A New Approach to Sparse Resultant Matrices for Bivariate Polynomials with Applications in Geometric Modeling

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Outline

1. Implicitization and Resultants
2. Multi–Sided Bezier Patches
3. Sparse Resultants and Multiplying Sets
4. Examples of Sparse Resultants for Bivariate Polynomials

Part I: Implicitization and Resultants

Implicitization

Parametric Equations (Rational Surface)

$$x = \frac{x(s,t)}{w(s,t)} \qquad y = \frac{y(s,t)}{w(s,t)} \qquad z = \frac{z(s,t)}{w(s,t)}$$

Implicit Equation (Algebraic Surface)

$$F(x,y,z) = 0$$

Theorem (Implicitization)

Every Rational Surface Lies on a Unique Irreducible Algebraic Surface.

Applications (Geometric Modeling)

Solid Modeling -- Inside/Outside Tests
Surface Modeling -- Intersection Algorithms

Sphere

Parametric Equations

$$x = \frac{2s}{1+s^2+t^2} \qquad y = \frac{2t}{1+s^2+t^2} \qquad z = \frac{1-s^2-t^2}{1+s^2+t^2}$$

$$x = \frac{2s(1-t^2)}{(1+s^2)(1+t^2)} \qquad y = \frac{2t(1+s^2)}{(1+s^2)(1+t^2)} \qquad z = \frac{(1-s^2)(1-t^2)}{(1+s^2)(1+t^2)}$$

Implicit Equation

$$x^2 + y^2 + z^2 - 1 = 0$$

Implicitization Methods

- Classical Resultants
- Grobner Bases
- Moving Surfaces (Syzygies, Correspondences)
- Sparse Resultants

Classical Resultants

Polynomials

Total Degree*n*

$$f(s,t) = \sum_{0 \le i+j \le n} a_{ij} s^i t^j$$

$$g(s,t) = \sum_{0 \le i+j \le n} b_{ij} s^i t^j$$

$$h(s,t) = \sum_{0 \le i+j \le n} c_{ij} s^i t^j$$

Bidegree*(m,n)*

$$f(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} s^i t^j$$

$$g(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{ij} s^i t^j$$

$$h(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} s^i t^j$$

Resultants

A resultant is an irreducible polynomial $Res(f,g,h)$ in the coefficients a_{ij}, b_{ij}, c_{ij} that vanishes if and only if the polynomials f,g,h have a common root.

Classical Resultants

Macaulay's Resultant (1916)

- Polynomials of Total Degree N ↔ Triangular Surface Patches
- Quotient of Two Determinants

$$\mathit{Size\ Numerator} = \big(3N(3N-1)/2\big) \times \big(3N(3N-1)/2\big)$$

$$\mathit{Size\ Denominator} = \mathit{Size\ Numerator} - 3N^2$$

- Entries are Coefficients of Original Polynomials

Dixon's Resultant (1908)

- Polynomials of Bidegree (M,N) ↔ Rectangular Tensor Product Patches
- Single Determinant of Size $6MN \times 6MN$ (also $2MN \times 2MN$)
- Entries are Coefficients of Original Polynomials

Dixon Example

Polynomials of Bidegree (1,1)

$$f(s,t) = a_{11}st + a_{10}s + a_{01}t + a_{00}$$

$$g(s,t) = b_{11}st + b_{10}s + b_{01}t + b_{00}$$

$$h(s,t) = c_{11}st + c_{10}s + c_{01}t + c_{00}$$

Dixon Resultant (6×6 *Matrix*)

$$Res(f,g,h) = Det\begin{pmatrix} a_{00} & 0 & b_{00} & 0 & c_{00} & 0 \\ a_{01} & 0 & b_{01} & 0 & c_{01} & 0 \\ a_{10} & a_{00} & b_{10} & b_{00} & c_{10} & c_{00} \\ a_{11} & a_{01} & b_{11} & b_{01} & c_{11} & c_{01} \\ 0 & a_{10} & 0 & b_{10} & 0 & c_{10} \\ 0 & a_{11} & 0 & b_{11} & 0 & c_{11} \end{pmatrix}$$

Alternative to Macaulay Resultant

Macaulay's Resultant

- Resultants are Quotients of Two Determinants

  $$\mathit{Size\ Numerator} = 3N(3N-1)/2 \times 3N(3N-1)/2$$

  $$\mathit{Size\ Denominator} = \mathit{Size\ Numerator} - 3N^2$$

- Entries are Original Polynomial Coefficients

Hybrid Resultant

- Resultants are Single Determinants of Size = $N(2N-1) \times N(2N-1)$
- Entries are Polynomials in the Original Polynomial Coefficients
- Cox, Little, O'Shea (1998), *Using Algebraic Geometry,* Springer

Example

Polynomials of Degree 2

$$f(s,t) = a_{20}s^2 + a_{11}st + a_{02}t^2 + a_{10}s + a_{01}t + a_{00}$$

$$g(s,t) = b_{20}s^2 + b_{11}st + b_{02}t^2 + b_{10}s + b_{01}t + b_{00}$$

$$h(s,t) = c_{20}s^2 + c_{11}st + c_{02}t^2 + c_{10}s + c_{01}t + c_{00}$$

Macaulay Resultant

$$Res(f,g,h) = \frac{Det(15\times 15)}{Det(3\times 3)}$$

Hybrid Resultant

$$Res(f,g,h) = Det(6\times 6)$$

Implicitization by Classical Resultants

Parametric Equations (Rational Surface)

$$x = \frac{x(s,t)}{w(s,t)} \qquad y = \frac{y(s,t)}{w(s,t)} \qquad z = \frac{z(s,t)}{w(s,t)}$$

Auxiliary Polynomials (Cross Multiply)

$$f(s,t) = x\,w(s,t) - x(s,t)$$

$$g(s,t) = y\,w(s,t) - y(s,t)$$

$$h(s,t) = z\,w(s,t) - z(s,t)$$

Implicit Equation (Resultant)

$$Res(f,g,h) = 0$$

Example of Implicitization

Auxiliary Polynomials (Tensor Product Surface Bidegree (1,1))

$$f(s,t) = (x d_{11} - a_{11})st + (x d_{10} - a_{10})s + (x d_{01} - a_{01})t + (x d_{00} - a_{00})$$

$$g(s,t) = (y d_{11} - b_{11})st + (y d_{10} - b_{10})s + (y d_{01} - b_{01})t + (y d_{00} - b_{00})$$

$$h(s,t) = (z d_{11} - c_{11})st + (z d_{10} - c_{10})s + (z d_{01} - c_{01})t + (z d_{00} - c_{00})$$

Implicit Equation (Dixon Resultant -- 6×6 *Matrix)*

$$Res(f,g,h) \equiv Det\begin{pmatrix} x d_{00} - a_{00} & 0 & \cdots & z d_{00} - c_{00} & 0 \\ x d_{01} - a_{01} & 0 & \cdots & z d_{01} - c_{01} & 0 \\ x d_{10} - a_{10} & x d_{00} - a_{00} & \cdots & z d_{10} - c_{10} & z d_{00} - c_{00} \\ x d_{11} - a_{11} & x d_{01} - a_{01} & \cdots & z d_{11} - c_{11} & z d_{01} - c_{01} \\ 0 & x d_{10} - a_{10} & \cdots & 0 & z d_{10} - c_{10} \\ 0 & x d_{11} - a_{11} & \cdots & 0 & z d_{11} - c_{11} \end{pmatrix} = 0$$

Classical Resultants and Base Points

Parametric Equations

$$x = \frac{x(s,t)}{w(s,t)} \qquad y = \frac{y(s,t)}{w(s,t)} \qquad z = \frac{z(s,t)}{w(s,t)}$$

