O of $\mathcal{U}$. Such a segment also lies in $\mathcal{H}(p_i,p_l,1)$. Let E be the part of $\mathcal{H}(p_i,p_l,1)$ lying outside of the sphere of radius $R+1-y^{\frac{2}{3}}$ and center O. See Figure 3.5a. Now, both p_j and p_k must be within distance 1 from E and inside $\mathcal{U}$. Denote this region by K. Then

$$\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right) \leqslant \left(\frac{\text{Volume of } K}{\text{Volume of } \mathcal{U}}\right)^2.$$

We need to bound the volume of K from above; here we give a very rough sketch of the proof. First notice that the "length" of K is at most $x+4$. Since K is enclosed in between a sphere of radius R and one of radius $R-y^{\frac{2}{3}}$, its "height" is at most $y^{\frac{2}{3}}$. For the "width", consider Figure 3.5b which shows a cross-section of K taken with a plane through O and perpendicular to p_ip_l. The width of K is no more than 4 times the width of E. The height of E can be bounded by some constant times $y^{\frac{2}{3}}$; then its width can be bounded by some constant times $y^{\frac{1}{3}}$. Thus the volume of K is smaller than $(x+4)y^{\frac{2}{3}}y^{\frac{1}{3}}=(x+4)y$, up to a constant, and the result follows. □

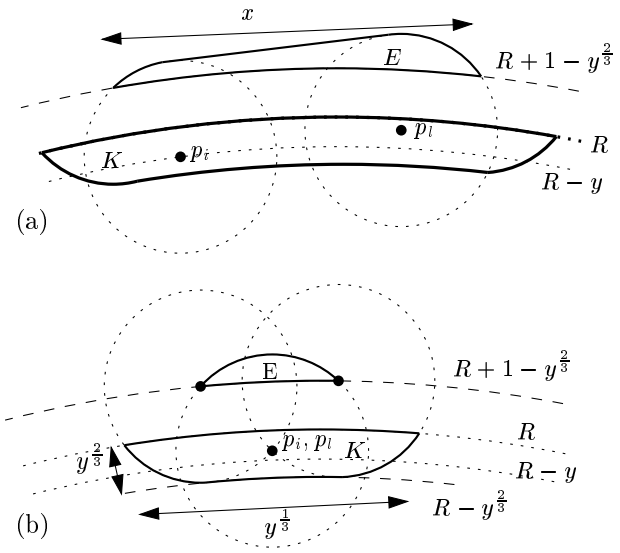


Figure 4. For the sketch of the proof of Lemma 19.

We can now bound the integral $I_{x>6,y<1}$ of Lemma 6.

Proposition 20 $I_{x>6,y<1}$ *is* $O\left(\frac{1}{n^3}\right)$.

[3] A complete proof is given in Appendix C.

PROOF. By Lemmas 6, 14, and 15, we get

$$I_{x>6,y<1} \leqslant \int_{x=6}^{2R}\int_{y=0}^{1} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y) \cdot \frac{9\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R},$$

where, by Lemma 13, $\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y)$ is equal to

$$\begin{aligned}
&\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right)\\
&\cdot \Pr\left(p\notin\mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}\\
&+\Pr\left(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right)\\
&\cdot \Pr\left(p\notin\mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}.
\end{aligned}$$

We split the integral at $x=6\sqrt{R}$. When $x\geqslant 6\sqrt{R}$, the distance from O to the tangent t_it_l is less than R (see the proof of Lemma 17), which is less than $R+1-y^{\frac{2}{3}}$ for any y in $(0,1)$. Thus, for any $y\in(0,1)$, the probability $\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y\right)$ is equal to 0. It then follows from Lemmas 17, 18 and 19 that $I_{x>6,y<1}\leqslant \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3$ with

$$\mathcal{I}_1=\int_{x=6}^{6\sqrt{R}}\int_{y=0}^{1} 81\frac{(x+4)^2y^2}{2R^6} \cdot 55\exp\left(-\frac{\mu}{12\sqrt{2}\,\pi}(x-5)y^2\right)\cdot\frac{9\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R},$$

$$\mathcal{I}_2=\int_{x=6}^{6\sqrt{R}}\int_{y=0}^{1}\left(\frac{3x+8}{R^3}\right)^2 \cdot 55\exp\left(-\frac{\mu}{12\sqrt{2}\,\pi}(x-5)y\right)\cdot\frac{9\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R},$$

$$\mathcal{I}_3=\int_{x=6\sqrt{R}}^{2R}\int_{y=0}^{1}\left(\frac{3x+8}{R^3}\right)^2 \cdot 55\cdot\exp\left(-\frac{\mu}{12\sqrt{2}\,\pi}(x-5)\right)\cdot\frac{9\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R}.$$

Computing these integrals with Maple, gives (assuming $\mu>0$), when R tends to infinity, that

$$\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3=O\left(\frac{1}{R^9}\right).$$

Since $R^3=n/\mu$, we get $I_{x>6,y<1}=O(\frac{1}{n^3})$. □

This concludes the proof of Theorem 1, because, by Lemmas 3, 6, and Propositions 9, 12, and 20, the expected number of $T4$-segments is smaller than

$$\sum_{(i,j,k,l)\in\mathcal{N}}\sum_{\omega=1}^{12}\left(O\left(\frac{1}{n^3}\right)+O\left(\frac{1}{n^3}\right)+O\left(\frac{1}{n^3}\right)\right)=O(n).$$

4. THE EXPECTED SIZE OF THE VISIBILITY COMPLEX IS LINEAR

In this section we prove Theorem 2, that the expected size of the visibility complex of a set of n uniformly distributed unit balls is linear.

We assume that the centers of the balls are in general position since this occurs with probability 1. This implies that no subset of five balls admits a common tangent. Thus, a k-dimensional face of the visibility complex is a connected set of maximal free segments with k degrees of freedom, that is, tangent to exactly $4-k$ balls.

We first give a lower bound on the expected size of the visibility complex under the distribution defined in Section 2.

Lemma 21 *The expected size of the visibility complex of n uniformly distributed unit balls is $\Omega(n)$.*

PROOF. We bound from below the expected number of 3-faces. Note that if a ball B_i is not intersected by any other ball, then there is at least one 3-face in the visibility complex formed by free segments tangent to B_i. Now, let us choose a ball B_i. The probability that another ball does not intersect B_i is greater than

$$1-\frac{\text{Volume of a ball of radius 2}}{\text{Volume of }\mathcal{U}}=1-\frac{8}{R^3}=1-\frac{8\mu}{n}.$$

Thus the probability that none of the $n-1$ other balls intersect B_i is greater than $\left(1-\frac{8\mu}{n}\right)^{n-1}$ which is greater than some constant $c>0$. It follows that the expected number of balls not intersected by any other ball is greater than $c\cdot n$. Thus, the expected number of 3-faces, which is a lower bound on the size of the complex, is at least linear. □

We now give an upper bound on the expected number of k-faces, for $k=0,\ldots,4$.

