Example

P: hyperboloid of two sheets of equation

$$x^2 - y^2 + z^2 - xy - y + 1 = 0$$

Q ellipsoid of equation

$$2\,x^2 + y^2 + 20\,z^2 - 4\,yz + 4\,y - 20\,z - 20 = 0$$

$\triangleleft$ Levin algorithm $\rightsquigarrow$ Solution: 20 megabytes

$\triangleleft$ Our solution:

$$\boxed{x_i(u) = \Upsilon_i(u) \pm \Phi_i(u)\sqrt{\Delta(u)}}$$

$u \in \mathbb{R}$ such that $\Delta(u) \geqslant 0$

$\Upsilon_i(u)$, $\Phi_i(u)$, $\Delta(u)$: polynomials of degree 3, 1 and 4, with coefficients in $\mathbb{Q}(\sqrt{\xi})$ for some $\xi \in \mathbb{N}$

$$\begin{aligned}
x(u) &= \left(-\tfrac{48\sqrt{14}}{77} - \tfrac{96}{77}\right)u^3 + \left(\tfrac{1556}{847} + \tfrac{250\sqrt{14}}{847}\right)u^2 + \left(-\tfrac{6\sqrt{14}}{9317} + \tfrac{4}{9317}\right)u \\
&\qquad + \left(-\tfrac{40}{14641} + \tfrac{40\sqrt{14}}{102487}\right) \pm \left(1 + \tfrac{\sqrt{14}}{7}\right)u\sqrt{\Delta}, \\
y(u) &= -\tfrac{96}{77}u^3 + \left(\tfrac{4}{7} + \tfrac{492\sqrt{14}}{847}\right)u^2 + \tfrac{4}{9317}u + \tfrac{40\sqrt{14}}{102487} \pm \tfrac{\sqrt{14}}{7}u\sqrt{\Delta}, \\
z(u) &= \left(\tfrac{306}{77} + \tfrac{6\sqrt{14}}{7}\right)u^3 + \left(\tfrac{108\sqrt{14}}{847} + \tfrac{8}{77}\right)u^2 - \tfrac{744}{9317}u + \tfrac{8\sqrt{14}}{102487} \\
&\qquad \pm \left(-\tfrac{4}{121} + \tfrac{2\sqrt{14}}{77}u\right)\sqrt{\Delta}, \\
w(u) &= \left(-\tfrac{40}{7} - \tfrac{8\sqrt{14}}{7}\right)u^3 + \left(\tfrac{52\sqrt{14}}{847} + \tfrac{4}{77}\right)u^2 - \tfrac{512}{9317}u + \tfrac{4\sqrt{14}}{102487} \\
&\qquad \pm \left(-\tfrac{2}{121} + \tfrac{\sqrt{14}}{77}u\right)\sqrt{\Delta}, \text{ where} \\
\Delta(u) &= \left(-\tfrac{452}{7} - \tfrac{100\sqrt{14}}{7}\right)u^4 + \left(-\tfrac{56\sqrt{14}}{11} - \tfrac{384}{77}\right)u^3 + \tfrac{744}{121}u^2 \\
&\qquad + \left(\tfrac{16}{9317} - \tfrac{48\sqrt{14}}{9317}\right)u + \left(\tfrac{80\sqrt{14}}{102487} - \tfrac{56}{14641}\right)
\end{aligned}$$

Memory space gain factor: 17.000

Our algorithm (1)

In projective space,
find a quadric R in the pencil $R(\lambda) = P - \lambda Q$, $\lambda \in \mathbb{R}$,
such that $\det R(\lambda) \geqslant 0$, $\lambda \in \mathbb{Q}$

$$P \cap Q = R \cap Q$$

Compute Gauss linear transformation that sends R into the form $\quad \alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 = 0$

Parameterise R in its local frame
Ex: signature $(2,2)$: $\alpha, \beta > 0, \quad \gamma, \delta < 0$.