Base Points

$$x(s_0,t_0) = y(s_0,t_0) = z(s_0,t_0) = w(s_0,t_0) = 0$$

Auxiliary Polynomials Vanish at Base Points

$$f(s_0,t_0) = x\,w(s_0,t_0) - x(s_0,t_0) = 0$$

$$g(s_0,t_0) = y\,w(s_0,t_0) - y(s_0,t_0) = 0$$

$$h(s_0,t_0) = z\,w(s_0,t_0) - z(s_0,t_0) = 0$$

Resultant Vanishes Identically in Presence of Base Points

$$Res(f,g,h) \equiv 0$$

Examples of Surfaces with Base Points

Triangular Surface -- Base Points at $(s,t)=(0,0)$

$$x(s,t)=a_{20}s^2+a_{11}st+a_{02}t^2+a_{10}s+a_{01}t$$

$$y(s,t)=b_{20}s^2+b_{11}st+b_{02}t^2+b_{10}s+b_{01}t$$

$$z(s,t)=c_{20}s^2+c_{11}st+c_{02}t^2+c_{10}s+c_{01}t$$

$$w(s,t)=d_{20}s^2+d_{11}st+d_{02}t^2+d_{10}s+d_{01}t$$

Tensor Product Surface -- Base Points at $(s,t)=(0,0)$

$$x(s,t)=a_{11}st+a_{10}s+a_{01}t$$

$$y(s,t)=b_{11}st+b_{10}s+b_{01}t$$

$$z(s,t)=c_{11}st+c_{10}s+c_{01}t$$

$$w(s,t)=d_{11}st+d_{10}s+d_{01}t$$

Dixon’s Resultant and Base Points

Auxiliary Polynomials -- Base Points at $(s,t)=(0,0)$

$$f(s,t)=(x\,d_{11}-a_{11})st+(x\,d_{10}-a_{10})s+(x\,d_{01}-a_{01})t$$

$$g(s,t)=(y\,d_{11}-b_{11})st+(y\,d_{10}-b_{10})s+(y\,d_{01}-b_{01})t$$

$$h(s,t)=(z\,d_{11}-c_{11})st+(z\,d_{10}-c_{10})s+(z\,d_{01}-c_{01})t$$

Dixon’s Resultant

$$Res(f,g,h)\equiv Det\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ x\,d_{01}-a_{01} & 0 & \cdots & z\,d_{01}-c_{01} & 0 \\ x\,d_{10}-a_{10} & 0 & \cdots & z\,d_{10}-c_{10} & 0 \\ x\,d_{11}-a_{11} & x\,d_{01}-a_{01} & \cdots & z\,d_{11}-c_{11} & z\,d_{01}-c_{01} \\ 0 & x\,d_{10}-a_{10} & \cdots & 0 & z\,d_{10}-c_{10} \\ 0 & x\,d_{11}-a_{11} & \cdots & 0 & z\,d_{11}-c_{11} \end{pmatrix}\equiv 0$$

Implicitization in the Presence of Base Points

Motivation

- Base Points Reduce the Implicit Degree of the Rational Surface
- Many Standard Models Contain Surfaces with Base Points
- Multi–Sided Bezier Patches Possess Base Points

Methods

- Perturbations (Canny, Manocha,…)
- Grobner Bases (Buchberger,…)
- Moving Surfaces (Sederberg, Chen,…)
- Sparse Resultants (Sturmfels, Zelevinsky,…)

Sparse Resultants

Sparse Polynomials

Polynomials with some coefficients set to zero.

Example

$$f(s,t) = a_{20}s^2 + a_{11}st + a_{02}t^2 + a_{10}s + a_{01}t$$

$$g(s,t) = b_{20}s^2 + b_{11}st + b_{02}t^2 + b_{10}s + b_{01}t$$

$$h(s,t) = c_{20}s^2 + c_{11}st + c_{02}t^2 + c_{10}s + c_{01}t$$

Sparse Resultant

A sparse resultant is an irreducible polynomial $Res(f,g,h)$ in the coefficients of f,g,h that vanishes if and only if the polynomials f,g,h have a common root other than the generic roots induced by the sparseness of the system.

Example

Polynomials

$$f(s,t) = a_{20}s^2 + a_{11}st + a_{02}t^2 + a_{10}s + a_{01}t$$

$$g(s,t) = b_{20}s^2 + b_{11}st + b_{02}t^2 + b_{10}s + b_{01}t$$

$$h(s,t) = c_{20}s^2 + c_{11}st + c_{02}t^2 + c_{10}s + c_{01}t$$

Sparse Resultant (9×9 *Matrix*)

$$Res(f,g,h) = Det\begin{pmatrix} a_{10} & 0 & 0 & b_{10} & 0 & 0 & c_{10} & 0 & 0 \\ a_{01} & a_{10} & 0 & b_{01} & b_{10} & 0 & c_{01} & c_{10} & 0 \\ a_{20} & 0 & 0 & b_{20} & 0 & 0 & c_{20} & 0 & 0 \\ a_{11} & a_{20} & 0 & b_{11} & b_{20} & 0 & c_{11} & c_{20} & 0 \\ a_{02} & a_{11} & a_{20} & b_{02} & b_{11} & b_{20} & c_{02} & c_{11} & c_{20} \\ 0 & a_{01} & a_{10} & 0 & b_{01} & b_{10} & 0 & c_{01} & c_{10} \\ 0 & a_{02} & a_{11} & 0 & b_{02} & b_{11} & 0 & c_{02} & c_{11} \\ 0 & 0 & a_{01} & 0 & 0 & b_{01} & 0 & 0 & c_{01} \\ 0 & 0 & a_{02} & 0 & 0 & b_{02} & 0 & 0 & c_{02} \end{pmatrix}$$

Implicitization by Sparse Resultants

Parametric Equations

$$x = \frac{x(s,t)}{w(s,t)} \qquad y = \frac{y(s,t)}{w(s,t)} \qquad z = \frac{z(s,t)}{w(s,t)}$$

Auxiliary Polynomials (Cross Multiply)

$$f(s,t) = x\,w(s,t) - x(s,t)$$

$$g(s,t) = y\,w(s,t) - y(s,t)$$

$$h(s,t) = z\,w(s,t) - z(s,t)$$

Implicit Equation (Sparse Resultant)

$$Res(f,g,h) = 0$$

Part II: Multi–Sided Bezier Patches

Bezier Patches

Surface Equations

- $R(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{ij}^{mn}(s,t)P_{ij} \qquad 0 \le s,t \le 1$ (Rectangular Patches)
- $T(s,t) = \sum_{0 \le i+j \le n} B_{ij}^{n}(s,t)P_{ij} \qquad 0 \le s+t \le 1$ (Triangular Patches)

Notation

- $P_{ij} = (x_{ij}, y_{ij}, z_{ij})$ = Control Points
- $B_{ij}^{mn}(s,t), B_{ij}^{n}(s,t)$ = Bivariate Bernstein Basis Functions

Properties and Algorithms for Standard Bezier Patches

Properties

1. Polynomial (Bernstein Basis)
2. Affine Invariant
3. Convex Hull
4. Boundaries are Bezier Curves Determined by Boundary Control Points
5. Non-Degenerate

Algorithms

1. Recursive Evaluation
2. Differentiation
3. Subdivision
4. Blossoming
5. Degree Elevation

Properties and Algorithms for Multi–Sided Bezier Patches

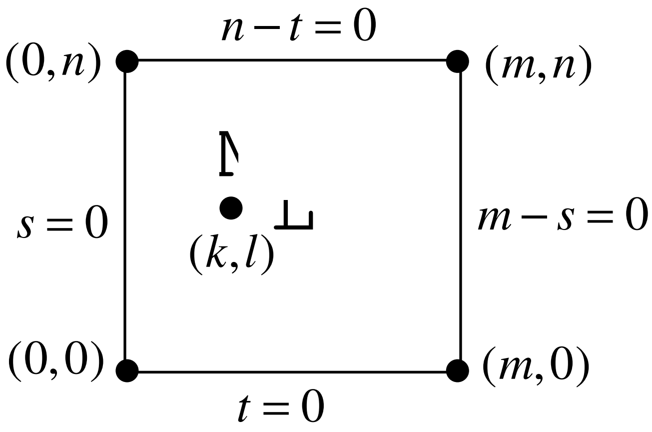
Properties

1. Rational (Bernstein Like Basis)
2. Affine Invariant
3. Convex Hull
4. Boundaries are Bezier Curves Determined by Boundary Control Points
5. Non-Degenerate