Lemma 22 *The expected number of 0-faces is at most linear.*

PROOF. A 0-face of the visibility complex is a maximal free line segment tangent to 4 balls. Each maximal free line segment tangent to 4 balls contains a $T4$-segment and each $T4$-segment is contained in one maximal free line segment. Thus, by Theorem 1, the expected number of 0-faces is at most linear. □

To deal with the faces of dimension $k\geqslant 1$, we divide them into two classes. A k-face is *open* if it is incident to at least one $(k-1)$-face, otherwise it is *closed*. Thanks to the general position, the number of k-faces incident to a $(k-1)$-face is constant. In the proof of the following lemmas, any upper bound on these constants can be used.

However, for completeness, we will use the exact values, but without justifying them.

Lemma 23 *The expected number of 1-faces is at most linear.*

PROOF. Note that a 0-face corresponds to a maximal free segment tangent to 4 balls and it is incident to those 1-faces corresponding to free segments tangent to 3 amongst those 4 balls. So, a 0-face is incident to exactly four 1-faces, which implies that the number of open 1-faces is bounded by 4 times the number of 0-faces, and is thus linear by the previous lemma.

Proving that the expected number of closed 1-faces is linear can be done in a way very similar to the proof of Theorem 1. The difference is that we consider now only

three balls and thus in all proofs, we forget ball B_k. We have to consider only $\binom{n}{3}$ triples of balls instead of $\binom{n}{4}$ quadruples, but we remove from the integral the probability $\Pr(p_k \in \mathcal{H}(p_i, p_l, 2) | \mathbf{x}_{i,l} = x) \leqslant \frac{3x+8}{R^3}$. Since $\frac{n}{R^3} = \mu$, this amounts to dividing the terms over which we integrate by $\mu(3x + 8)$ which does not change the general shape of the integrals (a polynomial multiplied by an exponential) which are convergent. Notice that B_i, B_j, B_l and ω now define a set of segments $t_i t_l$, and not just one anymore. However, those segments make a 3-face only if none of them is occluded by one of the $n-3$ remaining balls. Any particular choice of a tangent $t_i t_l$ in the 1-face will give a relevant cylinder $\mathcal{H}(t_i, t_l, 1)$ to use in the proofs. □

Lemma 24 *The expected number of 2-faces is at most linear.*

PROOF. Since a 1-face has three incident 2-faces, the linear bound on the number of 1-faces gives a linear bound on the number of open 2-faces. The closed case is solved similarly to the proof of Theorem 1. We now consider $O(n^2)$ pairs of balls B_i, B_l and we remove from the integrals the probability $\Pr(p_j, p_k \in \mathcal{H}(p_i, p_l, 2) | \mathbf{x}_{i,l} = x) \leqslant \left(\frac{3x+8}{R^3}\right)^2$ which gives an $O(n)$ bound. □

Lemma 25 *The expected number of 3-faces and 4-faces is at most linear.*

PROOF. A 3-face, corresponding to lines tangent to a ball, can only be closed if $n = 1$. The number of open 3-faces is linear by the fact that in general position a 2-face is incident to four 3-faces. The number of 4-faces is linear since a 3-face is incident to three 4-faces. □

5. WORST-CASE LOWER BOUND

Theorem 26 *The number of k-faces in the visibility complex of n disjoint unit balls in $\mathbb{R}^3$ is $\Omega(n^2)$ for all k between 0 and 4.*

PROOF. We first observe that the size of the visibility complex of n unit balls can trivially be quadratic by having the balls sparsely distributed in the space such that any pair of balls defines a closed 2-face.

Getting a quadratic number of free lines tangent to four balls amongst a set of n unit balls can be done by taking the balls B_i centered at $(2i, 0, 0)$ for $1 \leqslant i \leqslant \frac{n}{2}$ and the balls B'_j centered at $(2j-1, 10, 0)$ for $1 \leqslant j \leqslant \frac{n}{2}$. Then, for any i and j, the line through the points $(2i+1, 0, 1)$ and $(2j, 10, 1)$ is free and can be moved down so that it comes into contact with the four balls B_i, B_{i+1}, B'_j and B'_{j+1}. This argument proves that the number of k-faces ($0 \leqslant k \leqslant 2$) can be quadratic for all k.

The free segment $(2i, 1, 0)(2j-1, 9, 0)$ belongs to a 4-face linking B_i to B'_j, thus there is a quadratic number of 4-faces, and the bound applies also to 3-faces by considering lines tangent to B_i and stabbing B'_j.

In the above construction, the balls can be pushed together so that they fit in a spherical universe of radius $\sqrt[3]{n/\mu}$ without changing the result. Note also that the above construction can be slightly perturbed to obtain the same result for a set of n unit balls, disjoint or not, with no 4 centers coplanar. □

6. CONCLUSION

We conclude with several remarks and possible generalizations of our model.

- **Disjointness.** In our model, the balls are not disjoint. It would also be interesting to obtain a similar result in a model where the balls are disjoint, but unfortunately, in such a model the positions of the centers are no longer independent which makes the analysis much more intricate.

- **Smooth convex universe.** If the universe is no longer a ball, but a homothet of a smooth convex set with homothety factor proportional to $\sqrt[3]{n}$, the results can be generalized (the radius of curvature of the boundary of the universe plays the role of R in the proofs of the lemmas dealing with tangents outside the universe).

- **Balls of various radii.** The hypothesis that the balls have radii 1 is only really used in the analysis of the boundary of the universe. If we assume that the radii are in some interval $[r_{min}, r_{max}]$, the proof of the linear bound on the expected number of $T4$-segments inside the universe generalizes easily by considering the volumes $\mathcal{H}(p_i, p_l, 2r_{max})$ and $\mathcal{H}(t_i, t_l, r_{min})$.

- **Polyhedra of bounded aspect ratio.** This idea can be generalized further: consider polyhedra of constant complexity having concentric balls of radii r_{min} and r_{max} including and enclosing each polyhedron. In such a case, provided that the centers of the balls are uniformly distributed, the result on the expected number of $T4$-segments far from the boundary of the universe applies. This generalization is really powerful and interesting since such hypotheses on the shape of the objects are more realistic. In particular it suggests that the expected size of the visibility complex is linear for scenes arising in practice.