$$(x, y, z, w) = \left(\frac{ut+\alpha vs}{\alpha}, \frac{us-\beta vt}{\beta}, \frac{ut-\alpha vs}{\sqrt{|\alpha\gamma|}}, \frac{us+\beta vt}{\sqrt{|\beta\delta|}}\right)$$

$$(u, v), (s, t) \in \mathbb{P}^1$$

Compute the equation of Q in the local frame of R and substitute the parameterisation into this equation

We get a homogeneous degree 2 equation in (s, t)

$$a(u, v)\, s^2 + b(u, v)\, st + c(u, v)\, t^2 = 0$$

where a, b, c are homogeneous degree 2 polynomial in (u, v)
Solve for $(s, t) \quad \rightsquigarrow \quad (s^{\pm}(u, v), t(u, v))$
where $(u, v) \in \mathbb{P}^1$ is such that $b^2 - 4ac \geqslant 0$

Substitute (s, t) by $(s^{\pm}(u, v), t(u, v))$ into the parameterisation. Send the parameterisation back to the initial frame

Step 1

In projective space,
find a quadric R in the pencil $R(\lambda) = P - \lambda Q$, $\lambda \in \mathbb{R}$,
such that $\det R(\lambda) \geqslant 0$, $\lambda \in \mathbb{Q}$

$$P \cap Q = R \cap Q$$

Theorem: If $\det R(\lambda) < 0$ for any $\lambda \in \mathbb{Q}$, then P and Q intersect in at most two points.

Step 2

Compute Gauss linear transformation that sends R into the form $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 = 0$

$$\begin{aligned} a\,x^2 + b\,y^2 + 2c\,xy &= a\,(x + \frac{c}{a}y)^2 + (b - \frac{c^2}{a})\,y^2 \\ &= a\,X^2 + (b - \frac{c^2}{a})\,Y^2 \end{aligned}$$

$X = x + \frac{c}{a}y$
$Y = y$

Gauss' transformation of quadratic forms
into diagonal form
has rational coefficients

Parametrisations of quadric in canonical form
$(\alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 = 0)$
yield parametrisations of all quadrics

Step 3: Parameterization of quadrics in real projective space

$\triangleleft$ Signature $(4,0)$. In $\mathbb{R}^3$: $\emptyset$

$\triangleleft$ Signature $(3,1)$. In $\mathbb{R}^3$: ellipsoid, hyperboloid of 2 sheets, elliptic paraboloid

$$\Leftrightarrow \det R < 0$$

$\triangleleft$ Signature $(3,0)$. In $\mathbb{R}^3$: point, $\emptyset$

$\triangleleft$ Signature $(2,2)$. In $\mathbb{R}^3$: hyperboloid of 1 sheet, hyperbolic paraboloid

Canonical equation $\alpha\, x^2 + \beta\, y^2 - \gamma\, z^2 - \delta\, w^2 = 0$

$$\left[\tfrac{ut+\alpha vs}{\alpha}, \tfrac{us-\beta vt}{\beta}, \tfrac{ut-\alpha vs}{\sqrt{\alpha\gamma}}, \tfrac{us+\beta vt}{\sqrt{\beta\delta}}\right],\ (u,v),(s,t) \in \mathbb{P}^1$$

Worst-case optimal parameterization

$\omega = \sqrt{\alpha\beta\gamma\delta}$ is an invariant of the quadric (up to the square of a rational number)

Finding a rational parameterization over $\mathbb{Q}(\sqrt{\alpha\beta\gamma\delta})$

$$\Updownarrow$$

Finding a rational point on the quadric

$\triangleleft$ Signature $(2,1)$. In $\mathbb{R}^3$: cone, cylinders

Canonical equation $\alpha\, x^2 + \beta\, y^2 - \gamma\, z^2 = 0$

$$[uv, \tfrac{u^2-\alpha\beta v^2}{2\beta}, \tfrac{u^2+\alpha\beta v^2}{2\sqrt{\beta\gamma}}, s],\ (u,v,s) \in \mathbb{P}^{\star 2}$$

Worst-case optimal parameterization

Finding a rational parameterization (over $\mathbb{Q}$)

$\Updownarrow$

Finding a rational point on the quadric

$\triangleleft$ Signature $(2,0)$. In $\mathbb{R}^3$: line, $\emptyset$

Canonical equation $\alpha\, x^2 + \beta\, y^2 = 0$

$$[0,0,u,v],\ (u,v) \in \mathbb{P}^1$$

Optimal parameterization

$\triangleleft$ Signature $(1,1)$.