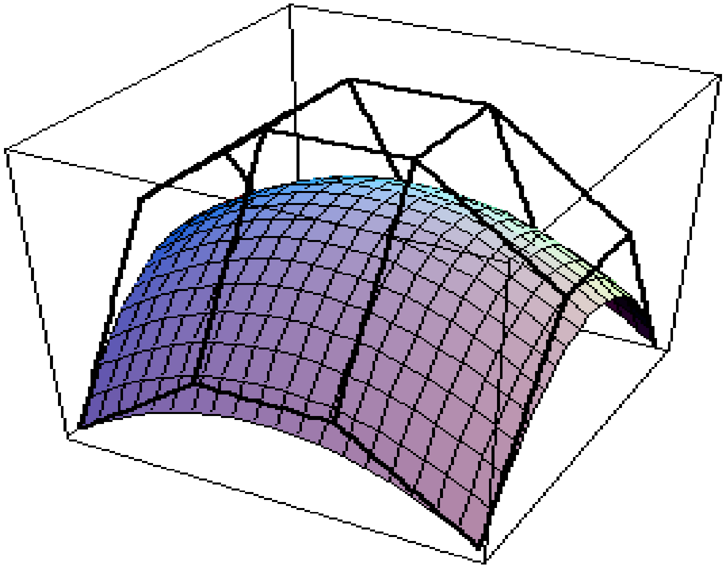
Algorithms

1. Recursive Evaluation
2. Differentiation
3. Subdivision
4. Blossoming
5. Depth Elevation

Rectangular Lattices and Tensor Product Patches



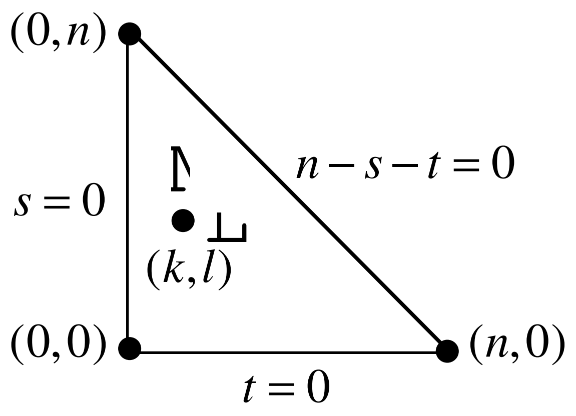
Newton Polygon



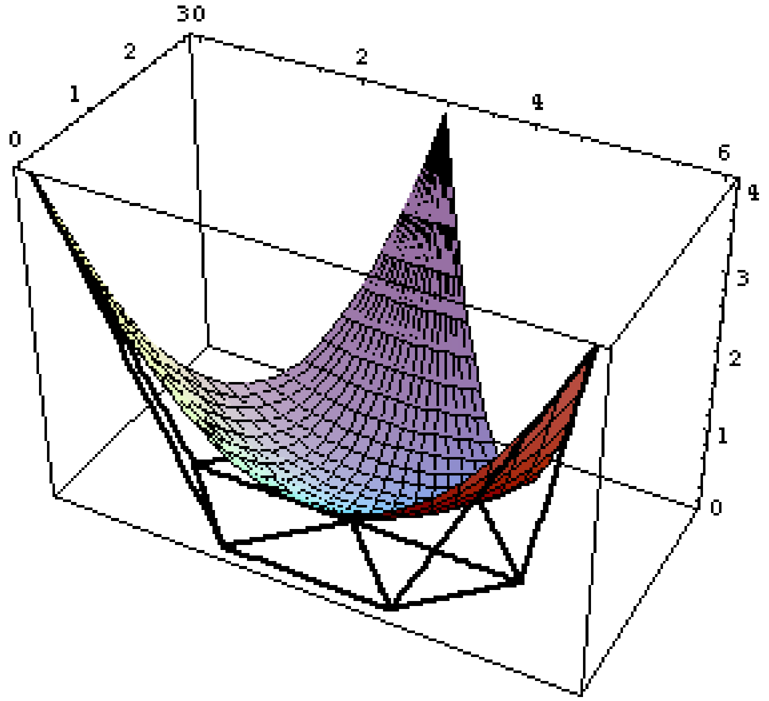
Four Sided Surface Patch

$$\mathit{Implicit}\;\; \mathit{Degree} = 2 \times \mathit{Area}(\mathit{NewtonPolygon}) = 2mn$$

Triangular Lattices and Three Sided Patches



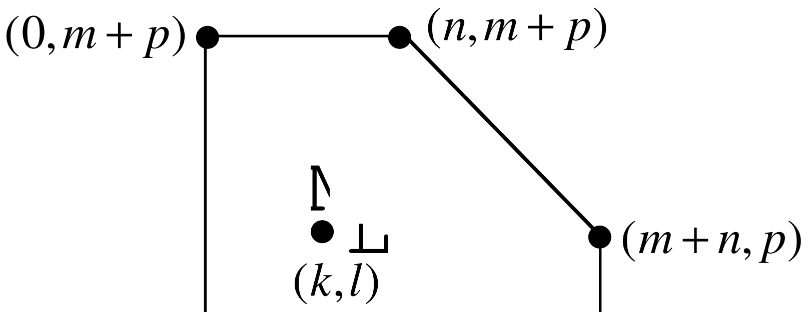
Newton Polygon



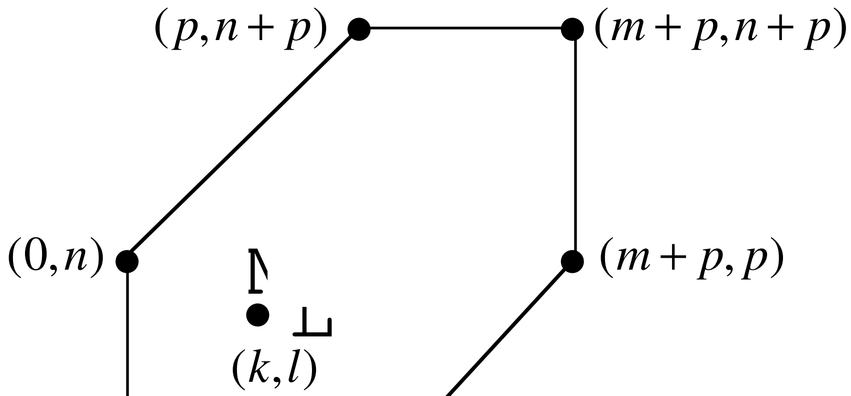
Three Sided Surface Patch

$$Implicit \;\; Degree = 2 \times Area(NewtonPolygon) = n^2$$

Pentagonal and Hexagonal Lattices



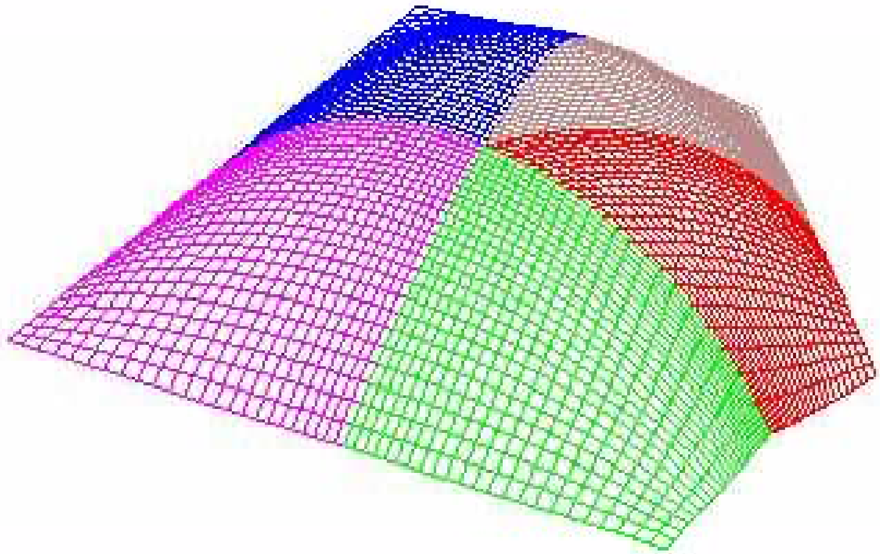
Lattice Pentagon



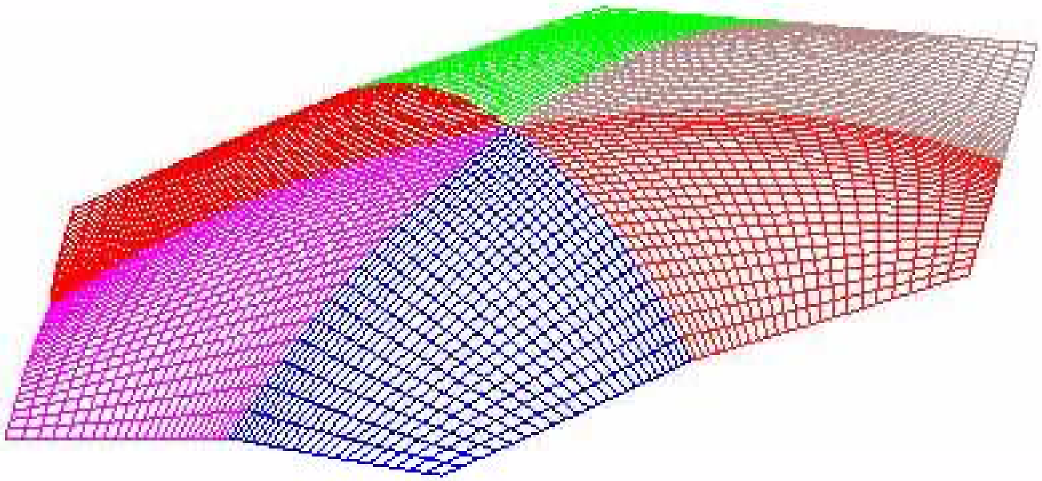
Lattice Hexagon

$$Implicit\ \ Degree = 2 \times Area(NewtonPolygon)$$

Pentagonal Bezier Patch



Hexagonal Bezier Patch



Part III: Sparse Resultants and Multiplying Sets

Sylvester's Resultant

Polynomials

$$f(t) = a_n t^n + \cdots + a_1 t + a_0$$

$$g(t) = b_n t^n + \cdots + b_1 t + b_0$$

Multiplying Set

$$1, t, \ldots, t^{n-1} \Rightarrow f(t), tf(t), \ldots, t^{n-1} f(t), g(t), tg(t), \ldots, t^{n-1} g(t)$$