- **Other visibility structures**. The bounds on the expected size of the visibility complex proved in this paper also provide new bounds on the complexity of other visibility structures. Consider for instance the aspect graph, a partition of viewpoint space into maximal connected regions by surfaces along which visibility events are observed. As explained in [15], the complexity of the aspect graph is dominated by δ^m, where δ is the degree of the surface corresponding to lines "tangent" to 3 objects and m the dimension of the viewpoint space. For a scene composed of n disjoint spheres, δ is trivially $O(n^3)$, so the aspect graph has $O(n^6)$ orthographic views and $O(n^9)$ perspective views. But the results of this paper show that the expected value of δ is $\Theta(n)$, since the expected number of families of lines tangent to three objects (related to the 1-faces of the visibility complex) is linear and the degree of each family is bounded. Thus the expected size of the aspect graph is $O(n^2)$ in the orthographic case and $O(n^3)$ in the perspective case. Similar observations hold for the polyhedral case.

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We would like to thank Luc Devroye for useful discussions.

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APPENDIX

These appendices contain the detailed proofs of Lemmas 15, 16 and 19 and are intended to enable the referees to verify the correctness of the result in the main paper.

A. PROOF OF LEMMA 15

We prove in this section Lemma 15 which states that

For $6 \leqslant x \leqslant 2R$ and $y \leqslant 1$, we have

$$\Pr((x < \mathbf{x}_{i,l} \leqslant x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y) \leqslant \frac{9\, x\, y\, dx}{R^3}.$$

Let $f(x, y)$ denote that probability and refer to Figure 5. Let us define the balls B_1 with center O and radius R, B_2 with center O and radius $R - y$, B_3 with center p_i and radius x and finally B_4 with center p_i and radius $x + dx$. The probability $f(x, y)$ is equal to the volume of $\mathcal{V} = (B1 \setminus B2) \cap (B4 \setminus B3)$ divided by $\frac{4}{3}\pi R^3$, the volume of the universe $\mathcal{U}$.

Let (p_i, X, Y, Z) be an orthogonal reference frame whose center is p_i and such that the X-axis is oriented along $\overrightarrow{Op_i}$ (see Figure 5). To compute the volume of $\mathcal{V}$, we integrate over X the area of the section of $\mathcal{V}$ by planes $\mathcal{P}_X$ that are orthogonal to the X-axis (Op_i) and intersect it at the X-coordinate X.

Let C_1 (resp. C_2, C_3, C_4) denote the circle that is the boundary of the intersection of B_1 (resp. B_2, B_3, B_4) and the plane (p_i, X, Y). The equations of these circles are, in the frame (p_i, X, Y),

$$\begin{array}{lrl} C_1: & (X + R - y)^2 + Y^2 & = R^2, \\ C_2: & (X + R - y)^2 + Y^2 & = (R - y)^2, \\ C_3: & X^2 + Y^2 & = x^2, \\ C_4: & X^2 + Y^2 & = (x + dx)^2. \end{array}$$

Note that C_3 and C_4 properly intersect C_1 and C_2 (see Figure 5), or only C_1 (see Figure 6), or neither C_1 nor C_2. Indeed, first C_3 and C_4 either intersect C_2 or contain it, since they are centered at $p_i \in C_2$. Second, C_3 and C_4 either intersect C_1 or contain it, since they are centered at $p_i \in C_2$ and their radii are greater than or equal to $x \geqslant 6$, which is bigger than $y \leqslant 1$, the distance from p_i to C_1. Third, C_3 and C_4 cannot intersect C_2 but not C_1 because it would imply that C_3 and C_4 are entirely contained inside C_1, contradiction.

In the case where C_3 and C_4 intersect neither C_1 nor C_2, the result is obvious because, then, the volume of $\mathcal{V}$ is 0, thus the probability $f(x, y) = 0 \leqslant \frac{9\, x\, y\, dx}{R^3}$. We consider separately the two other cases.

Case 1: C_3 and C_4 properly intersect C_1 and C_2.

Refer to Figure 5 and let X_1, X_2, X_3 and X_4 be defined as follows: X_1 is the X-coordinate of the points of intersection of circles C_2 and C_4, X_2 is the X-coordinate of the points of intersection of circles C_2 and C_3, X_3 is the X-coordinate of the points of intersection of circles C_1 and C_4, X_4 is the X-coordinate of the points of intersection of circles C_1 and C_3. Note that $X_1 < X_2 < X_3 < X_4$.

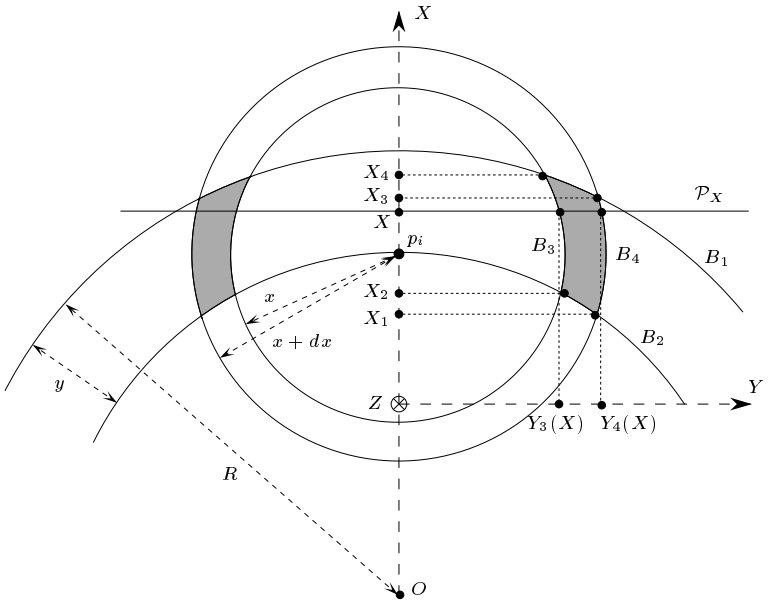


Figure 5. For the proof of Lemma 15: Case 1.

We now compute the area of the section of $\mathcal{V}$ with the plane $\mathcal{P}_X$, distinguishing three cases depending on the value of X with respect to $X_1 < X_2 < X_3 < X_4$. Let us first define $Y_m(X)$, $m = 1, \ldots, 4$, to be the positive Y-coordinate of a point on

the circle C_m with X-coordinate X. The equations of the circles $C_1, \ldots, C_4$ give

$$\begin{array}{ll} Y_1^2(X) & = R^2 - (X + R - y)^2, \\ Y_2^2(X) & = (R-y)^2 - (X + R - y)^2, \\ Y_3^2(X) & = x^2 - X^2, \\ Y_4^2(X) & = (x + dx)^2 - X^2. \end{array}$$

We now distinguish the three cases depending on the value of X.