In $\mathbb{R}^3$: planes (intersecting, parallel, simple)

Canonical equation $\alpha\, x^2 - \beta\, y^2 = 0$

$$[u, \tfrac{\sqrt{\alpha\beta}}{\beta}u, v, s], \quad [u, -\tfrac{\sqrt{\alpha\beta}}{\beta}u, v, s] \quad (u,v,s) \in \mathbb{P}^2$$

Optimal parameterization

$\triangleleft$ Signature $(1,0)$. In $\mathbb{R}^3$: double plane, $\emptyset$

Canonical equation $x^2 = 0$

$$[0,u,v,s],\ (u,v,s) \in \mathbb{P}^2$$

Optimal parameterization

Computing a quadric in the pencil and its parameterisation with coefficients in $\mathbb{Q}(\sqrt{\xi}),\ \xi \in \mathbb{Q}$

- **Step 1:** If signature$(R(\lambda)) = (2,2)$ then

◁ Find a point $M \in \mathbb{P}^3(\mathbb{R})$ on $R(\lambda)$
(e.g. $(0,0,-b+\sqrt{\delta},2a)$)

◁ Approximate M by a point $N \in \mathbb{P}^3(\mathbb{Q})$

◁ Compute $\lambda' \in \mathbb{Q}$ such that $N \in R(\lambda')$
Refine the approximation of M until $\det R(\lambda') > 0$

- **Step 2:** Let N' be such that N and N' are the 2 distinct points of intersection of $R(\lambda')$ and a line

◁ Compute the transformation sending N, N' onto $(1,\pm 1,0,0)$

◁ Apply Gauss' transformation
$\hookrightarrow R(\lambda')$ has equation $x^2 - y^2 + a\,z^2 + b\,w^2 = 0$

◁ Deduce the local frame where
$R(\lambda')$ has equation $x^2 + y^2 - z^2 - \xi\,w^2 = 0$

- **Step 3:** Parameterize in the local frame $R(\lambda')$ by

$$(x,y,z,w) = \left(ut+vs, us-vt, ut-vs, \tfrac{us+vt}{\sqrt{\xi}}\right)$$
$$(u,v),(s,t) \in \mathbb{P}^1$$

Result
If $\mathbf{R}(\lambda)$ has no double root

Parameterization

$$x_i(u) = \Upsilon_i(u) \pm \Phi_i(u)\sqrt{\Delta(u)}$$

$$u \in \mathbb{R} \text{ such that } \Delta(u) \geqslant 0$$

$\Upsilon_i(u)$, $\Phi_i(u)$, $\Delta(u)$: polynomials of degree 3, 1 and 4, with coefficients in $\mathbb{Q}(\sqrt{\xi})$ for some $\xi \in \mathbb{N}$

Worst-case optimal parameterization

There exists no rational parameterization

Finding a parameterization in $\mathbb{Q}[u, \sqrt{\Delta}]$, ($\Delta \in \mathbb{Q}[u]$)
$\Updownarrow$
Finding a rational point on the surface
$z^2 =$ degree 8 polynomial in (x, y)

If $\mathbf{R}(\lambda)$ admits double roots

Rational parameterization over a field $\mathbb{K}$

where the degree of the extension of $\mathbb{K}$ over $\mathbb{Q}$ is minimal in the worst-case, for each type of intersection (cubic+line, or 2 conics, or conic+2 lines, or 4 lines, or conic+double point)