Sylvester Resultant

$$Syl(f,g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ \vdots & a_0 & \cdots & \vdots & \vdots & b_0 & \cdots & \vdots \\ a_n & \vdots & \cdots & \vdots & b_n & \vdots & \cdots & \vdots \\ 0 & a_n & \cdots & \vdots & 0 & b_n & \cdots & \vdots \\ \vdots & 0 & \cdots & a_0 & \vdots & 0 & \cdots & b_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_n \end{pmatrix}$$

Dixon's Resultant

Polynomials

$$f(s,t) = a_{mn}s^m t^n + \cdots + a_{10}s + a_{01}t + a_{00}$$

$$g(s,t) = b_{mn}s^m t^n + \cdots + b_{10}s + b_{01}t + b_{00}$$

$$h(s,t) = c_{mn}s^m t^n + \cdots + c_{10}s + c_{01}t + c_{00}$$

Multiplying Set

$$R = \left\{1, s, t, \ldots, s^{2m-1}t^{n-1}\right\}$$

Dixon Resultant $(m = n = 1 \Rightarrow R = \{1, s\})$

$$Dix(f,g,h) = \begin{pmatrix} a_{00} & 0 & b_{00} & 0 & c_{00} & 0 \\ a_{01} & 0 & b_{01} & 0 & c_{01} & 0 \\ a_{10} & a_{00} & b_{10} & b_{00} & c_{10} & c_{00} \\ a_{11} & a_{01} & b_{11} & b_{01} & c_{11} & c_{01} \\ 0 & a_{10} & 0 & b_{10} & 0 & c_{10} \\ 0 & a_{11} & 0 & b_{11} & 0 & c_{11} \end{pmatrix}$$

Sparse Resultants

Polynomials

$$f(s,t)=\sum_{i,j} a_{ij}s^i t^j \qquad g(s,t)=\sum_{i,j} b_{ij}s^i t^j \qquad h(s,t)=\sum_{i,j} c_{ij}s^i t^j$$

$$A=\left\{(i,j)\,|\,s^i t^j \text{ has a non-zero coefficient in } f,g,h\right\}$$

Multiplying Set

$$R=\left\{(k,l)\,|\,s^k t^l \text{ is a collection of multiplying monomials}\right\}$$

Sparse Resultant

$$Res(f,g,h)=Coeffs\left(s^k t^l f \quad s^k t^l g \quad s^k t^l h \;\;|\,(k,l)\in R\right)$$

- $\#\,Columns = 3(\#R)$
- $\#\,Rows = \#(A\oplus R) = \#\left\{s^{i+k}t^{j+l}\,|\,(i,j)\in A \text{ and } (k,l)\in R\right\}$
- *Common Root* $\Rightarrow$ *Dependent Rows*

The Size of the Multiplying Set

Theorem

Let

$$f(s,t) = \sum_{i,j} a_{ij} s^i t^j \qquad g(s,t) = \sum_{i,j} b_{ij} s^i t^j \qquad h(s,t) = \sum_{i,j} c_{ij} s^i t^j$$

$$A = \left\{ (i,j) \mid s^i t^j \text{ has a non-zero coefficient in } f, g, h \right\}$$

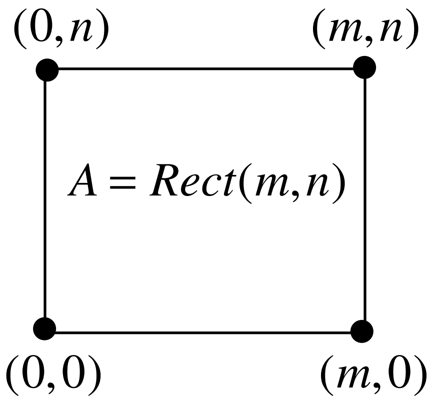
Then

$$deg_{a,b,c} \left| Res(f,g,h) \right| = 2Area(ConvexHull(A))$$

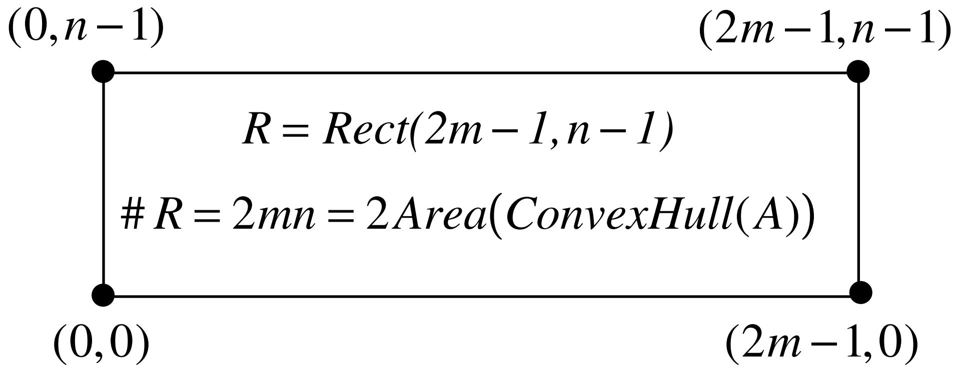
Zube's Constraint Equations

- $\# R = 2Area(ConvexHull(A))$ (Columns)
- $\#(A \oplus R) = 3(\# R) = 6Area(ConvexHull(A))$ (Rows)

Dixon’s Resultant

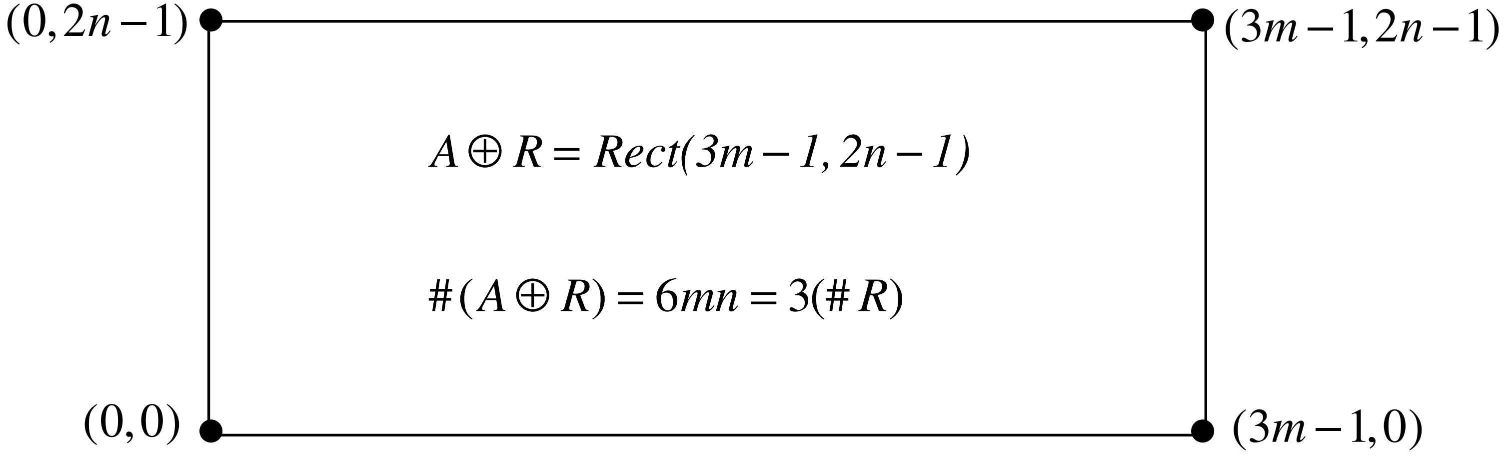


$$A = Rect(m,n)$$



$$R = Rect(2m-1, n-1)$$

$$\# R = 2mn = 2Area(ConvexHull(A))$$



$$A \oplus R = Rect(3m-1, 2n-1)$$

$$\#(A \oplus R) = 6mn = 3(\# R)$$

The Shape of the Multiplying Set

Strategy

1. Decompose $A = \left\{ (i,j) \mid s^i t^j \text{ has a non-zero coefficient in } f, g, h \right\}$

   into a Minkowski sum of simpler shapes -- typically lines and triangles.