- If $X_2 < X < X_3$ then $\mathcal{V} \cap \mathcal{P}_X$ is the annulus that is the disk $B_4 \cap \mathcal{P}_X$ minus the disk $B_3 \cap \mathcal{P}_X$ (see Figure 5). The area of this annulus is equal to

$$\pi Y_4^2(X) - \pi Y_3^2(X) = \pi((x+dx)^2 - X^2) - \pi(x^2 - X^2) = \pi(2\,x\,dx + dx^2).$$

- If $X_1 < X < X_2$ then, similarly, $\mathcal{V} \cap \mathcal{P}_X$ is the annulus $B_4 \cap \mathcal{P}_X \setminus B_2 \cap \mathcal{P}_X$, and its area is

$$\pi Y_4^2(X) - \pi Y_2^2(X) = \pi(x+dx)^2 + 2\pi X(R-y).$$

- Finally, if $X_3 < X < X_4$ then $\mathcal{V} \cap \mathcal{P}_X$ is the annulus $B_1 \cap \mathcal{P}_X \setminus B_3 \cap \mathcal{P}_X$, and its area is

$$\pi Y_1^2(X) - \pi Y_3^2(X) = \pi(2Ry - y^2 - 2X(R-y) - x^2).$$

The volume of $\mathcal{V}$ is then given by

$$\pi \left(\int_{X_1}^{X_2} (x+dx)^2 + 2X(R-y)dX + \int_{X_2}^{X_3} 2\,x\,dx + dx^2 dX \; + \int_{X_3}^{X_4} 2Ry - y^2 - 2X(R-y) - x^2 dX \right)$$

which is equal to

$$\pi[(x+dx)^2(X_2 - X_1) + (R-y)(X_2^2 - X_1^2) + (2\,x\,dx + dx^2)(X_3 - X_2) \\ + (2Ry - y^2 - x^2)(X_4 - X_3) - (R-y)(X_4^2 - X_3^2)].$$

We now compute the values of X_1, X_2, X_3 and X_4 which come directly from the equations of the circles. We get

$$X_1 = \frac{-(x+dx)^2}{2(R-y)}, \qquad X_3 = \frac{R^2 - (x+dx)^2 - (R-y)^2}{2(R-y)},$$
$$X_2 = \frac{-x^2}{2(R-y)}, \qquad X_4 = \frac{R^2 - x^2 - (R-y)^2}{2(R-y)}.$$

By substituting X_1, X_2, X_3, X_4 into the above equation, we get that the volume of $\mathcal{V}$ is equal to

$$\begin{array}{rcl} \pi \dfrac{4Rx + 2Rdx - 2yx - ydx}{R-y} y dx & \leqslant & \pi \dfrac{4Rx + 2Rdx}{R-y} y dx \\ & \leqslant & \pi \dfrac{6Rx}{R-y} y dx \end{array}$$

since $dx < x$. Now, since $x \geqslant 6$, $R \geqslant 3$, thus $y < R/2$ and $R - y > R/2$. Thus, the volume of $\mathcal{V}$ is smaller than

$$\pi \frac{6\,R\,x}{R/2}\, y\, dx = 12\,\pi\, x\, y\, dx$$

and the probability

$$f(x,y) \leqslant \frac{12\,\pi\, x\, y\, dx}{\frac{4}{3}\,\pi\, R^3} = \frac{9\, x\, y\, dx}{R^3}.$$

Case 2: C_3 and C_4 properly intersect C_1 but not C_2.

Refer to Figure 6 and let $X_1 = -(x+dx), X_2 = -x$, and X_3, X_4 and $Y_i^2(X)$, $i = 1, \ldots, 4$, be defined as in Case 1. Similarly as before, we get that the volume of $\mathcal{V}$ is then given by

$$\pi \left(\int_{X_1}^{X_2} Y_4^2(X)\, dX + \int_{X_2}^{X_3} (Y_4^2(X) - Y_3^2(X))\, dX \; + \int_{X_3}^{X_4} (Y_1^2(X) - Y_3^2(X))\, dX \right).$$

Computing these integrals with Maple gives that the volume of $\mathcal{V}$ is equal to

$$x(x+y)(2\,R - x - y)\frac{\pi\, dx}{(R-y)} + (2\,(R-y)\,(2x+y) - 3x^2 + y^2)\frac{\pi\, dx^2}{2(R-y)} + (2\,(R-y) - 3x)\frac{\pi\, dx^3}{3(R-y)} - \frac{\pi\, dx^4}{4(R-y)}. \qquad (3)$$

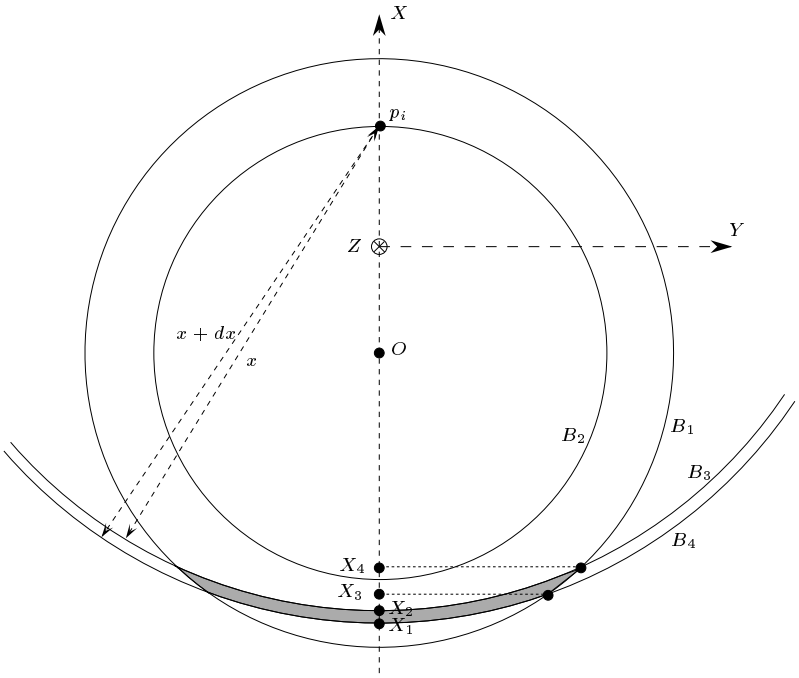


Figure 6. For the proof of Lemma 15: Case 2.

All but the first term of (3) are negative. Indeed, $R - y > 0$ ($y \leqslant 1$ and $R \geqslant 3$) and, since C_3 does not properly intersect C_2, we have $2(R-y) \leqslant x$, thus $(2\,(R-y) - 3x) \leqslant -2\,x < 0$ and

$$2\,(R-y)\,(2x+y) - 3x^2 + y^2 \leqslant x\,(2x+y) - 3x^2 + y^2 = -x^2 + x\,y + y^2 \leqslant -x^2 + x + 1 < 0$$

for any $x \geqslant 6$.

Consider now the first term of (3). It follows from $2(R-y) \leqslant x$ that $(2\,R - x - y) \leqslant y$. Moreover, since C_3 properly intersects C_1, we have $x \leqslant 2(R-y) + y$, thus $(x+y) \leqslant 2R$. Since, as before, $\frac{1}{R-y} \leqslant \frac{2}{R}$, we get that the first term of (3) is smaller than $x\,2R\,y\,\pi\,dx\frac{2}{R} = 4\pi\,x\,y\,dx$. It follows that

$$f(x,y) \leqslant \frac{4\,\pi\,x\,y\,dx}{\frac{4}{3}\,\pi\,R^3} = \frac{3\,x\,y\,dx}{R^3} \leqslant \frac{9\,x\,y\,dx}{R^3},$$

ending the proof of Lemma 15.

B. PROOF OF LEMMA 16

We prove in this section Lemma 16 which states that

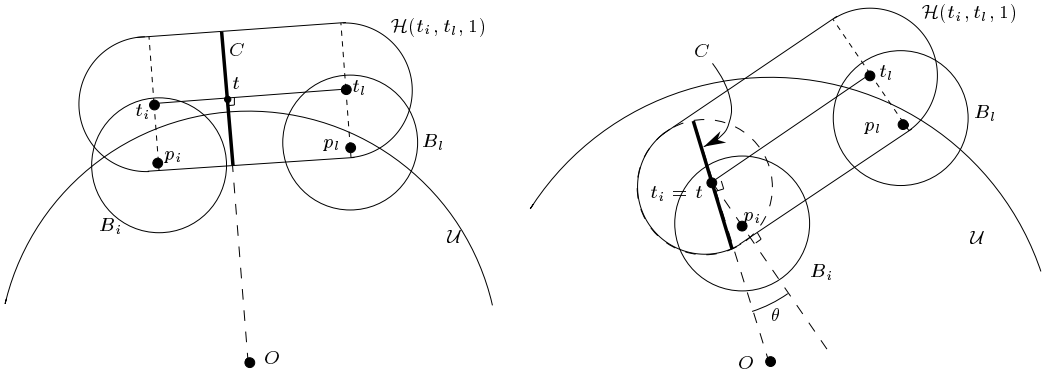
> *For any i, j, k, l, ω and $0 \leqslant s \leqslant 1$ such that $\mathbf{x}_{i,l} \geqslant 6$, $\mathbf{y}_l \leqslant \mathbf{y}_i \leqslant 1$ and segment $t_i t_l$ is at distance less than $R + 1 - s$ from the center of $\mathcal{U}$, the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is larger than $\frac{1}{9\sqrt{2}}\,(x-5)\,s\,\sqrt{s}$.*

Recall that O denotes the center of $\mathcal{U}$. Let v denote the length such that the distance from O to the segment $t_i t_l$ is $R+1-v$, and let t be the point on segment $t_i t_l$ realizing this distance (see Figure 7) Assume without loss of generality that t is closer to t_i than to t_l. Let C (resp. D) be the unit radius circle (resp. disk) centered at t in the plane, denoted $\mathcal{P}$, containing the vectors $\overrightarrow{Ot}$ and $\overrightarrow{Ot} \times \overrightarrow{t_i t_l}$. Let θ be the angle between the plane orthogonal to $t_i t_l$ and $\mathcal{P}$. We first prove the following lemma.

Lemma 27 *The volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{3}\min(2, \frac{v\sqrt{v}}{\sqrt{2}}) \cdot \min(\frac{x-2}{2}, (x-2)\cos\theta - 1)$.*

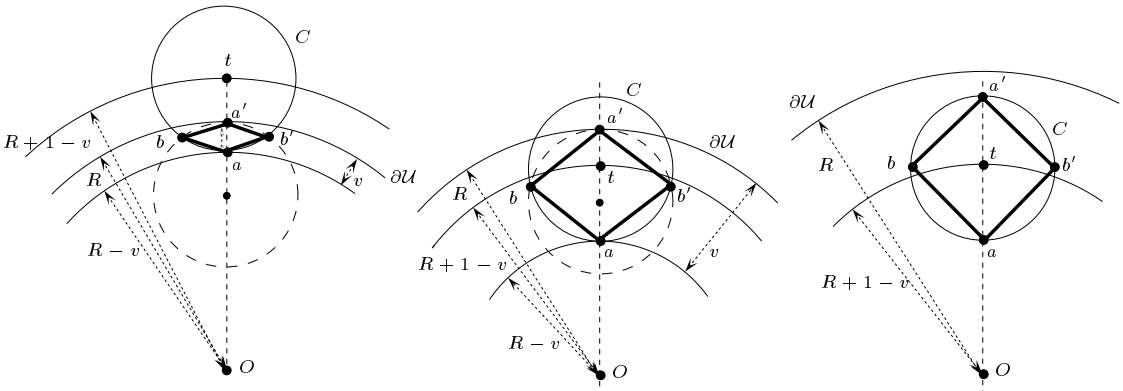
Proof. Let a denote the closest point on C from O, a' the farthest point in $D \cap \mathcal{U}$ from O, and b and b' the two points of intersection of C and the perpendicular bisector of segment aa' (see Figure 8).

The volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than the volume of the convex hull whose vertices are a, b, a', b' and p_l because $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is (strictly) convex and contains these five points. The volume of this polyhedron is equal to one third of the area of its base, the quadrilateral with vertices a, b, a', b', times its height, the distance from p_l to the plane $\mathcal{P}$ containing a, b, a', b'.



(a) t is not equal to t_i or t_l; $\theta = 0$. (b) t is equal to t_i.

Figure 7. For the definition of t and C (C is shown from the side view).



(a) $v \in [0, 1]$ (b) $v \in [1, 2]$ (c) $v > 2$

Figure 8. For the definition of a, a', b, b'.

We first compute a lower bound on the area of the quadrilateral with vertices a, b, a', b'. First, if $v \leqslant 2$ (see Figure 8a and b), then the length of aa' is equal to v, and a simple calculation gives that the length of bb' is equal to $2\sqrt{v - \frac{v^2}{4}} \geqslant \sqrt{2v}$ (for $v \leqslant 2$). Thus the area of the quadrilateral a, b, a', b' is greater than $\frac{v\sqrt{v}}{\sqrt{2}}$. Second, if $v > 2$ (see Figure 8c), then C is entirely included in $\mathcal{U}$ and the area of the quadrilateral a, b, a', b' is equal to 2. Thus, the area is at least $\min(2, \frac{v\sqrt{v}}{\sqrt{2}})$.

The volume of the polyhedron is thus greater than $\frac{1}{3}\min(2, \frac{v\sqrt{v}}{\sqrt{2}})$ times the distance from p_l to the plane $\mathcal{P}$. We consider two cases.