2. Seek a multiplying set $R = \left\{ (k,l) \mid s^k t^l \text{ is a collection of monomials} \right\}$

   that is a Minkowski sum of these same simple shapes.

Multiplying Sets

Decomposition

$$A = n_1 I_1 \oplus \cdots \oplus n_p I_p \equiv I(n_1, \ldots, n_p)$$

$$R = x_1 I_1 \oplus \cdots \oplus x_p I_p = I(x_1, \ldots, x_p)$$

$$A \oplus R = (x_1 + n_1) I_1 \oplus \cdots \oplus (x_p + n_p) I_p = I(x_1 + n_1, \ldots, x_p + n_p)$$

Zube's Constraint Equations

$$\# R = 2 Area\left(ConvexHull(A)\right)$$

$$\#(A \oplus R) = 3(\# R) = 6 Area\left(ConvexHull(A)\right)$$

Diophantine Constraint Equations

$$\# I(x_1, \ldots, x_p) = 2 Area\left(ConvexHull(I(n_1, \ldots, n_p))\right)$$

$$\# I(x_1 + n_1, \ldots, x_p + n_p) = 6 Area\left(ConvexHull(I(n_1, \ldots, n_p))\right)$$

Solving and Testing

Solve Diophantine Constraint Equations

$$\underbrace{\#I(x_1,\ldots,x_p)}_{quadratic} = 2\underbrace{Area\left(ConvexHull(I(n_1,\ldots,n_p))\right)}_{constant}$$

$$\underbrace{\#I(x_1+n_1,\ldots,x_p+n_p)-\#I(x_1,\ldots,x_p)}_{linear} = 4\underbrace{Area\left(ConvexHull(I(n_1,\ldots,n_p))\right)}_{constant}$$

Test Determinants and Discard Degenerate Solutions

$$\left|Res(f,g,h)\right| \equiv 0$$

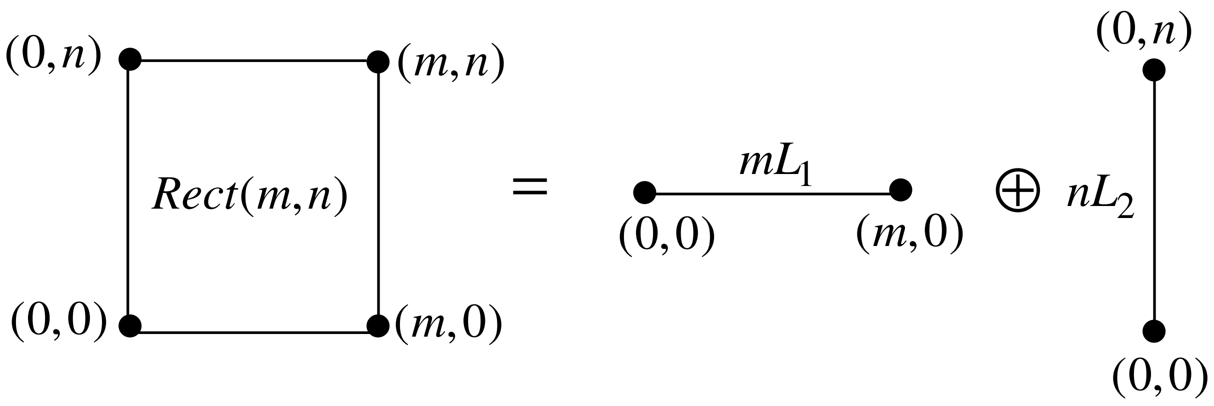
Example: $p = 2$

- Generate 2 Equations -- 1 Linear, 1 Quadratic -- in 2 Unknowns
- Eliminate 1 Variable -- Reduce to 1 Quadratic Equation in 1 Unknown
- Find 2 Closed Form Solutions (Possibly Not Integer, Possibly Not Real)
- Test Only Integer Solutions

Part IV: Examples of Sparse Resultants for Bivariate Polynomials

Rectangles

Decomposition



Constraint Equations

$(x+1)(y+1) = 2mn$

$n(x+1) + m(y+1) = 3mn$

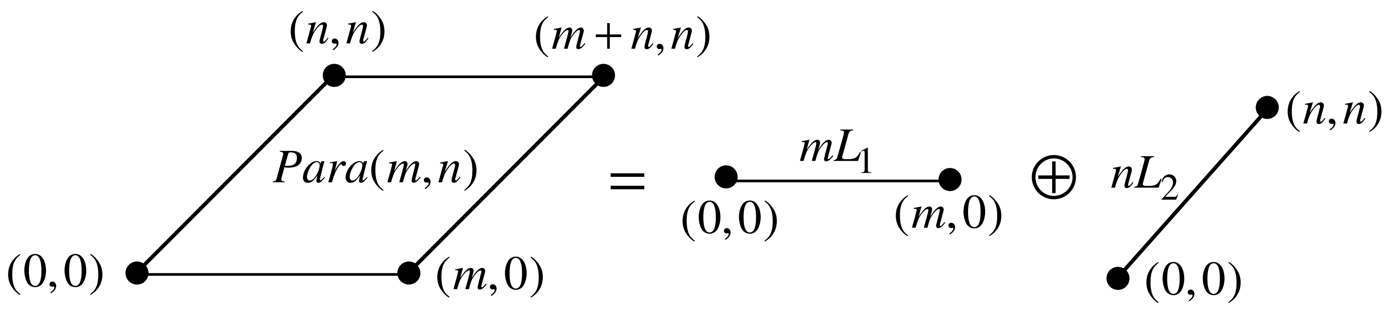
Solutions (Dixon's Resultant)