First, suppose that t belongs to the interior of the segment $t_i t_l$ (see Figure 7a). Then, the height is equal to the distance from t_l to t since p_l and a, b, a', b' belong, respectively, to the two planes, orthogonal to $t_i t_l$ and passing through t_l and t, respectively. Since t_i and t_l belong to B_i and B_l, they are at least distance $x - 2$ apart, thus t and t_l are at least distance $\frac{x-2}{2}$ apart. Thus, the height from p_l to $\mathcal{P}$ is at least $\frac{x-2}{2}$.

Second, suppose that $t = t_i$ (see Figure 7b); $t \neq t_l$ since we assumed that t is closer to t_i than to t_l. Refer to Figure 9. Let A and B be the orthogonal projections of p_l and t_l onto $\mathcal{P}$, respectively. Note that the lengths of Ap_l and Bt_l are the distances from p_l and t_l to the plane $\mathcal{P}$, respectively. Considering the triangle $\triangle Ap_l t_l$, we see that the length of Ap_l is greater than the length of At_l minus the length of $t_l p_l$, and since the distance between t_l and p_l is less than 1, the length of Ap_l is greater than the length of At_l minus 1. Since $A \in \mathcal{P}$ and the length of Bt_l is the distance from t_l to the plane $\mathcal{P}$, the length of At_l is greater than that of Bt_l. So we have that the length of Ap_l is greater than the length of Bt_l minus 1. To bound the length of Bt_l, we now consider the triangle $\triangle Bt_l t$. The angle $\angle Bt_l t$ is the angle between the normal of the plane $\mathcal{P}$ and $t_i t_l$, that is, by definition, θ. So the length of Bt_l is the length of $t_i t_l$ times $\cos\theta$ and, since $t_i t_l$ is at least $x - 2$, is greater than $(x - 2)\cos\theta$. Thus the length of Ap_l is greather than $(x - 2)\cos\theta - 1$.

Finally, the distance from p_l to the plane $\mathcal{P}$ is greater than $\min(\frac{x-2}{2}, (x - 2)\cos\theta - 1)$, so we just proved that V is greater than $\frac{1}{3}\min(2, \frac{v\sqrt{v}}{\sqrt{2}}) \cdot \min(\frac{x-2}{2}, (x - 2)\cos\theta - 1)$. □

The following lemma bounds $\cos\theta$.

Lemma 28 *For any two balls B_i and B_l whose centers are distance $x \geqslant 6$ apart, and at distance less than 1 from the boundary*

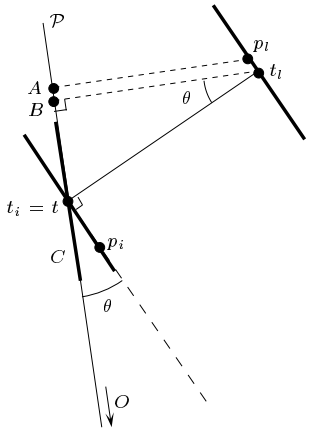


Figure 9. The height from p_l to $\mathcal{P}$ is greater than $t_i t_l \cos\theta - 1$.

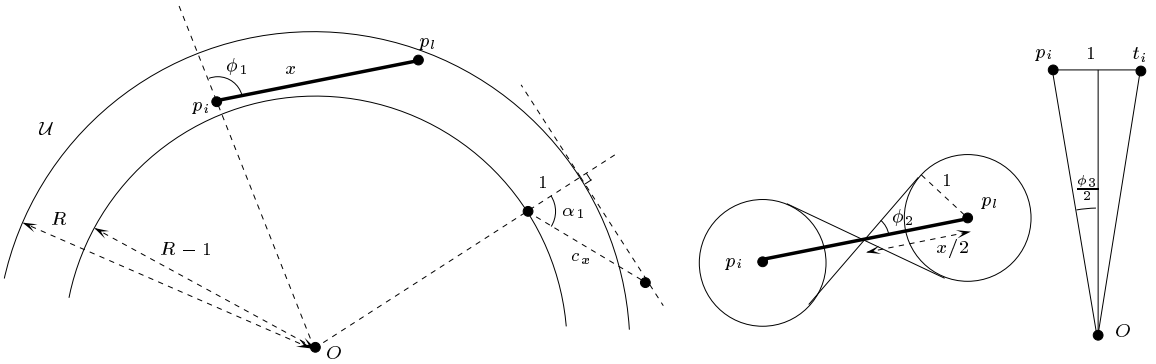


Figure 10. For the proof of Lemma 28.

of $\mathcal{U}$, and for any line segment $t_i t_l$ tangent to B_i and B_l at t_i and t_l, respectively, the angle θ is such that $\cos\theta > \frac{1}{3}$.

Proof. Note first that this lemma is intuitively obvious: indeed (see Figure 7b), if x is sufficiently large, and if t_i is the closest point on segment $t_i t_l$ to O, then the angle between the plane supporting C and the segment $t_i t_l$ is necessarily close to $\pi/2$, which implies that θ is close to 0. We now prove the lemma.

Let c_x be a positive constant (that we will later choose equal to 6) and refer to Figure 10. Let α_1 denote the angle in $(0, \pi/2)$ equal to $\arccos(1/c_x)$. If p_i and p_l are farther than c_x apart and within distance 1 from the boundary of $\mathcal{U}$, then angle $\phi_1 = \angle(\overrightarrow{Op_i}, \overrightarrow{p_i p_l})$ belongs to $[\alpha_1, \pi] = [\arccos(1/c_x), \pi]$.

On the other hand, if p_i and p_l are distance $x \geqslant c_x$ apart, then the angle $\phi_2 = \angle(\overrightarrow{t_i t_l}, \overrightarrow{p_i p_l})$ belongs to $[0, \arcsin(2/x)] \subset [0, \arcsin(2/c_x)]$. Since $\angle(\overrightarrow{Op_i}, \overrightarrow{p_i p_l}) \leqslant \angle(\overrightarrow{Op_i}, \overrightarrow{t_i t_l}) + \angle(\overrightarrow{t_i t_l}, \overrightarrow{p_i p_l})$, we get

$$\angle(\overrightarrow{Op_i}, \overrightarrow{t_i t_l}) \geqslant \angle(\overrightarrow{Op_i}, \overrightarrow{p_i p_l}) - \angle(\overrightarrow{t_i t_l}, \overrightarrow{p_i p_l}) = \phi_1 - \phi_2 \in [\arccos(1/c_x) - \arcsin(2/c_x), \pi].$$

One can show that the function $\arccos(1/u) - \arcsin(2/u)$ in u is increasing and is positive for $u > \sqrt{5} \sim 2.24$. Thus, for $x \geqslant c_x = 6$, the angle $\angle(\overrightarrow{Op_i}, \overrightarrow{t_i t_l}) \geqslant \arccos(1/6) - \arcsin(2/6) \sim 1.06 > 1$.

Now, $\theta = \frac{\pi}{2} - \angle(\overrightarrow{Ot_i}, \overrightarrow{t_i t_l})$ and $-\angle(\overrightarrow{Ot_i}, \overrightarrow{t_i t_l}) \leqslant \angle(\overrightarrow{Op_i}, \overrightarrow{Ot_i}) - \angle(\overrightarrow{Op_i}, \overrightarrow{t_i t_l}) < \angle(\overrightarrow{Op_i}, \overrightarrow{Ot_i}) - 1$, for $x \geqslant c_x = 6$. The angle $\phi_3 = \angle(\overrightarrow{Op_i}, \overrightarrow{Ot_i})$ is maximum when the triangle $\triangle O p_i t_i$ is isosceles at O; the angle is then equal to $2\arcsin(1/2d(O, p_i))$ which is smaller than $2\arcsin(\frac{1}{2(R-1)})$ since $d(O, p_i) \geqslant R - 1$. Thus, $\phi_3 = \angle(\overrightarrow{Op_i}, \overrightarrow{Ot_i}) \leqslant 2\arcsin(\frac{1}{2(R-1)})$. It follows that $\theta < \frac{\pi}{2} + 2\arcsin(\frac{1}{2(R-1)}) - 1$. Furthermore, if $x \geqslant c_x = 6$, then $R \geqslant 3$, thus $\frac{1}{2(R-1)} \leqslant \frac{1}{4}$, and $\arcsin(\frac{1}{2(R-1)}) \leqslant \arcsin(\frac{1}{4})$. It follows that $2\arcsin(\frac{1}{2(R-1)}) - 1 \leqslant 2\arcsin(\frac{1}{4}) - 1 < -0.4$. Thus $\theta < \frac{\pi}{2} - 0.4 \in (0, \pi/2)$ and $\cos\theta > \cos(\frac{\pi}{2} - 0.4) > \frac{1}{3}$. □

We can now conclude the proof of Lemma 16. For any $0 \leqslant s \leqslant 1$, if segment $t_i t_l$ is at distance $R+1-v \leqslant R+1-s$ from the center of $\mathcal{U}$, then $v \geqslant s$ and, by Lemma 28, $(x-2)\cos\theta - 1 > \frac{x-5}{3}$ which means that $\min(\frac{x-2}{2}, (x-2)\cos\theta - 1) = \frac{x-5}{3}$. Thus, combining these facts with Lemma 27 gives that the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{9\sqrt{2}}(x-5)\min(2\sqrt{2}, v\sqrt{v}) \geqslant \frac{1}{9\sqrt{2}}(x-5)\min(2\sqrt{2}, s\sqrt{s}) \geqslant \frac{1}{9\sqrt{2}} s\sqrt{s}(x-5)$ since $s \leqslant 1$. So the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{9\sqrt{2}} s\sqrt{s}(x-5)$.

C. PROOF OF LEMMA 19

We prove here Lemma 19 which states that

If $y < 1$ *then* $\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant y\right) \leqslant 81\dfrac{(x+4)^2y^2}{2R^6}$.

Let F be the open ball with center O and radius $R+1-y^{\frac{2}{3}}$, F_b its frontier, and E the part of $\mathcal{H}(p_i,p_l,1)$ that is outside F (see Figure 11). Note that p_j and p_k are at distance 1 from any tangent they define, and that a far tangent is, by definition, contained in E. It follows that a far tangent can only exist if p_j and p_k are at distance at most 1 from E, and, of course, inside the universe.

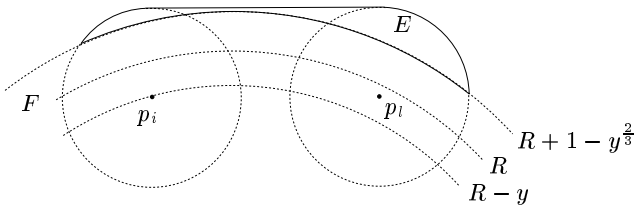


Figure 11. The part E of $\mathcal{H}(p_i,p_l,1)$ outside F.

Lemma 29 *If* $z \in \mathcal{U}$ *is at distance less than 1 from* E*, then* z *is at distance less than* 1 *from* $E \cap F_b$.

Proof. Let z be a point in $\mathcal{U}$ whose distance to E is smaller than 1, and let w be a point of E such that $|zw| \leqslant 1$. Note that E is the union of the parts outside F of the unit balls whose centers belong to $[p_i,p_l]$, so there exists p in $[p_i,p_l]$ such that w is in $B(p,1)$. This ensures that F_b meets $B(p,1) \cap B(w,1)$ (Figure 12 shows the situation in the plane (z,p,w), in which F_b appears as a circle separating w from z and p). Hence, there is at least one point in $B(p,1) \cap F_b$ at distance less than 1 from z. Since $B(p,1) \cap F_b$ is contained in $E \cap F_b$, this proves our claim. □

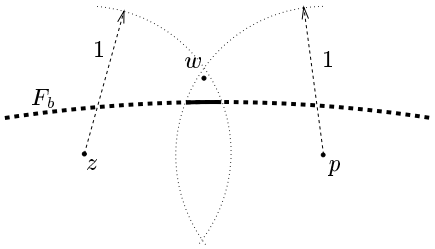


Figure 12. $B(p,1) \cap B(w,1)$ meets F_b.

So the existence of a far tangent requires that p_j and p_k are at distance at most 1 from $E \cap F_b$. To bound the probability of such an event, we now enclose $E \cap F_b$ in a subset of F_b easier to deal with. Let S be a sphere whose radius is greater than 1, Δ a ray starting at its center and B a unit ball centered on Δ. The intersection $B \cap S$, when not empty, is a spherical cap centered on $\Delta \cap S$. There exists a position for the center of B on Δ such that the caps $B \cap S$ is maximal for the inclusion. Now, for $p \in \mathcal{U}$, let $\pi(p)$ be the point of the segment $[Op] \cap \mathcal{U}$ such that $B(\pi(p),1) \cap F_b$ is maximal. Note that π is also the projection on the sphere with center O and radius $R' = \min(R, \sqrt{(R+1-y^{\frac{2}{3}})^2 - 1})$. We define $G = \bigcup_{p \in [p_i,p_l]} B(\pi(p),1) \cap F_b$ (see Figure 13)).

By construction, G contains $E \cap F_b$. Let H denote the points of $\mathcal{U}$ at distance less than 1 from G. We will bound the probability that a far tangent exists by the probability that both p_j and p_k are in H. To bound the volume of H from above, we first estimate its section by planes containing O and orthogonal to (O,p_i,p_l). The section of G by such a plane is a circle arc on F_b symmetrical with respect to the plane (O,p_i,p_l) (see Figure 14).

Lemma 30 *The area of* $P \cap H$ *is less than* $(6\pi\sqrt{2}) \cdot y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}$.