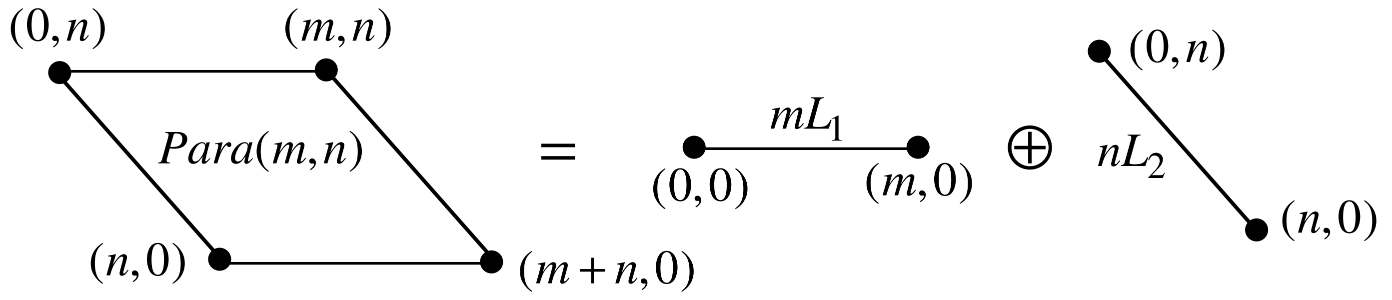
$(x,y) = (m-1, 2n-1)$

$(x,y) = (2m-1, n-1)$

Isomorphic Parallelograms

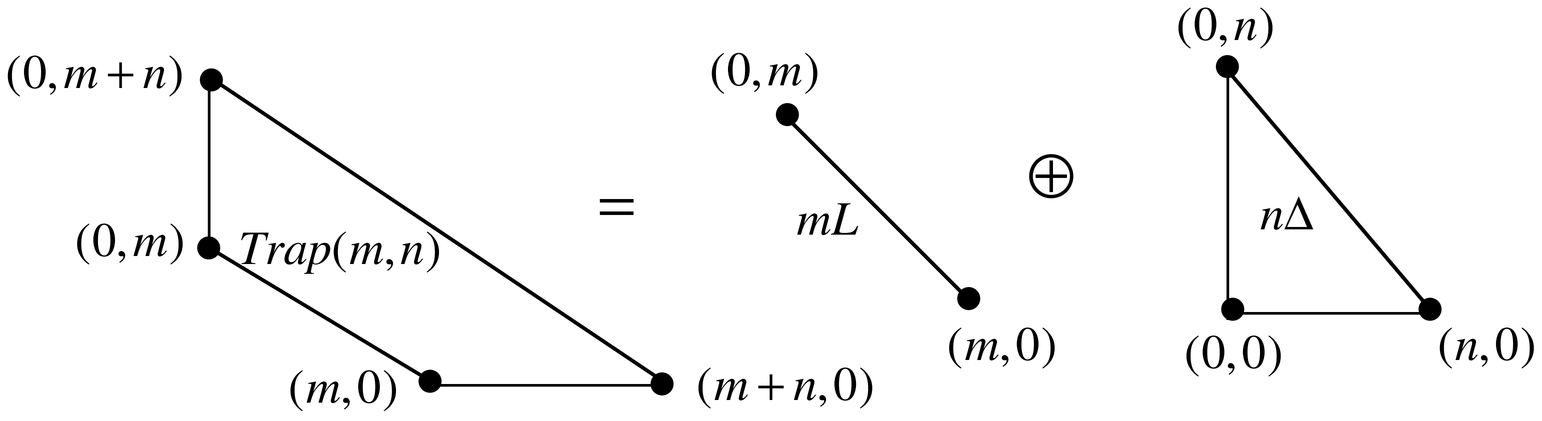
Decomposition





Trapezoids

Decomposition



Constraint Equations	*Solutions*
$(y+1)(2x+y+2)=4mn+2n^2$	$(x,y)=\left(\frac{4m+n-1}{2},n-1\right)$
$n(x+1)+2(m+n)(y+1)=6mn+3n^2$	$(x,y)=\left(\frac{2m-n-1}{2},2n-1\right)$

Valid Solutions

Solutions

$$(x,y)=\left(\frac{4m+n-1}{2},n-1\right)\Rightarrow n\ \ odd$$

$$(x,y)=\left(\frac{2m-n-1}{2},2n-1\right)\Rightarrow n\ \ odd,\ \ 2m\geq n+1$$

Case: $n=1$

- Polynomials of Total Degree $m+1$
- Zero of Multiplicity m at the Origin
- Both Solutions Valid

Case: $m=0$

- Polynomials of Total Degree n
- Neither Solution Valid -- $|Res(f,g,h)|\equiv 0$

Example

Sparse Polynomials ($m = n = 1 \Rightarrow$ *total degree* $= 2$, *constant term* $= 0$)

$$f(s,t) = a_{20}s^2 + a_{11}st + a_{02}t^2 + a_{10}s + a_{01}t$$

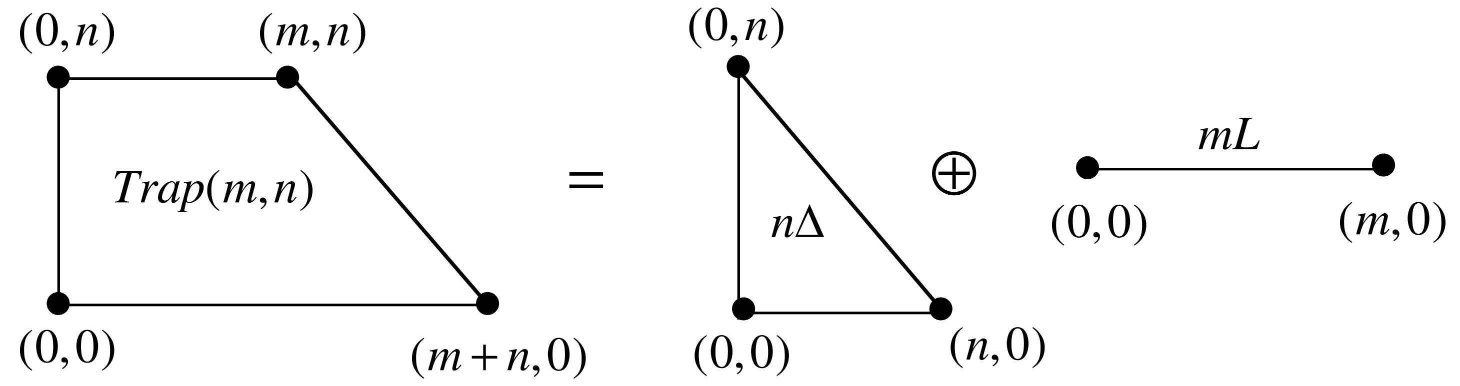
$$g(s,t) = b_{20}s^2 + b_{11}st + b_{02}t^2 + b_{10}s + b_{01}t$$

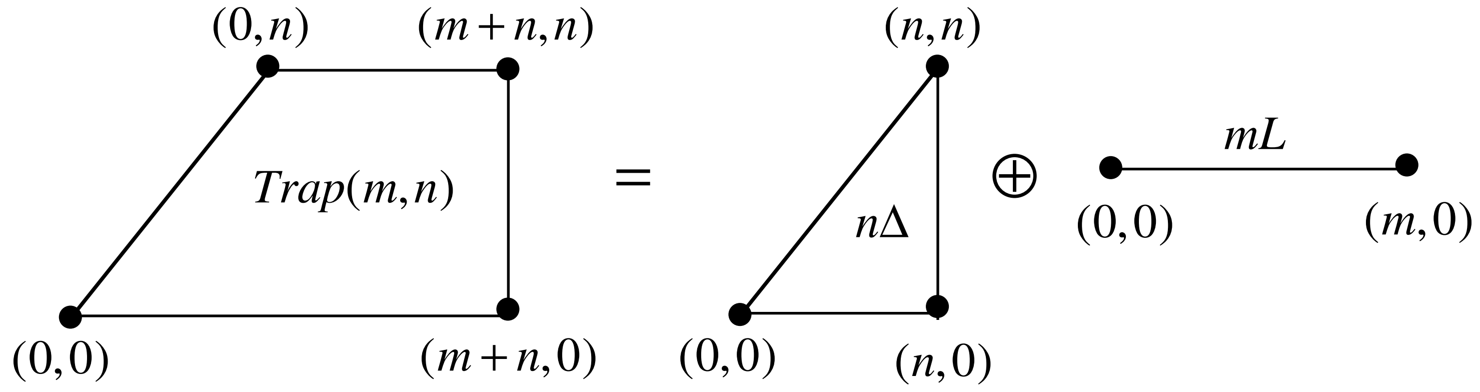
$$h(s,t) = c_{20}s^2 + c_{11}st + c_{02}t^2 + c_{10}s + c_{01}t$$

Sparse Resultant ($(x,y) = (2,0) \Rightarrow R = \{s^2, st, t^2\}$)

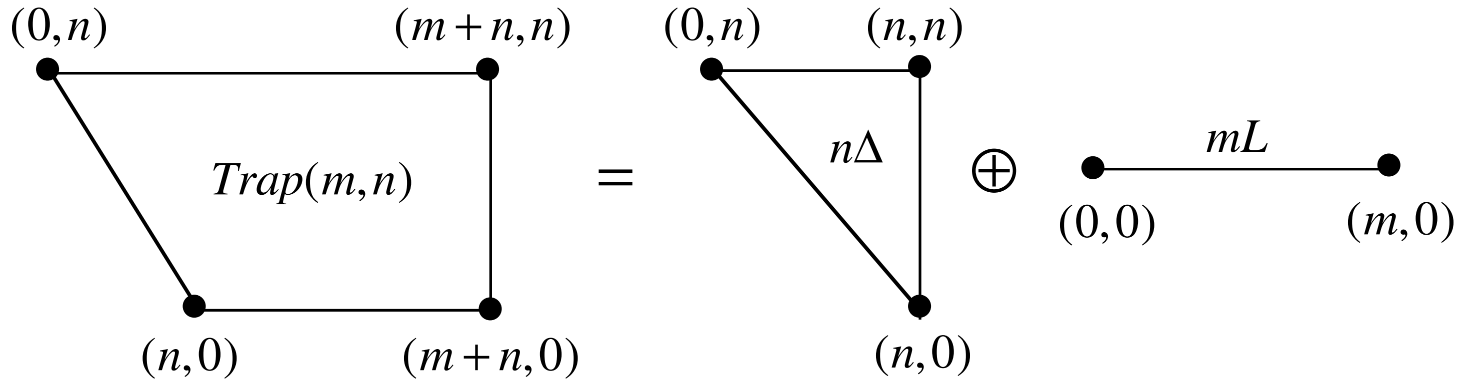
$$Res(f,g,h) = Det\begin{pmatrix} a_{10} & 0 & 0 & b_{10} & 0 & 0 & c_{10} & 0 & 0 \\ a_{01} & a_{10} & 0 & b_{01} & b_{10} & 0 & c_{01} & c_{10} & 0 \\ a_{20} & 0 & 0 & b_{20} & 0 & 0 & c_{20} & 0 & 0 \\ a_{11} & a_{20} & 0 & b_{11} & b_{20} & 0 & c_{11} & c_{20} & 0 \\ a_{02} & a_{11} & a_{20} & b_{02} & b_{11} & b_{20} & c_{02} & c_{11} & c_{20} \\ 0 & a_{01} & a_{10} & 0 & b_{01} & b_{10} & 0 & c_{01} & c_{10} \\ 0 & a_{02} & a_{11} & 0 & b_{02} & b_{11} & 0 & c_{02} & c_{11} \\ 0 & 0 & a_{01} & 0 & 0 & b_{01} & 0 & 0 & c_{01} \\ 0 & 0 & a_{02} & 0 & 0 & b_{02} & 0 & 0 & c_{02} \end{pmatrix}$$