Proof. The circle arc $G \cap P$ is maximal when P meets $[p_i,p_l]$, so we will assume that P is such a plane. Let a and b be the endpoints of $G \cap P$; they are the intersection points of F_b and a circle of radius 1 centered in c, the intersection point of P, the plane (O,p_i,p_l) and the sphere of center O and radius R'. The lines (Oa) and (Ob) split $P \cap H$ into three parts.

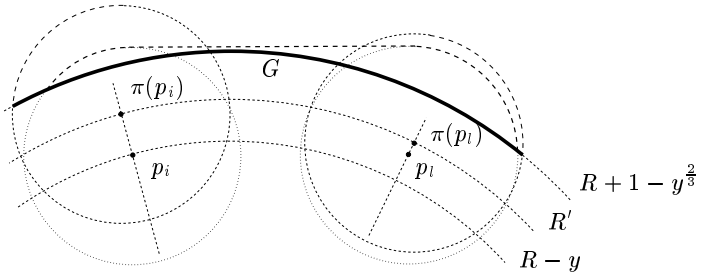


Figure 13. G, a part of F_b enclosing $E \cap F_b$.

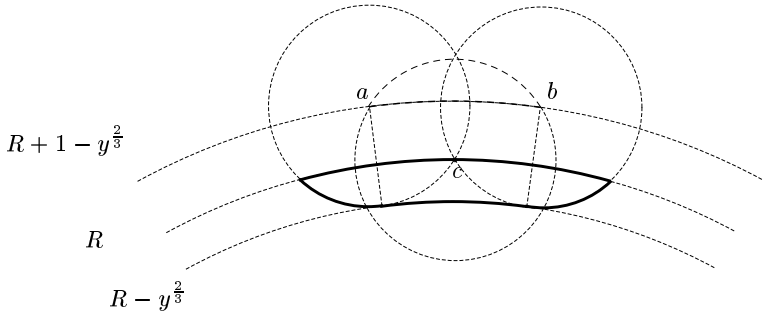


Figure 14. A section of H by a plane P when $R' = R$.

Symmetries with respect to the lines (Oa) and (Ob) send the left and right parts into the central one. Hence, the area of $P \cap H$ is bounded by 3 times the area of its central part. This part is delimited by the rays $[Oa)$ and $[Ob)$ and two circles with center O and radii R and $R - y^{\frac{2}{3}}$. So, if α denotes the length of the circle arc ab, the area A of the central part is given by

$$A = \frac{\alpha}{2\pi(R+1-y^{\frac{2}{3}})} \cdot \pi(R^2 - (R-y^{\frac{2}{3}})^2) = \frac{2Ry^{\frac{2}{3}} - (y^{\frac{2}{3}})^2}{2(R+1-y^{\frac{2}{3}})}\alpha \leqslant \alpha y^{\frac{2}{3}}.$$

Now, we want to bound the length α. We choose an orthonormal frame (c, X, Y) such that O has coordinates $(0, -R')$. Note that a is at the intersection of the circles with respective centers c and O and radii 1 and R'. Suppose that $R' = R$. Then, using the equations of those two circles we get:

$$Y_a = 1 - \frac{y^{\frac{2}{3}}}{2}(2 - \frac{1-y^{\frac{2}{3}}}{R}) \geqslant 1 - \frac{1}{2}y^{\frac{2}{3}} \qquad \text{which implies that} \qquad X_a = \sqrt{1-Y_a^2} \leqslant \sqrt{y^{\frac{2}{3}}}.$$

Now, suppose that $R' \neq R$. Then $(R+1-y^{\frac{2}{3}})^2 - 1 \leqslant R^2$, which implies $y^{\frac{2}{3}} \geqslant \dfrac{R}{R+1} \geqslant \dfrac{1}{2}$ which in turn ensures that $X_a \leqslant 1 \leqslant \sqrt{2y^{\frac{2}{3}}}$.

In both cases we find that $X_a \leqslant \sqrt{2y^{\frac{2}{3}}}$, which leads to $\alpha \leqslant (2\pi\sqrt{2})\sqrt{y^{\frac{2}{3}}}$. Hence, the area of the middle part is bounded by $(2\pi\sqrt{2})y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}$. This shows that $P \cap H$ is contained in a wedge sector whose area is less than $(6\pi\sqrt{2}) \cdot y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}$ □

To complete this analysis, let us express the volume of H by an integral using spherical coordinates with the ϕ-axis passing through O and orthogonal to the plane (O, p_i, p_l). This leads to

$$\text{Volume of } H = \int_\phi \int_r \int_\theta 1_H(r,\theta,\phi) \cdot r^2 \cos\phi \, dr \, d\theta \, d\phi.$$

Since H is inside the sphere of radius R we can bound this by

$$\text{Volume of } H \leqslant R \int_\phi \left(\int_r \int_\theta 1_H(r,\theta,\phi) \cdot r dr d\theta \right) \cos\phi \, d\phi.$$

Note that the double integral between parentheses represents the area of the section of H by a plane P_ϕ that is orthogonal to (O, p_i, p_j) and goes through O. Let $\Delta\phi$ denote the angle between the two extreme ϕ-planes that intersect H. Using the

bound we just computed on such an area, and bounding $\cos\phi$ from above by 1, we can write

$$\text{Volume of } H \leqslant R\Delta\phi \cdot A \leqslant R\Delta\phi \cdot (6\pi\sqrt{2}) \cdot y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}.$$

Note that $R\Delta\phi$ is bounded by the length λ of the intersection of G with the ϕ-plane that contains p_i and p_l. The endpoints of this circle arc are (see Figure 13) on circles with centers $\pi(p_i)$ and $\pi(p_l)$ and radius 1. Henceforth, we have

$$R\Delta\phi \leqslant \lambda \leqslant 1 + |\pi(p_i)p_i| + |p_i p_l| + |p_l\pi(p_l)| + 1 \leqslant x + 2 + 2y \leqslant x + 4.$$

This ensures that Volume of $H \leqslant (x+4) \cdot (6\pi\sqrt{2}) \cdot y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}$.

To conclude, we simply use that a far tangent defined by (i,j,k,l) only occurs if p_j and p_k fall in the volume H defined by p_i and p_l. Since p_j and p_k are independently and identically distributed, with a uniform distribution inside $\mathcal{U}$, the probability of the latter event is

$$\left(\frac{\text{Volume of } H}{\text{Volume of } \mathcal{U}}\right)^2 \leqslant \left(\frac{9(x+4)\sqrt{2} \cdot y^{\frac{2}{3}}\sqrt{y^{\frac{2}{3}}}}{2R^3}\right)^2 = 81\frac{(x+4)^2(y^{\frac{2}{3}})^3}{2R^6}.$$

And that finally proves that $\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant y\right) \leqslant 81\frac{(x+4)^2(y^{\frac{2}{3}})^3}{2R^6}$.