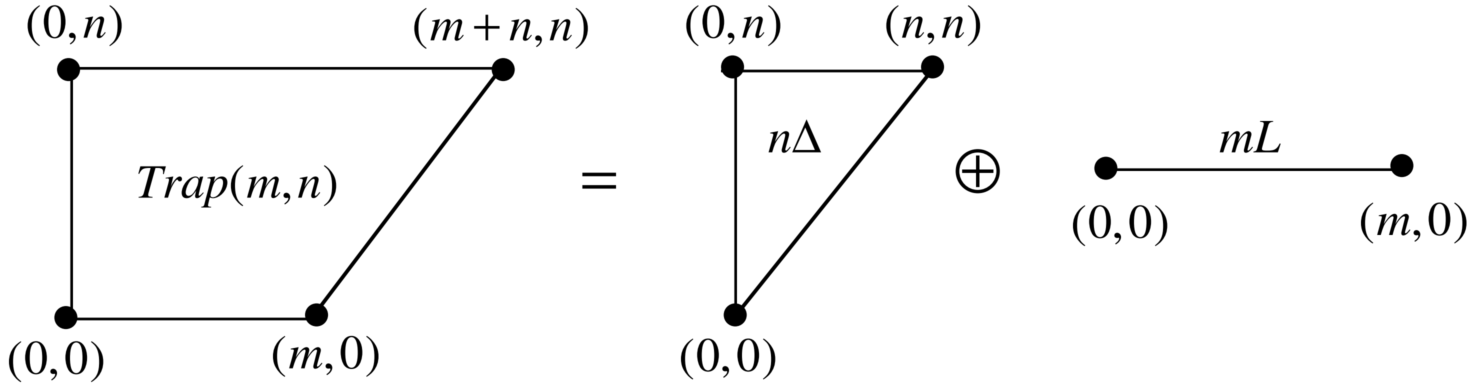
Isomorphic Trapezoids





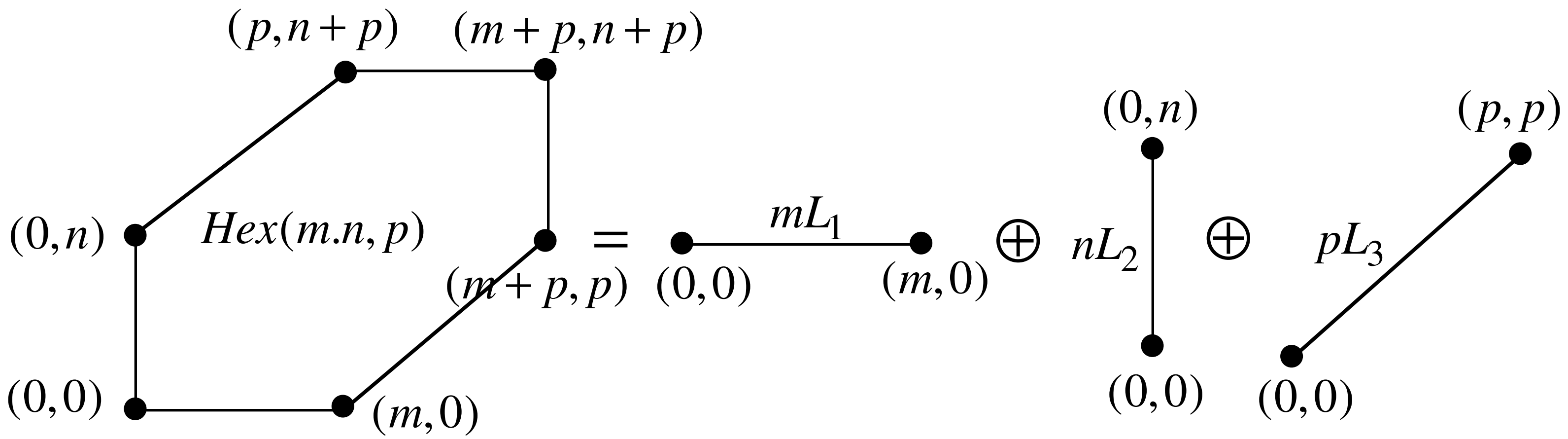
More Isomorphic Trapezoids





Hexagons

Decomposition



Constraint Equations

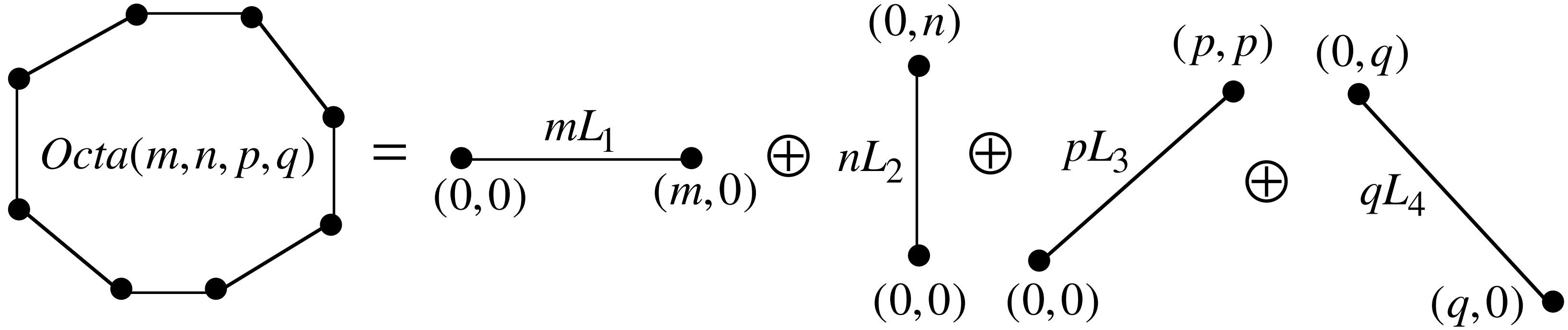
- $xy+yz+xz+x+y+z+1=2(mn+mp+np)$
- $(n+p)x+(m+p)y+(m+n)z=3(mn+mp+np)-(m+n+p)$

Solutions

- $(m,n,p)=(1,n,p)\Rightarrow(x,y,z)=(1,n-1,2p),\ (1,2n,p-1),\ldots$

Octagons

Decomposition



Constraint Equations

$$(w+y+z+1)(x+y+z+1)-y(y+1)-z(z+1)=2(m+p+q)(n+p+q)-2p^2-2q^2$$

$$(w+y+z+m+p+q+1)(x+y+z+n+p+q+1)-(y+q)(y+q+1)-(z+q)(z+q+1)$$

$$=6(m+p+q)(n+p+q)-6p^2-6q^2$$

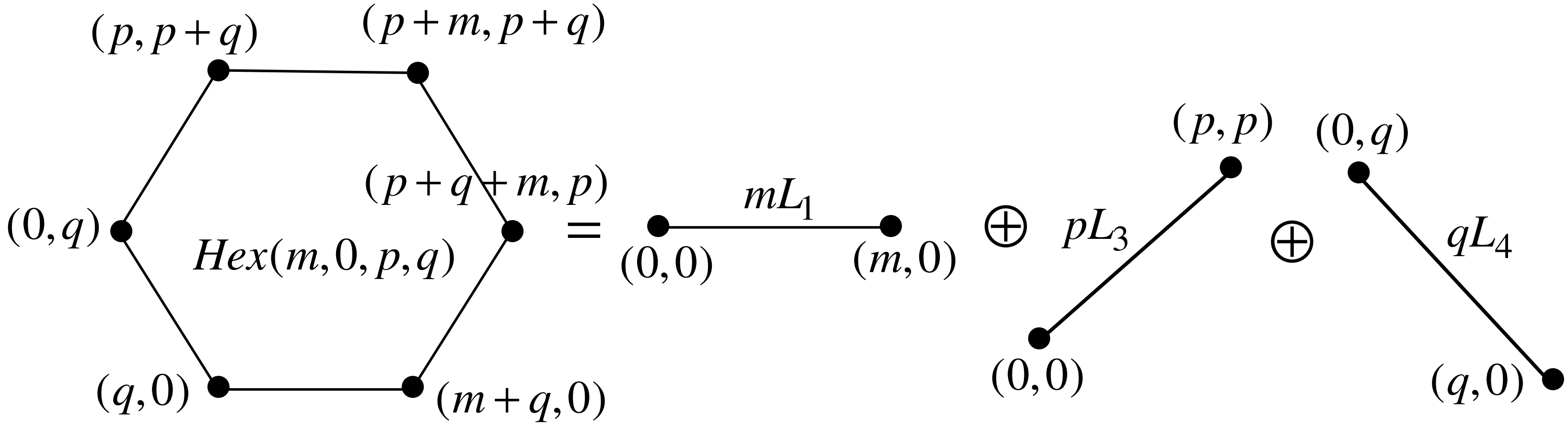
Solutions

$(m,n,p,q)=(1,1,1,1)$

$(w,x,y,z)=(1,2,2,0),(1,2,0,2),(2,1,2,0),(2,1,0,2),(0,3,1,1),(3,0,1,1)$

Hexagons

Decomposition



Solutions

$\underline{(m,n,p,q)=(1,0,2,1)}$

$(x,y,z,w)=(3,0,1,1),\ (0,0,4,1)$ -- Yes

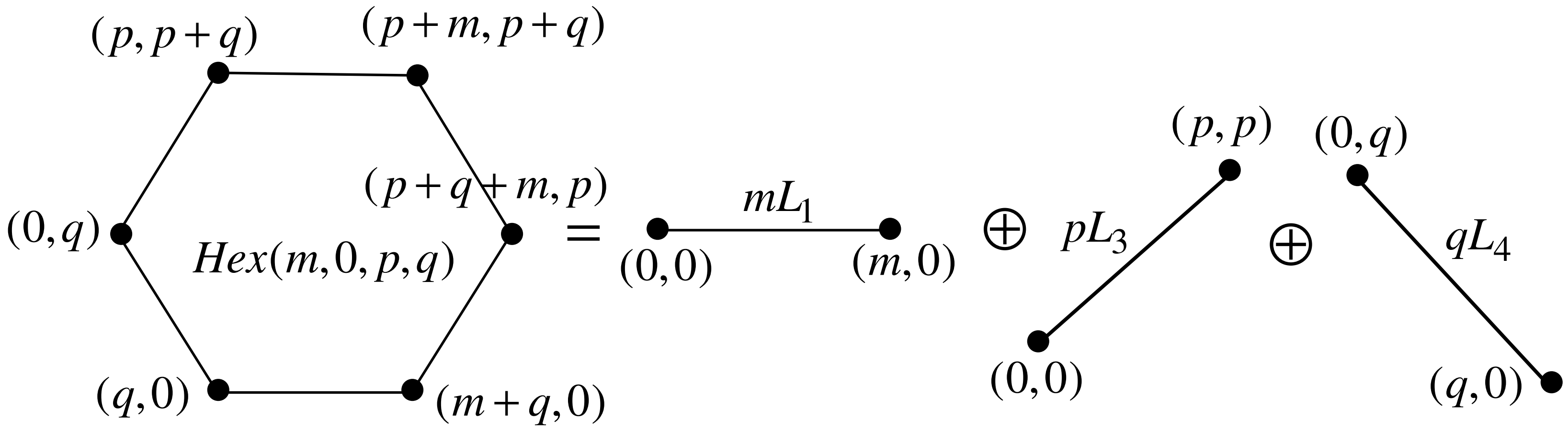
$(x,y,z,w)=(1,2,2,0),\ (2,2,1,0)$ -- No

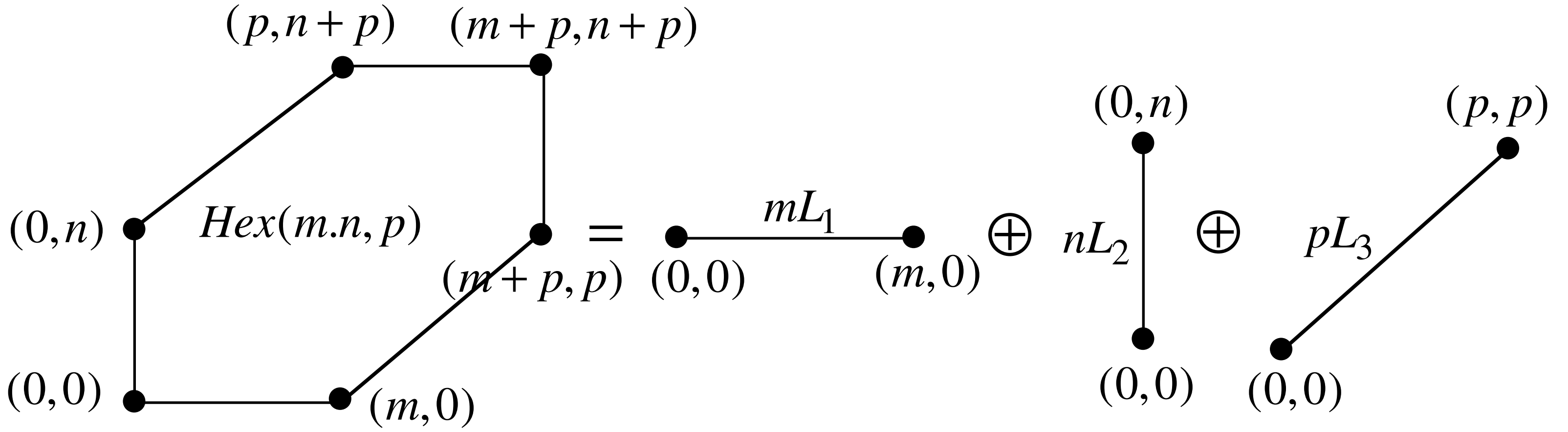
$\underline{(m,n,p,q)=(2,0,2,2)}$

$(x,y,z,w)=(3,0,4,1),\ (3,0,1,4)$ -- Yes

$(x,y,z,w)=(0,0,4,3),\ (0,0,3,4),$
$(0,1,3,3),\ (3,3,1,1),$
$(6,0,2,1),\ (6,0,1,2),$
$(6,6,1,1)$ – *No*

Non Isomorphic Hexagons



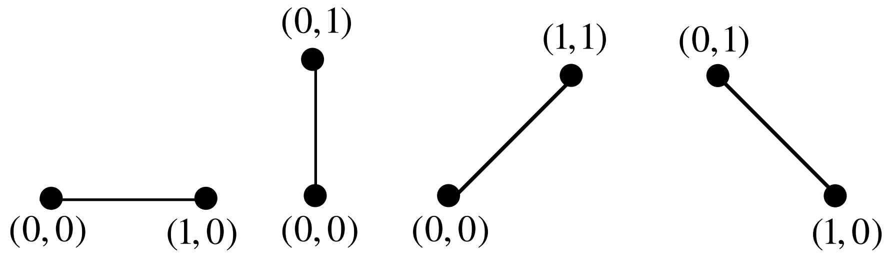


Future Work

- Generate a Catalogue of Sparse Resultant Determinants for Many Different Collections of Bivariate Systems of Sparse Polynomials
    - Pentagons, Hexagons, Heptagons, Octagons
- Develop the Theory Surrounding the Method
    - Solve Diophantine Equations
    - Characterize when the Method Succeeds
    - Identify Isomorphic Cases
- Apply Sparse Resultants to the Analysis of Multi–Sided Bezier Patches
    - Implicitization, Inversion, Intersection
- Compare Sparse Resultant Methods to Other Techniques
    - Dixon–Cayley Methods
    - Method of Moving Surfaces
- Investigate Numerical Robustness
- Extend the Method to the Multivariate Setting

Primitives for Minkowski Sums

Lines



Triangles